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A CHARACTERIZATION OF SEPARATING PAIRS AND TRIPLETS IN A GRAPH

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A characterization of separating pairs and triplets in a graph

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We obtain tight upper bounds of \( n(n-3) \) and \( (n-2)(n-4) \) for the number of separating pairs and triplets in an undirected biconnected and triconnected graph, respectively, where \( n \) is the number of vertices in a graph. We present worst-case graphs that exactly achieve our upper bounds. Finally, we give an \( O(n) \) characterization for the separating pairs in a biconnected graph.
A Characterization of Separating Pairs and Triplets in a Graph

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ABSTRACT

We obtain tight upper bounds of \( \frac{n(n-3)}{2} \) and \( \frac{(n-1)(n-4)}{2} \) for the number of separating pairs and triplets in an undirected biconnected and triconnected graph, respectively, where \( n \) is the number of vertices in a graph. We present worst-case graphs that exactly achieve our upper bounds. Finally, we give an \( O(n) \) characterization for the separating pairs in a biconnected graph.

1. Introduction

Connectivity is an important graph property and there has been a considerable amount of work on algorithms for determining connectivity of graphs\(^1\). An undirected graph \( G = (V,E) \) is \( k \)-connected if for any subset \( V' \) of \( k-1 \) vertices of \( G \) the subgraph induced by \( V-V' \) is connected. A subset \( V' \) of \( k \) vertices is a separating \( k \)-set if the subgraph induced by \( V-V' \) is not connected. For \( k=1 \) the set \( V' \) becomes a single vertex which is called an articulation point, and for \( k=2,3 \) the set \( V' \) is called a separating pair and separating triplet, respectively. Efficient algorithms are available for finding all separating \( k \)-sets in \( k \)-connected undirected graphs for \( k \leq 3 \)\(^2\).

We address the following question: what is the maximum number of separating pairs and triplets in biconnected and triconnected undirected graphs, respectively?

An undirected graph \( G \) on \( n \) vertices has a trivial upper bound of \( \left[ \frac{n^2}{4} \right] \) on the number of separating \( k \)-sets, \( k \geq 1 \). The graph that achieves this bound for all \( k \) is a graph on \( n \) vertices without any edges. For \( k=1 \) the maximum number of articulation points in a connected graph is \( (n-2) \) and a graph that achieves it is a path on \( n \) vertices.
In this paper we show that for \( k=2 \) the maximum number of separating pairs in an undirected biconnected graph is \( \frac{n(n-3)}{2} \) and a graph that achieves it is a cycle on \( n \) vertices. Further, we observe that there is an \( O(n) \) representation for the separating pairs in any biconnected graph (although the number of such pairs could be \( \Theta(n^2) \)). Finally, we prove that for \( k=3 \) the maximum number of separating triplets in a triconnected graph is \( \frac{(n-1)(n-4)}{2} \) and we present a graph, namely the wheel [Tu], that achieves it.

In a companion paper [Ka1] we prove that the number of separating \( k \)-sets in a \( k \)-connected graph is \( O(c^k n^2) \) and we show that the bound is tight up to the constant \( c \).

2. Graph-theoretic definitions

An undirected graph \( G=(V,E) \) consists of a vertex set \( V \) and an edge set \( E \) containing unordered pairs of distinct elements from \( V \). A path \( P \) in \( G \) is a sequence of vertices \( <v_0, \ldots, v_k> \) such that \( (v_i,v_{i+1}) \in E, i=1, \ldots, k \). The path \( P \) contains the vertices \( v_0, \ldots, v_k \) and the edges \( (v_0,v_1), \ldots, (v_{k-1},v_k) \) and has endpoints \( v_0, v_k \) and internal vertices \( v_1, \ldots, v_{k-1} \).

We will sometimes specify a graph \( G \) structurally without explicitly defining its vertex and edge sets. In such cases, \( V(G) \) will denote the vertex set of \( G \) and \( E(G) \) will denote the edge set of \( G \). Also, if \( V' \subseteq V \) and \( v \in V \) we will use the notation \( V' \cup v \) to represent \( V' \cup \{v\} \).

