Let \((X_t)\) be a Markov process, not assumed to be time homogeneous. It is well known that \(X_t = (t, X_t)\) is a time homogeneous Markov process. Let \(A\) be its generator. The Feynman-Kac's formula for \(X_t\) takes the following form if the equation

\[
(1.1) \quad A\psi + \psi = 0
\]

admits a solution \(\psi\), then \(\psi\) has the representation, for \(s < t\):

\[
(1.2) \quad \psi(s, X_s) = E \left[ \psi(t, X_t) \exp \left( \int_s^t c(u, X_u) \, du \right) q(X_s) \right].
\]

We prove this under general conditions on \((X_t)\).
ON THE FEYNMAN-KAC'S FORMULA AND ITS APPLICATIONS TO FILTERING THEORY

by

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1. Introduction: Let \((X_t)\) be a Markov process, not assumed to be time homogeneous. It is well known that \(\mathcal{X}_t = (t, X_t)\) is a time homogeneous Markov process. Let \(A\) be its generator. The Feynman-Kac's formula for \(X_t\) takes the following form if the equation

\[ Av + cv = 0 \tag{1.1} \]

admits a solution \(v\), then \(v\) has the representation, for \(s < t\)

\[ v(s, X_s) = E \left[ v(t, X_t) \exp \left( \int_s^t c(u, X_u) du \right) | \sigma(X_s) \right] . \tag{1.2} \]

We prove this under general conditions on \((X_t)\).

Then we come to the question of existence of solution to (1.1). We show that under some regularity conditions on \((X_t)\), (1.1) has a solution for a rich class of boundary conditions. This implies that the 'dual' equation to (1.1) admits a unique solution. The 'dual' equation is an equation for measures on the state space of \((X_t)\) and its unique solution is the distribution of \(X_t\) under an absolutely continuous change of the underlying probability measure by a multiplicative functional.

These results on the measure valued equations significantly extend results given in [3] on the conditional distributions for the nonlinear filtering problem (in the white noise approach).
2. Let \((S, \mathcal{S})\) be a measurable space. Let \((X_t)\) be an \((S, \mathcal{S})\) valued Markov process on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with transition probability function \(P\), i.e.

\[
\{w : X_t(w) \in B\} \in \mathcal{F}
\]

and

\[
E_x \left[ 1_B(X_t) | F_s \right] = P(s, x, t-s, B) \quad \text{a.s.} \quad w
\]

for all \(0 \leq s < t < \infty, B \in \mathcal{S}\). Here, the function \(P(s, x, t, B)\) on \(\{0 \leq s < t < \infty, x \in S, B \in \mathcal{S}\}\) is assumed to satisfy the following conditions.

(2.2) For \(s > 0, t > 0, x \in S; P(s, x, t, \cdot)\) is a countably additive probability measure on \((S, \mathcal{S})\).

(2.3) For \(s > 0, x \in S, B \in \mathcal{S}; P(s, x, 0, B) = 1_B(x)\).

(2.4) For \(t > 0, B \in \mathcal{S}; (s, x) \mapsto P(s, x, t, B)\) is a \(B([0, \infty)) \otimes \mathcal{S}\) measurable function \((\mathbb{B}(E)\) denotes the Borel \(\sigma\)-field of a topological space \(E\) and \(\otimes\) denotes the product of \(\sigma\)-fields).

(2.5) For \(s > 0, u \geq 0, t \geq 0, x \in S, B \in \mathcal{S}\) we have

\[
\int_S P(s+t, x, u, B) P(s, x, t, dz) = P(s, x, t+u, B)
\]

Throughout, \(F_t^X\) denotes the smallest \(\sigma\)-field with respect to which the family \(\{X_u : 0 \leq u \leq t\}\) is measurable. We also assume that

(2.6) the process \((X_t)\) is \(F_t^X\) - progressively measurable, i.e. for all \(t_0 < \infty\), the mapping \((t, w) \mapsto X_t(w)\) from \([0, t_0] \times \Omega \to S\) is \(\mathbb{B}(\Omega) \otimes \mathbb{B}(X_t)\) measurable.
Let $\hat{S} = \mathbb{R} \times S$, $\hat{S} = B(\mathbb{R}) \otimes S$ and $\mathcal{Y}$ be the class of bounded real valued measurable functions on $\hat{S}$.

**Definition:** A sequence $\{ f_k \} \subseteq \mathcal{Y}$ is said to converge weakly to $f \in \mathcal{Y}$, written as $w-lim f_k = f$, if $f_k(x)$ is uniformly bounded and for each $x \in S$, $f_k(x)$ converges to $f(x)$.

