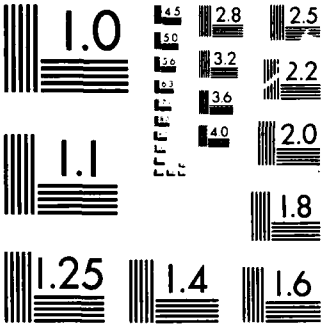
AD-A184 258 LOWER BOUNDS ON BEARING ACCURACY FOR CLOSELY SPACED SOURCES(U) AMERICAN SOCIETY FOR ENGINEERING EDUCATION WASHINGTON DC M HAMILTON 01 SEP 87 N00014-83-D-0689 1/1

UNCLASSIFIED F/G 17/1 NL



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

DTIC FILE COPY

(12)

AD-A184 258

LOWER BOUNDS ON BEARING ACCURACY FOR CLOSELY SPACED SOURCES

By Miriam Hamilton
Naval Underwater Systems Center
New London, Connecticut

DTIC
ELECTE
SEP 0 2 1987
S D

N00014-83-D-0089

Final report presented to the
American Society of Engineering Education
September 1, 1987

DISTRIBUTION STATEMENT A
Approved for public release;
Distribution Unlimited

87 9 1 054

LOWER BOUNDS ON BEARING ACCURACY FOR CLOSELY SPACED SOURCES

Abstract

Resolution of multiple sources that are closely spaced in bearing is a topic of considerable interest in passive localization applications. Many of the available studies have been devoted to the implementation of algorithms with less emphasis on the absolute lower bounds on attainable accuracy. This report presents Cramer-Rao lower bounds on bearing estimation error for two closely spaced incoherent sources which radiate spectrally overlapping broadband Gaussian noise. The radiated signals are assumed to occupy the same frequency band which is taken be to be lowpass and rectangular in shape. The signals have (possibly different) unknown spectrum levels. In the most general problem setting, e.g., when prior knowledge of signal levels or a single source bearing is unavailable, estimation accuracy is shown to degrade in proportion to the second power of the inverse angular spacing, resulting in large degradations when compared to the accuracy attainable in a single source setting. There is evidence that in certain situations, a form of apriori knowledge concerning signal power levels can be used to significant advantage in joint bearing estimation.

The detectability of a second source from a signal field containing two closely spaced sources that are overlapping in both time and frequency depends on the resolving power of the array and is expected to be poor at low signal-to-noise ratios and short observation times when the angular spacing is small. The problem is addressed by considering the following binary hypothesis test :

H_0: There are two closely spaced sources present, which radiate power SNR_1 and SNR_2, respectively;

H_1: There is one source present radiating the total received power $SNR_1 + SNR_2$.

where upper bounds on the false alarm and miss probabilities are computed. The results indicate that below a well defined threshold, defined by a combination of the two signal-to-noise ratios, observation time and angular spacing, the probability of error in guessing the wrong hypothesis tends to one half. The reachability of the earlier computed Cramer-Rao lower bounds for this regime is questionable.

per ca

A-1

Introduction

Joint bearing estimation of multiple radiating sources is a problem which has generated considerable interest in the signal processing community in recent years. A large majority of the studies have been concerned with the implementation of so-called high resolution algorithms, which have been known to be successful in resolving multiple closely spaced sources [1]-[5]. There have been fewer general studies concerned with absolute lower bounds on accuracy. Without relevant performance figures to serve as benchmarks against which the accuracy of practical algorithms may be compared, the degree of optimality for different approaches is difficult to assess. In addition, most of the studies (both analytical and algorithm-based) have treated the narrowband problem exclusively [3] -[5].

Lower bounds on bearing error in a multiple source setting have been computed by other authors and most notable of these are references [6] - [10]. Reference [6] considers Cramer-Rao lower bounds on bearing error as well as the structure of maximum likelihood bearing estimators for in the cases of both closely spaced and well separated sources. When the bearing difference is smaller than a beamwidth over the entire (overlapping) frequency band, bearing accuracy associated with a given source is shown to degrade seriously as result of the presence of an interfering signal at unknown bearing. The author demonstrates that the result holds for both broadband and narrowband signals. The analysis yields considerable insight into the structure of the likelihood function as well as the lower bounds under the two extreme differential bearing situations, but it employs the assumption of known spectral properties for the sources. Since the high resolution methods do not require prior knowledge of spectral properties to determine bearing, and because generally the spectral properties are unknown apriori, a reliable basis for comparison of the lower bounds with practical estimation performance would seem to require extension of the unknown parameter set in lower bound computations to include spectral parameters.

Reference [7] presents analytical forms for the complete Fisher matrix corresponding to estimation of both bearing and power levels for two incoherent narrowband sources. The authors point out that for bearing differences which exceed a beamwidth, the impact of uncertainty in power levels on bearing accuracy could be minimized with the use of arrays which exhibit smooth beampattern values in the neighborhood of an interference. The discussion was confined to source separations that exceed a beamwidth, leaving open the question of estimation accuracy degradations for sources in a closely spaced configuration.

Ng in reference [8] computes analytical forms of the Fisher Information matrix for a two receiver array, leaving open the question of performance for larger arrays, as well as the performance degradations resulting from unknown signal levels. Boehme in reference [9] demonstrates with extensive simulation results the statistical coupling of bearing, range and signal level estimates in a multiple source setting. His study considered both well separated and closely spaced

sources but presented no performance figures.

This study considers the bearing estimation accuracy of two closely spaced sources under conditions in which prior knowledge of the radiated signal powers is not necessarily available. The source signals are modelled as spectrally and temporally overlapping. The study investigates the performance of line arrays of equally spaced sensors under conditions of practical interest. The power ratio of the two source is demonstrated to play a major role in determining accuracy. The discussion is largely focussed on the situation in which the angular difference between the sources is extremely small.

At low signal to noise ratios and short observation times, the ability of the array to distinguish the two sources from a single source is expected to deteriorate, and this suggests that without apriori knowledge of the number of sources present, the Cramer-Rao lower bounds are simply unreachable. The final portion of the study addresses the resolvability of the two sources when the angular difference becomes small.

The report is organized as follows. In section 1, the general theory is presented which includes analytical forms of the lower bounds on source bearings. Section 2 discusses features of the bounds when the signal levels are known apriori. The relative signal to noise ratios of the individual sources is shown to play an important role in determining overall accuracy. Unknown signal power levels are introduced in section 3 where accuracy reductions in the performance computed in section 2 are shown to be quite large. Section 4 considers the resolvability of the sources. A summary of the important results is given in section 5.

1. General Theory

An equally spaced line array is located in the far field relative to a pair of incoherent radiating sources (Figure 1). The sources are oriented at bearings α_1 (source 1) and α_2 (source 2) with respect to the array axis. The signal received by any given sensor in the array is equal to the sum of the two source signals in addition to a noise signal that is assumed to be locally generated or produced by the receiver itself :

$$x_i(t) = s_1(t-t_i) + s_2(t-z_i) + n_i(t) \quad ; \qquad 0 \le t \le T, \quad i = 1, \ldots, M \tag{1}$$

It is assumed that the signals $s_1(t)$ and $s_2(t)$ are mutually incoherent but individually (perfectly) coherent over the receiving array, while the noise signals are statistically independent from sensor to sensor. The variable t_i denotes the propagation delay for source signal 1 to travel to sensor i. Similarly z_i refers to the delay associated with source

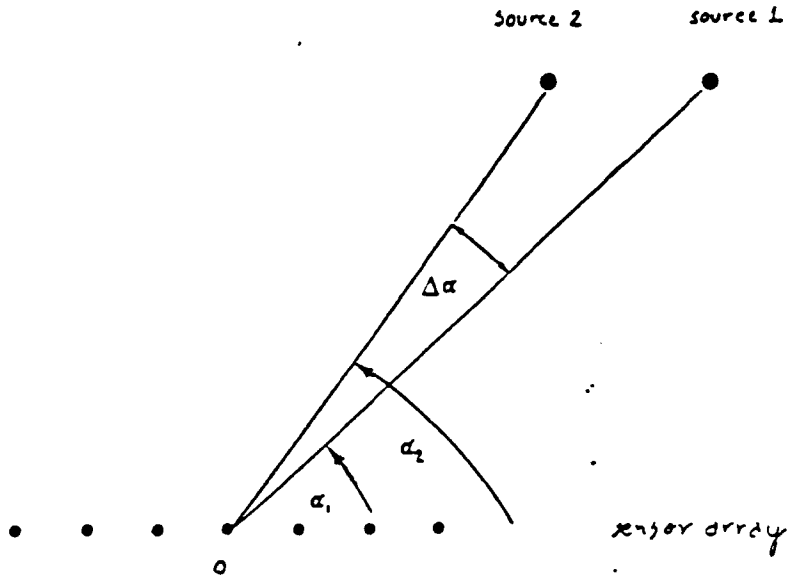


Figure 1

The sensor spacing, d, is uniform over the array. Since there are a total of M receivers in the array, the array has length (M-1)d = L. The angular difference between the sources measured with respect to the array axis at the midpoint is $\Delta\alpha$.

Based on signal observations from the sensors, we wish to find the lower bound on estimates of α_1 and α_2. If the spectral properties of the signal and noise processes are known apriori, the parameters of interest include only bearings and the relevant parameter vector $\underline{\theta}$ is given by :

$$\underline{\theta} = [\ \alpha_1, \alpha_2]^T$$

The inverse of the well known Fisher Information matrix [11] yields the error variances of estimates for parameters in $\underline{\theta}$:

$$J_{ij} = E\left\{ \frac{\partial \ln p(\underline{X}/\underline{\theta})}{\partial \theta_i} \; \frac{\partial \ln p(\underline{X}/\underline{\theta})}{\partial \theta_j} \right\} \tag{2}$$

$$\mathrm{Cov}\,(\,\hat{\alpha}_i, \hat{\alpha}_j\,) \geq J^{-1}\Big|_{ij} \tag{3}$$

If the spectral parameters are unknown, $\underline{\theta}$ will be suitably enlarged. In the first part of this discussion, the effects of uncertain spectral parameters will be ignored.

The following assumptions are used in the analysis :

1. The signals $s_1(t)$ and $s_2(t)$ are sample functions from statistically independent stationary Gaussian processes with zero means. The processes have lowpass, rectangular spectra of equal width, W, but with possibly different heights.

2. The noise signals appearing at the various receivers are sample functions from statistically independent, stationary Gaussian processes with zero means and identical spectral properties.

3. The observation time, T, is much longer than the (common) signal correlation time.

The observed data is represented by the vector of Fourier coefficients of the signals received by the various sensors concatenated:

$$\underline{X} = [\; \underline{X}(\omega_1)^T, \; \underline{X}(\omega_2)^T, \; . \; . \; . \; , \; \underline{X}(\omega_N)^T \;]^T$$

where

$$\underline{X}(\omega_k) = [\; X_1(\omega_k), X_2(\omega_k), \; . \; . \; . \; . \; , \; X_M(\omega_k) \;]^T$$

$$X_i(\omega_k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} x_i(t)\; e^{-j\omega t} dt$$

The probability density function of the data vector, $\underline{X}$, conditioned on $\underline{\theta}$, is complex and Gaussian with zero mean. If the time-bandwidth product TW is much larger than one, the data Fourier coefficients associated with different frequencies are approximately uncorrelated. Under these cond-itions, the pdf of the data vector $\underline{X}$ has the form:

TW

$$p(\underline{X}/\underline{\theta}) = \prod_{k=1}^{TW} [\ \det \pi K_k\ ]^{-1} \exp(\ -\underline{X}_k\ K_k^{-1} \underline{X}_k\ ) \tag{4}$$

K_k is the covariance matrix of the sensor data at frequency ω_k. Define the signal steering vectors $\underline{v_1}$ and $\underline{v_2}$ as follows :

$$\underline{v_1} = [\ e^{-j\omega_k t_1}\ ,\ .\ .\ .\ .\ ,\ e^{-j\omega_k t_M}\ ]^T \tag{5}$$

$$\underline{v_2} = [\ e^{-j\omega_k z_1}\ ,\ .\ .\ .\ .\ ,\ e^{-j\omega_k z_M}\ ]^T \tag{6}$$

K_k then has the form :

$$K_k = S_1(\omega_k)\underline{v_1}\underline{v_1}^{+} + S_2(\omega_k)\underline{v_2}\underline{v_2}^{+} + N(\omega_k)I \tag{7}$$

where I is the identity matrix. $S_1(\omega_k)$, $S_2(\omega_k)$ and $N(\omega_k)$ are the power spectra of $s_1(t)$, $s_2(t)$ and $n_i(t)$, respectively.