An undirected graph \( G=(V,E) \) is connected if there exists a path between every pair of vertices in \( V \). For a graph \( G \) that is not connected, a connected component of \( G \) is an induced subgraph of \( G \) which is maximally connected.

A vertex \( v \in V \) is an articulation point of a connected undirected graph \( G=(V,E) \) if the subgraph induced by \( V-\{v\} \) is not connected. \( G \) is biconnected if it contains no articulation point.

Let \( G=(V,E) \) be a biconnected undirected graph. A pair of vertices \( v_1,v_2 \in V \) is a separating pair for \( G \) if the induced subgraph on \( V-\{v_1,v_2\} \) is not connected. \( G \) is triconnected if it contains no separating pair.

A triplet \( (v_1,v_2,v_3) \) of distinct vertices in \( V \) is a separating triplet of a triconnected graph if the subgraph induced by \( V-\{v_1,v_2,v_3\} \) is not connected. \( G \) is four-connected if it contains no separating triplets.
Let $G=(V,E)$ be an undirected graph and let $V' \subseteq V$. A graph $G'=(V',E')$ is a subgraph of $G$ if $E' \subseteq E \cap \{(v_i, v_j)\mid v_i, v_j \in V'\}$. The subgraph of $G$ induced by $V'$ is the graph $G''=(V'',E'')$ where $E'' = E \cap \{(v_i, v_j)\mid v_i, v_j \in V'\}$.

3. The tight upper bound for $k=2$

**Theorem 1** The maximum number of separating pairs in an undirected biconnected graph is $\frac{n(n-3)}{2}$.

**Proof:** Let $(v_1, v_2)$ be a separating pair of a biconnected graph $G$ on $n$ vertices and $m$ edges, whose removal separates $G$ into nonempty $G_1$ and $G_2$ (see Figure 1).

Let $g(n)$ be the maximum number of separating pairs in a graph on $n$ vertices. Then we can divide all separating pairs into four types:

1. Separating pairs completely inside $G_1 \cup \{v_1, v_2\}$.

2. Separating pairs completely inside $G_2 \cup \{v_1, v_2\}$.

3. Separating pairs with one vertex from $G_1$ and one vertex from $G_2$.

4. The separating pair $(v_1, v_2)$.

The number of separating pairs of type one and type two are upper bounded by $g(l+2)$ and $g(n-l)$, respectively, where $l$ is the cardinality of $V(G_1)$ and $n-l-2$ is the cardinality of $V(G_2)$. The number of separating pairs of type three is trivially upper bounded by $l(n-l-2)$. Hence, any function $g(n)$ that satisfies the recurrence

![Figure 1](image)

Separating $G$ into nonempty $G_1$ and $G_2$ by separating pair $(v_1, v_2)$.
is an upper bound on the number of separating pairs in a graph on \( n \) vertices.

We note that \( g(n) = \frac{n(n-3)}{2} \) satisfies this recurrence.

Graph \( C_n \), the cycle on \( n \) vertices, has \( \frac{n(n-3)}{2} \) separating pairs, so the bound is worst-case optimal.

Even though the number of separating pairs in a biconnected \( n \)-node graph \( G = (V,E) \) can be as large as \( \Theta(n^2) \), we observe that there are more succinct representations for them.

1. The tree of triconnected components of a biconnected graph has size \( O(n+m) \), where \( |E| = m \) [HoTa,MiRa], and this is a representation for all separating pairs together with the triconnected components of the graph.

2. The algorithm in [MiRa] enumerates the separating pairs as a collection \( C = \{V_1, \ldots, V_s\} \) of subsets of \( V \), with the interpretation that any pair of vertices within a single \( V_i \) is either a separating pair for \( G \) or the endpoints of an edge in a specified 'ear' in \( G \), and further, every separating pair for \( G \) appears in at least one of the \( V_i \)’s. It is not difficult to establish that \( \sum_{i=1}^{s} |V_i| = O(n) \); thus this gives an \( O(n) \) representation for separating pairs. We omit the proof of this result here since it requires extensive background material from [MiRa]. It will appear in [Ka2].