For $f \in \mathcal{Y}$, $t \geq 0$, let $T_t f : S \rightarrow X$ be defined by

$$
(T_t f)(s,x) = \int f(s+t,z) P(s,x,t,dz), \quad (s,x) \in \hat{S}.
$$

Using the properties of $P$, it can be checked that $T_t f \in \mathcal{Y}$ and that for $u \geq 0$, $t \geq 0$,

$$
T_u \left[ T_t f \right] = T_{t+u} f, \quad f \in \mathcal{Y}.
$$

Thus $\{ T_t : t \geq 0 \}$ is a semigroup of operators (from $\mathcal{Y}$ into itself).
Remark: It is well known and easy to check that $\hat{X}_t = (t, X_t)$ is a Markov process with stationary transition probability function $\hat{P}$ given by

$$\hat{P}(t, (s,x), B) = P(s,x,t,B(s+t)), \quad B \in \hat{S}$$

where $B^u$ denotes the $u$-section of $B \subseteq S$. The semigroup $(T_t: t \geq 0)$ defined above is the usual semigroup associated with the transition function $P$ (as in [2], section 2.1).

We will now recall the definition and some properties of the weak generator $A$ of $(T_t: t \geq 0)$. Let $\mathcal{D}_0$ be given by

$$\mathcal{D}_0 = \{ f \in \hat{V} : \text{w-lim}_{t \to 0} T_t f = f \}$$

Definition: Let $\mathcal{D}_A$ be the class of $f \in \hat{V}$ for which the

$$(2.9) \quad \text{w-lim}_{t \to 0} \frac{T_t f - f}{t} = g$$

exists and belongs to $\mathcal{D}_0$ and for $f \in \mathcal{D}_A$, define $Af = g$, where $g$ is given by (2.9).

The following properties are easy to prove. We will only state them here. For a proof see chapter 1 in [2].

$$(2.10) \quad T_t(\mathcal{D}_A) \subseteq \mathcal{D}_A \quad \text{and for} \quad f \in \mathcal{D}_A, \quad A(T_t f) = T_t Af$$

$$(2.11) \quad \text{For} \quad f \in \mathcal{D}_0, \quad t + (T_t f)(s,x) \text{ is a right continuous function for all} \quad (s,x) \in \hat{S},$$

$$(2.12) \quad \text{For} \quad f \in \mathcal{D}_A, \quad \text{we have, for all} \quad (s,x) \in \hat{S}, \quad t \geq 0 \quad (T_t f)(s,x) = f(s,x) + \int_0^t (T_u Af)(s,x) du$$
(2.13) Given \( f \in L_1 \) there exists a sequence \( \{ f_k \} \subseteq \mathcal{D}_A \) such that

\[
\lim_{k \to \infty} f_k = f.
\]

In (2.13) above, \( f_k \) can be taken to be

\[
f_k(s,x) = \int_0^\infty k \, e^{-kt} (T_t f)(s,x) \, dx.
\]

The property (2.12) has the following important consequence.

**Proposition 1:** For \( f \in \mathcal{D}_A \), \( M_t \) given by

(2.14) \[ M_t(w) = f(t, X_t(w)) - \int_0^t (Af)(u, X_u(w)) \, du \]

is a martingale with respect to the \( \sigma \)-fields \( \mathcal{F}_t^X \).

**Proof:** The progressive measurability of \( (X_t) \) implies the \( \mathcal{F}_t^X \)-measurability of \( M_t \). Since \( f, Af \in \mathcal{D}_A \), they are bounded and hence \( M_t \) is itself bounded for each \( t \). Now (2.1) implies

(2.15) \[ E_\omega \left[ f(t, X_t) \mathcal{F}_s^X \right] = \int f(t, z) \, P(s, X_s, t-s, dz) \]

for \( s \leq t \). Similarly for \( s \leq u \), we have

(2.16) \[ E_\omega \left[ (Af)(u, X_u) \mathcal{F}_s^X \right] = (T_{u-s} Af)(s, X_s). \]

Using (2.11), (2.12), (2.15) and (2.16), it can be checked that

\[ E_\omega \left[ M_t - M_s \mathcal{F}_s^X \right] = 0. \]
We now turn our attention to the Feynman-Kac's formula. Our next result is a step in this direction.