The elements of J can be obtained from equation (2), using the following well known results (11) :

$$J_{ij} = \sum_{k=1}^{TW} Tr(\ K_k^{-1} \frac{\partial K_k}{\partial \theta_i} K_k^{-1} \frac{\partial K_k}{\partial \theta_j}\ ) \tag{8}$$

$$= \sum_{k=1}^{TW} -Tr(\ \frac{\partial}{\partial \theta_i} (K_k^{-1})\ \frac{\partial K_k}{\partial \theta_j}\ ) \tag{9}$$

K_k can be inverted with the use of Bartlett's Identity:

$$K_k^{-1} = \frac{1}{N_k} \left[I - G_k \left\{ P_1 \underline{v}_1 \underline{v}_1^{\dagger} + P_2 \underline{v}_2 \underline{v}_2^{\dagger} - P_1 P_2 \left\{ \underline{v}_1 \underline{v}_1^{\dagger} \underline{v}_2 \underline{v}_2^{\dagger} + \underline{v}_2 \underline{v}_2^{\dagger} \underline{v}_1 \underline{v}_1^{\dagger} \right\} \right\} \right] \tag{10}$$

The k - subscripts have been omitted in the quantities P_1 and P_2 in (10) to simplify subsequent algebraic expressions :

$$P_1 = \frac{S_1(\omega_k)/N(\omega_k)}{1 + MS_1(\omega_k)} \quad ; \quad P_2 = \frac{S_2(\omega_k)/N(\omega_k)}{1 + MS_2(\omega_k)} \tag{11}$$

$$G_k = \frac{1}{1 - P_1 P_2 \, |\underline{v}_1^{\dagger} \underline{v}_2|^2} \quad . \tag{12a}$$

$$|\underline{v}_1^{\dagger} \underline{v}_2|^2 = \sum_{i,j=1}^{M} \cos \left[\frac{\omega_k \, (t_i - t_j - z_i + z_j)}{c} \right] \tag{12b}$$

Note the dependence of the function $|\underline{v}_1^{\dagger} \underline{v}_2|^2$ on the angular frequency ω and the differential bearing and range between sources which is expressed by the quantity

$$(t_i - t_j - z_i + z_j).$$

Calculations carried out in Appendix 1 yield the following expressions for elements of the Fisher Information matrix :

$$J_{11} = \sum_{k=1}^{TW} G_k \left\{ P_1 \, S_1(\omega_k)/N(\omega_k) \left\{ \mathcal{Z}_1 - P_2 \mathcal{Y}_1 \right\} + P_1^2 P_2^2 G_k \eta_1^2 \right\} \tag{13}$$

$$J_{12} = \sum_{k=1}^{TW} G_k P_1 P_2 \left\{ \frac{\partial \, |\underline{v}_1^{\dagger} \underline{v}_2|^2}{\partial \alpha_1 \, \partial \alpha_2} + G_k P_1 P_2 \eta_1 \eta_2 \right\} \tag{14}$$

$$J_{22} = \sum_{k=1}^{TW} G_k \left\{ P_2 \, S_2(\omega_k)/N(\omega_k) \left\{ \mathcal{Z}_2 - P_1 \mathcal{Y}_2 \right\} + P_1^2 P_2^2 G_k \eta_2^2 \right\} \tag{15}$$

The variables $\eta_1, \eta_2, \mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Y}_1, \mathcal{Y}_2$ are defined as follows :

$$\eta_i \triangleq \frac{\partial}{\partial \alpha_i} \, |\underline{v}_1^{\dagger} \underline{v}_2|^2 \qquad ; \; i = 1,2 \tag{16}$$

$$z_i \triangleq \frac{\partial^2}{\partial\alpha_i \partial\alpha_i} \left| \underline{v}_i^+ \underline{v}_i^- \right|^2 \Big|_{\alpha_i^- = \alpha_i} \qquad ; \; i = 1,2 \tag{17}$$

$$y_i \triangleq 2 \, \mathrm{Tr} \left\{ \left(\frac{\partial}{\partial\alpha_i} \underline{v}_i \underline{v}_i^+ \right) \left(\underline{v}_j \underline{v}_j^+ \right) \left(\frac{\partial}{\partial\alpha_i} \underline{v}_i \underline{v}_i^+ \right) \right\} \qquad ; \; i \neq j = 1,2 \tag{18}$$

z_i is recognized as the familiar pattern function derivative. From an analytical point of view, z_i is the only geometrical quantity that is important in determining the accuracy of a location parameter for a single source. One can show this by setting $S_2(\omega_k)$ equal to zero over the frequency band in (13). It then follows that $G_k \rightarrow (1 + MS_1(\omega_k)/N(\omega_k))^{-1}$ and one obtains for J_{11}^{-1} :

$$\mathrm{Var}(\hat{\alpha}_1) \geq J_{11}^{-1} \triangleq \left\{ \sum_{k=1}^{TW} \frac{(S_1(\omega_k)/N(\omega_k))^2}{1 + MS_1(\omega_k)/N(\omega_k)} \, z_1 \right\}^{-1} \tag{19a}$$

It can be shown (see Appendix 2) that when the sensors are arranged in a line array configuration of total length L, (19a) is equivalent to:

$$\mathrm{Var}(\hat{\alpha}_1) \geq \left\{ \sum_{k=1}^{WT} \frac{(S_1(\omega_k)/N(\omega_k))^2}{1 + MS_1(\omega_k)/N(\omega_k)} \left(\frac{2\pi L \sin(\alpha_1)}{\lambda_k} \right)^2 \right\}^{-1} \tag{19b}$$

where λ_k is the signal wavelength at frequency k. The above expression for the variance of bearing error indicates that the important geometrical quantity determining accuracy is the ratio $L\sin(\alpha_1)/\lambda_k$. $L\sin(\alpha_1)$ is the array baseline (length of the array component in the direction orthogonal to the line of sight to the source). Accurate bearing estimates will therefore result when the baseline exceeds a signal period. Since the Cramer-Rao lower bound is a local error bound, it will not yield information about estimation accuracies resulting from ambiguous estimates (noise generated peaks in the likelihood function which occur far away from the true parameter value, or multiple solutions to the likelihood equation resulting from ambiguous delay estimates for narrowband signals). If the possibility of ambiguity error can be safely discounted, then equation (19b) states that reductions in the $L\sin(\alpha_1/\lambda_k)$ ratio must be offset by simultaneous increases in the signal to noise ratio or observation time, if operation within reasonable accuracy limits is to be acheived. The remainder of this report will be confined to a discussion of local errors only.

Returning now to equations (13) -(18), it is clear that the remaining quantities appearing in the Fisher matrix describe the interactions of the two sources resulting from effects of interference. Of prominence

are derivatives of the array beampattern,

$$|\underline{v}_1^{+}\underline{v}_2|^2 .$$

The beampattern is defined in equation (12b) and will be shown to depend on the signal frequency, the array configuration and the angles of orientation of the two sources. The array beampattern is the measured response at a particular frequency for an array which has been focussed on source 1 but which receives signal energy from the direction of an interference (source 2). At frequencies for which the beampattern is large, the sources appear to radiate from nearly the same bearing angle. At frequencies for which the beampattern is low the sources are easily distinguishable. Of particular interest to us is the case in which the sources appear to emanate from nearby bearings for all signal frequencies in the signal band.

It can be easily shown with a few steps of algebra that if the array beampattern happened to be equal to zero over the frequency band, then the beampattern, its derivatives and the γ functions would be iden--

tically zero. According to (13) - (15), this situation would result in a diagonal Fisher matrix, uncorrelated estimation errors for α_1 and α_2, and no degradation in bearing accuracy for either source - not a surprising result for well-separated sources.

For the case of closely spaced sources, however, by inspecting equations (13) - (15), the dependence of the lower bound on the two signal to noise ratios as well as the angular spacing and array configuration is quite complicated. For example, the function G_k varies with frequency and the resulting k-sums must be evaluated with care. It will be convenient to consider bearing accuracy under specific conditions of interest and for these cases the behavior of the k-sums can be made simple. The next section considers the problem of closely spaced sources.

2. Bearing Accuracy for Closely Spaced Sources

Consider now the problem of closely spaced sources, where $\alpha_1 - \alpha_2$ is sufficiently small so that

$$|\underline{v}_1^{+}\underline{v}_2|^2 \cong M^2 . \tag{20}$$

If the source signals impinge on the array in the form of plane waves, elements of the Fisher matrix can be simplified considerably. Choosing the array midpoint as the coordinate system origin, for plane wave arrivals the travel times are given by :

$$t_i = id\cos\alpha_1/c \quad ; \quad z_i = id\cos\alpha_2/c \quad ; \quad i = -(M-1)/2,\ldots,(M-1)/2 \tag{21a}$$

$$t_i - z_i = id(\cos\alpha_i - \cos\alpha_j)/c \tag{21b}$$

The beampattern becomes :

$$|\underline{v}_1^{\dagger}\underline{v}_2|^2 = \sum_{i,j=-\frac{M-1}{2}}^{\frac{M-1}{2}} \cos\left\{ (\omega_k/c)(i-j)d(\cos\alpha_1 - \cos\alpha_2) \right\} \tag{22}$$

Suppose $|\alpha_1 - \alpha_2| << 2\pi$:

$$t_i - z_i \cong id\Delta\alpha\sin\alpha; \qquad \Delta\alpha = \alpha_1-\alpha_2 \; ; \qquad \alpha = (\alpha_1+\alpha_2)/2; \tag{23a}$$

Under these conditions the cosine argument of the function in (22) will vary negligibly over the i - index provided $\Delta\alpha L\sin(\alpha)W/c << 2\pi$. In such a case

$$1 - |\underline{v}_1^{\dagger}\underline{v}_2|^2 / M^2 << 1 \tag{23b}$$

Note that the condition (23b) would be satisfied in general when

$$\max_{i,j} \left|(\omega_k d/c)(i-j)(\cos\alpha_1-\cos\alpha_2)\right| << 2\pi \qquad \forall\, k \tag{24}$$

In addition to the case $\alpha_1 \cong \alpha_2$, the criterion of equation (24) can also be met whenever $L << \lambda_{min}$. We assume that this is not the case, that is, we wish to be in an estimation regime in which accurate estimates of α_1 or α_2 are possible when the measurements are obtained without the presence of the second signal arrival. When $L << \lambda_{min}$ the array simply has no means of determining bearing even for a single source since differential delays which are fractions of a signal period cannot be exploited in bearing measurement.

Consider then the lower bound for closely spaced sources, i.e., equation (24) holds but $L > \lambda_{min}$. Computations carried out in Appendix 2 yield the following approximate expressions for the Fisher matrix elements. When contributions to the lower bound of order

$$(\beta^2 W^2)$$

and smaller are neglected one obtains the following :

$$J_{11} = \sum_{k=1}^{TW} H_k(MS_1(\omega_k)/N(\omega_k))^2 x_1^2 \omega_k^2 \left[1 + \beta\omega_k^2 H_k(MS_2(\omega_k)/N(\omega_k)^2 \right] \tag{25}$$

$$J_{12} = \sum_{k=1}^{TW} H_k(M^2 S_1(\omega_k) S_2(\omega_k)/(N(\omega_k))\ )\ x_1 x_2 \omega_k^2 \Big[\ 1 - 6/5\beta\omega_k^2\ -$$

$$\beta\omega_k^2 M^2 H_k S_1(\omega_k) S_2(\omega_k)/(N(\omega_k)^2)\Big] \tag{26}$$

$$J_{22} = \sum_{k=1}^{TW} H_k(MS_2(\omega_k)/N(\omega_k))^2 x_2^2 \omega_k^2 \Big[\ 1 + \beta\omega_k^2 H_k(MS_1(\omega_k)/N(\omega_k))^2\Big] \tag{27}$$

$$x_i^2 \triangleq (L\sin\alpha_i/c)^2/6 \tag{28}$$

$$\beta \triangleq (\ \Delta\alpha L\sin\alpha/c)^2/6 \tag{29a}$$

$$\xi \triangleq 1 - |\underline{v}_1^{+}\underline{v}_2|^2/M^2 \tag{29b}$$

$$H_k \triangleq [\ 1 + M(S_1+S_2)/N + {}^2M\ S_1S_2{}^2/N\ \ \xi\ ]^{-1} \tag{29c}$$

If the TW product is large compared to one, the k - sums appearing in the above expressions can be approximated by integrals. Let $S \triangleq S_1 + S_2$. Define the following

$$I_1 \triangleq \frac{T}{2\pi}\int_0^W \frac{\omega^2\, d\omega}{(1 + MS/N + M^2S_1S_2/N^2\ \xi\ )} \tag{30}$$

$$I_2 \triangleq \frac{T}{2\pi}\int_0^W \frac{\omega^4\, d\omega}{(1 + MS/N + M^2S_1S_2/N^2\ \xi\ )} \tag{31}$$

$$I_3 \triangleq \frac{T}{2\pi}\int_0^W \frac{\omega^4\, d\omega}{(1 + MS/N + M^2S_1S_2/N^2\ \xi\ )^2} \tag{32}$$

In terms of the above integrals, the F.I.M. (Fisher Information matrix) elements can be expressed as follows :

$$J_{11} = x_1^2 (MS_1/N)^2 \left\{ I_1 + \beta (MS_2/N)^2\ I_3 \right\} \tag{33}$$

$$J_{12} \triangleq x_1 x_2 M^2 S_1 S_2/N^2 \left\{ I_1 - (6/5)\beta I_2 - \beta M^2 S_1 S_2/N^2 I_3 \right\} \tag{34}$$

$$J_{22} \triangleq x_2^2 (MS_2/N)^2 \left\{ I_1 + \beta (MS_1/N)^2\ I_3 \right\} \tag{35}$$

Consider equations (33)-(35), which are explicit functions of I_1, I_2 and I_3. If

$$1 + MS/N \quad >> \quad M^2S_1S_2/N^2\ \xi \quad \cong \quad M^2S_1S_2/N\ \frac{\beta W^2}{2}\ (\ 1 - \frac{\beta W^2}{5} + \ .\ .\ .\ ) \tag{36}$$

over the frequency band then the denominators of the integrands are dominated by the constant term, 1 + MS/N. Under this condition, the integrands of I_1, I_2 and I_3 will grow (approximately) with ω^2, ω^4 and ω^0, respectively. If, on the other hand

$$1 + MS/N \quad << \quad M^2S_1S_2/N^2\xi \quad \cong \quad M^2S_1S_2/N^2\ \frac{\beta W^2}{2}\ (\ 1 - \frac{\beta W^2}{5} + \ .\ .\ .\ ) \tag{37}$$

over some frequency band, then integrands of I_1, I_2 and I_3 will tend to vary (approximately) with ω^0, ω^2 and ω^0, respectively. Note that there is no other possibility for the behavior of H_k.

Obviously (37) will hold only at large signal to noise ratios. In fact in order for (37) to hold

$$MS_i/N \quad >> \quad (\beta\omega_k^2)^{-1} \qquad i = 1 \text{ and } 2$$

For example, if $(\beta\omega_k^2) \leq .1$, then $MS_i/N > 10$ for $i = 1,2$. On the other hand, equation (36) will hold when

$$MS_i/N < (\beta\omega_k^2)^{-1} \qquad \text{for } i = 1 \text{ or } 2.$$

Define the variable $\mathcal{R}$:

$$\mathcal{R} \triangleq \frac{M^2 S_1 S_2/N^2}{1 + MS/N} \tag{38a}$$

We will consider two regimes of interest occurring at the extreme values :

$$\mathcal{R}\xi_{max} \begin{cases} << 1 & (38b) \\ >> 1 & (38c) \end{cases}$$

where ξ_{max} is defined as the value of ξ occurring at the largest (common) signal frequency, W.