4. The upper bound for \( k=3 \)

The wheel \( W_n \) [Tu] is \( C_{n-1} \) together with a vertex \( v \) and an edge between \( v \) and every vertex on \( C_{n-1} \). It is easy to see that \( W_n \) is triconnected and has \( \frac{(n-1)(n-4)}{2} \) separating triplets. In the following theorem we prove that this is the worst-case for the number of separating triplets in a triconnected graph.

**Theorem 3** The number of separating triplets in an undirected triconnected graph is \( \leq \frac{(n-1)(n-4)}{2} \) for any \( n \).

**Proof:** Assume there exists a separating triplet \( (v_1,v_2,v_3) \) in \( G \), which separates \( G \) into nonempty \( G_1 \) and \( G_2 \) (see Figure 2). Now, we can divide separating triplets in \( G \) into 6 distinct types:

1. Separating triplets completely inside \( G_1 \cup \{v_1,v_2,v_3\} \).
2. Separating triplets completely inside \( G_2 \cup \{v_1,v_2,v_3\} \).
3. Separating triplets with one vertex from \( G_1 \), one vertex from \( G_2 \) and one vertex from \( \{v_1,v_2,v_3\} \).
Figure 2.
Separating $G$ into $G_1$ and $G_2$ by separating triplet $\{v_1,v_2,v_3\}$

4). Separating triplets with one vertex from $G_1$ and two vertices from $G_2$.

5). Separating triplets with two vertices from $G_1$ and one vertex from $G_2$.

6). The separating triplet $\{v_1,v_2,v_3\}$.

Let the number of vertices in $G_1$ be $k$, then the number of vertices in $G_2$ is $n-k-3$. Let $g(n)$ be the maximum number of separating triplets in a graph on $n$ vertices, $h(k,n-k)$ be the number of separating triplets of the third type and $f(k,n-k)$ and $f(n-k,k)$ be the number of separating triplets of the fourth and fifth types respectively.

Then any $g(n)$ that satisfies the recurrence

$$g(n) = \max_k (g(k+3) + g(n-k) + h(k,n-k) + f(k,n-k) + f(n-k,k) + 1)$$

is an upper bound on the number of separating triplets in $G$.

Let us now find the upper bounds for the functions $h$ and $f$.

Lemma 2: $f(k,n-k) + f(n-k,k) \leq \frac{3}{2} (3n-14)$.

Proof: Let $\{w_1,w_2,w_3\}$ be a separating triplet with $w_1 \in G_1$ and $w_2,w_3 \in G_2$. The separating triplet $\{w_1,w_2,w_3\}$ separates $G_1$ into $L_1$ and $L_2$, and separates $G_2$ into $L_3$ and $L_4$ (see Figure 3). Let us see how the original separating triplet $\{v_1,v_2,v_3\}$ is separated by the separating triplet $\{w_1,w_2,w_3\}$.

All $v_i, i=1,2,3$ cannot belong to one separated component of $G$ with respect to the separating triplet $\{w_1,w_2,w_3\}$, otherwise either $w_1$ would be an articulation point, or $\{w_2,w_3\}$ would be a separating pair, or both. W.L.O.G. assume that $v_1$ belongs to one separated component and $v_2,v_3$ to the other.
Separating $G_1$ into $L_1$ and $L_2$ and $G_2$ into $L_3$ and $L_4$ by $(w_1, w_2, w_3)$

Subgraph $L_1$ must be empty, otherwise $(w_1, v_1)$ becomes a separating pair. Since the graph is triconnected, $(w_1, v_1) \in E$, $\exists x, y \in L_3 \cup w_2 \cup v_3: (x, v_1) \in E$, $(y, v_1) \in E$ and $\forall z \in L_2 \cup L_4 \cup v_2 \cup v_3: (z, v_1) \in E$. Hence, vertex $w_1$ is unique up to a division of the original separating triplet $(v_1, v_2, v_3)$ into $v_1$ and $v_2$. So, if there is a separating triplet of the fourth type which separates $v_1$ from $v_2$ and $v_3$ then there is no separating triplet of the fifth type which separates $v_1$ from $v_2$ and $v_3$.

Let us see how many separating triplets of the fourth type there are in $G$ that separate the original separating triplet $(v_1, v_2, v_3)$ into $v_1$ and $v_2$. The vertex $w_1$ must belong to all of them. Let us see the choices for $(w_2, w_3)$, such that $(w_1, w_2, w_3)$ is a separating triplet of the fourth type.

