Research Report ONR-87-1

ASYMPTOTIC NORMALITY OF POLY-T DENSITIES
WITH BAYESIAN APPLICATIONS

George Y. Wong
Division of Biostatistics
Memorial Sloan-Kettering Cancer Center
New York, New York 10021

October, 1987

Prepared under contract NO. N00014-85-K-0485, NR 150-536
with the Personnel and Training Research Programs
Psychological Sciences Division
Office of Naval Research

Approved for public release; distribution unlimited.
Reproduction in whole or in part is permitted for
any purpose of the United States Government.
Asymptotic Normality of Poly-t Densities With Bayesian Applications

George Y. Wong

A poly-t density is a density which is proportional to a product of at least two t-like factors, each of which is of the form \( (x - y)^2 + \beta (x - y)^2 \) where \( \beta \) is a positive number, \( y \) is an arbitrary location vector and \( \beta \) is a symmetric semi-positive definite scale matrix. In general, \( \beta \) is a function of \( d \). Such a density arises, for example, in the Bayesian analysis of a linear model with a normal error term, independent normal priors on the linear parameters and inverted-gamma priors on the variance components. A theorem about the asymptotic normality of the density as a subset of the individual \( d \)'s tend to infinity is proved under very general conditions. A corollary specifically related to the Bayesian linear model is also given. Detailed results are illustrated in the familiar Bayesian multiple linear regression model with two variance components. The Tiao-Zellner expansion for approximating the particular poly-t form involving two proper multivariate t factors is extended to the case of two arbitrary t-like factors.
ASYMPTOTIC NORMALITY OF POLY-T DENSITIES
WITH BAYESIAN APPLICATIONS

GEORGE Y. WONG
DIVISION OF BIOSTATISTICS
MEMORIAL SLOAN-KETTERING CANCER CENTER
NEW YORK, NY 10021

Key Words and Phrases: poly-t density; asymptotic normality; generalized Tiao-Zellner expansion; Bayesian linear model; posterior inference of linear parameters.

ABSTRACT

A poly-t density is a density which is proportional to a product of at least two t-like factors, each of which is of the form \( [1 + (x - \mu)^T \Sigma (x - \mu)]^{-d/2} \) where \( d \) is a positive number, \( \mu \) is an arbitrary location vector and \( \Sigma \) is a symmetric semi-positive definite scale matrix. In general, \( \Sigma \) is a function of \( d \). Such a density arises, for example, in the Bayesian analysis of a linear model with a normal error term, independent normal priors on the linear parameters and inverted-gamma priors on the variance components. A theorem about the asymptotic normality of the density as a subset of the individual \( d \)'s tend to infinity is proved under very general conditions. A corollary specifically related to the Bayesian linear model is also given. Detailed results are illustrated in the familiar Bayesian multiple linear regression model with two variance components. The Tiao-Zellner expansion for approximating the particular poly-t form involving two proper multivariate t factors is extended to the case of two arbitrary t-like factors.
1. INTRODUCTION

A p-dimensional random vector $\mathbf{X}$ is said to have a poly-t distribution if its density is proportional to a product of $L>2$ "t-like" factors, or

$$f(x; d_1, \ldots, d_L) \propto \prod_{k=1}^{L} \left\{ 1 + (x - \mu_k)^T M_k(d_k) (x - \mu_k) \right\},^{-d_k/2}$$

(1)

where $d_k > 0$, $\mu_k$ is a px1 location vector and $M_k(d_k)$ is a pxp symmetric semi-positive definite scale matrix, $k=1, \ldots, L$. To ensure that the right-hand side of (1), which we denote by $g(x; d_1, \ldots, d_L)$, is normalizable, we must require that $d_1 + \ldots + d_L > p$ and $M_1(d_1) + \ldots + M_L(d_L)$ be positive definite (Dickey 1986).

The normalizing constant $\int g(x; d_1, \ldots, d_L) dx$, however, cannot be expressed in a simple closed form. When $L$ is smaller than $p$, Dickey showed that the normalizing constant can be expressed in an $(L-1)$-fold integral.

A useful form of density (1) is obtained by letting

$$d_k = \nu_k + m_k, \quad M_k(d_k) = M_k/\nu_k$$

(2)

for some $\nu_k > 0$, $m_k$ such that $\nu_k + m_k > 0$, and $M_k$ symmetric semi-positive definite.

When $m_k = p$ and $M_k$ is positive definite, then the $k$th t-like factor is proportional to a proper multivariate t density with $\nu_k$ degrees of freedom.

Such a poly-t density form plays an important role in the Bayesian analysis of a general linear model with a normal error term. The Bayesian approach considered here is described as follows: (1) A vague prior is imposed on a subset of the linear parameters; (2) The rest of the parameters are partitioned into different subsets, and an exchangeable normal prior with mean 0 and an unknown variance component is imposed on each subset independently; and (3) The prior variances and the error variance are given independent inverted gamma distributions. It can be shown that the posterior density of the linear parameters is of the above poly-t density form. Details about poly-t posterior
distributions can be found in Dickey (1974), Dreze (1977), Rajagopalan and Broemeling (1983), Broemeling and Abdullah (1984), and Broemeling (1985).