Let \( g: [0,t_o] \times S \rightarrow \mathbb{R} \) be a \( \mathcal{B}( [0,t_o] ) \times \mathbb{R} \) measurable function such that

\[
(2.17) \quad E_w \left[ \int_0^t |g(u, X_u)| du \right] < \infty
\]

and for a positive integrable function \( a: [0, t_o] \rightarrow \mathbb{R} \),

\[
(2.18) \quad g(u, x) < a(u) \quad \text{for all } x \in S, u \in [0, t_o].
\]

Fix \( 0 < s < t \) and let

\[
(2.19) \quad B_t(w) = \exp \left( \int_s^t g(u, X_u(w)) du \right).
\]

Then we have

**Theorem 2:** Let \( f \in \mathcal{D}_A \) and \( g \) satisfy (2.17), (2.18). Then

\[
(2.20) \quad Z_t = f(t, X_t) \cdot B_t - \int_s^t \left[ (Af)(u, X_u) + g(u, X_u) \right] \cdot B_u du
\]

is an \( \mathcal{F}_t^Y \) martingale for \( t > s \) (where \( B \) is given by (2.19)).

**Proof:** It is easy to see that \( Z_t \) is \( \mathcal{F}_t^Y \) measurable. The condition (2.18) implies that \( B_t \) is bounded. Since \( f, Af \) are also bounded the condition (2.17) gives the integrability of \( Z_t \). To prove the martingale property, suffices to prove that for \( s < r < t, C \in \mathcal{F}_r^Y \),

\[
(2.20) \quad E_w \left[ (Z_t - Z_r) \cdot 1_C \right] = 0.
\]

Let \( f_t(t, w) = f(t, X_t(w)) - \int_t^{t_o} (Af)(u, X_u(w)) du \). Then by Proposition 1, it follows that for \( 0 \leq t \leq t_o \)
(2.21) \[ E \left[ f_1(t,.)|\mathcal{F}^X_t \right] = f(t,X_t) \]

and hence

(2.22) \[ E \left[ l_c \cdot (Z_t - Z_r) \right] = E \left[ 1_c \cdot (f_1(t,.)B_t - f_1(r,.)B_r) - \int_r^t ((Af+gf)(u,X_u)du) \right]. \]

Now for each \( \omega \), \( f_1(t,\omega), B_t(\omega) \) are absolutely continuous functions and hence

\[
\begin{align*}
  f_1(t,\omega)B_t(\omega) - f_1(r,\omega)B_r(\omega) &= \int_r^t \frac{d}{du} \left[ f_1(u,\omega)B_u(\omega) \right] du \\
  &= \int_r^t \left[ f_1(u,\omega)g(u,X_u(\omega))B_u(\omega) \right. \\
  &\quad + (Af)(u,X_u(\omega))B_u(\omega) \left. \right] du.
\end{align*}
\]

Thus

(2.23) \[ E \left[ f_1(t,.)B_t - f_1(r,.)B_r \right] = \left[ E \left[ 1_c \cdot f_1(u,.)g(u,X_u)B_u \right] du \\
  + E \left[ 1_c \cdot \int_r^t (Af)(u,.)B_u du \right] \\
  = E \left[ 1_c \cdot \int_r^t (Af+gf)(u,X_u)du \right]. \]

using (2.21) once again. Now (2.22) and (2.23) give the required equality

\[ E \left[ l_c (Z_t - Z_r) \right] = 0. \]

Remark: It can be verified that

\[ Z_t = M_t B_t - \int_s^t M_u dB_u \]

where \( M \) is given by (2.14). Hence if \( M \) were right continuous, it would follow from the "integration by parts formula for martingale"
(See [5]) that $(Z_t, \frac{X}{t})$ is a martingale. However, in general $M_t$ need not be right continuous and hence we have given a direct proof.

The following is the Feynman-Kac's formula for a time inhomogeneous Markov process.

**Theorem 3**: Let $0 < t_c < \infty$ be fixed. Let\( c : [0, t_c] \times S \to \mathbb{R} \)
and \( g : S \to \mathbb{R} \) be bounded measurable functions. Suppose that \( v \in \mathcal{D}_A \)
is a solution to

\[
(2.24) \quad [Av + cv](u, x) 1_{\{u < t_c\}} = 0
\]

and

\[
(2.25) \quad v(t_c, x) = g_c(x).
\]

Then \( v \) admits a representation, for \( s < t_c \)

\[
(2.26) \quad v(s, X_s) = E_s \left[ g_c(X_{t_c}) \exp \left( \int_s^{t_c} c(u, X_u) du \right) \right] \text{ a.e. } \pi.
\]

**Proof**: Fix \( s < t_c \). Take \( f = v \) and \( g = c \) in Theorem 2 to obtain that \( (Z_t, \frac{X}{t})_{s \leq t \leq t_c} \) is a martingale, where

\[
(2.27) \quad Z_t = v(t, X_t) \exp \left( \int_s^t v(u, X_u) du \right).
\]

Here we have used the fact that \( v \) satisfies (2.24) so that the second term appearing in the expression for \( Z_t \) is zero. Thus

\[
E_s \left[ Z_{t_c} \mid \frac{X}{t} \right] = Z_s \quad \text{ a.s. } \pi.
\]

This is same as (2.26) since \( v(t_c, x) = g_c(x) \).
1. In this section we consider the question as to under what conditions \( t, x \), \( (X_t) \) does the problem (2.24), (2.25) admit a solution. Of course, if the solution exists, it has to satisfy (2.26) and this gives a clue as to what conditions one should put on \( c_g(x_t) \).