Assume there is a separating triplet of the fourth type $(w_1, u_1, u_2)$, where $u_1 \in L_3$, $u_2 \in L_4$. The separating triplet $(w_1, u_1, u_2)$ separates $L_3$ into $L_{\cdot 3}$ and $L_{\cdot 3}$, and separates $L_4$ into $L_{\cdot 4}$ and $L_{\cdot 4}$ (see Figure 4).

The vertex $v_1$ is connected by an edge to only one of the $L_{\cdot 3} \cup u_1$ and $L_{\cdot 3}$, otherwise $(w_1, u_1, u_2)$ is not a separating triplet. If $v_1$ is not connected to the $L_{\cdot 3} \cup u_1$ and $L_{\cdot 3}$ then $(w_2, w_3)$ is a separating pair. W.L.O.G. assume $\forall x \in L_{\cdot 3}: (x, v_1) \in E$. By the symmetry $(v_2, v_3)$ is connected to only one of the $L_{\cdot 4}$ and $L_{\cdot 4}$. Let us see how the separating triplet $(w_1, u_1, u_2)$ separates $(w_2, w_3)$.

If vertices $w_2$ and $w_3$ are not separated by $(w_1, u_1, u_2)$ then there are four cases to consider.

When $w_2$ and $w_3$ belong to the same component as $L_{\cdot 3}$ and $L_{\cdot 4}$ with respect to the separating triplet $(w_1, u_1, u_2)$ and $(v_2, v_3)$ is connected by an edge to $L_4$ then $(w_1, u_2)$ is a separating pair which separates $L_2 \cup (v_2, v_3) \cup L_4$ from $v_1 \cup L_3 \cup (w_2, w_3) \cup L_{\cdot 4}$. 

Figure 3.
Figure 4.
Separating $L_3$ into $L'_3$ and $L_3$ and $L_4$ into $L'_4$ and $L_4$ by $(w_1,u_1,u_2)$

When $w_2$ and $w_3$ belong to the same component as $L'_3$ and $L'_4$ with respect to the separating triplet $(w_1,u_1,u_2)$ and $(v_2,v_3)$ is connected by an edge to $L'_4$ then $(u_1,u_2)$ is a separating pair which separates $L_3 \cup L_4$ from the rest of the graph.

When $w_2$ and $w_3$ belong to the same component as $L_3$ and $L_4$ with respect to the separating triplet $(w_1,u_1,u_2)$ and $(v_2,v_3)$ is connected by an edge to $L'_4$ then $(u_1,u_2)$ is a separating pair which separates $L_3 \cup (w_2,w_3) \cup L_4$ from the rest of the graph.

When $w_2$ and $w_3$ belong to the same component as $L_3$ and $L_4$ with respect to the separating triplet $(w_1,u_1,u_2)$ and $(v_2,v_3)$ is connected by an edge to $L_4$ then $(w_1,u_1)$ is a separating pair which separates $L_3 \cup v_1$ from the rest of the graph.

Hence, $w_2$ and $w_3$ belong to different components with respect to the separating triplet $(w_1,u_1,u_2)$. Subgraph $L_3$ must be empty; otherwise $(u_1,w_3)$ becomes a separating pair. Hence, $(u_1,w_3) \in E$, otherwise $(w_1,w_2)$ is a separating pair. If $(v_2,v_3)$ is connected to $L'_4$ then $(u_1,u_2)$ is a separating pair or $(w_1,u_1,u_2)$ is not a separating triplet. So, $\forall x \in L'_4: (x,v_2) \in E, (x,v_3) \in E, \exists y,z \in L_4 \cup (w_2,w_3): (y,v_2) \in E, (z,v_3) \in E$. Subgraph $L'_4$ must be empty, otherwise $(w_2,u_2)$ is a separating pair or $(w_1,u_1,u_2)$ is not a separating triplet. Hence, $(u_2,w_2) \in E$, otherwise $(w_1,w_3)$ is a separating pair (see Figure 5).