When $\mathcal{R}\xi_{max} << 1$ it is shown in Appendix 3 that elements of the F.I.M. can be approximated by :

$$J_{11} = \frac{TW}{2\pi} \frac{(MS_1/N)^2 x_1^2}{1 + MS/N} \frac{W^2}{3} \left(1 + \frac{3\mathcal{R}W^2}{5}\left(\frac{2S_2}{S_1} - 1\right)\right) \tag{38}$$

$$J_{12} = \frac{TW}{2\pi} \frac{M^2 S_1 S_2/N^2 x_1 x_2}{1 + MS/N} \frac{W^2}{3}\left(1 - \frac{6}{25}\beta W^2 - \frac{9}{5}\mathcal{R}W^2\right) \tag{39}$$

$$J_{22} = \frac{TW}{2\pi} \frac{(MS_2/N)^2 x_2^2}{1 + M(S_1+S_2)/N} x_2^2 \frac{W^2}{3}\left(1 + \frac{3\mathcal{R}W^2}{5}\left(\frac{2S_1}{S_2} - 1\right)\right) \tag{40}$$

After a few steps of algebra,

$$\det J = J_{11}J_{22} - J_{12}^2 = \left[\frac{TW}{2\pi} x_1 x_2 \frac{W^2}{3} \frac{M^2 S_1 S_2/N^2}{1 + M(S_1+S_2)/N}\right]^2 \times \Big[$$

$$\frac{3RW^2}{5}\left\{\frac{2(MS/N)^2}{M^4S_1S_2/N^4}\left(1-\frac{6}{5}\beta W^2\right)-\frac{36}{25}\beta W^2\right\}+\frac{12}{25}\beta W^2\Big)\Big] \qquad (41)$$

The error in estimating α_i is given by equation (42) which employs the convention $i \neq j$; $i,j = 1,2$.

$$\mathrm{Var}(\hat{\alpha}_i) \geq \frac{J_{ii}}{\det J} = \left[\frac{TW}{2\pi} x_i^2 \frac{W^2}{3} \frac{(MS_i/N)^2}{1+MS/N}\right]^{-1} \times$$

$$\left[\frac{1+\frac{3}{5}\beta W^2\left(\frac{(MS_i/N)^2-(1/2)M^2S_1S_2/N^2}{1+MS/N}\right)}{\frac{3}{5}RW^2\left\{\frac{2(MS/N)^2}{M^4S_1S_2/N^4}\left(1-\frac{6}{5}\beta W^2\right)-\frac{36}{25}\beta W^2\right\}+\frac{12}{25}\beta W^2}\right] \qquad (42)$$

It will be useful to consider the above expression for bearing error under the following conditions: It was shown earlier that when $R\xi_{max} << 1$, one but not both signal to noise ratios, MS_1/N and MS_2/N, could exceed $(\beta W^2)^{-1}$. However, both signal to noise ratios could be quite small. Suppose $MS_i/N << (\beta W^2)^{-1}$:$i = 1,2$. One can show that

$$\mathrm{Var}(\hat{\alpha}_i) \geq \left[\frac{TW}{2\pi} x_i^2 \beta W^2 (MS_i/N)^2 W^2\right]^{-1} \times \begin{cases} 6.25 \ ; & MS/N << 1 \\ 5 \ ; & (\beta W^2)^{-1} >> MS/N >> 1 \end{cases} \qquad (43)$$

Equation (43) states that when $MS/N << 1$ the lower bound varies inversely with the product βW^2. It tends to infinity as $\Delta\alpha$ tends to zero. When $(\beta W^2)^{-1} >> MS/N >> 1$ the lower bound exhibits similar behavior. The bearing accuracy is nearly independent of the signal to noise ratio within this regime since the bound varies only slightly over the allowed range.

However, if either signal to noise ratio happens to exceed $(\beta W^2)^{-1}$ the bound behaves quite differently..

Let $MS_i/N >> (\beta W^2)^{-1}$ but $MS_j/N << (\beta W^2)^{-1}$. It is straightforward to show that

$$\mathrm{Var}(\hat{\alpha}_i) \geq \left[\frac{TW}{2\pi} x_i^2 \frac{W^2}{3} MS_i/N\right]^{-1} \ ; \ R\xi_{max} << 1, \ (MS_i/N)\beta W^2 >> 1 \qquad (44)$$

$$\mathrm{Var}(\hat{\alpha}_j) \geq \left[\frac{TW}{2\pi} x_j^2 \frac{W^2}{3} \frac{\beta W^2}{5} (MS_j/N)^2\right]^{-1} ; \ R\xi_{max} << 1, \ (MS_i/N)\beta W^2 >> 1 \qquad (45)$$

By definition, $MS_j/N \ll (\beta W^2)^{-1}$ so that the bearing error for source j is much larger than for source i. In order to obtain insight into the effects of the presence of an interference on the ability to determine bearing, it will be useful to compute the fractional error, $\Delta Var(\hat{\alpha}_i)$, defined as the ratio of bearing error for estimates of α_i obtained under the assumption that source j (the interference) is present and located at an unknown bearing, to the equivalent performance figure obtained with source j absent. The lower bound corresponding to the latter situation is given by expression (19b). With the definition for x_i, (19b) is equivalent to :

$$Var(\hat{\alpha}_i/\text{source j absent}) \geq \left[(MS_i/N)^2 I_1 \Big|_{S_2 = 0} x_i^2 \right]^{-1}$$

$$\geq \left[\frac{TW}{2\pi} \frac{(MS_i/N)^2}{1 + MS_i/N} \frac{W^2}{3} x_i^2 \right]^{-1}$$

Therefore we have :

$$\Delta Var(\hat{\alpha}_i) \triangleq \frac{Var(\hat{\alpha}_i/\text{source j present})}{Var(\hat{\alpha}_i/\text{source j absent})}$$

For the low SNR case : $MS/N \ll (\beta W^2)^{-1}$

$$\Delta Var(\hat{\alpha}_i) = (MS_i/N\ \beta W^2)^{-1} \times \begin{cases} 2.03 & ; MS/N \ll 1 \\ 1.66 & ; 1 \ll MS/N \ll (\beta W^2)^{-1} \end{cases} \tag{46}$$

For $MS_i/N \gg (\beta W^2)^{-1}$; i = 1 or 2 :

$$\Delta Var(\hat{\alpha}_i) = 1$$

$$\Delta Var(\hat{\alpha}_j) = 5(MS_j/N\ \beta W^2)^{-1} \qquad j \neq i \tag{47}$$

Equations (43) and (46) show that the value of the bearing error associated with α_1 is insensitive to the signal to noise ratio S_2/N, since the total variation of the bound over the summed SNR is smaller than 2 db. The resulting bearing error, however, is larger by the factor $(\beta W^2)^{-1}$ when compared to the case of source 2 absent. Equations (44) and

(47), on the other hand, shows that one can obtain undegraded bearing accuracy in the estimation of α_i if S_i is sufficiently large so that $(MS_i/N)*\beta W^2 >> 1$ provided that S_j is much smaller than S_i.

The case $R\xi_{max} >> 1$ is considered next. Appendix 4 contains the details leading to the following expression for the Fisher matrix :

$$J \cong \frac{TW}{2\pi} \frac{W^2L^2}{6c^2} (\beta W^2/2)^{-1} *$$

$$\begin{bmatrix} (S_1/S_2)\sin^2(\alpha_1)\left\{1 + 2S_2/S_1\right\} & -\sin(\alpha_1)\sin(\alpha_2)\left\{1 + 2/5\beta W^2\right\} \\ -\sin(\alpha_1)\sin(\alpha_2)\left\{1 + 2/5\beta W^2\right\} & (S_2/S_1)\sin^2(\alpha_2)\left\{1 + 2S_1/S_2\right\} \end{bmatrix} \tag{48}$$

It is straightforward to show that

$$\mathrm{Var}(\hat{\alpha}_i) \geq \frac{(\beta W^2/2)\left(\frac{S_iS_j}{S^2} + \frac{S_j^2}{2S^2}\right)}{\left[\frac{TW}{2\pi} x_i^2 W^2\right]} \tag{49}$$

Equation (49) decreases monotonically as the ratio S_i/Sj tends to zero. The fractional error or penalty on bearing accuracy resulting from the presence of a second source in the estimation of either α_1 or α_2 is given by :

$$\Delta\mathrm{Var}(\hat{\alpha}_i) = (\beta W^2/12)\left(1 - \frac{S_j^2}{S^2}\right) \tag{50}$$

As S_j/S_i tends to zero,

$$\Delta\mathrm{Var}(\hat{\alpha}_i) \rightarrow \frac{\beta W^2 \, MS_j/N}{6} \tag{51}$$

$$\Delta \mathrm{Var}(\hat{\alpha}_i) \quad - \quad \frac{\beta W^2 \; MS_i/N}{12} \tag{52}$$

Since both products on the right sides of (51) and (52) are larger than one by assumption, the fractional increase in bearing error for $R\xi >> 1$ is quite large.

Figure 2 depicts the fractional error for bearing $\hat{\alpha}_i$ with $MS_i/N << (\beta W^2)^{-1}$ (solid curve) and $MS_i/N >> (\beta W^2)^{-1}$ (dashed curve) as a function of MS_j/N. The accuracy reductions are significant at all values of MS_j/N for the solid curve, but present only at the high end for MS_j/N in the dashed curve.

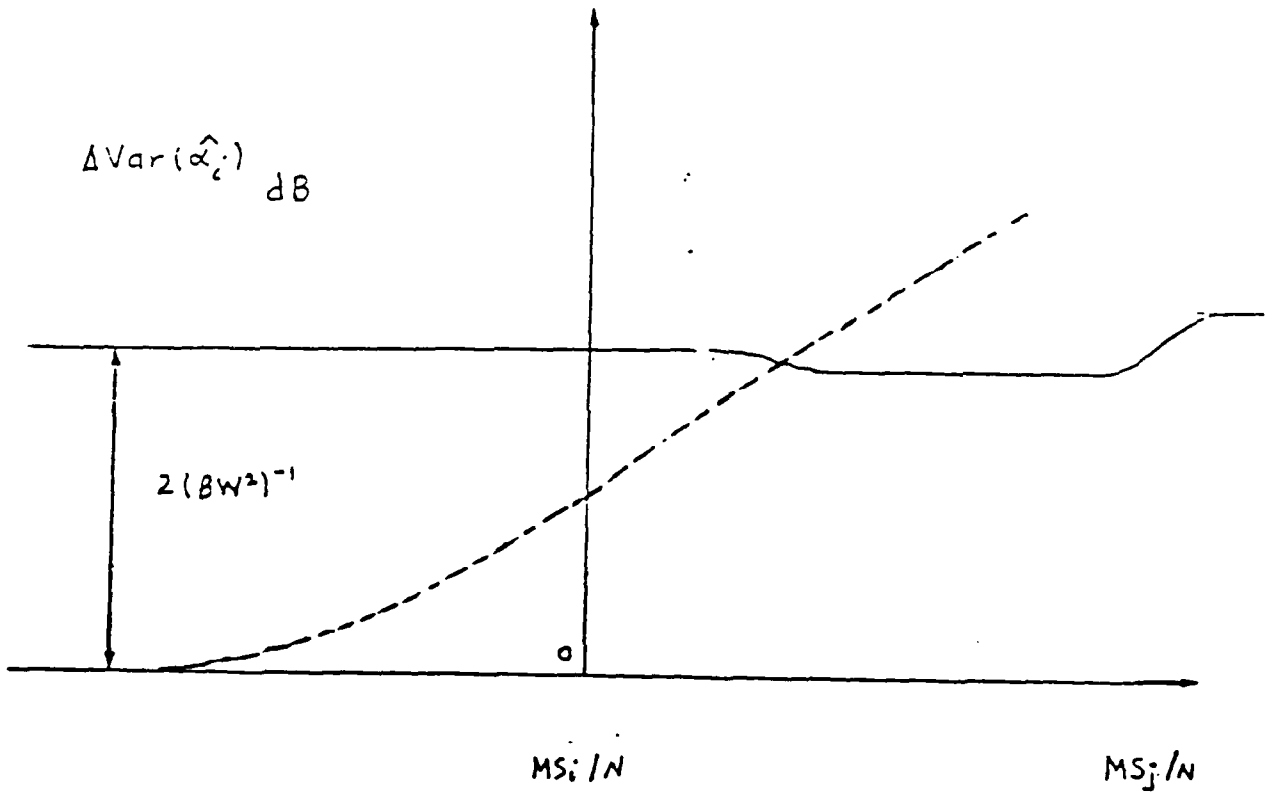


Figure 2

I t is worth pointing out that if one of the sources happened to be at a known bearing, the lower bound on estimates of the unknown source bearing has the following interesting properties. Let source j be located at known bearing and source i be at unknown bearing. If source j has a signal to noise ratio which is much smaller than $(\beta W^2)^{-1}$ the degradation factor is:

$$\Delta Var(\hat{\alpha}_i) = 1 + S_j/S_i \tag{53a}$$

Judging from (53a), the incremental bearing error rises linearly with the ratio of interference to signal powers. However, if the SNR of source j, MS_j/N exceeds $(\beta W^2)^{-1}$ as well as MS_i/N, the degradation factor is:

$$\Delta Var(\hat{\alpha}_i) = \frac{1 + S_j/S_i}{MS_j/N\ \beta W^2} \cong (MS_i/N\ \beta W^2)^{-1} >> 1 \tag{53b}$$

Since $(MS_j/N)\ \beta W^2 >> 1$, (53b) is smaller than (53a) but large degradations in bearing error remain as result of the presence of source 2. If the signal to noise ratio for source i is large, $MS_i/N >> (\beta W^2)^{-1}$, but $MS_j/N << (\beta W^2)^{-1}$, no degradations in accuracy are observed.

The relationship between (53a) and (53b) can be explained as follows. Suppose $\Delta\alpha = 0$ -- this corresponds to (53a). Any finite MS_j/N must be smaller than βW^2 since β is zero. As $\Delta\alpha$ increases, a large value of MS_j/N exists for which $MS_j/N >> (\beta W^2)^{-1}$. However, as $\Delta\alpha$ increases sufficiently the beampattern will tend to zero and the function ξ will tend to one. At this point there is no accuracy reduction in estimating α_i because the sources have become well separated. Hence the transition from (53a) to (53b) describes the effect of increasing separation on bearing accuracy improvement.

When both signal to noise ratios exceed $(\beta W^2)^{-1}$ one can compute analogous degradation factors :

$$\Delta Var(\hat{\alpha}_i) = \frac{3M^2 S_1^2 S_2/N^3 \beta W^2}{2MS/N(MS/N + MS_j/N)} - \begin{cases} 1.5\ MS_2/N\ \beta W^2 ; & S_i >> S_j \\ .75\ MS_1/N\ \beta W^2 ; & S_j >> S_i \end{cases}$$

With $MS_i/N >> (\beta W^2)^{-1}$, accuracy reductions resulting from the presence of a second source are large in this setting also.