In this article, we are concerned with the asymptotic normality of a poly-t density as some or all of \( d_1, \ldots, d_L \) become large. The asymptotic normal density can be used to approximate the poly-t density directly. Also, it can serve as an important sampling distribution for the Monte Carlo evaluation of any probability statement about \( \mathbf{x} \), or any moment of \( \mathbf{x} \). Despite its usefulness in Bayesian linear modeling, the asymptotic normality of a poly-t density has never been formally established. In Section 2, we prove the normality result for the poly-t density (1) under fairly general conditions. As will be seen there, the proof is much harder than that in the case of a single multivariate t density; the difficulty is mainly due to the fact that the normalizing constant of a poly-t density is in terms of an intergral expression.

In Section 3, we illustrate the asymptotic results on a Bayesian linear model with two variance components. There we show that the appropriate asymptotic normal density may be made the leading term in a generalization of the Tiao-Zellner expansion (1964) to approximate the posterior density of the linear parameters.

2. MAIN RESULTS

We show that under some fairly general conditions, the poly-t density (1) converges to a proper multivariate noraml density. It follows from a result of Scheffe (1947) that the poly-t variable actually converges in distribution to the multivariate normal variable with the limiting normal density. We also give a corollary on a special form of (1) which is particularly useful in Bayesian linear modeling. For convenience, we write \( A>0 \) and \( A>0 \) to mean that the symmetric matrix \( A \) is semi-positive definite and positive definite, respectively. Also, we denote the density of a multivariate normal density
with the mean \( \mu \) and covariance matrix \( \Sigma \) by \( f_N(\mathbf{x}; \mu, \Sigma) \).

Theorem. For the general poly-t density (1), assume that

1. \( d_k M_k(d_k) \) converges to \( M_k > 0 \) as \( d_k \to \infty \), \( k = 1, \ldots, L \), and \( M = M_1 + \cdots + M_L > 0 \), and

2. there exist semi-positive definite matrices \( B_1, \ldots, B_L \) such that \( B_1 + \cdots + B_L > 0 \) and \( d_k M_k(d_k) - B_k > 0 \) for all \( d_k \) sufficiently large.

Then as \( d_1, \ldots, d_L \) all tend to infinity, \( f(\mathbf{x}; d_1, \ldots, d_L) \) will tend to the limiting density \( f_N(\mathbf{x}; \mu, M^{-1}) \), where \( \mu = M^{-1}(M_1 \mu_1 + \cdots + M_L \mu_L) \).

Proof. Without any loss of generality, we prove the theorem for the case \( L = 2 \). To show that \( f(\mathbf{x}; d_1, d_2) \) converges to \( f_N(\mathbf{x}; \mu, M^{-1}) \) as \( d_1, d_2 \to \infty \), it suffices to show that (i) \( g(\mathbf{x}; d_1, d_2) \) tends to a limit \( g(x) \) proportional to \( f_N(\mathbf{x}; \mu, M^{-1}) \), and (ii) \( \int g(\mathbf{x}; d_1, d_2) \, \mathrm{d}x \) as \( d_1, d_2 \to \infty \).

To prove (i), we make use of assumption (1) and obtain

\[
g(\mathbf{x}; d_1, d_2) = \prod_{k=1}^{2} \left[ 1 + Q_k/d_k + o(1/d_k) \right]^{-d_k/2},
\]

where \( Q_k = (x - \mu_k)^T M_k (x - \mu_k) \), \( k = 1, 2 \). It follows that \( g(\mathbf{x}; d_1, d_2) \) tends to

\[
g(x) = \exp \{- (\mu_1 - \mu_2)^T M_2 (\mu_1 - \mu_2)/2 \} \times \exp \{- (x - \mu)^T M (x - \mu)/2 \},
\]

which is proportional to \( f_N(\mathbf{x}; \mu, M^{-1}) \).

To prove (ii), we show that for \( d_1 \) and \( d_2 \) sufficiently large, \( g(\mathbf{x}; d_1, d_2) \) is dominated by an integrable function \( h(\mathbf{x}; d_1, d_2) \), which monotonically decreases to an integrable function \( h(x) \); moreover, \( \int h(\mathbf{x}; d_1, d_2) \, \mathrm{d}x \) also decreases monotonically to \( \int h(x) \, \mathrm{d}x \). Then using a generalization of the Lebesgue Dominated Convergence Theorem (see, for example, Royden, p.89), we conclude that the limit of \( \int g(\mathbf{x}; d_1, d_2) \, \mathrm{d}x \) is \( \int g(x) \, \mathrm{d}x \).