The above means that for each separating triplet $(w_1,w_2,w_3)$ there exists at most one separating triplet $(w_1,u_1,u_2)$ such that $u_1 \in L_3$ and $u_2 \in L_4$. So, $\forall x \in L'_3, \forall y \in L_4 \{w_1,x,w_3\}, \{w_1,x,u_2\}, \{w_1,y,w_2\}, \{w_1,y,u_1\}$.
Figure 5. Illustrating the configuration between separating triplets \( \{w_1, w_2, w_3\} \) and \( \{w_1, u_1, u_2\} \) and \( \{w_1, y, x\} \) are not separating triplets.

Let the number of vertices in \( L_3 \) be \( l \) then the number of vertices in \( L_4 \) will be \( n-k-l-4 \). Then the maximum number of separating triplets that use \( w_1 \) is

\[
r(n-k-3) = \max_i \left[ r(n-k-l-5) - 1 + r(l+2) - 1 - 4 \right] = \max_i \left[ r(n-k-l-5) + r(l+2) \right] + 2, \quad r(2) = 1, \quad r(1) = 0,
\]

where \( r(n-k-l-5) - 1 \) counts all separating triplets which use \( w_1 \) and two vertices from \( L_3 \cup u_2 \cup w_3 \), \( r(l+2) - 1 \) counts all separating triplets which use \( w_1 \) and two vertices from \( L_3 \cup u_1 \cup w_2 \) and \( 4 \) counts \( \{w_1, u_1, u_2\} \), \( \{w_1, w_2, w_3\} \), \( \{w_1, u_1, w_2\} \) and \( \{w_1, u_2, w_3\} \).

The solution for this recurrence is \( r(n-k-3) \leq \frac{3}{2} (n-k-3) - 2 \). Since there exists a unique \( w_1 \), for every separation of \( v_i \), \( i = 1, 2, 3 \) from the other two \( v_i \)'s, the upper bound for the separating triplets of the fourth and fifth types together is:

\[
f(k, n-k) + f(n-k, k) \leq 3 \left( \max \frac{3}{2} \max (n-k-3) - 2 \right) \leq \frac{3}{2} \left[ 3(n-4)-2 \right] = \frac{3}{2} (3n-14).
\]

\[ \square \]

Corollary The maximum number of separating triplets of the fourth type which separate \( \{v_i\} \) from \( \{v_1, v_2, v_3\} \setminus \{v_i\} \) is \( \leq \frac{3}{2} (n-k-3) - 2 \).

Analogously, we can state corollary for the fifth type separating triplet.
**Lemma 3** \( h(k, n-k) \leq k(n-k-3) \).

**Proof:** Assume there is separating triplet \((w_1, v_2, w_2)\) of the third type in \(G\), where \(w_1 \in G_1\) and \(w_2 \in G_2\). It separates \(G\) into \(K_1\) and \(K_2\), and separates \(G_2\) into \(K_3\) and \(K_4\). Vertices \(v_1\) and \(v_2\) must belong to the different components with respect to separating triplet \((w_1, v_2, w_2)\), otherwise either \((w_1, v_2)\) is a separating pair, or \((w_2, v_2)\) is a separating pair, or both.

**Claim 1** Vertex \(v_2\) has a direct edge to every nonempty subgraph \(K_1, K_2, K_3, K_4\).

W.L.O.G. assume that \(K_1\) is not empty and \(\forall x \in K_1, (x, v_2) \in E\). Then \((v_1, w_1)\) is a separating pair of \(G\), which separates \(K_1\) from the rest of the graph.

Now, we will prove that there are no separating triplets of the third type which use \(v_1\) or \(v_3\). We will prove this by contradiction. W.L.O.G. assume there is a separating triplet \((u_1, v_1, u_2)\), where \(u_1 \in G_1\) and \(u_2 \in G_2\) \((u_1\) may be equal to \(w_1\) and \(u_2\) may be equal to \(w_2\)).