Suppose that \( S \) is a topological space, \( \mathcal{B} \) is its Borel \( \sigma \)-field. Let \( \mathcal{X} \) be the space of all right continuous mappings \( X \) from \([0,\infty)\) into \( S \). We will denote by \( X_t \) the value of \( X \) at \( t \). Let \( S = \sigma(X_t : s < u < t) \). We assume that

\[
\text{(1.1)} \quad \text{for all } \omega, \ X_t(\omega) \in \mathcal{X}
\]

and that for all \((s, x) \in S\), there exists a probability measure \( P_{s, x} \) on \((\mathcal{X}, \mathcal{B})\) such that for \( 0 \leq t_s < t_1 < \cdots < t_k, y \in S, A_1, A_2, \ldots, A_k \in \mathcal{B}; k \geq 1 \), we have

\[
\text{(2.2)} \quad P_{t_s, y} (X_{t_1} \in A_1 \cap \cdots \cap X_{t_k} \in A_k) = \prod_{i=1}^{k} P_t (y_i | t_{i-1} \leq y_i \leq t_i, y_{i-1} \in A_1, \ldots, y_{i-1} \in A_{i-1} \cap \cdots \cap y_k \in A_k).
\]

Remark: The main thrust of this assumption is that \( P_{s, x} \) is realized on \( \mathcal{X} \). The relation (2.1) and (3.2) imply that for \( \{t_i\}, \{A_i\} \) as in (3.2), we have

\[
\text{(3.3)} \quad P_\pi \left( \prod_{i=1}^{k} P_t (X_{t_i} | t_{i-1} \leq X_{t_i} \leq t_i, y_{i-1} \in A_1, \ldots, y_{i-1} \in A_{i-1} \cap \cdots \cap y_k \in A_k) \right).
\]

and hence by standard arguments, we have for \( B \in \mathcal{B}_\pi \)

\[
\text{(3.4)} \quad \pi(X \in B | t_t) = \int X \in B \quad \text{a.s. } \pi.
\]

Similarly, it can be proved that for \( s < t, B \in \mathcal{B}_\pi, x \in S \),

\[
\text{(3.5)} \quad P_{s, x}(B | t_t) = P_{t_s, x}(B) \quad \text{a.s. } P_{s, x}.
\]
We are now in a position to prove a 'converse' to the Feynman-Kac's formula.

**Theorem 4:** Let $0 < t_0 < T$ be fixed. Let $c : \left[0, t_0\right] \times S \rightarrow \mathbb{R}$ be a bounded continuous function. Let $f \in \mathbb{D}_A$. Let $v : S \rightarrow \mathbb{R}$ be defined by

$$v(s, x) = \mathbb{E}_{P_{s, x}} \left[ f(t_0, X_t) \exp(\int_s^{t_0} c(u, X_u) \, du) \right], \quad s < t_0$$

$$= f(s, x), \quad s > t_0.$$

Then $v \in \mathbb{D}_A$ and $\Delta v = f_1$ where

$$f_1(s, x) = -c(s, x)v(s, x), \quad s < t_0$$

$$= (\Delta f)(s, x), \quad s > t_0.$$

**Proof** Since $v(s, x) = f(s, x)$ for $s > t_0$, we have

$$(T_t v)(s, x) = (T_t f)(s, x)$$

for $s > t_0$, $x \in S$, $t > 0$. Hence for $s > t_0$, $x \in S$,

$$\lim_{t \downarrow 0} \frac{(T_t v)(s, x) - v(s, x)}{t} = (\Delta f)(s, x) = f_1(s, x).$$