At low signal to noise ratios for the source at unknown bearing, increases in the signal to noise ratio for the source at known bearing tends to improve bearing accuracy. At large signal to noise ratios for the source at unknown bearing, reductions in accuracy are present only when the SNR of the other source is large as well. In this setting, the optimal condition is $S_1 = S_2$, where the error increases monotonically as the power ratio deviates from one. Hence, even when the bearing of one source is known apriori , the ability to determinine the bearing of a second, closely spaced source can be quite poor.

The results of this section indicate that a potentially useful regime exists in the estimation of a particular source bearing provided that the SNR for that source is sufficiently large and the interfering source SNR is sufficiently small. When the conditions fail to hold,

bearing accuracy is greatly reduced in the multiple source setting. In the next section it will be shown that unknown spectral levels compromise bearing accuracy further and that under these conditions one is always forced to operate with large accuracy reductions.

3. Bearing Estimation Accuracy Without Prior Knowledge of Source Power Levels

When prior knowledge of the source power levels is unavailable the relevant parameter vector $\underline{\theta}$ becomes :

$$\underline{\theta} = [\ \alpha_1,\ \alpha_2,\ S_1,\ S_2\ ]^T. \tag{56a}$$

Consider the following partitioned matrix :

$$J = \left[\begin{array}{c|c} J_1 & J_2 \\ \hline J_2^T & J_3 \end{array}\right] \tag{56b}$$

where J_1 is the Fisher matrix associated with bearings only

$$[J_1]_{ij} \triangleq E\left\{\frac{\partial \ln p(\underline{X}/\underline{\theta})}{\partial \alpha_i} \frac{\partial \ln p(\underline{X}/\underline{\theta})}{\partial \alpha_j}\right\} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \tag{57}$$

and

$$[J_2]_{ij} \triangleq E\left\{\frac{\partial \ln p(\underline{X}/\underline{\theta})}{\partial \alpha_i} \frac{\partial \ln p(\underline{X}/\underline{\theta})}{\partial S_j}\right\} = \begin{bmatrix} J_{13} & J_{14} \\ J_{23} & J_{24} \end{bmatrix} \tag{58}$$

$$[J_3]_{ij} \triangleq E\left\{\frac{\partial \ln p(\underline{X}/\underline{\theta})}{\partial S_i} \frac{\partial \ln p(\underline{X}/\underline{\theta})}{\partial S_j}\right\} = \begin{bmatrix} J_{33} & J_{34} \\ J_{43} & J_{44} \end{bmatrix} \tag{59}$$

The total bearing error can be expressed as follows :

$$\mathrm{Cov}(\hat{\alpha}_i, \hat{\alpha}_j) \geq \left[\ J_1 - J_2^T J_3^{-1} J_2\ \right]_{ij}^{-1} \tag{60}$$

Define the matrix X :

$$X \triangleq J_2^T J_3^{-1} J_2 \tag{61}$$

The elements of X are as follows :

$$X_{11} = \frac{J_{13}^2 J_{44} - 2J_{13}J_{34}J_{14} - J_{14}^2 J_{33}}{\det(J_3)} \tag{62}$$

$$X_{12} = \frac{J_{13}J_{44}J_{23} - J_{14}J_{34}J_{23} - J_{34}J_{24}J_{13} + J_{14}J_{24}J_{33}}{\det(J_3)} \tag{63}$$

$$X_{22} = \frac{J_{23}^2 J_{44} - 2J_{34}J_{24}J_{23} + J_{24}^2 J_{33}}{\det(J_3)} \tag{64}$$

Appendix 5 contains the details of the following derivations of the complete Fisher matrix. Define

$$\xi \triangleq 1 - \frac{|\underline{v}_1^{\dagger}\underline{v}_2|^2}{M^2}$$

$$J_{13} = -\sum_{k=1}^{WT} \frac{M^3 S_1 S_2/N^3 \ (1 + MS_2/N \ \xi)^2}{(1 + MS/N + M^2 S_1 S_2/N^2 \ \xi)^2} \ \frac{\eta_1}{M^2} \tag{65}$$

$$J_{14} = \sum_{k=1}^{WT} \frac{M^2 S_1/N^2 (1 + MS_1/N)}{(1 + MS/N + M^2 S_1 S_2/N^2 \ \xi)^2} \ \frac{\eta_1}{M^2} \tag{66}$$

$$J_{23} = \sum_{k=1}^{WT} \frac{M^2 S_2/N^2 (1 + MS_2/N)}{(1 + MS/N + M^2 S_1 S_2/N^2 \ \xi)^2} \ \frac{\eta_2}{M^2} \tag{67}$$

$$J_{24} = -\sum_{k=1}^{WT} \frac{M^2 S_1 S_2/N^3 (1 + MS_1/N \ \xi)}{(1 + MS/N + M^2 S_1 S_2/N^2 \ \xi)^2} \ \frac{\eta_2}{M^2} \tag{68}$$

$$J_{23} \cong - M^2 S_2/N^2 (1 + MS_2/N) \; (I_4 - (2/5\beta) \; I_5) \; \frac{L^2 \Delta\alpha \sin(\alpha_2) \sin(\alpha)}{6c^2} \tag{77}$$

$$J_{33} \cong (M/N)^2 (I_5 + (MS_2/N)\beta \; I_4 + \beta^2 MS_2/N \; (MS_2/4N - 1/5) \; I_3) \tag{78}$$

$$J_{34} \cong (M/N)^2 (I_5 - (\beta/2) \; I_4 + (\beta^2/10) \; I_3) \tag{79}$$

$$J_{44} \cong (M/N)^2 (I_5 + (\beta MS_1/N) I_4 + \beta^2 MS_1/N \; (MS_1/4N - 1/5) \; I_3) \tag{80}$$

Appendices 6 - 8 contain details leading to expressions for the F.I.M. under various conditions. It is useful to consider properties of the J_3 matrix alone before discussing the problem of joint bearing and power estimates. The following statements are based on derivations taken from the appendices.

1. When the signal field contains only a single arrival (e.g., $S_2 = 0$), estimates of bearing and signal power are statistically uncorrelated. The lower bound on the variance of signal power estimates for a signal of strength S is given by :

$$\mathrm{Var}(\hat{S}_i) \geq \left[\frac{TW}{2\pi} \frac{(M/N)^2}{(1 + MS/N)^2} \right]^{-1} \tag{81}$$

$$\propto \frac{1}{TW} \begin{cases} N^2/ M^2 & MS/N \ll 1 \\ S^2 & MS/N \gg 1 \end{cases} \tag{82}$$

2. When the signal field contains arrivals from two incoherent sources, bearing and signal power estimates are in general statistically correlated. For signal to noise ratios, MS_i/N, smaller than $(\beta W^2)^{-1}$:

$$\mathrm{Var}(\hat{S}_i) \geq \left[\frac{TW}{2\pi} \frac{(M/N)^2}{1 + MS/N} \frac{\beta W^2}{3} \right]^{-1} \tag{83}$$

The degradation factor is proportional to $(\beta W^2 (1 + MS/N))^{-1} \gg 1$.

3. If one signal to noise ratio exceeds $(\beta W^2)^{-1}$ (for example, source i) estimates of signal powers are statistically uncorrelated. The lower bounds are given by :

$$J_{33} = \sum_{k=1}^{WT} \frac{(M/N)^2 (1 + MS_2/N\ \xi)^2}{(1 + MS/N + M^2 S_1 S_2/N^2\ \xi)^2} \tag{69}$$

$$J_{34} = J_{43} = \sum_{k=1}^{WT} \frac{(M/N)^2\ (1 - \xi)}{(1 + MS/N + M^2 S_1 S_2/N^2\ \xi)^2} \tag{70}$$

$$J_{44} = \sum_{k=1}^{WT} \frac{(M/N)^2 (1 + MS_1/N\ \xi)^2}{(1 + MS/N + M^2 S_1 S_2/N^2\ \xi)^2} \tag{71}$$

If one defines integrals I_4 and I_5 as follows, the elements of J_2 and J_3 can be expressed entirely in terms of integrals I_1 - I_5.

$$I_4 \triangleq \frac{T}{2\pi} \int_0^W \frac{\omega^2\, d\omega}{(1 + MS/N + M^2 S_1 S_2/N^2\ \xi)^2} \tag{72}$$

$$I_5 \triangleq \frac{T}{2\pi} \int_0^W \frac{d\omega}{(1 + MS/N + M^2 S_1 S_2/N^2\ \xi)^2} \tag{73}$$

With the following expressions from Appendix 3 for η_1 and η_2 ;
:

$$\eta_i = x_i(\beta)^{1/2}\ \omega^2 (\ 1 - 2/5\ \beta\omega^2 + \ .\ .\ .\ )$$

one can show that

$$J_{13} \cong - M^3 S_1 S_2/N^3 (\ I_4 + (MS_2/2N - 2/5)\beta\ I_3\ )\ \frac{L^2 \Delta\alpha \sin(\alpha_1)\sin(\alpha)}{6c^2} \tag{74}$$

$$J_{14} \cong M^2 S_1/N^2\ (1 + MS_1/N)\ (\ I_4 - 2/5\beta\ I_3\ )\ \frac{L^2 \Delta\alpha \sin(\alpha_1)\sin(\alpha)}{6c^2} \tag{75}$$

$$J_{24} \cong M^3 S_1 S_2/N^3\ (\ I_4 + (MS_1/2N - 2/5)\beta\ I_3\ )\ \frac{L^2 \Delta\alpha \sin(\alpha_2)\sin(\alpha)}{6c^2} \tag{76}$$

2 2 $L^2 \Delta\alpha \sin(\alpha_2)\sin(\alpha)$

$$\mathrm{Var}(\hat{S}_i) \geq \left[\frac{TW}{2\pi} \frac{(M/N)^2}{20} (\beta W^2)^2 \right]^{-1} \tag{84}$$

$$\mathrm{Var}(\hat{S}_j) \geq \left[\frac{TW}{2\pi} \frac{(1/N)^2}{S_i^2} \right]^{-1} \tag{85}$$

Since

$$MS_i/N \gg (\beta W^2)^{-1}$$

the lower SNR source has poorer accuracy in spectrum level estimation than the large SNR source. Both estimation errors exhibit degradations resulting from the multiple source setting, in spite of the fact that the errors are actually uncorrelated.

4. If both signal to noise ratios exceed the value $(\beta W^2)^{-1}$, the estimation errors of S_i and S_j are also statistically uncorrelated :

$$\mathrm{Var}(\hat{S}_i) \geq \left[\frac{TW}{2\pi} \frac{(M/N)^2}{20} (\beta W^2)^2 \right]^{-1} \quad ; \; i = 1,2 \tag{86}$$

We consider now the joint parameter estimation problem. Suppose $MS_i/N \ll (\beta W^2)^{-1}$; $i = 1,2$. Computations carried out in Appendix 6 for the elements of the X matrix are as follows :

$$X_{11} = \frac{TW}{2\pi} \frac{(MS_1/N)^2}{(1 + MS/N)} x_1^2 \frac{W^2}{3} \left\{ 1 - \frac{12}{25} \beta W^2 + \frac{61 \beta W^2}{300(1 + MS/N)} \right.$$
$$+ \frac{3\beta W^2}{5} \frac{(MS_2/N)(MS_2/N - 2MS_1/N)}{1 + MS/N} - \frac{3 \beta W^2}{20} \left[\frac{(MS/N)^2 - 6M^2 S_1 S_2/N^2}{1 + MS/N} \right]$$
$$\left. - \frac{6\beta W^2 M^2 S_1 S_2/N^2}{5(1 + MS/N)^2} \right\} \tag{87}$$

$$X_{21} = \frac{TW}{2\pi} \frac{M^2 S_1 S_2/N^2}{(1 + MS/N)} x_1 x_2 \frac{W^2}{3} \left\{ 1 - \frac{12}{25} \beta W^2 + \frac{11 \beta W^2}{300 (1 + MS/N)} - \right.$$
$$- \frac{3\beta W^2}{20} \left[\frac{(MS/N)^2 - 6M^2 S_1 S_2/N^2}{1 + MS/N} \right] - \frac{6M^2 S_1 S_2/N^2}{5(1 + MS/N)} \left[\frac{2 + MS/N}{1 + MS/N} \right]$$
$$\left. + \frac{3\beta W^2}{10} \frac{(MS_2/N)^2 + (MS_1/N)^2}{1 + MS/N} \right\} \tag{88}$$

$$X_{22} = \frac{TW}{2\pi} \frac{(MS_2/N)^2}{(1 + MS/N)} x_2^2 \frac{W^2}{3} \left\{ 1 - \frac{12}{25} \beta W^2 + \frac{61 \beta W^2}{300 (1 + MS/N)} + \right.$$
$$+ \frac{3\beta W^2}{5} \frac{(MS_1/N)(MS_1/N - 2MS_2/N)}{(1 + MS/N)} - \frac{3\beta W^2}{20} \left[\frac{(MS/N)^2 - 6M^2 S_1 S_2/N^2}{1 + MS/N} \right]$$

$$- \frac{6M^2 S_1 S_2/N^2}{5(1 + MS/N)^2} \Bigg\} \tag{89}$$

One obtains for the elements of $J_1 - X$:

$$[\, J_{11} - X_{11} \,] = \frac{TW}{2\pi} \frac{(MS_1/N)^2}{1 + MS/N} \frac{W^2}{3} x_1^2 \beta W^2 \frac{W^2}{3} \left\{ \frac{6\, M^2 S_1 S_2/N^2}{5\,(1 + MS/N)^2} \right.$$

$$\left. \frac{12}{25} - \frac{61/300}{(1 + MS/N)} + \frac{3/20\,(MS/N)^2}{1 + MS/N} \right\}$$

$$[\, J_{21} - X_{21} \,] = \frac{TW}{2\pi} \frac{M^2 S_1 S_2/N^2}{1 + MS/N} \frac{W^2}{3} x_1 x_2 \beta W^2 \left\{ \frac{6}{25} - \frac{11/300}{1 + MS/N} \right.$$

$$\left. + \frac{6}{5} \frac{M^2 S_1 S_2/N^2}{(1 + MS/N)^2} - \frac{3}{20} \frac{(MS/N)^2}{1 + MS/N} \right\}$$

$$[\, J_{22} - X_{22} \,] = \frac{TW}{2\pi} \frac{(MS_2/N)^2}{1 + MS/N} \frac{W^2}{3} \beta W^2 \left\{ \frac{12}{25} - \frac{61/300}{1 + MS/N} + \right.$$

$$\left. + \frac{6M^2 S_1 S_2/N^2}{5(1 + MS/N)^2} + \frac{3/20\,(MS/N)^2}{1 + MS/N} \right\}$$

At low signal to noise ratios, i.e., $MS/N \ll 1$, the lower bound on bearing error is given by:

$$\mathrm{Var}(\hat{\alpha}_i) \geq \left\{ \frac{TW}{2\pi} \frac{W^2}{75} \frac{24}{7} \quad (MS_i/N)^2 \ \beta W^2 \right\}^{-1} \tag{90}$$

The incremental error due to uncertainty concerning signal power levels is defined as follows :

$$\delta\mathrm{Var}(\hat{\alpha}_i) = \frac{\mathrm{Var}(\hat{\alpha}_i / \ S_1, S_2 \text{ unknown})}{\mathrm{Var}(\hat{\alpha}_i / \ S_1, S_2 \text{ known})} = \frac{7}{2} . \tag{91}$$

At large signal to noise ratios, $MS/N \gg 1$:

$$\mathrm{Var}(\hat{\alpha}_i) \geq \left[\frac{TW}{2\pi} \quad \frac{(MS_i/N)^2 W^2}{MS/N} \frac{}{3} \beta W^2 \right]^{-1} \times \left[\frac{25}{36} \left(1 + \frac{5 \ S_1 S_2}{3S^2} \right)^{-1} \right] \tag{92}$$

The incremental error due to unknown power levels is given by:

$$\delta\mathrm{Var}(\hat{\alpha}_i) \cong \frac{(5/12)MS/N}{1 + 5S_1S_2/3S^2} \tag{93}$$

which grows with MS/N.