**Case 1:** \(u_1 \in K_2\), if \(K_2\) is not empty (see Figure 6).

![Figure 6](image)

**Illustrating Case 1 in the proof of Lemma 3**

By Claim 1 for \(v_1\) and the existence of separating triplet \((u_1, v_1, u_2)\), \(K_1, w_1, K_2 - u_1\) belong to the same connected component with respect to separating triplet \((u_1, v_1, u_2)\). If \(v_2\) belongs to the same component then \((v_1, u_1)\) is a separating pair which separates \(K_3 \cup w_2 \cup K_4 \cup v_3\) from the rest of the graph. If \(v_2\) does not belong to the same component then \((v_1, u_1)\) is a separating pair which separates \(K_1 \cup w_1 \cup K_2 - u_1\) from the rest of the graph.
Analogously, \( u_2 \in K_4 \).

Case 2: \( u_1 = w_1 \).

Since \( \{u_1,v_1,u_2\} \) is a separating triplet then \( v_2 \) does not have any edges to \( K_1 \) and hence, \( K_1 \) is empty by Claim 1. But then \( \{v_1,u_2\} \) is a separating pair, if \( \{u_1,v_1,u_2\} \) is a separating triplet.

Analogously, \( u_2 \neq w_2 \).

Case 3: \( u_1 \in K_1 \) and \( u_2 \in K_3 \).

If \( \{u_1,v_1,u_2\} \) is a separating triplet then either \( \{u_1,u_2\} \), or \( \{u_1,v_1\} \), or \( \{v_1,u_2\} \) is a separating pair.

That means that if there is a separating triplet of the third type which uses one of the \( v_i, i = 1,2,3 \) then there are no separating triplets of the third type that use the other \( v_j, j = 1,2,3, j \neq i \).

Since the number of choices for \( w_1 \) is \( |V(G_1)| = k \) and the number of choices for \( w_2 \) is \( |V(G_2)| = (n-k-3) \), the number of separating triplets of the third type is \( h(k,n-k) \leq k(n-k-3) \).

Let us now tighten the upper bound for the number of separating triplets in the triconnected graph \( G \). Assume that \( \{v_1,v_2,v_3\} \) divides the graph such that the ratio \( \frac{|V(G_1)|}{|V(G_2)|} \) is as close to one as possible over all separating triplets in \( G \). From Lemma 3 we know that there is a unique vertex among \( \{v_1,v_2,v_3\} \) that participates in the separating triplets of the third type. W.L.O.G., let this vertex be \( v_2 \).

Lemma 4: If there is a separating triplet of the fourth type or the fifth type that separates \( v_2 \) from \( v_1 \) and \( v_3 \) then there are no separating triplets of the third type.

Proof: W.L.O.G., assume there exists a separating triplet of the fourth type \( \{w_1,w_2,w_3\} \), with \( w_1 \in G_1 \) and \( w_2,w_3 \in G_2 \), which separates \( v_2 \) from \( v_1 \) and \( v_3 \). It separates \( G_1 \) into \( K_1 \) and \( K_2 \), and separates \( G_2 \) into \( K_3 \) and \( K_4 \).

From the proof of Lemma 2, \( K_1 \) is empty. \((w_1,v_2) \in E \) and \((x,v_2) \in E \). \( \forall x \in G_1 \cup v_1 \cup v_3 - w_1 \) (see Figure 7).

Assume there is a separating triplet of the third type \( \{u_1,v_2,u_2\} \), where \( u_1 \in G_1 \) and \( u_2 \in G_2 \). By Claim 1 \( v_2 \) must be connected by an edge to every nonempty component of \( G_1,G_2 \) which is created by the separator \( \{u_1,v_2,u_2\} \). By the proof of Lemma 3 \( u_1 = w_1 \). If \( v_1 \) and \( v_3 \) are separated by \( \{w_1,w_2,w_3\} \) then \( (v_2,w_3) \in E \) and \((x,v_2) \in E, \forall x \in G_2 - w_2 - w_3 \). Furthermore, by Claim 1, no separating triplet of the third type exists. If \( v_1 \) and \( v_3 \) are not separated by \( \{w_1,w_2,w_3\} \) then \( (v_2,u_2) \) is a separating pair. These two contradictions prove the
Figure 7. Illustrating the proof of Lemma 4

Now we will do a case by case analysis of trade-offs between separating triplets of the third type and the separating triplets of the fourth type and the fifth type.

Case 1: There are no separating triplets of the fourth type or the fifth type.