We now make use of assumption (2) to prove the above claims. First, it is an immediate consequence of the assumption that for sufficiently large \( d_1 \) and \( d_2 \),

\[
g(\mathbf{x}; d_1, d_2) \leq h(\mathbf{x}; d_1, d_2) = \prod_{k=1}^{2} \left[ 1 + (x - \mu_k)^T B_k (x - \mu_k)/d_k \right]^{-d_k/2}.
\]

We now make use of assumption (2) to prove the above claims. First, it is an immediate consequence of the assumption that for sufficiently large \( d_1 \) and \( d_2 \),

\[
g(\mathbf{x}; d_1, d_2) \leq h(\mathbf{x}; d_1, d_2) = \prod_{k=1}^{2} \left[ 1 + (x - \mu_k)^T B_k (x - \mu_k)/d_k \right]^{-d_k/2}.
\]
Also, it can be directly verified that as \( \mathbf{d}_1, \mathbf{d}_2 \to +\infty \), \( h(\mathbf{x}|\mathbf{d}_1, \mathbf{d}_2) \) monotonically decreases to an integrable function

\[
h(\mathbf{x}) = \exp \left\{ - (\mathbf{u}_1 - \mathbf{u}_2)^T \mathbf{B}_1 (\mathbf{B}_1 + \mathbf{B}_2)^{-1} \mathbf{B}_2 (\mathbf{u}_1 - \mathbf{u}_2)/2 \right\}
\]

\[
\times \exp \left\{ - (\mathbf{x} - \mathbf{v})^T (\mathbf{B}_1 + \mathbf{B}_2) (\mathbf{x} - \mathbf{v})/2 \right\},
\]

where \( \mathbf{v} = (\mathbf{B}_1 + \mathbf{B}_2)^{-1} (\mathbf{B}_1 \mathbf{u}_1 + \mathbf{B}_2 \mathbf{u}_2) \). To establish the integrability of \( h(\mathbf{x}|\mathbf{d}_1, \mathbf{d}_2) \), let \( m = \min(d_1, d_2) \). Then the monotonicity of the function implies that

\[
h(\mathbf{x}d_1, d_2) \leq \prod_{k=1}^{2} \left\{ 1 + (\mathbf{x} - \mathbf{u}_k)^T \mathbf{B}_k (\mathbf{x} - \mathbf{u}_k)/m \right\}^{-m/2}
\]

\[
\leq \left\{ 1 + (\mathbf{x} - \mathbf{v})^T (\mathbf{B}_1 + \mathbf{B}_2) (\mathbf{x} - \mathbf{v})/m \right\}^{-m/2}.
\]

For sufficiently large \( d_1 \) and \( d_2 \), the \( t \)-like factor on the right-hand side, denoted by \( h_m(\mathbf{x}) \), is proportional to a multivariate \( t \) density with \( m - p \) degrees of freedom, mean \( \mathbf{v} \) and scale matrix \( \mathbf{B}_1 + \mathbf{B}_2 \); therefore, \( h(\mathbf{x}|\mathbf{d}_1, \mathbf{d}_2) \) is integrable. Let \( m \) be fixed. Define \( \mathbf{h}(\mathbf{x}; \mathbf{d}_1, \mathbf{d}_2) = h_{m}(\mathbf{x}) - h(\mathbf{x}|\mathbf{d}_1, \mathbf{d}_2) \), for \( d_1, d_2 > m \).

Applying the Monotone Convergence Theorem to this sequence of functions together with the fact that \( h_{m}(\mathbf{x}) \) is integrable, we conclude that

\[
\int h(\mathbf{x}|\mathbf{d}_1, \mathbf{d}_2) d\mathbf{x} \text{ tends to } \int h(\mathbf{x}) d\mathbf{x}.
\]

This completes the proof.

We point out that if only the elements of a subset of \( \mathbf{d}_1, \ldots, \mathbf{d}_L \) tend to infinity, then the theorem will apply to the product of the corresponding \( t \)-like factors while the remaining factors will stay put.

We now consider a special form of the poly-\( t \) density (1) with scale matrix

\[
\mathbf{M}_k(\mathbf{d}_k) = \mathbf{w}_k(\mathbf{d}_k) \mathbf{N}_k
\]

for some positive function \( \mathbf{w}_k \) and semi-positive definite matrix \( \mathbf{N}_k \). The following corollary concerns the asymptotic behavior of such a poly-\( t \) density.

**Corollary** For the particular poly-\( t \) density with \( \mathbf{M}_k(\mathbf{d}_k) \) of the form (3), \( k=1, \ldots, L \), assume that \( d_k \mathbf{w}_k(\mathbf{d}_k) > c_k > 0 \) as \( d_k \to +\infty \), \( k=1, \ldots, L \) and \( \mathbf{M} = \mathbf{M}_1 + \ldots + \mathbf{M}_L > 0 \),
where $M_k = c_k N_k$ is the limit of $d_k^{-1}M_k(d_k)$. Then $f(x_i d_{1}, ..., d_L) = f_N(x_i \mu, \Sigma^{-1})$ as $d_{1}, ..., d_L \to \infty$.