At low signal to noise ratios the incremental errors due to unknown amplitude factors is a constant value of approximately 5.4 db. In the large signal to noise ratio case the errors are likely to be even larger and this suggests that apriori knowledge of signal levels might be used to advantage in improving overall accuracy.

Next consider the following. Suppose $MS_1/N \ll (\beta W^2)^{-1}$ but $MS_2/N \gg (\beta W^2)^{-1}$. Appendix 7 computes the following lower bounds. At low signal to noise ratios for source 1, i.e., $MS_1/N \ll 1$:

$$\mathrm{Var}(\hat{\alpha}_1) \geq \left\{ \frac{TW}{2\pi} x_1^2 \ W^2 \ \beta W^2 \ \frac{12}{75} MS_1/N \ \frac{S_1}{S_2} \right\}^{-1} \tag{94}$$

$$\mathrm{Var}(\hat{\alpha}_2) \geq \left\{ \frac{TW}{2\pi} x_2^2 \ W^2 \ \beta W^2 \ \frac{12}{75} MS_2/N \right\}^{-1} \tag{95}$$

from which the incremental bearing errors can be obtained :

$$\delta\mathrm{Var}(\hat{\alpha}_1) = \frac{S_2}{S_1} = \delta\mathrm{Var}(\hat{\alpha}_2) \tag{96}$$

At large signal to noise ratios for source 1 ($MS_1/N \gg 1$) :

$$\mathrm{Var}(\hat{\alpha}_1) \geq \left\{ \frac{TW}{2\pi} x_1^2 \ W^2 \ \frac{2}{5} \ \beta W^2 \ (MS_1/N)^2 \ \frac{S_1}{S_2} \right\}^{-1} \tag{97}$$

$$\mathrm{Var}(\hat{\alpha}_2) \geq \left\{ \frac{TW}{2\pi} x_2^2 \ W^2 \ \frac{2}{5} \ \beta W^2 \ M^2 S_1 S_2/N^2 \right\}^{-1} \tag{98}$$

The incremental bearing errors are

$$\delta Var(\hat{\alpha}_1) = \delta Var(\hat{\alpha}2) = \frac{S_2}{2S_1} \tag{99}$$

The effects of uncertain signal levels produce large performance degradations, particularly for $\hat{\alpha}_2$ which previously exhibited no incremental bearing errors resulting from the presence of source 1.

Finally we consider the last case :

$$MS_i/N \gg (\beta W^2)^{-1} \quad ; \; i = 1 \text{ and } 2.$$

From Appendix 8:

$$Var(\hat{\alpha}_i) \geq \left[\frac{TW}{2\pi} \; x_i^2 \; W^2 \right]^{-1} \times \left[(R\beta W^2)^{1/2} \frac{\beta W^2}{(S_i/S_j)^2 + 1} \right] \tag{100}$$

The incremental bearing error resulting from lack of knowledge of power levels is

$$\delta Var(\hat{\alpha}_i) = (R\beta W^2)^{1/2} \frac{(S_i/S_j)^2 + 2(S_i/S_j)}{((S_i/S_j)^2 + 1)(S_i/S_j + 1)^2}$$

$$- (R\beta W^2)^{1/2} \times \begin{cases} (S_j/S_i)^2 & ; \; S_i \gg S_j \\ .375 & ; \; S_i = S_j \\ 2 & ; \; S_i \ll S_j \end{cases} \tag{101}$$

The above degradation factor increases monotonically with the ratio S_j/S_i. It is also proportional to $(R\beta W^2)^{1/2}$, which is by definition, a factor exceeding one.

Figure 3 depicts the incremental bearing error due to unknown signal levels for $\hat{\alpha}_i$ as a function of MS_j/N for $MS_i/N \ll (\beta W^2)^{-1}$ (solid curve) and $MS_i/N \gg (\beta W^2)^{-1}$ (dashed curve). In the lower SNR case for source i, the incremental error increases monotonically with MS_j/N, unlike the nearly constant behavior of the fractional error for known signal levels of Figure 2. In the large SNR case for source i the incremental error decreases as MS_j/N increases.

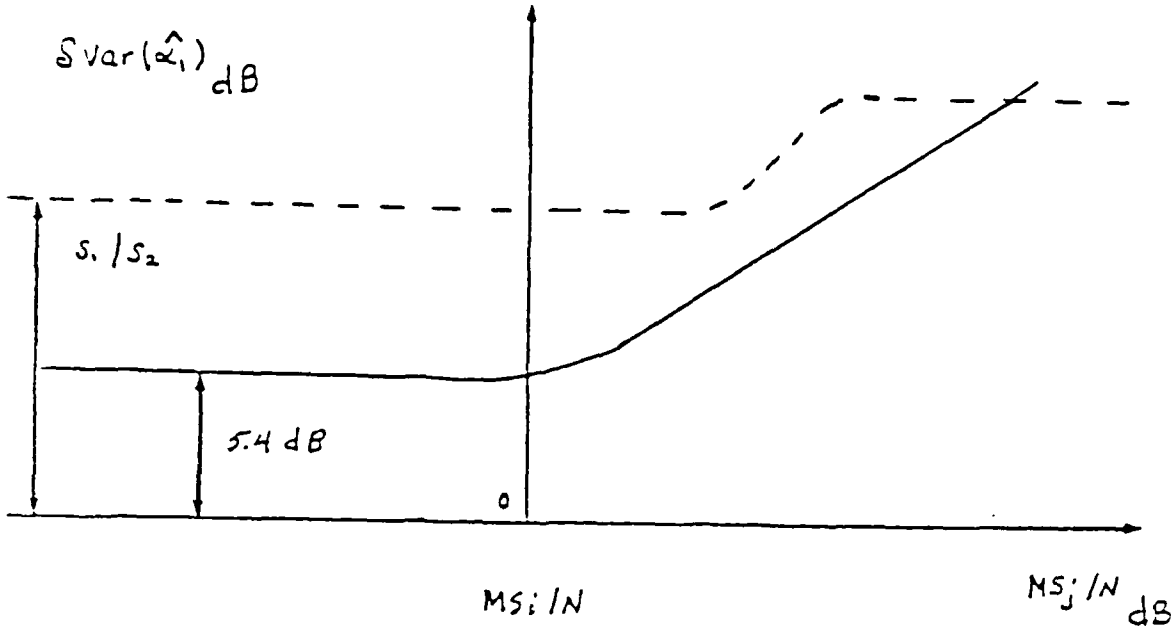


Figure 3

Figure 4 plots the total fractional bearing error for source i , in the case of unknown signal levels relative to the accuracy attainable when source j is absent.

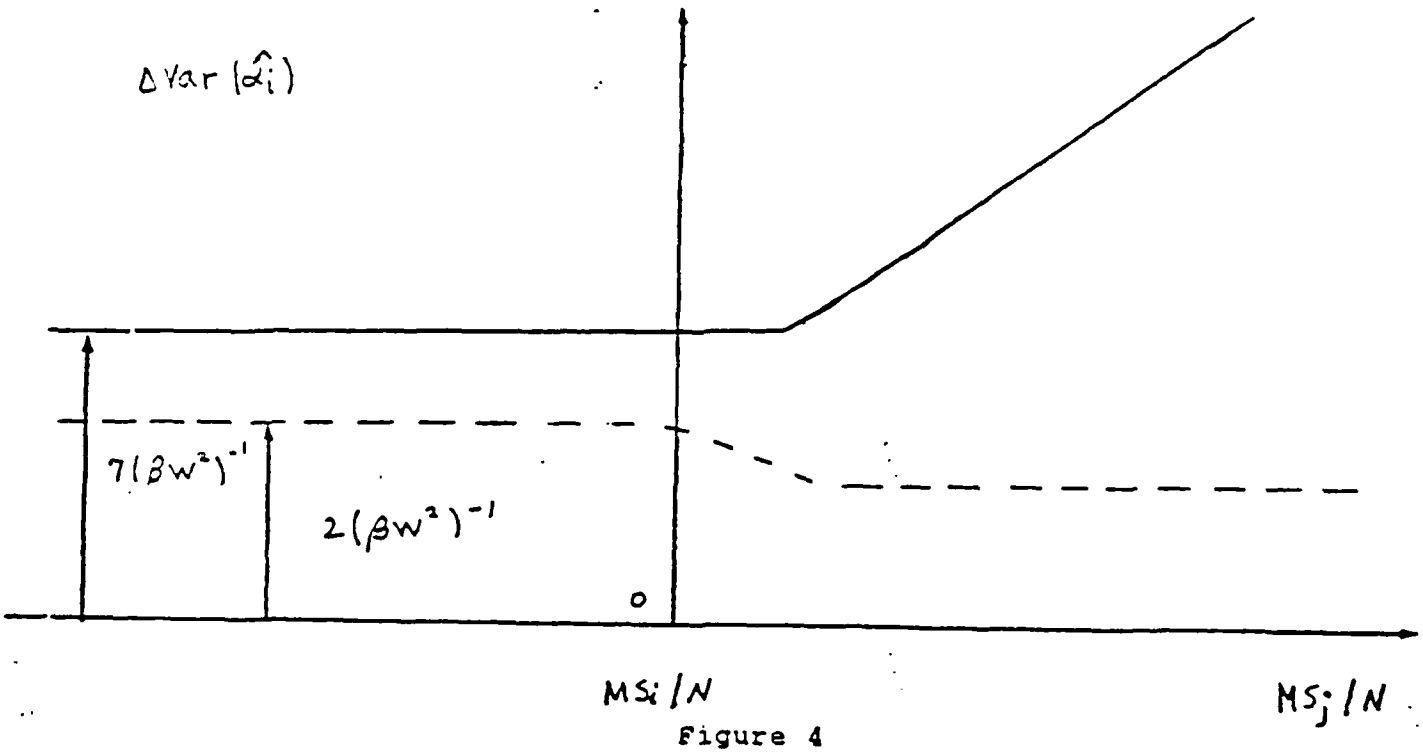


Figure 4

It has [illegible] estimation [illegible]

sors, the lower bounds, however, become unreachable, if the correct number of sources present is not known to the observer or cannot be determined from the data. It is expected that when the two sources become very close in bearing, the ability of the data to correctly predict the number of sources present will deteriorate unless one has access to very large signal to noise ratios or long observation times. On the other hand, for large separations the two sources are expected to be distinguishable at much lower signal to noise ratios and shorter observation times. The former situation is referred to as operation below threshold, where resolvability of the sources and reachability of the Cramer-Rao lower bounds is questionable. The next section is concerned with the error associated with a binary hypothesis test, where there is one source present under the first hypothesis and two present under the second. If the probability of error in choosing the incorrect hypothesis is low, the sources are expected to be distinguishable. When the error is large, resolution of the sources becomes unlikely and the ability to reach the Cramer-Rao lower bounds will be poor.

4. Probability of Error in Resolving the Two Sources

Determination of the correct number of sources present using signal observations is of interest to this study because the problem sheds light on the issue of resolution. It is reasonable to expect that with sufficiently high SNR and long observation time, two sources can be distinguished even if they are located at arbitrarily close bearings. However, one expects that the ability to predict the existence of the second nearby source will deteriorate as the SNR and observation time become small. This section considers this problem from the viewpoint of determining the probability of error in choosing the wrong decision in the following binary hypothesis test :

$$H_0 : \quad x_i(t) = (S_1+S_2)^{1/2} \times s(t - v_i) + n_i(t)$$

$$H_1 : \quad x_i(t) = (S_1)^{1/2} \; s_1(t-t_i) + (S_2)^{1/2} \; s_2(t-z_i) + n_i(t)$$

Under H_1 there are two incoherent sources present, which are located at bearing angles α_1 and α_2. Under H_0 there is a single source located at bearing α_0, where α_0 is the "weighted average" bearing:

$$\alpha_0 = \frac{S_1\alpha_1 + S_2\alpha_2}{S} \quad ; \text{ and } v_i = d\sin(\alpha_0)i/c.$$

Note that if $S_1 = S_2$, $\alpha_0 = \alpha$, the true average. If $S_1 >> S_2$ so that $S \cong S_1$ then $\alpha_0 \cong \alpha_1$. The signals $s_1(t)$, $s_2(t)$ and $s(t)$ as assumed to have identical lowpass spectra of width W and with unity height.

The following material is contained in reference [7], pp. - where lower bounds on the false alarm and miss probabilities in a binary hypothesis test are presented.