Let \( g(n) \) be the maximum number of separating triplets of \( G \) on \( n \) vertices. Then, using Lemma 3 we obtain the following recurrence relation

\[
g(n) = \max_{1 \leq k \leq n-4} (g(k+3) + g(n-k) + k(n-k-3) + 1)
\]

The smallest function satisfying this recurrence is \( g(n) = \frac{1}{2}n^2 - \frac{5}{2}n + 2 \). Note that, with this solution, equality holds since the wheel \( W_n \) has this number of separating triplets.

By Lemma 2, if there exists a separating triplet of the fourth type that separates \( v_1 \) from \( v_2 \) and \( v_3 \), then no separating triplet of the fifth type exists which separates \( v_1 \) from \( v_2 \) and \( v_3 \). Since the separating triplets of the fourth type and the fifth type are analogous, we need only consider one of them in the case analysis.

Case 2: There is a separating triplet of the fourth type that separates \( v_1 \) from \( v_2 \) and \( v_3 \).

Let \( (w_1, w_2, w_3) \) be such a separating triplet, with \( w_1 \in G_1 \) and \( w_2, w_3 \in G_2 \). It separates \( G_2 \) into \( G_2' \) and \( \tilde{G}_2 \) and \( G_1 = \{w_1\} \cup \tilde{G}_1 \). Furthermore, suppose \( \{w_1, w_2, w_3\} \) maximizes \( |V(G_2')| \), where \( G_2' \) is the part of \( G_2 \)
separated by \((v_1, w_2, w_3)\). Define \(G_2 = G_2 - G_2' - w_2 - w_3\) and let \(|V(G_2')| = l\). Now we will consider three cases depending on whether separating triplets of the fourth and fifth types exist, which separate \(v_3\) from \(v_1, v_2\). We do not restrict separating triplets which involve \(v_2\).

**Case A:** There are no separating triplets of the fourth type or the fifth type that separate \(v_3\) from \(v_1\) and \(v_2\).

If there is a separating triplet \((u_1, v_2, u_2)\), of the third type where \(u_1 \in G_1\) and \(u_2 \in G_2\), then \(u_2 \in \tilde{G}_2\) by Claim 1.

Hence, the following recurrence relation is obtained using the corollary to lemma 2:

\[
g(n) = \max_{1 \leq k \leq n-3} \left( g(k+3) + g(n-k) + \max_{0 \leq l \leq k} \left( k(n-k-l-5) + \frac{3}{2}(l+2) - 1 \right) \right).
\]

Since the function to be maximized is linear in \(l\), the maximum is reached at one of the endpoints of the interval for \(l\). If \(k \leq 1\) then the maximum is reached when \(l = n-k-6\). But in this case \((v_1, w_2, w_3)\) would be chosen instead of \((v_1, v_2, v_3)\). If \(k > 1\) then the maximum is reached when \(l = 0\) and the recurrence becomes

\[
g(n) = \max_{1 \leq k \leq n-3} \left( g(k+3) + g(n-k) + k(n-k-5) + 2 \right),
\]

whose solution is no greater than the bound of Case 1.

**Case B:** There is a separating triplet of the fourth type which separates \(v_3\) from \(v_1\) and \(v_2\).

Let \((x_1, x_2, x_3)\) be such a separating triplet, with \(x_1 \in G_1\) and \(x_2, x_3 \in G_2\). Furthermore, suppose \((x_1, x_2, x_3)\) maximizes \(|V(G_2')|\), where \(G_2\) is the part of \(G_2\) separated by \((v_3, x_2, x_3)\).

Vertices \(x_2, x_3 \in \tilde{G}_2 \cup w_2 \cup w_3\), otherwise \(G\) is not triconnected. Define \(G_2 = G_2 - G_2' - x_2 - x_3\) and let \(|V(G_2')| = \bar{l}\). If there is a separating triplet of the third type \((u_1, v_2, u_2)\), where \(u_1 \in G_1\) and \(u_2 \in G_2\), then by Claim 1 \(u_2 \in \tilde{G}_2\).