**Proof** We need to check that the two assumptions of the theorem hold for this particular form of poly-t density. Assumption (1) is obviously satisfied. For assumption (2), we take $B_k = \epsilon_k M_k$ for some appropriately chosen positive number $\epsilon_k$. Clearly, $B_1 + ... + B_L = \epsilon_1 M_1 + ... + \epsilon_L M_L > 0$. We must choose $\epsilon_k$ such that $d_k^{-1} M_k(d_k) - B_k = [d_k w_k(d_k) - \epsilon_k c_k] N_k > 0$ for $d_k$ sufficiently large. Since $d_k w_k(d_k) \to c_k$ as $d_k \to \infty$, it follows that for $d_k$ large enough, $d_k w_k(d_k) > c_k - n_k > 0$ where $n_k$ is an arbitrary positive number. Therefore, we can choose $\epsilon_k = (c_k - n_k)/c_k$. This completes the proof.

An example of a poly-t density with scale matrices of the specific form (3) is when $d_k$ and $M_k(d_k)$ are given by (2). Here we may choose $w_k(d_k) = 1/(d_k - m_k)$, and $N_k = M_k$. Note that the $M_k$ matrix in (2) is the limit of $d_k^{-1}M_k(d_k)$ since $c_k = 1$.

3. **APPLICATION TO BAYESIAN LINEAR MODELING**

The results of the previous section are now applied to a multiple regression model with an exchangeable normal prior on the regression coefficients and independent inverted gamma priors on the variance components. The sampling model is represented by

$$y = X \beta + \epsilon,$$

$$\epsilon \sim N(0, \sigma^2 I_n),$$

where $y$ is an $n \times 1$ vector of observations, $X$ is an $n \times p$ fixed design matrix in correlation form, and $I_n$ is the $n \times n$ identity matrix. The prior specification is given by
\[ B \sim N(\nu_1 \mathbf{1}_p, \sigma_B^2 \mathbf{1}_p), \]
\[ \nu \lambda / \sigma^2 \sim \chi_v^2, \quad \nu \beta \lambda / \sigma_B^2 \sim \chi_v^2, \]

where \( \mathbf{1}_p \) is a \( p \times 1 \) vector of ones, and the \( \lambda \)'s and \( \nu \)'s are hyperparameters that determine the prior means and variances of the variance components. For example \( E(\sigma^2) = \nu \lambda / (\nu - 2) \), and \( V(\sigma^2) = 2 \nu^2 \lambda^2 / (\nu - 2)^2 (\nu - 4) \).

Therefore, a large value of \( \nu \) reflects a strong prior certainty that \( \lambda \) is a correct guess of \( \sigma^2 \). Additionally, we assume that prior knowledge about \( \mu \) is vague.

In the special case of the one-way random effects model with a block diagonal matrix \( X = \text{diag}(\mathbf{1}_{n_1}, ..., \mathbf{1}_{n_p}) \), where \( n_i \) is the sample size of the \( i \)th group, Hill (1965, 1977, 1980) presented exact and approximate posterior inference of \( \sigma_B^2 \) and \( \sigma^2 \). His results can be used to obtain the posterior moments of the group means \( \mathbf{B} \). Lindley and Smith (1972) estimated the parameters of the above general model using a computationally simple posterior joint modal approach. None of these authors, however, discussed the approximation of the posterior density of \( \mathbf{B} \), which we denote by \( f(B \mid Y) \). In the following, we show how the results of Section 2 may be used to approximate this density.

From Lindley and Smith, the posterior density of \( \mathbf{B} \) is

\[ f(B \mid Y) \propto \left(1 + \mathbf{H} \mathbf{B} / \nu_B \lambda \right)^{- (\nu_B + p - 1) / 2} \times \left[1 + \left(B - \hat{B}\right)^T X^T X (B - \hat{B}) / (\nu \lambda + n s^2)\right]^{- (\nu + n) / 2}, \]

where \( \mathbf{H} = \mathbf{I}_p - (1/p) \mathbf{1}_p \mathbf{1}_p^T \), \( \hat{B} \) is a least squares estimate of \( B \) and \( s^2 \) is the residual mean square. The posterior density of \( \mathbf{B} \) is a poly-t density of the special form (2) with

\[ \nu_1 = \nu_B, \quad \nu_2 = \nu + ns^2 / \lambda, \]
\[ m_1 = p - 1, \quad m_2 = n - ns^2 / \lambda, \]
\[ M_1 = \mathbf{H} / \lambda_B, \quad M_2 = X^T X / \lambda. \]
When there is prior near certainty concerning \( \sigma^2_o \) and \( \sigma^2 \) so that both \( \nu_o \) and \( \nu \) are large, a simple application of the corollary shows that \( f(\theta_i | x) \) is approximately normal with mean \( \mu \) and covariance matrix \( \Sigma \) given by

\[
\mu = (H/\lambda + X^T \lambda)^{-1} X^T y,
\]
\[
\Sigma = (H/\lambda + X^T \lambda)^{-1}.
\]

When there is a considerable amount of uncertainty concerning the prior value \( \lambda_o \) (or \( \lambda \)) so that \( \nu_o \) (or \( \nu \)) is not assigned a large value, the normal approximation may be improved upon by using a generalization of the Tiao-Zellner expansion. An alternative was proposed by Dickey (1967) using an expansion based on Appell's polynomials (1880) in the univariate case \((p=1)\); however, the multivariate version has not been studied. The original Tiao-Zellner expansion is designed for a special case of a poly-t density with \( L=2 \) proper multivariate t factors with mean \( \mu_k \), scale matrix \( M_k \) and \( \nu_k \) degrees of freedom, \( k=1,2 \). In the expansion, the poly-t density is approximated by a density in the form of \( f_N(x_i, M^{-1}) \), where \( M = M_1 + M_2 \), multiplied by a double infinite series in the inverse powers of \( \nu_1 \) and \( \nu_2 \).