Define the probability density functions of the received data

dentical lowpass spectra of width W and with unity height.

The following material is contained in reference [7], pp. - where lower bounds on the false alarm and miss probabilities in a binary hypothesis test are presented.

Define the probability density functions of the received data vector $\underline{X}$ under the two hypotheses :

$$p(\underline{X}/H_1),\ p(\underline{X}/H_0),$$

the log-likelihood :

$$\ell(\underline{X}) \triangleq \ln \frac{p(\underline{X}/H_1)}{p(\underline{X}/H_0)}$$

The likelihood ratio test compares $\ell(\underline{X})$ with a threshold γ, where

$$\ell(\underline{X}) \underset{H_0}{\overset{H_1}{\gtrless}} \gamma \qquad \text{and} \qquad \gamma = \ln \frac{P_0(C_{10} - C_{00})}{P_1(C_{01} - C_{11})}$$

C_{ij} (i,j = 1,2) is the cost of choosing hypothesis H_i when hypothesis H_j occurred. P_i the probability of hypothesis H_i occurring. In this problem, the hypotheses are chosen to be equally likely and the costs equal so that $\gamma = 0$.

The moment-generating function of $\ell(\underline{X})$ on hypothesis H_0 is

$$\Phi_{\ell/H0}(s) \triangleq E\left\{ e^{s\ell}/H_0 \right\} = \int_{-\infty}^{\infty} e^{s\ell}\, p_{\ell/H0}(L/H_0)\, dL$$

Define $\mu(s)$ as follows :

$$\mu(s) \triangleq \ln \Phi_{\ell/H0}(s)$$

Because $\ell(\underline{X})$ is a function of $\underline{X}$, the above integral can be expressed

$$\Phi_{\ell/H0}(s) = \int_{-\infty}^{\infty} e^{s\ell}\, p(\underline{X}/H_0)\, d\underline{X}$$

Then

$$\mu(s) = \ln \int_{-\infty}^{\infty} \left[\frac{p(\underline{X}/H_1)}{p(\underline{X}/H_0)} \right]^s p(\underline{X}/H_0)\, d\underline{X} \tag{102}$$

Upper bounds on the false alarm and miss probabilities, denoted by P_F and P_M, respectively, are derived in reference. The bounds depend on the value of s for which $\dot{\mu}(s) = 0$. The bounds are given by :

$$P_F \le \mathrm{erfc}_*\left[s \quad \mu(s_m)^{1/2} \right] \exp\left[\mu(s_m) + \frac{s_m^2}{2} \ddot{\mu}(s_m) \right] \tag{103}$$

$$P_M \le \mathrm{erfc}_*\left[(1 - s_m)\ \mu(s_m)^{1/2})\right] \exp\left[\mu(s_m) + \frac{(1 - s_m)^2}{2} \ddot{\mu}(s_m) \right] \tag{104}$$

$$\dot{\mu}(s_m) = 0 \tag{105}$$

where $\mathrm{erfc}_*(\ )$ is the complementary error function :

$$\mathrm{erfc}_*(x) \triangleq \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp\ (-\ u^2/\ 2)\ du$$

Under hypothesis H_0, there are two signals present and the covariance matrix of the sensor data has the form :

$$K_0 = \begin{bmatrix} K_0(\omega_1) & & & \\ & K_0(\omega_2) & & 0 \\ & & \cdot & \\ 0 & & & \cdot \\ & & & K_0(\omega_N) \end{bmatrix}$$

$$K_0(\omega_k) = N_k I + S_1(\omega_k)\underline{v}_1\underline{v}_1^* + S_2(\omega_k)\underline{v}_2\underline{v}_2^*$$

Under H_1, K_1 is also block diagonal but with the form

$$K_1(\omega_k) = N_k I + (S_1 + S_2)\underline{v}\underline{v}^*.$$

The steering vector $\underline{v}_0$ representing the single source under H_1 is at this point perfectly general. A particular choice of $\underline{v}_0$ will be included at a later point.

According to equation (102),

$$\mu(s) = \ln \int_{-\infty}^{\infty} \left[\frac{p(\underline{X}/H_1)}{p(\underline{X}/H_0)}\right]^s p(\underline{X}/H_0)\, d\underline{X}$$

$$= \ln \left\{ \left[\frac{\det(\bar{K}_0)}{\det(\bar{K}_1)}\right]^s [\det(\bar{K}_0)]^{-1} \int_{-\infty}^{\infty} \exp\left\{ -\underline{X}^*(s\bar{K}_1^{-1} + (1-s)\bar{K}_0^{-1})\,\underline{X} \right\} d\underline{X} \right\}$$

$$= \ln \left[\left[\frac{\det(\bar{K}_0)}{\det(\bar{K}_1)}\right]^s \frac{\det(s\bar{K}_1^{-1} + (1-s)\bar{K}_0^{-1})^{-1}}{\det\bar{K}_0} \right] \tag{106}$$

Based on calculations carried out in Appendix 8 one obtains the following expression for $\mu(s)$. Define :

$$\xi_{12} \triangleq \xi = 1 - \frac{1}{M^2} |\underline{v}_1^* \underline{v}_2|^2$$

$$\xi_{01} \triangleq 1 - \frac{1}{M^2} |\underline{v}_0^* \underline{v}_1|^2$$

$$\xi_{02} \triangleq 1 - \frac{1}{M^2} |\underline{v}_0^* \underline{v}_2|^2$$

$$\chi \triangleq 1 - \frac{1}{M^3} \mathrm{Re}\left((\underline{v}_0^* \underline{v}_1)(\underline{v}_1^* \underline{v}_2)(\underline{v}_2^* \underline{v}_0) \right)$$

$$\bar{R} \triangleq \frac{R}{4} = \frac{M^2 S_1 S_2 / N^2}{1 + MS/N}$$

$$\mu(s) = \sum_{k=1}^{WT} \ln\left[1 + \bar{R}\xi_{12}\right]^s - \ln\Big[1 + \bar{R}\big((S/S_2)\xi_{01} + (S/S_1)\xi_{02} \big) s(1-s)$$

$$+ \bar{R}\xi_{12} s^2 + s^2(1-s)\, \bar{R}\, MS/N\, (\xi_{12} + \xi_{01} + \xi_{02} - 2\chi)\Big] \tag{107}$$

Equation (107) is a function of the two steering vectors $\underline{v}_1$ and $\underline{v}_2$ as well as the third vector $\underline{v}_0$, which is still perfectly general. Next let $\underline{v}_0$ represent the steering vector of delays for a signal radiated at bearing α_0 :

$$\alpha_0 = \frac{S_1\alpha_1 + S_2\alpha_2}{S} .$$

After simple computations carried out in Appendix 8 one can show that $\mu(s)$ takes the form :

$$\mu(s) = \sum_{k=1}^{-\infty} s \ln(1 - \bar{R}\beta\omega_k^2) - \ln(1 + \bar{R}\beta\omega_k^2 s) \qquad (108)$$

In order to find the value s_m, the above equation is differentiated and set equal to zero :

$$\dot{\mu}(s) = \sum_{k=1}^{TW} \ln(1 + \bar{R}\beta\omega_k^2) - \frac{\bar{R}\beta\omega_k^2}{1 + \bar{R}\beta\omega_k^2 s} = 0$$

In order to obtain the solution, it is useful to consider the value of the quantity $\bar{R}\beta W^2$. Suppose the angle difference, $\Delta\alpha$, upon which β depends happened to be extremely small. For any finite signal to noise ratios (and hence finite value of $\bar{R}$) there exists a value of $\Delta\alpha$ sufficiently small so that the product :

$$\bar{R}\beta W^2$$

will be smaller than one. It will become apparent later that a threshold region, above which the error probability tends to zero, will depend upon a variety of parameters. A low probability of error indicates that resolution of the sources is likely. An error near one half indicates that the data is essentially useless in determining the number of sources present. Since the minimum angular separation for which the two sources can be distinguished is of ultimate interest to this study, we choose $\Delta\alpha$ so that $\bar{R}\beta W^2$ is much smaller than unity. Then the parameter combinations necessary to cause the probability of error to vanish can be determined for any particular $\Delta\alpha$.

With a few steps of algebra it is possible to show that the solution $\dot{\mu}(s_m) = 0$ under the stated condition of $\bar{R}\beta W^2 \ll 1$ is given by :

$$s_m = 1/2.$$

Appendix 10 contains details leading to the derivation.

It is then a simple matter to demonstrate that the false alarm and miss probabilities, defined in equations (103) and (104), are as follows. Appendix 11 contains the details of the derivation.

$$P_F = P_M = P_r(\varepsilon) = \exp\left\{-\frac{TW}{80\pi}(\bar{R}\beta W^2)^2\right\} * \operatorname{erfc}*\left\{\left[\frac{TW}{20\pi}\right]^{1/2}\bar{R}\beta W^2\right\} \qquad (109)$$

Equation (109) reveals a dependence of the error probability on two different functions. If one defines the new parameter

$$\Omega \triangleq \left[\frac{TW}{20\pi}\right]\bar{R}\beta W^2$$

then $Pr(\varepsilon)$ becomes

$$Pr(\varepsilon) = \exp\left(- 5\,\Omega^{2} \right) \times \mathrm{erfc}_{*}(\,\Omega\,)$$

Note that as the argument of the exponential function drops below minus one, $Pr(\varepsilon)$ will tend to zero. The error probability will also tend to zero if the argument of the complementary error function exceeds one. When $\Omega \ll 1$, one obtains

$$Pr(\varepsilon) = \frac{1}{2}\left(1 - \pi^{-1/2}\,\Omega - 5\,\Omega^{2} + \ldots \right)$$

which is nearly equal to one half. Under these conditions the two sources cannot be distinguished and one is therefore operating below the threshold which is defined by :

$$\Omega = \left[\frac{TW}{20\pi} \right]^{1/2} \frac{M^{2}\,S_1S_2/N^{2}}{1 + MS/N} \; \frac{(L\Delta\alpha\sin(\alpha)W)^{2}}{6} \ll 1. \qquad (110)$$

At the other extreme, it is clear that if $\Omega \gg 1$ the probability of error will tend to zero. Rearranging terms in Ω and solving for $\Delta\alpha$, one obtains the following minimum angular separation to acheive above threshold operation :

$$\Delta\alpha > 2.26\left[\frac{M^{2}\,S_1S_2/N^{2}\;L^{2}\sin^{2}(\alpha)}{(1 + MS/N)\;\lambda_{min}^{2}}\,(TW)^{.5} \right] = \left[\frac{M^{2}\,S_1S_2/N^{2}\;L^{2}\sin^{2}(\alpha)}{(1 + MS/N)\;\lambda_{min}^{2}}\,(TW)^{.5} \right]^{-1/2} 2.26 \qquad (111)$$

If $(L/\lambda_{min}) = 10$, $\alpha = \pi/2$, $TW = 100$, and $MS_1/N = MS_2/N = .05$, resolution is possible when $\Delta\alpha > .15$ radians = 8.6 degrees. If the signal to noise ratios increase to .25 each but with the remaining parameters kept fixed, the minimum resolvable angle difference is 5.79 degrees.

Detectability of two sources from a signal field containing two closely spaced sources that are overlapping both temporally and spectrally is related to the resolving ability of the array. This section has determined the necessary parameter combinations that lead to accurate detection. Under such conditions, the two signals are distinguishable and it is believed that the Cramer-Rao lower bounds can be reached. When the conditions are not satisfied (when one is operating below the threshold), the two sources are essentially indistinguishable. Without apriori knowledge of the correct number of sources present, one simply cannot hope to reach the Cramer-Rao lower bounds.

5. Summary

This study has been concerned with determining absolute lower bounds on bearing accuracy when two closely spaced incoherent sources radiate broadband Gaussian noise and are received by an equally spaced line array in the presence of spatially white Gaussian noise. The sources are taken to be closely spaced whenever their difference in bearing is much smaller than a beamwidth. The sources are overlapping both temporally and spectrally. The spectral shapes of the signals are taken to lowpass, rectangular functions of identical width, W, but with possibly different heights. The uniform sensor spacing, d, øs taken to be one half the signal wavelength at the highest frequency (λ_{min}) in the overlapping signal band, with total length, $L \gg \lambda_{min}$. The approach of the study is to isolate interference effects resulting from the closely spaced configuration of sources, and to do this in a setting in which reasonably accurate es-timates or either bearing may be expected in the absence of the interfering source.

In order to assess the performance loss resulting from the closely spaced source configuration, the ratio of Cramer-Rao lower bounds obtained in the two source setting to the bounds computed for the bearing estimate with a single source only was computed. The major results for the situation of known signal levels are as follows :

1. Even when the bearing of one of the sources is known apriori, bearing accuracy for the source at unknown bearing is strongly degraded if the SNR of the known source exceeds that of the unknown source. The degradation factor reduces as the SNR of the known source increases beyond a well defined limit and then saturates at a nonzero value.

2. When both source bearings are unknown and the two signal to noise ratios are much smaller than the quantity

$$\left\{ 1 - \frac{1}{M^2} \left| \underline{v}_1^* \underline{v}_2 \right|^2 \right\}^{-1}$$

then the accuracy of either bearing estimate is degraded by the factor $(\Delta\alpha)^{-2} \gg 1$. In this regime, the accuracy of either source bearing is relatively unaffected by the SNR of the interfering source, provided both signal to noise ratios are less than the quantity referred to above.

3. If one of the signal to noise ratios exceeds

$$\left\{ 1 - \frac{1}{M^2} \left| \underline{v}_1^* \underline{v}_2 \right|^2 \right\}^{-1}$$

while the other does not, the situation is quite different. In this case the larger SNR source has no performance degradations resulting from the presence of the second nearby source. Hence one can obtain good estimation accuracy even in the presence of interference for the large SNR

source. The lower SNR source, on the other hand, has large degradations in accuracy which are relatively insensitive to further increases in the interference SNR (as discussed in item 2).

4. If both signal to noise ratios exceed the above quantity, the estimation of either source bearing is again degraded as in the " low signal to noise ratio " regime, with optimal performance acheived when the power ratio is one. The error for both sources is proportional to the interference source power.

These results suggest that if the signal levels are known apriori, the power level ratio is an important quantity in determining bearing accuracy.