Hence, the following recurrence relation is obtained using the corollary to lemma 2:

\[
g(n) = \max_{1 \leq k \leq n-3} \left( g(k+3) + g(n-k) + \max_{0 \leq l \leq k} \left( k(n-k-l-6) + \frac{3}{2}(l+2) - 4 + 1 \right) \right).
\]

As in Case A, the maximum is reached when \(l = \bar{l} = 0\), if \(k > 1\). Hence, the equality becomes

\[
g(n) = \max_{1 \leq k \leq n-3} \left( g(k+3) + g(n-k) + k(n-k-5) + 3 \right),
\]

which again gives a worse upper bound than the bound of Case 1. If \(k = 1\) then the maximum is reached when either \(l = n-k-5\) and \(\bar{l} = 0\) or \(\bar{l} = n-k-5\) and \(l = 0\). But in this case either \((v_1, w_2, w_3)\) or \((v_3, x_2, x_3)\) would be chosen instead of \((v_1, v_2, v_3)\).

**Case C:** There is a separating triplet of the fifth type which separates \(v_3\) from \(v_1\) and \(v_2\).
Let \((x_1,x_2,x_3)\) be such a separating triplet, with \(x_1 \in G_2\) and \(x_2,x_3 \in G_1\). Furthermore, suppose \((x_1,x_2,x_3)\) maximizes \(|V(G_1)|\), where \(G_1\) is the part of \(G\) separated by \((v_3,x_2,x_3)\). Define \(G'_1 = G_1 - x_2 - x_3 - w_1\) and let \(|V(G'_1)| = \tilde{l}\). Since \((v_1,v_2,v_3)\) was chosen as the initial separating triplet instead of \((v_1,v_2,x_1)\) or \((w_1,v_2,v_3)\), 
\(|V(G_1)| - |V(G_2)| \leq 1\). Therefore, \(k = \lceil \frac{n-3}{2} \rceil\) or \(\lceil \frac{n-3}{2} \rceil\). Since these two cases are analogous, assume \(k = \lceil \frac{n-3}{2} \rceil\).

If there is a separating triplet of the third type \((u_1,v_2,u_2)\), where \(u_1 \in G_1\) and \(u_2 \in G_2\), then by Claim 1 \(u_1 \in G'_1 \cup w_1\) and \(u_2 \in G'_2 \cup x_1\). Hence, the recurrence relation obtained is using the corollary to lemma 2:

\[ g(n) = g\left(\left\lfloor \frac{n+3}{2} \right\rfloor\right) + g\left(\left\lceil \frac{n+3}{2} \right\rceil\right) + \max_{\text{osd } \frac{n+3}{2}-2} \left(\left\lfloor \frac{n-3}{2} \right\rfloor - l - 1\right) \left(\left\lceil \frac{n-3}{2} \right\rceil - \tilde{l} - 1\right) + \frac{3}{2} (l + \tilde{l} + 4 - 3).\]

The right hand side is bilinear in \(l\) and \(\tilde{l}\), hence the maximum is reached at the endpoints of the intervals. If \(l\) or \(\tilde{l}\) is equal to 0 then we get a degenerate case that is equal to case A. If \(l = \left\lfloor \frac{n-3}{2} \right\rfloor - 2\) and \(\tilde{l} = \left\lceil \frac{n-3}{2} \right\rceil - 2\) then the equality becomes

\[ g(n) = g\left(\left\lfloor \frac{n+3}{2} \right\rfloor\right) + g\left(\left\lceil \frac{n+3}{2} \right\rceil\right) + \frac{3}{2} (n - 3) - 2.\]

The solution to this recurrence is \(\leq \frac{3}{2} n \log_2 n + \frac{13}{2}\). For any \(n \geq 19\) this solution gives an upper bound smaller than \(\frac{(n-1)(n-4)}{2}\). All triconnected graphs on \(5 \leq n \leq 18\) vertices with constraints of Case C have less number of separating triplets than the wheel on \(n\) vertices. Hence, for case 2

\[ g(n) \leq \frac{(n-1)(n-4)}{2}\]

for all \(n\).

Note: Case 2 includes the case when no separating triplet of the third type exists.

This concludes the case by case analysis of the trade-offs between separating triplets of \(G\) of the third type and the separating triplets of the fourth and fifth types.

The established upper bound on the number of separating triplets of \(G\) for all \(n\) is

\[ g(n) \leq \frac{(n-1)(n-4)}{2}.\]
REFERENCES


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