When most of the probability mass is concentrated in the intersection of two ellipsoids \( \{x: (x- \mu_k)^T M_k (x- \mu_k) \leq \nu_k \} \), \( k=1,2 \), the expansion will yield reasonable approximation with only a few terms in the series expansion.

The Tiao-Zellner expansion may be generalized to the case of a poly-t density of the form (2). Lindley (1971) mentioned the use of this expansion in the case of the one-way model. However, he did not specify the limiting normal density. Also, the degrees of freedom \( \nu_1 \) and \( \nu_2 \) were not correctly stated. We now sketch the essential steps in the generalization using \( f(\theta_i | x) \) in (4) as an example.

Let \( \nu_k \) and \( m_k \) be as defined in (5), \( Q_1 = \hat{\theta}^T H \hat{\theta} / \lambda_o \), and \( Q_2 = (\hat{\theta} - \hat{\theta})^T X^T X (\hat{\theta} - \hat{\theta}) / \lambda \). Following Tiao and Zellner, we obtain for the first factor
\[(1+Q_1/\nu_1)^{-\nu_1 m_1}/2 = \exp(-Q_1/2) \exp[Q_1/2-((\nu_1+m_1)/2) \log (1+Q_1/\nu_1)].\]

This is of the same form as that obtained in (3.2) of Tiao and Zellner except that we replace \(p\) there by \(m_1\). Therefore, the functional forms of the polynomials \(p_i = p_i(Q_1)\) in the expansion

\[(1+Q_1/\nu_1)^{-\nu_1 m_1}/2 = \exp(-Q_1/2) \sum_{j=0}^{\infty} p_i \nu_1^{-i}\]

are of the same forms as those given in Tiao and Zellner except that the constant \(p\) is replaced by \(m_1\) (for instance, \(p_1 = (Q_1-m_1 Q_1)/4\)). Similarly, for the second factor,

\[(1+Q_2/\nu_2)^{-\nu_2 m_2}/2 = \exp(-Q_2/2) \sum_{j=0}^{\infty} q_j \nu_2^{-j}\]

where the constant \(p\) which appears in the polynomials \(q_j = q_j(Q_2)\) in Tiao and Zellner is now replaced by \(m_2\).

Following the rest of the derivation of Tiao and Zellner, we obtain the generalized expansion

\[f(\theta; \chi) = f_N(\theta; \nu, M^{-1}) \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} d_{ij} \nu_1^{-i} \nu_2^{-j},\]

where the leading normal term is the one guaranteed by the corollary, and the \(d_{ij}\)'s are polynomials in \(p_i\) and \(q_j\) given in (3.9) of Tiao and Zellner. The coefficients which appear in the polynomial \(d_{ij}\), namely \(b_{rs} = E[p_r(Q_1)q_s(Q_2)]\) for \(r \leq i, s \leq j\), also have to be modified accordingly. The modification essentially consists of the following steps: (1) Change the constant \(p\) in \(p_r(Q_1)\) given in Tiao and Zellner to \(m_1\); (2) Change the constant \(p\) in \(q_s(Q_2)\) given in Tiao and Zellner to \(m_2\), and (3) Use the bivariate moment-cumulant inversion formulas as given by Cook (1951) to express \(b_{rs}\) in terms of the mixed cumulants of \(Q_1\) and \(Q_2\). Expressions for these cumulants are derived in Tiao and Zellner. Following these four steps, the coefficients \(b_{rs}\) can be evaluated.
in a straightforward manner. We omit the details here.

In the case of a product of two t-like factors of the form (2), it is also feasible to approximate the density by direct one-dimensional numerical integration. Dickey (1968) showed that integration can be facilitated if the scale matrices are simultaneously diagonalized first (more details are presented in Box and Tiao (1973, chapter 9)). The generalized Tiao-Zellner expansion offers a simple approximation alternative in terms of familiar multivariate normal calculations. When there are many such t-like factors, such as in the posterior analysis of a general mixed linear model with many variance components, it will be virtually impossible to approximate the density by high-dimensional numerical integration. The generalized expansion, however, can be extended to the case of many t-like factors in a straightforward manner. Obviously, the computational complexity will increase rapidly as the number of t-like factors increases.

ACKNOWLEDGEMENTS

Preparation of this article was supported by the office of Naval Research Contract No. N00014-85-K-0485.
BIBLIOGRAPHY

Appell, P. (1880). Développement en série entière de $(1+ax)^{1/x}$. Archiv der Mathematik und Physik, 65, 171-175.