5. When the signal levels are unknown and both signal to noise ratios are low in the sense of the above, i.e.,

$$MS_i/N \ll \left\{ 1 - \frac{1}{M^2} \left| \underline{v}_1^* \underline{v}_2 \right|^2 \right\}^{-1},$$

the accuracy reduction over the known signal level case for either source is independent of signal to noise ratio and equal to 5.4 db.

6. If either SNR is larger than the above factor, both estimates are degraded over the single source accuracy, unlike the result determined earlier where the larger SNR source suffered no loss in bearing accuracy. Incremental errors resulting from unknown signal levels are proportional to the ratio of the larger to smaller source SNR. The absolute error for the lower SNR source increases with the interference power.

7. If both signal to noise ratios exceed the value described earlier, the absolute bearing errors rise with the -3/2 power of the angular spacing and tend to zero as the ratio of source to interference power tends to infinity. The incremental error resulting from unknown signal levels increases with the interference power up to a defined value and then saturates at a finite value.

9. Reachability of the Cramer-Rao lower bounds is unlikely if the two sources cannot be distinguished (i. e., the correct number of sources indeterminable from the data). This question was pursued by determining the probability of error in choosing the wrong hypothesis in the following binary hypothesis test :

Under H_0, there are two signals present;

Under H_1, there is one signal present radiating the total received signal power from bearing angle α_0 located at the weighted average bearing :

$$\alpha_0 = \frac{S_1\alpha_1 + S_2\alpha_2}{S}$$

The procedure in the study is to determine the minimum angular differ-

ence, associated with a given set of parameters (SNR, T, L/λ_{min}), for which the error probability in choosing the incorrect hypothesis will tend to zero. For these combinations of parameters the correct number of sources can be determined from the data, and one expects that the earlier computed Cramer-Rao lower bounds can be reached.

The results show that the minimum angular difference associated with a small error probability can be reduced by increases in SNR and T. Quantitative expressions for the required parameter combinations are presented which lead to existence of a threshold, below which the data is essentially useless in determining the number of sources present. In this regime, reachability of the Cramer-Rao lower bounds is highly suspect.

Appendix 1

Derivation of equations (13) - (15)

The elements of the Fisher Information Matrix can be computed from the following expression :

$$J_{ij} = - \mathrm{Tr} \left\{ \frac{\partial}{\partial \theta_i}(K^{-1}) \frac{\partial K_k}{\partial \theta_j} \right\} = - \sum_{k=1}^{TW} \mathrm{Tr} \left\{ \frac{\partial}{\partial \theta_i}(K_k^{-1}) \frac{\partial K_k}{\partial \theta_j} \right\}$$

K_k and K_k^{-1} are given by :

$$K_k = N_k I + S_1 \underline{v}_1 \underline{v}_1^{\dagger} + S_2 \underline{v}_2 \underline{v}_2^{\dagger}$$

$$K_k^{-1} = \frac{1}{N_k} \left\{ I - G_k \left[P_1 \underline{v}_1 \underline{v}_1^{\dagger} + P_2 \underline{v}_2 \underline{v}_2^{\dagger} - P_1 P_2 (\underline{v}_1 \underline{v}_1^{\dagger} \underline{v}_2 \underline{v}_2^{\dagger} + \underline{v}_2 \underline{v}_2^{\dagger} \underline{v}_1 \underline{v}_1^{\dagger}) \right] \right\}$$

Since $\underline{v}_1$ is a function of α_1 and $\underline{v}_2$ is a function of α_2, let $R_i \triangleq \frac{\partial}{\partial \alpha_i}(\underline{v}_i \underline{v}_i^{\dagger})$ so that :

$$\frac{\partial}{\partial \alpha_1}(K_k^{-1}) = - \frac{1}{N_k} \left\{ G_k \left[P_1 R_1 - P_1 P_2 (R_1 \underline{v}_2 \underline{v}_2^{\dagger} + \underline{v}_2 \underline{v}_2^{\dagger} R_1) \right] + \frac{\partial G_k}{\partial \alpha_1} \left[\underline{v}_1 \underline{v}_1^{\dagger} P_1 + \underline{v}_2 \underline{v}_2^{\dagger} P_2 - P_1 P_2 \, (\underline{v}_1 \underline{v}_1^{\dagger} \underline{v}_2 \underline{v}_2^{\dagger} + \underline{v}_2 \underline{v}_2^{\dagger} \underline{v}_1 \underline{v}_1^{\dagger}) \right] \right\}$$

Then

$$\frac{\partial G_k}{\partial \alpha_1} = G_k^2 \left\{ P_1 P_2 \frac{\partial}{\partial \alpha_1} |\underline{v}_1^{\dagger} \underline{v}_2|^2 \right\} = G_k^2 \, P_1 P_2 \eta_1^2$$

and

$$\frac{\partial}{\partial \alpha_1}(K_k^{-1}) = - \frac{G_k}{N_k} \left\{ P_1 R_1 - P_2 P_1 (R_1 \underline{v}_2 \underline{v}_2^{\dagger} + \underline{v}_2 \underline{v}_2^{\dagger} R_1) - G_k P_1 P_2 \eta_1 \times (\underline{v}_1 \underline{v}_1^{\dagger} P_1 + \underline{v}_2 \underline{v}_2^{\dagger} P_2 - P_1 P_2 (\underline{v}_1 \underline{v}_1^{\dagger} \underline{v}_2 \underline{v}_2^{\dagger} + \underline{v}_2 \underline{v}_2^{\dagger} \underline{v}_1 \underline{v}_1^{\dagger}) \right\}$$

With $\frac{\partial K_k}{\partial \alpha_1} = S_1(\omega_k) \, R_1$, it follows that

$$-\mathrm{Tr}\left(\frac{\partial}{\partial \alpha_1} K_k^{-1} \frac{\partial K_k}{\partial \alpha_1}\right) = S_1(\omega_k)/N(\omega_k) \mathrm{Tr} \left\{ (P_1 R_1^2 - P_1 P_2 (R_1 \underline{v}_2 \underline{v}_2^{\dagger} R_1 + \underline{v}_2 \underline{v}_2^{\dagger} R_1^2) - P_1 P_2 G_k \eta_1 \left[P_1 \underline{v}_1 \underline{v}_1^{\dagger} R_1 + P_2 \underline{v}_2 \underline{v}_2^{\dagger} R_1 - P_1 P_2 (\underline{v}_1 \underline{v}_1^{\dagger} \underline{v}_2 \underline{v}_2^{\dagger} + \underline{v}_2 \underline{v}_2^{\dagger} \underline{v}_1 \underline{v}_1^{\dagger}) R_1 \right] \right\}$$

With the identities :

$$\mathrm{Tr}\ (R_1^{\,2}) = 2\left\{ (\underline{v}_1^{\,+}\underline{v}_1)\left(\frac{\partial \underline{v}_1}{\partial \alpha_1}^{+}\frac{\partial \underline{v}_1}{\partial \underline{v}_1} \right) - \left(\underline{v}_1^{\,+}\frac{\partial \underline{v}_1}{\partial \underline{v}_1}\right)\left(\frac{\partial \underline{v}_1}{\partial \underline{v}_1}^{+}\underline{v}_1\right) \right\}$$

$$\mathrm{Tr}\ (R_1\underline{v}_2\underline{v}_2^{\,+}R_1 + \underline{v}_2\underline{v}_2^{\,+}R_1^{\,2}) = 2\left\{ \left(\frac{\partial \underline{v}_1}{\partial \alpha_1}^{+}\underline{v}_2 \right)\left[(\underline{v}_2^{\,+}\underline{v}_1)\left(\frac{\partial \underline{v}_1}{\partial \alpha_1}^{+}\underline{v}_2\right) + \left(\underline{v}_2^{\,+}\frac{\partial \underline{v}_1}{\partial \alpha_1}\right)\left(\underline{v}_1^{\,+}\underline{v}_1\right) \right] + (\underline{v}_1^{\,+}\underline{v}_2)\left[(\underline{v}_2^{\,+}\underline{v}_1)\left(\frac{\partial \underline{v}_1}{\partial \alpha_1}^{+}\frac{\partial \underline{v}_1}{\partial \alpha_1}\right) + \left(\underline{v}_2^{\,+}\frac{\partial \underline{v}_1}{\partial \alpha_1}\right)\left(\underline{v}_1^{\,+}\frac{\partial \underline{v}_1}{\partial \alpha_1}\right) \right] \right\}$$

$$\mathrm{Tr}\ (\underline{v}_1\underline{v}_1^{\,+}R_1) = 0$$

$$\mathrm{Tr}\ (\underline{v}_2\underline{v}_2^{\,+}R_1) = \frac{\partial}{\partial \alpha_1}(\underline{v}_2^{\,+}\underline{v}_1\underline{v}_1^{\,+}\underline{v}_2) = \eta_1$$

$$\mathrm{Tr}\ (\underline{v}_1\underline{v}_1^{\,+}\underline{v}_2\underline{v}_2^{\,+}R_1 + \underline{v}_2\underline{v}_2^{\,+}\underline{v}_1\underline{v}_1^{\,+}R_1) = \mathrm{Tr}\ \left(\frac{\partial}{\partial \alpha_1}(\underline{v}_2\underline{v}_2^{\,+}\underline{v}_1\underline{v}_1^{\,+}\underline{v}_1\underline{v}_1^{\,+})\right)$$

$$= (\underline{v}_1^{\,+}\underline{v}_1)\eta_1$$

With the use of the above expressions, one can show that :

$$-\mathrm{Tr}\ \left(\frac{\partial}{\partial \alpha_1}K_k^{-1}\ \frac{\partial K_k}{\partial \alpha_1}\right) = G_k\ (S_1(\omega_k)/N(\omega_k))\ \left\{ P_1 z_1 - P_2 y_1 + P_1 P_2^{\,2}\eta_1^{\,2}\ (1 - P_1) \right\}$$

where $\quad z_1 = (\underline{v}_1^{\,+}\underline{v}_1)\left(\frac{\partial \underline{v}_1}{\partial \alpha_1}^{+}\frac{\partial \underline{v}_1}{\partial \alpha_1}\right) - \left(\underline{v}_1^{\,+}\frac{\partial \underline{v}_1}{\partial \alpha_1}\right)\left(\frac{\partial \underline{v}_1}{\partial \alpha_1}^{+}\underline{v}_1\right)$

$$= \frac{\partial^2}{\partial\alpha_1\partial\alpha_1'} \left| \underline{v}_1^{\dagger}\underline{v}_1' \right|^2 \Big|_{\alpha_1' = \alpha_1}$$

$$y_1 = 2\ \mathrm{Tr}\ (\underline{v}_2\underline{v}_2^{\dagger}R_1^2)$$

J_{11} then becomes :

$$J_{11} = \sum_{k=1}^{TW} G_k \left\{ S_1(\omega_k)/N(\omega_k)P_1 \left\{ z_1 - P_2 y_1 \right\} + P_1^2\ P_2^2\ G_k \eta_1^2 \right\} \quad \text{(A3)}$$

Using analogous computations for element J_{12} :

$$- \mathrm{Tr}\ \left(\frac{\partial}{\partial\alpha_1} K_k^{-1} \frac{\partial K_k}{\partial\alpha_2} \right) = G_k(S_2(\omega_k)/N(\omega_k))\ \mathrm{Tr}\Big\{ [\ P_1R_1R_2 - P_1P_2[$$

$$R_1\underline{v}_2\underline{v}_2^{\dagger}R_2 + \underline{v}_2\underline{v}_2^{\dagger}R_1R_2\ ] - G_kP_1P_2\eta_1\ ] \quad [- P_1\underline{v}_1\underline{v}_1^{\dagger}R_2$$

$$- P_2\underline{v}_2\underline{v}_2^{\dagger}R_2 + P_1P_2(\underline{v}_1\underline{v}_1^{\dagger}\underline{v}_2\underline{v}_2^{\dagger}R_2 + \underline{v}_2\underline{v}_2^{\dagger}\underline{v}_1\underline{v}_1^{\dagger}R_1)]\ \Big\}$$

Inserting the identities :

$$\mathrm{Tr}\ (R_1R_2) = \frac{\partial^2}{\partial\alpha_1\partial\alpha_2} |\underline{v}_1^{\dagger}\underline{v}_2|^2$$

$$\mathrm{Tr}\ (R_1\underline{v}_2\underline{v}_2^{\dagger}R_2 + \underline{v}_2\underline{v}_2^{\dagger}R_1R_2) = (\underline{v}_2^{\dagger}\underline{v}_2)\ \frac{\partial^2}{\partial\alpha_1\partial\alpha_2}\ |\underline{v}_1^{\dagger}\underline{v}_2|^2$$

$$\mathrm{Tr}\ (\ \underline{v}_1\underline{v}_1^{\dagger}R_2) = \frac{\partial}{\partial\alpha_2}\mathrm{Tr}\ (\underline{v}_1\underline{v}_1^{\dagger}\underline{v}_2\underline{v}_2) = \eta_2 \triangleq \frac{\partial}{\partial\alpha_2}|\underline{v}_1^{\dagger}\underline{v}_2|^2$$

$$\mathrm{Tr}\ (\underline{v}_2\underline{v}_2^{+}R_2) = 0$$

$$\mathrm{Tr}\ (\ \underline{v}_1\underline{v}_1^{+}\underline{v}_2\underline{v}_2^{+}R_2 + \underline{v}_2\underline{v}_2^{+}\underline{v}_1\underline{v}_1^{+}R_2\ ) = \frac{\partial}{\partial\alpha_2}\ (\underline{v}_1\underline{v}_1^{+}\underline{v}_2\underline{v}_2^{+}\underline{v}_2\underline{v}_2^{+})$$

$$= (\underline{v}_2^{+}\underline{v}_2)\eta_2$$

it then follows that:

$$-\ \mathrm{Tr}\ (\ \frac{\partial}{\partial\alpha_1}K_k^{-1}\ \frac{\partial K_k}{\partial\alpha_2}\ ) = G_k(S_2(\omega_k)/N(\omega_k))\ \left\{\ \frac{\partial^2}{\partial\alpha_1\partial\alpha_2}|\underline{v}_1^{+}\underline{v}_2|^2\right.$$

$$\left.(\ 1 - P_2)\ P_1 + G_kP_1^2P_2\ (1 - P_2)\eta_1\eta_2\ \right\} \qquad \text{(A4)}$$

J_{22} is identical to element J_{11} with the exchange of subscripts (1) with (2) in (A4) :

$$J_{22} = \sum_{k=1}^{TW} G_k\ \left\{\ P_2S_2(\omega_k)/N(\omega_k)\left\{\ z_2 - P_1y_2\ \right\} + P_1^2P_2^2G_k\eta_2^2\ \right\} \qquad \text{(A5)}$$

Equations (A3) - (A5) are equations (13) - (15).