For $x \in X$, $s < t_0$ let us define

$$C_s(X) = \exp(\int_s^{t_0} c(u, X_u) \, du).$$

Then for $s < t_0$, we have

$$v(s, x) = \mathbb{E}_{P_{s, x}} \left[ f(t_0, X_t) C_s(X) \right].$$
For \( s < t_0, \ s + t < t_0, \) we thus have

\[
(T_t \psi)(s, x) = \int \psi(s + t, z) P(s, x, t, ds)
\]

\[
= E_{s, x} [\psi(s + t, X_{s+t})]
\]

\[
= E_{s, x} \left[ E_{s+t, X_{s+t}} \left( f(t_o, X_{t_o}) C_{s+t}(X) \right) \right]
\]

\[
= E_{s, x} \left[ f(t_o, X_{t_o}) C_{s+t}(X) \right].
\]

by (3.5). Hence, for \( s < t_0, \ x \in S, \ s + t < t_0, \) we have

\[
\frac{(T_t \psi)(s, x) - \psi(s, x)}{t} = E_{s, x} \left[ f(t_o, X_{t_o}) \cdot \frac{C_{s+t}(X) - C_s(X)}{t} \right].
\]

For all \( x \in X, \) we have from (3.9)

\[
\lim_{t \to 0} \frac{C_{s+t}(X) - C_s(X)}{t} = -c(s, x) \cdot C_s(X).
\]

Further

\[
\frac{C_{s+t}(X) - C_s(X)}{t} = | -c(s, x) \cdot C_s(X)|
\]

\[
\leq K
\]

where \( K \) depends only on \( t_0 \) and the upper bound of \( |c| \). The dominated convergence theorem gives that for \( s < t_0, \)

\[
\lim_{t \to 0} \frac{(T_t \psi)(s, x) - \psi(s, x)}{t} = E_{s, x} \left[ f(t_o, X_{t_o}) \cdot [-c(s, x) C_s(X)] \right]
\]

\[
= -c(s, x) \psi(s, x)
\]

\[
= f_1(s, x)
\]
as $P_{s,x}(X = x) = 1$. Also, (3.13) implies that the left hand expression in (3.11) is uniformly bounded (in $s,x,t$). Thus we have

$$W-lim_{t \to 0} \frac{(T_t v)(s,x) - v(s,x)}{t} = f_1(s,x).$$

Remains to prove that $f_1 \in \mathcal{J}_0$. This will prove that $v \in \mathcal{J}_A$ and that $Av = f_1$. If $s \geq t_0$, $f_1(s,x) = (\Delta f)(s,x)$ and hence for $s \geq t_0$,

$$t \geq 0, \quad (T_t f_1)(s,x) = (T_t \Delta f)(s,x).$$

Since $\Delta f \in \mathcal{J}_0$, this gives

$$\lim_{t \to 0} (T_t f_1)(s,x) = f_1(s,x) \text{ for } s \geq t_0, \ x \in S.$$  

For $s < t_0$, we have

$$T_t f_1(s,x) - f_1(s,x) = -E_{P_{s,x}} \left[ c(s+t, X_{\infty+t})v(s+t, X_{\infty+t}) - c(s,x)v(s,x) \right]$$

$$= -E_{P_{s,x}} \left[ \left( v(s+t, X_{\infty+t}) - v(s,x) \right) \left( c(s+t, X_{\infty+t}) - c(s,x) \right) \right]$$

$$- c(s,x) E_{P_{s,x}} \left[ v(s+t, X_{\infty+t}) - v(s,x) \right].$$

Now as $t \to 0$, $c(s+t, X_{\infty+t}) = c(s,x)$ a.e. as $y_{s,x}^+$ as $c$ is continuous and $X_{\infty}$ is right continuous. Hence by the dominated convergence theorem,

$$\lim_{t \to 0} E_{P_{s,x}} \left[ v(s+t, X_{\infty+t}) - c(s,x) \right] = 0.$$  

The relation (3.14) implies that

$$-c(s,x) E_{P_{s,x}} \left[ v(s+t, X_{\infty+t}) - v(s,x) \right] = -c(s,x) \left[ (T_t v)(s,x) - v(s,x) \right]$$

$$+ 0$$

as $t \to 0$. These observations give

12:
\[(3.17) \quad \lim_{t \to 0} (T_t f_1)(s,x) = f_1(s,x) \quad \text{for} \quad s < t_0, x \in S. \]

Now (3.16), (3.17) and the fact that \( T_t f_1 \) is uniformly bounded yield

\[ w-\lim_{t \to 0} (T_t f_1) = f_1. \]

**Remark:** Under the conditions assumed in this section and Theorem 4, the equations (2.24), (2.25) for \( g_0(x) = f(t_0,x) \) have a unique solution \( v \) on \([0,t_0] \times S\) which is given by (3.6). To see this, let \( v' \) be any solution. Apply Theorem 3 to the process \( (X_t : t \geq s) \) on the probability space \((\mathcal{F}, \mathcal{B}_t, P_s)\) to obtain, for \( s < t_0,\)

\[ v'(s,X_s) = \mathbb{E}_{P_{s,x}} \left[ f(t_0,X_{t_0}) \mathbb{C}_{S}(X) | \mathcal{B}_S \right] \quad \text{a.s.} \quad P_{s,x}. \]