Dr. Terry Ackerman
American College Testing Programs
P.O. Box 168
Iowa City, IA 52243

Dr. Robert Ahlers
Code N711
Human Factors Laboratory
Naval Training Systems Center
Orlando, FL 32813

Dr. James Algina
University of Florida
Gainesville, FL 32605

Dr. Erling B. Andersen
Department of Statistics
Studiestraede 6
1455 Copenhagen
DENMARK

Dr. Eva L. Baker
UCLA Center for the Study of Evaluation
145 Moore Hall
University of California
Los Angeles, CA 90024

Dr. Isaac Bejar
Educational Testing Service
Princeton, NJ 08540

Dr. Menucha Birenbaum
School of Education
Tel Aviv University
Tel Aviv, Ramat Aviv 69978
ISRAEL

Dr. Arthur S. Blaiwes
Code N711
Naval Training Systems Center
Orlando, FL 32813

Dr. Bruce Bloxom
Defense Manpower Data Center
550 Camino El Estero,
Suite 200
Monterey, CA 93943-3231

Dr. R. Darrell Bock
University of Chicago
NORC
6030 South Ellis
Chicago, IL 60637

Cdt. Arnold Bohrer
Sectie Psychologisch Onderzoek
Rekruterings-En Selectiecentrum
Kwartier Koningen Astrid
Bruijnstraat
1120 Brussels, BELGIUM

Dr. Robert Breaux
Code N-095R
Naval Training Systems Center
Orlando, FL 32813

Dr. Robert Brennan
American College Testing Programs
P.O. Box 168
Iowa City, IA 52243

Dr. Lyle D. Broemeling
ONR Code 1111SP
800 North Quincy Street
Arlington, VA 22217

Dr. James Carlson
American College Testing Program
P.O. Box 168
Iowa City, IA 52243

Dr. John B. Carroll
409 Elliott Rd.
Chapel Hill, NC 27514

Dr. Robert Carroll
OP 0187
Washington, DC 20370

Mr. Raymond E. Christal
AFHRL/MOE
Brooks AFB, TX 78235
Dr. Norman Cliff  
Department of Psychology  
Univ. of So. California  
University Park  
Los Angeles, CA 90007

Dr. Fritz Drasgow  
University of Illinois  
Department of Psychology  
603 E. Daniel St.  
Champaign, IL 61820

Director,  
Manpower Support and  
Readiness Program  
Center for Naval Analysis  
2000 North Bearegarden Street  
Alexandria, VA 22311

Defense Technical  
Information Center  
Cameron Station, Bldg 5  
Alexandria, VA 22314  
Attn: TC  
(12 Copies)

Dr. Stanley Collyer  
Office of Naval Technology  
Code 222  
800 N. Quincy Street  
Arlington, VA 22217-5000

Dr. Stephen Dunbar  
Lindquist Center  
for Measurement  
University of Iowa  
Iowa City, IA 52242

Dr. Hans Crombag  
University of Leyden  
Education Research Center  
Boerhaavelaan 2  
2334 EN Leyden  
The NETHERLANDS

Dr. James A. Earles  
Air Force Human Resources Lab  
Brooks AFB, TX 78235

Mr. Timothy Davey  
University of Illinois  
Educational Psychology  
Urbana, IL 61801

Dr. Kent Eaton  
Army Research Institute  
5001 Eisenhower Avenue  
Alexandria, VA 22333

Dr. Doug Davis  
Chief of Naval Education  
and Training  
Naval Air Station  
Pensacola, FL 32508

Dr. John M. Eddins  
University of Illinois  
252 Engineering Research  
Laboratory  
103 South Mathews Street  
Urbana, IL 61801

Mr. Timothy Davey  
University of Illinois  
Educational Psychology  
Urbana, IL 61801

Dr. Susan Embretson  
University of Kansas  
Psychology Department  
426 Fraser  
Lawrence, KS 66045

Dr. Hans Crombag  
University of Leyden  
Education Research Center  
Boerhaavelaan 2  
2334 EN Leyden  
The NETHERLANDS

Dr. John M. Eddins  
University of Illinois  
252 Engineering Research  
Laboratory  
103 South Mathews Street  
Urbana, IL 61801

Mr. Timothy Davey  
University of Illinois  
Educational Psychology  
Urbana, IL 61801

Dr. Susan Embretson  
University of Kansas  
Psychology Department  
426 Fraser  
Lawrence, KS 66045

Dr. Dattprasad Divgi  
Center for Naval Analysis  
4401 Ford Avenue  
P.O. Box 16268  
Alexandria, VA 22302-0268

Dr. Benjamin A. Fairbank  
Performance Metrics, Inc.  
5825 Callaghan  
Suite 225  
San Antonio, TX 78228

Dr. Hei-Ki Dong  
Ball Foundation  
800 Roosevelt Road  
Building C, Suite 206  
Glen Ellyn, IL 60137

Dr. Hsi-Ki Dong  
Center for Naval Analysis  
4401 Ford Avenue  
P.O. Box 16268  
Alexandria, VA 22302-0268

Dr. Pat Federico  
Code 511  
NPRDC  
San Diego, CA 92152-6800
Dr. Leonard Feldt  
Lindquist Center for Measurement  
University of Iowa  
Iowa City, IA 52242

Dr. Richard L. Ferguson  
American College Testing Program  
P.O. Box 168  
Iowa City, IA 52240

Dr. Gerhard Fischer  
Liebiggasse 5/3  
A 1010 Vienna  
AUSTRIA

Dr. Myron Fischl  
Army Research Institute  
5001 Eisenhower Avenue  
Alexandria, VA 22333

Prof. Donald Fitzgerald  
University of New England  
Department of Psychology  
Armidale, New South Wales 2351  
AUSTRALIA

Mr. Paul Foley  
Navy Personnel R&D Center  
San Diego, CA 92152-5690

Dr. Alfred R. Fregly  
AFOSR/NL  
Bolling AFB, DC 20332

Dr. Robert D. Gibbons  
University of Illinois-Chicago  
P.O. Box 6998  
Chicago, IL 60680

Dr. Janice Gifford  
University of Massachusetts  
School of Education  
Amherst, MA 01003

Dr. Robert Glaser  
Learning Research & Development Center  
University of Pittsburgh  
3939 O’Hara Street  
Pittsburgh, PA 15260

Dr. Bert Green  
Johns Hopkins University  
Department of Psychology  
Charles & 34th Street  
Baltimore, MD 21218

Dipl. Pad. Michael W. Habon  
Universitat Dusseldorf  
Erziehungswissenschaftliches Universitaetsstr. 1  
D-4000 Dusseldorf 1  
WEST GERMANY

Dr. Ronald K. Hambleton  
Prof. of Education & Psychology  
University of Massachusetts at Amherst  
Hills House  
Amherst, MA 01003

Dr. Delwyn Harnisch  
University of Illinois  
51 Gerty Drive  
Champaign, IL 61820

Ms. Rebecca Hetter  
Navy Personnel R&D Center  
Code B2  
San Diego, CA 92152-6800

Dr. Paul W. Holland  
Educational Testing Service  
Rosedale Road  
Princeton, NJ 08541

Prof. Lutz F. Hornke  
Institut fur Psychologie  
RWTH Aachen  
Jaegerstrasse 17/19  
D-5100 Aachen  
WEST GERMANY

Dr. Paul Horst  
677 G Street, #184  
Chula Vista, CA 90010

Mr. Dick Hoshaw  
OP-135  
Arlington Annex  
Room 2834  
Washington, DC 20350
Dr. Lloyd Humphreys
University of Illinois
Department of Psychology
603 East Daniel Street
Champaign, IL 61820

Dr. Steven Hunka
Department of Education
University of Alberta
Edmonton, Alberta
CANADA

Dr. Huynh Huynh
College of Education
Univ. of South Carolina
Columbia, SC 29208

Dr. Robert Jannarone
Department of Psychology
University of South Carolina
Columbia, SC 29208

Dr. Dennis E. Jennings
Department of Statistics
University of Illinois
1409 West Green Street
Urbana, IL 61801

Dr. Douglas H. Jones
Thatcher Jones Associates
P.O. Box 6640
10 Trafalgar Court
Lawrenceville, NJ 08648

Dr. Milton S. Katz
Army Research Institute
5001 Eisenhower Avenue
Alexandria, VA 22333

Prof. John A. Keats
Department of Psychology
University of Newcastle
N.S.W. 2308
AUSTRALIA

Dr. G. Gage Kingsbury
Portland Public Schools
Research and Evaluation Department
501 North Dixon Street
P. O. Box 3107
Portland, OR 97209-3107

Dr. William Koch
University of Texas-Austin
Measurement and Evaluation Center
Austin, TX 78703

Dr. James Krantz
Computer-based Education Research Laboratory
University of Illinois
Urbana, IL 61801

Dr. Leonard Kroeker
Navy Personnel R&D Center
San Diego, CA 92152-6800

Dr. Daryll Lang
Navy Personnel R&D Center
San Diego, CA 92152-6800

Dr. Thomas Leonard
University of Wisconsin
Department of Statistics
1210 West Dayton Street
Madison, WI 53705

Dr. Michael Levine
Educational Psychology
210 Education Bldg.
University of Illinois
Champaign, IL 61801

Dr. Charles Lewis
Educational Testing Service
Princeton, NJ 08541

Dr. Robert Linn
College of Education
University of Illinois
Urbana, IL 61801

Dr. Robert Lockman
Center for Naval Analysis
4401 Ford Avenue
P.O. Box 16268
Alexandria, VA 22302-0268

Dr. Frederic M. Lord
Educational Testing Service
Princeton, NJ 08541
Dr. James Lumsden  
Department of Psychology  
University of Western Australia  
Nedlands W.A. 6009  
AUSTRALIA

Dr. Milton Maier  
Center for Naval Analysis  
4401 Ford Avenue  
P.O. Box 16268  
Alexandria, VA 22302-0268