Since under \( P_{s,x} \) any set in \( \mathcal{B}_S \) has measure zero or one, the conditional expectation appearing above is the unconditional expectation and thus equals \( v(s,x) \). Also \( X_s = x \) a.s. \( P_{s,x} \). Hence we have

\[ v'(s,X_s) = v(s,x) \quad \text{a.s.} \quad P_{s,x}. \]

Those observation imply

\[ v'(s,x) = v(s,x). \]

4. We now consider an equation dual to (2.24), namely

\[ \frac{d}{dt} K_t = A^* K_t + \xi(t,s) K_t \]

where \( \{K_t \} \subseteq \mathcal{M}(S) \) - the class of finite signed measures on \((S, \mathcal{S})\). The equation (4.1) is purely formal and is to be interpreted as
\begin{align}
\langle f(t, \cdot), K_t \rangle &= \langle f(0, \cdot), K_0 \rangle + \int_0^t \langle Af(u, \cdot), K_u \rangle du + \int_0^t \langle g(u, \cdot) f(u, \cdot), K_u \rangle du \\
\text{for } f \in D. \quad \text{Here, } \langle \theta, \mu \rangle \text{ denotes } \int \theta \, d\mu \text{ for } \mu \in \mathcal{M}(S) \text{ and a function } \\
\theta: S \rightarrow \mathbb{R}. \quad \text{Thus } \langle f(t, \cdot), \mu \rangle = \int f(t, x) \, d\mu(x) \text{ for } f \in \mathcal{D}. \quad \text{We will show that this equation with boundary condition} \\
\langle f(t, \cdot), \mu \rangle &= \int f(t, x) \, d\mu(x) \text{ for } f \in \mathcal{D}. \quad \text{We will show that this equation with boundary condition} \\
(4.3) & \quad K_0 = \Pi \circ X_C^{-1} \\
\text{admits a unique solution which is given by} \\
(4.4) & \quad K_t(B) = \mathbb{E}_{\mu} \left[ 1_B(X_t) \exp \left( \int_0^t g(u, X_u) \, du \right) \right], \quad B \in \mathcal{B} \\
\text{The uniqueness will be proved in the class of } \{K_t\} \text{ satisfying} \\
(4.5) & \quad \{K_t\} \subseteq \mathcal{M}(S), \quad t + K_t(B) \text{ is a Borel measurable function} \\
& \quad \text{for all } B \in \mathcal{B} \text{ and } K_t \ll \Pi \circ X_C^{-1} \text{ with} \\
& \quad \left| \frac{dK_t}{d\mu X_C^{-1}} \right| \leq M \\
\text{for all } t, \text{ for a fixed constant } M. \\
\text{We continue to assume that the conditions imposed on } (X_t) \text{ in} \\
\text{Section 3, are valid. We further assume that } S \text{ is a complete separable} \\
\text{metric space. We begin with a Lemma.} \\
\text{Lemma 5: Let } 0 < t < \infty \text{ be fixed. Let } \mu \in \mathcal{M}(S) \text{ be such that} \\
(4.6) & \quad \langle f(t, \cdot), \mu \rangle = 0 \quad \forall f \in D. \\
\text{Then } \mu \equiv 0. 
\end{align}
Proof : Let $E$ be the class of $f \in J$ for which (4.6) holds. Easy to see that if $f_k \in F$, w-lim $f_k = f$, then $f \in F$. Hence by (2.13), $J_0 \subseteq \hat{F}$.

For $f \in C_b(S)$, (i.e. $f : S \to \mathbb{R}$ is bounded continuous), we have

$$(T_t f)(s,x) = E_{s,t} f(s+t, X_{s+t}) \to f(s,x) \text{ as } t \to 0,$$

since $X$ is right continuous. Thus $C_b(S) \subseteq J \subseteq \hat{F}$.

Given $f_0 \in C_b(S)$, taking $f(s,x) = f_0(x)$, we have $f \in C_b(S) \subseteq J$ and hence

$$(4.7) \quad \langle f_0, u \rangle = 0.$$  

The validity of (4.7) for all $f_0 \in C_b(S)$ implies $u = 0$ because $S$ - the Borel $\sigma$ field - is also the smallest $\sigma$ field with respect to which $C_b(S)$ is measurable.

We are now in a position to prove the assertions made at the beginning of this section. This result may be considered as a dual Feynman-Kac's formula.