Dr. William L. Maloy  
Chief of Naval Education and Training  
Naval Air Station  
Pensacola, FL 32508

Dr. Gary Marco  
Stop 31-E  
Educational Testing Service  
Princeton, NJ 08451

Dr. Clessen Martin  
Army Research Institute  
5001 Eisenhower Blvd.  
Alexandria, VA 22333

Dr. James McBride  
Psychological Corporation  
c/o Harcourt, Brace, Jovanovich Inc.  
1250 West 6th Street  
San Diego, CA 92101

Dr. Clarence McCormick  
HQ, MEPCOM  
MEPCT-P  
2500 Green Bay Road  
North Chicago, IL 60064

Dr. Robert McKinley  
Educational Testing Service  
Princeton, NJ 08541

Dr. James McMichael  
Technical Director  
Navy Personnel R&D Center  
San Diego, CA 92152

Dr. Barbara Means  
Human Resources Research Organization  
1100 South Washington  
Alexandria, VA 22314

Dr. Robert Mislevy  
Educational Testing Service  
Princeton, NJ 08541

Dr. William Montague  
NPRDC Code 13  
San Diego, CA 92152-6800

Ms. Kathleen Moreno  
Navy Personnel R&D Center  
Code 62  
San Diego, CA 92152-6800

Headquarters, Marine Corps  
Code MPI-20  
Washington, DC 20380

Dr. W. Alan Nicewander  
University of Oklahoma  
Department of Psychology  
Oklahoma City, OK 73069

Deputy Technical Director  
NPRDC Code 01A  
San Diego, CA 92152-6800

Director, Training Laboratory, NPRDC (Code 05)  
San Diego, CA 92152-6800

Director, Manpower and Personnel Laboratory, NPRDC (Code 06)  
San Diego, CA 92152-6800

Director, Human Factors & Organizational Systems Lab, NPRDC (Code 07)  
San Diego, CA 92152-6800

Fleet Support Office, NPRDC (Code 301)  
San Diego, CA 92152-6800

Library, NPRDC  
Code P201L  
San Diego, CA 92152-6800
Dr. Mary Schratz  
Navy Personnel R&D Center  
San Diego, CA 92152-6800

Dr. Dan Segall  
Navy Personnel R&D Center  
San Diego, CA 92152

Dr. W. Steve Sellman  
OASD(MRA&L)  
28269 The Pentagon  
Washington, DC 20301

Dr. Kazuo Shigemasu  
7-9-24 Kugenuma-Kaigan  
Fujusawa 251  
JAPAN

Dr. William Sims  
Center for Naval Analysis  
4401 Ford Avenue  
P.O. Box 16268  
Alexandria, VA 22302-0268

Dr. H. Wallace Sinaiko  
Manpower Research and Advisory Services  
Smithsonian Institution  
801 North Pitt Street  
Alexandria, VA 22314

Dr. Richard E. Snow  
Department of Psychology  
Stanford University  
Stanford, CA 94306

Dr. Richard Sorensen  
Navy Personnel R&D Center  
San Diego, CA 92152-6800

Dr. Paul Speckman  
University of Missouri  
Department of Statistics  
Columbia, MO 65201

Dr. Judy Spray  
ACT  
P.O. Box 168  
Iowa City, IA 52243

Dr. Martha Stocking  
Educational Testing Service  
Princeton, NJ 08541

Dr. Peter Stoloff  
Center for Naval Analysis  
200 North Beauregard Street  
Alexandria, VA 22311

Dr. William Stout  
University of Illinois  
Department of Mathematics  
Urbana, IL 61801

Maj. Bill Strickland  
AF/MPXOA  
4E168 Pentagon  
Washington, DC 20330

Dr. Hariharan Swaminathan  
Laboratory of Psychometric and Evaluation Research  
School of Education  
University of Massachusetts  
Amherst, MA 01003

Mr. Brad Symson  
Navy Personnel R&D Center  
San Diego, CA 92152-6800

Dr. John Tangney  
AFOSR/NL  
Bolling AFB, DC 20332

Dr. Kikumi Tatsuoka  
CERL  
252 Engineering Research Laboratory  
Urbana, IL 61801

Dr. Maurice Tatsuoka  
220 Education Bldg  
1310 S. Sixth St.  
Champaign, IL 61820

Dr. David Thissen  
Department of Psychology  
University of Kansas  
Lawrence, KS 66044

Mr. Gary Thomasson  
University of Illinois  
Educational Psychology  
Champaign, IL 61820
Dr. George Wong
Biostatistics Laboratory
Memorial Sloan-Kettering Cancer Center
1275 York Avenue
New York, NY 10021

Dr. Wallace Wulfeck, III
Navy Personnel R&D Center
San Diego, CA 92152-6800

Dr. Kentaro Yamamoto
Computer-based Education Research Laboratory
University of Illinois
Urbana, IL 61801

Dr. Wendy Yen
CTB/McGraw Hill
Del Monte Research Park
Monterey, CA 93940

Dr. Joseph L. Young
Memory & Cognitive Processes
National Science Foundation
Washington, DC 20550
END
12-87
DTC