Theorem 6 : Suppose that $\phi$ satisfies (2.17) and (2.18). Then the equation (4.2) with boundary condition (4.3) admits a unique solution in the class of $\{K_t\}$ satisfying (4.5). The unique solution is given by (4.4).

Proof : First we will prove that $\{K_t\}$ defined by (4.4) satisfies (4.2). Let $\{K_t\}$ be defined by (4.4). Easy to see that (4.3) and (4.5) are satisfied.
Taking \( s = 0 \) in Theorem 2, it follows from the martingale property of \( Z_t \) that \( E_t Z_t = E Z_0 \). Here, \( Z_t \) is given by (2.20) where in turn \( B_t \) is given by (2.19), with \( s = 0 \). Noting that with these notations,

\[ <\theta, K_t> = E_t \theta(X_t) B_t \]

we conclude from the relation \( E_t Z_t = E Z_0 \) that

\[ <f(t,\cdot), K_t> - \int_0^t <(Af+Qf)(u,\cdot), K_u>du = <f(0,\cdot), K_0> \]

Hence \( \{K_t\} \) satisfies (4.2).

To prove the uniqueness part, we will prove the following. Suppose \( \{K_t\} \) satisfies (4.2), (4.5) and \( K_t \equiv 0 \). Then \( K_t \equiv 0, \ t \geq 0 \).

For this fix \( t_c < \infty \) and \( f \in \mathbb{D}_A^k \). Let \( \nu \) be the measure defined on \( S' = [0, t_c] \times S \) by

\[ \nu(B) = E_t \int_0^t 1_B(u, X_u)du, \quad B \in \mathcal{B}(S') \]

Then note that (2.17) implies \( \int_{S'} |g| \, d\nu < \infty \). Hence if \( g_k : S' \to \mathbb{R} \) is defined by

\[ g_k(s, x) = g(s, x) 1(|g(s, x)| < k) \]

then we have

\[ \int_{S'} |g_k - g| \, d\nu \to 0 \quad \text{as} \quad k \to \infty. \]

For each \( k, \ g_k \) is bounded by \( k \). By Lusin's theorem (see [1], p. 187) we can pick \( c_{k, l} \in \mathbb{C}_b(S') \), bounded by \( k \), such that

\[ c_{k, l} + r_k \quad \text{a.e.} \nu \quad \text{as} \quad l \to \infty. \]
Hence

\[ \lim_{i \to \infty} \int |c_k,i - g_k| \, \text{d}v = 0. \]

Let \( v_{k,i} \) be given by (3.6) for \( c = c_{k,i} \). Then \( A v_{k,i} = -c_{k,i} v_{k,i} \) in \([0, t_0) \times S\) by Theorem 4. Using (4.2) for \( v_{k,i} \) and recalling that \( K_0 = 0 \), we have

\[ (4.13) \quad \langle f(t_0, \cdot), K_0 \rangle = \langle v_{k,i}(t_0, \cdot), K_0 \rangle \]

\[ = \int_0^{t_0} \langle (g - c_{k,i} v_{k,i})(u, \cdot), K_0 \rangle \, \text{d}u \]

Thus

\[ (4.14) \quad |\langle f(t_0, \cdot), K_0 \rangle| \leq M \int_0^{t_0} |\langle g - c_{k,i} v_{k,i}(u, \cdot), \varphi \rangle| \, \text{d}u \]

As \( i \to \infty \), \( v_{k,i} \) converges pointwise to \( v_k \) and is bounded by \( k \), where \( v_k \) is given by (3.6) for \( c = c_k \). This and (4.11), (4.12), (4.14) imply

\[ |\langle f(t_0, \cdot), K_0 \rangle| \leq M \int |g - c_k| |v_k| \, \text{d}v. \]

Since (4.9) implies \( c_k(u, x) \leq a(u) \), it follows that

\[ |v_k| \leq M_1 \cdot \exp \left( \int_0^{t_0} a(u) \, \text{d}u \right) = M_2 \]

where \( |f| \leq M_1 \). Hence
This and (4.10) imply \( <f(t_o,.), K_{t_o}> = 0 \). Since \( f \in D_A \) is arbitrary, Lemma 5 gives \( K_{t_o} = 0 \). This completes the proof.

We will briefly consider the equation for normalized measures

\[
N_t(B) = \frac{K_t(B)}{K_t(S)}, \quad B \in S
\]

where \( K_t \) is given by (4.4). It is easy to see, using (4.2) that \( \{N_t\} \) satisfies.

\[
<f(t, .), N_t> = \int_0^t \langle (Af + gf)(u, .), N_u \rangle du - \int_0^t <f(u, .), N_u \rangle \langle g(u, .), N_u \rangle du
\]

We will now prove that \( \{N_t\} \) is the unique solution to this equation.

**Theorem 7:** The equation (4.16) with boundary condition \( N_0 = \pi \circ X^{-1}_0 \) admits a unique solution in the class of \( \{N_t\} \) satisfying (4.5). The solution is given by (4.15).

**Proof:** We need to prove uniqueness of the solution. Let \( N_t' \) be any other solution, i.e. satisfying (4.5), (4.16) and \( N_0' = \pi \circ X^{-1}_0 \). Then it can be checked that \( N_t'(S) = 1 \) for all \( t \geq 0 \). Further, if \( K_t' \) is defined by

\[
K_t'(B) = N_t'(B) \cdot \exp(\int_0^t \langle g(u, .), N_u' \rangle du)
\]

then \( K_t' \) is a solution to (4.2) and that it satisfies (4.3), (4.5).

Hence by Theorem 6, \( K_t' = K_t \). This and the observation that

\[
K_t'(S) = \frac{K_t'(B)}{K_t'(S)} = \frac{K_t(B)}{K_t(S)}
\]

give us the required equality, namely \( N_t' = N_t \).
5. We will now give applications of the results in the previous sections to filtering theory.

We refer the reader to [4] for a detailed discussion and background on the white noise approach to filtering theory.

We assume that the signal process \( (X_t) \) is a Markov process satisfying the conditions imposed in the previous sections.

Let \( \mathcal{K} \) be a separable Hilbert space. Let \( h : [0,T] \times S \to \mathcal{K} \) be a measurable function such that

\[
E \left[ \int_0^T \| h_u(X_u) \|_{\mathcal{K}}^2 \, du \right] < \infty
\]

Let \( H = L^2([0,T], \mathcal{K}) \) and let \( \xi : \Omega \to H \) be defined by

\[
(\xi(\omega))_u = h_u(X_u(\omega)), \quad 0 \leq u \leq T.
\]

Consider the model

\[
y = \xi + \varepsilon
\]

where \( \varepsilon = (\varepsilon_t) \) is \( \mathcal{K} \)-valued white noise independent of \( (X_t) \). Here \( y \) is the observation process and \( y, \xi, \varepsilon \) are realised on a quasi cylinder probability space \( (\mathcal{E}, \mathcal{F}, \mathbb{P}) \) (See [4] section 6). We now state the Bayes formula. For the relevant definitions and proof, see [4].

**Theorem 8**: For \( g : S \to \mathbb{R} \) bounded, measurable,

\[
E_a(g(X_t) | y_u : u < t) = \int_S g(x) \, dF_t(x, y)(x)
\]

where

\[
\Gamma_t(y)(B) = E \left[ I_B(X_t) \exp \left( \int_0^t h_u(X_u) y_u \, du - \frac{1}{2} \int_0^t \| h_u(X_u) \|_{\mathcal{K}}^2 \, du \right) \right]
\]
\[ f(t, u, y) = f(0, u, y) + \int_0^t \langle (Af + y^t f)(u, s), f_u(s) \rangle ds, \quad f \in D_A \]

with the condition \( f(0, u, y) = n_c \cdot \chi^{-1} \) in the class of \( K_t \) satisfying (4.5).

(ii) For all \( y \in H \), \( F_t(y) \) is the unique solution to the equation

\[ f(t, u, y) = f(0, u, y) + \int_0^t \langle (Af + y^t f)(u, s), F_u(s) \rangle ds, \quad f \in D_A \]

with the initial condition \( F(0, u, y) = n_c \cdot \chi^{-1} \) in the class of \( K_t \) satisfying (4.5).

Proof: Since

\[ |c_y(t, x)| \leq \frac{1}{T} ||h_t(x)||^2 + \frac{1}{T} ||y_t||^2 \]

and

\[ r_y(t, x) \leq \frac{1}{T} ||y_t||^2 \]

it follows that for all \( v \in H \), \( c_y \) satisfies (2.17) and (2.18). Thus (i) follows from Theorem 6 and (ii) from Theorem 7.
Remark: Theorem 9 was proved in [3] under the much stronger condition

\[ ||h_t(x)|| \leq a_t \quad \text{with} \quad \int_0^T a_t^2 \, dt < \infty. \]

The equations (5.5) and (5.6) are analogues of the Zakai and Fujisaki-Kallianpur-Kunita equations. In [3], \( \Gamma_t(y) \) and \( F_t(y) \) were also characterized as unique solutions to another type of equations (equations (3.4) and (3.11) in [3]) under the condition (5.7). With a little bit of work, it can be shown that (5.7) can be replaced by (5.1) in these results as well.

References

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