Appendix 2

Derivation of equations (25) - (27)

According to equation (16):

$$\eta_i = \frac{\partial}{\partial \alpha_i} |\underline{v}_1 \dagger \underline{v}_2|^2$$

With

$$|\underline{v}_1 \dagger \underline{v}_2|^2 = \sum_{i,j=-N}^{N} \cos(\omega_k d(i-j)(\cos(\alpha_1)-\cos(\alpha_2))/c) \qquad \text{(A2.1)}$$

it follows that

$$\frac{\partial}{\partial \alpha_1} |\underline{v}_1 \dagger \underline{v}_2|^2 = \sum_{i,j=-N}^{N} \frac{\omega_k d(i-j)}{c} \sin(\alpha_1)\sin(\omega_k d(i-j)(\cos(\alpha_1)-\cos(\alpha_2))/c) \qquad \text{(A2.2)}$$

Introducing the notation $\Delta\alpha = (\alpha_1 - \alpha_2)$, $\alpha = (\alpha_1 + \alpha_2)/2$, along with the assumption that the argument of (A2.1) varies negligibly over the the i and j sums one obtains:

$$\frac{\partial}{\partial \alpha_1} |\underline{v}_1 \dagger \underline{v}_2|^2 \cong \sum ((\omega_k d(i-j)/c)^2 - \frac{(\omega_k d(i-j)/c)^4}{3!} \Delta\alpha^2 \sin^2(\alpha) + \ldots) \times \Delta\alpha \sin(\alpha)\sin(a_1) \qquad \text{(A2.2a)}$$

With

$$\sum_{i,j=-N}^{N} (i-j)^2 = \frac{4MN(N+1)(2N+1)}{6}.$$

For large M, $N \cong M/2$ and

$$\sum_{i,j=-N}^{N} (i-j)^2 \cong \frac{M^4}{3}.$$

Truncating the series of (A2.2a) after a single term:

$$\frac{\partial}{\partial \alpha_1} |\underline{v}_1 \dagger \underline{v}_2|^2 \cong \frac{M^2 L^2 \Delta\alpha \sin(\alpha)\sin(\alpha_1)}{6} \omega_k^2 \qquad \text{(A2.3)}$$

The expression for y_1 is given by equation (18):

$$y_1 = 2\mathrm{Tr}\left\{ \left(\frac{\partial}{\partial\alpha_1}(\underline{v}_1\underline{v}_1^{+})\right)(\underline{v}_2\underline{v}_2^{+})\left(\frac{\partial}{\partial\alpha_1}(\underline{v}_1\underline{v}_1^{+})\right) \right\}$$

$$= 2\left\{ (\underline{v}_1^{+}\underline{v}_1)\left(\frac{\partial\underline{v}_1}{\partial\alpha_1}^{+}\underline{v}_2\right)\left(\underline{v}_2^{+}\frac{\partial\underline{v}_1}{\partial\alpha_1}\right) + (\underline{v}_1^{+}\underline{v}_2)\left(\underline{v}_2^{+}\frac{\partial\underline{v}_1}{\partial\alpha_1}\right)\left(\underline{v}_1^{+}\frac{\partial\underline{v}_1}{\partial\alpha_1}\right) \right.$$

$$\left. (\underline{v}_2^{+}\underline{v}_1)\left(\frac{\partial\underline{v}_1}{\partial\alpha_1}^{+}\underline{v}_2\right)\left(\frac{\partial\underline{v}_1}{\partial\alpha_1}^{+}\underline{v}_1\right) + |\underline{v}_1^{+}\underline{v}_2|^2\,\frac{\partial\underline{v}_1}{\partial\alpha_1}^{+}\,\frac{\partial\underline{v}_1}{\partial\alpha_1}\right\} \qquad \text{(A2.4)}$$

For far field sources, $\underline{v}_1^{+}\underline{v}_2 = \underline{v}_2^{+}\underline{v}_1$. Also $\frac{\partial\underline{v}_1}{\partial\alpha_1}^{+}\underline{v}_1 = -\,\underline{v}_1^{+}\frac{\partial\underline{v}_1}{\partial\alpha_1}$.

Hence the middle two terms of equation (A2.4) are identically zero. Carrying out a few steps of algebra the remaining terms become :

$$y_1 = 2\sum_{i,j,l=-N}^{N} \omega_k^2\,\frac{d^2\sin^2(\alpha_1)}{c^2}\,(l^2 + ij)\left(1 - \frac{1}{2!}\,\frac{(\Delta a\,\sin\alpha\,d\,\omega_k)^2}{c^2}\right. \times$$

$$\left.(i-j)^2 + \ldots\right)$$

$$= \frac{M^2\omega_k^2L^2\sin^2(\alpha_1)}{6c^2} + O(\beta\omega_k^2)^2 \qquad \text{(A2.5)}$$

Finally

$$z_1 = -\left.\frac{\partial^2}{\partial\alpha_1\partial\alpha_1'}|\underline{v}_1^{+}\underline{v}_1'|^2\right|_{\alpha_1'=\alpha_1}$$

$$= 2\left\{(\underline{v}_1^{+}\underline{v}_1)\left(\frac{\partial\underline{v}_1}{\partial\alpha_1}^{+}\frac{\partial\underline{v}_1}{\partial\alpha_1}\right) - \left(\underline{v}_1^{+}\frac{\partial\underline{v}_1}{\partial\alpha_1}\right)\left(\frac{\partial\underline{v}_1}{\partial\alpha_1}^{+}\underline{v}_1\right)\right\}$$

$$= 2\,\frac{\omega_k^2 d^2\sin^2(\alpha_1)}{c^2}\sum_{i,j=-N}^{N}(i^2-ij) = \frac{4MN(N+1)(2N+1)}{6c^2}\,d^2\sin^2(\alpha_1)$$

For large M we have

$$z_1 = \frac{M^2L^2\sin^2(\alpha_1)}{6c^2}\,\omega_k^2. \qquad \text{(A2.6)}$$

$$\frac{\partial^2}{\partial\alpha_1\partial\alpha_2}|\underline{v}_1^{+}\underline{v}_2|^2 = \frac{\omega_k^2 d^2\sin(\alpha_1)\sin(\alpha_2)}{c^2}\sum_{i,j=-N}^{N}(i-j)^2\left(1 - \frac{1}{2!}\right.($$

$$\frac{(\Delta\alpha\sin(\alpha)(i - j))^2 d^2 \omega_k^2)}{c^2}) + \ldots)$$

$$= \frac{M^2 L^2 \sin(\alpha_1)\sin(\alpha_2)\omega_k^2}{6c^2}(1 + \frac{\beta\omega_k^2}{5} 6 + \ldots) \qquad \text{(A2.6)}$$

If one exchanges the index 1 with the index 2 in equations (A2.3) - (A2.6), expressions for z_2, y_2 and η_2 can be obtained. Substituting the variables

$$x_1 = (\frac{L^2 \sin^2(\alpha_1)}{6c^2})^{1/2} \quad , \quad x_2 = (\frac{L^2 \sin^2(\alpha_2)}{6c_2})^{1/2} \quad , \quad \xi = 1 - \frac{|\underline{v}_1 \dot{+} \underline{v}_2|}{M^2}$$

into the above results and with the definition

$$H_k \triangleq [1 + MS/N + M^2 S_1 S_2/N^2 \ \xi]^{-1}$$

substitution yields (25) - (27) :

$$J_{11} = \sum_{k=1}^{WT} H_k (MS_1/N)^2 x_1^2 \omega_k^2 \left[1 + \beta\omega_k^2 H_k (MS_2/N)^2 \right] \qquad \text{(A2.5)}$$

$$J_{12} = \sum_{k=1}^{WT} H_k \ M^2 S_1 S_2/N^2 x_1 x_2 \omega_k^2 \left[1 - 6/5\beta\omega_k^2 - \beta\omega_k^2 H_k M^2 S_1 S_2/N^2 \right] \qquad \text{(A2.6)}$$

$$J_{22} = \sum_{k=1}^{TW} H_k (MS_2/N)^2 x_2^2 \omega_k^2 \left[1 + \beta\omega_k^2 H_k (MS_2/N)^2 \right] \qquad \text{(A2.8)}$$

Appendix 3

Derivation of equations (39) - (41)

Elements of the Fisher matrix expressed in terms of integral I_1, I_2 and I_3 are given by equations (33) - (35) :

$$J_{11} = x_1^2 (MS_1/N)^2 \left\{ I_1 + \beta (MS_2/N)^2 I_3 \right\} \tag{A3.1}$$

$$J_{12} = x_1 x_2 M^2 S_1 S_2 / N^2 \left\{ I_1 - (6/5)\beta I_2 - \beta M^2 S_1 S_2 / N^2 I_3 \right\} \tag{A3.2}$$

$$J_{22} = x_2^2 (MS_2/N)^2 \left\{ I_1 + \beta (MS_1/N)^2 I_3 \right\} \tag{A3.3}$$

By direct integration

$$I_1 = \frac{T}{2\pi} \int_0^W \frac{\omega^2 d\omega}{(1 + MS/N + M^2 S_1 S_2 / N^2 \ \xi)}$$

With $\xi = \frac{\beta W^2}{2} (1 - \frac{\beta W^2}{5} + . . .)$, terminating the Taylor series after a single term will not materially affect the value of the integral, since the denominator of the integrand is dominated by the term 1 + MS/N. An approximation for I_1 is as follows :

$$I_1 \cong \frac{T}{2\pi} \int_0^W \frac{\omega^2 d\omega}{1 + MS/N + M^2 S_1 S_2 / N^2 (\beta W^2/2)}$$

Changing variables: $x \triangleq \omega R^{1/2}$, one obtains the simpler integral

$$= \frac{T}{2\pi} \frac{1}{1 + MS/N} R^{-3/2} \int_0^{WR^{1/2}} \frac{x^2 dx}{(1 + x^2)}$$

$$= \frac{TW}{2\pi} \frac{1}{1 + MS/N} \frac{1}{R} \left\{ 1 - (WR^{1/2})^{-1} \tan^{-1}(WR^{1/2}) \right\}$$

For $RW^2 << 1$

$$\tan^{-1}(WR^{1/2}) = WR^{1/2} - \frac{(WR^{1/2})^3}{3} + \frac{(WR^{1/2})^5}{5} + \ldots$$

One then obtains for I_1 :

$$I_1 \cong \frac{TW}{2\pi} \frac{1}{1 + MS/N} \frac{W^2}{3} \left(1 - \frac{3}{5} RW^2 + \ldots \right) \qquad (A3.4)$$

After similar calculations for I_2 and I_3 :

$$I_2 = \frac{T}{2\pi} \frac{1}{1 + MS/N} \frac{W^2}{3R} \left(1 - 3(W^2R)^{-1}(1 - (W^2R))^{-1/2}\tan^{-1}(WR^{1/2}) \right)$$

$$= \frac{TW}{2\pi} \frac{1}{1 + MS/N} \frac{W^2}{5} \left(1 - \frac{W^2R}{7} + \ldots \right) \qquad (A3.5)$$

$$I_3 = \frac{T}{2\pi} \frac{1}{(1 + MS/N)^2} \frac{W^4}{5} \left(1 - \frac{10}{7} W^2R + \ldots \right) \qquad (A3.6)$$

Substituting (A3.4) - (A3.6) into (A3.1) - (A3.3) one obtains :

$$J_{11} = \frac{TW}{2\pi} \frac{(MS_1/N)^2}{1 + MS/N} \frac{W^2}{3} \left(1 + \frac{3}{5} RW^2 \left(\frac{2S_2}{S_1} - 1 \right) \right)$$

$$J_{12} = \frac{TW}{2\pi} \frac{M^2S_1S_2/N^2}{1 + MS/N} \frac{W^2}{3} \left(1 - \frac{6}{25} \beta W^2 - \frac{9}{5} RW^2 \right)$$

$$J_{22} = \frac{TW}{2\pi} \frac{(MS_2/N)^2}{1 + MS/N} \frac{W^2}{3} \left(1 + \frac{3}{5} RW^2 \left(\frac{2S_1}{S_2} - 1 \right) \right)$$

which are equations (39) - (41).

Appendix 4

Derivation of equations (48) - (50)

The elements of the F.I.M can be expressed in terms of integrals I_1, I_2 and I_3, defined in equations (30) - (32) :

$$I_1 = \frac{T}{2\pi} \int_0^W \frac{\omega^2 d\omega}{1 + MS/N + M^2 S_1 S_2/N^2 - \xi} \qquad (A4.1)$$

With $\xi = \frac{\beta W^2}{2} - \frac{\beta^2 W^4}{10} + \ldots$, substituting the definition for R :

$$R \triangleq \frac{M^2 S_1 S_2/N^2 \ \beta/2}{1 + MS/N}$$

and changing variables : $x \triangleq \omega R^{1/2}$, write

$$I_1 \triangleq \frac{T}{2\pi} \frac{1}{1 + MS/N} R^{-3/2} \int_0^{\ } \frac{x^2 dx}{1 + x^2 - \beta R^{-1} x^4/5 + \ldots} \qquad (A4.2)$$

For $WR^{1/2} \ll 1$, terminating the ξ series after a single term affects the integral value only marginally since the denominator of the integrand is dominated by the constant term, 1. When $WR^{1/2} \gg 1$, the situation is potentially different because the integration is carried over a region in which powers of x contribute significantly to the denominator of the integrand.

The integrand of (A4.2) is strictly positive and monotonic; since the integral is taken over the positive line segment ($0, WR^{1/2}$) the integral is also strictly increasing. Let $g(\Delta\alpha) \triangleq \xi$:

$$g(\Delta\alpha) = 1 - |\underline{v}_1^* \underline{v}_2|^2/M^2 = 1 - \frac{1}{M^2} \sum_{i,j=-(M-1)/2}^{(M-1)/2} \cos(\omega_k (i - j) d\Delta\alpha \sin(\tau)/c)$$

A Taylor series for g as function of $\Delta\alpha$ is given by :

$$g(\Delta\alpha) = g(0) + \Delta\alpha\, g'(0) + \Delta\alpha^2 g''(0)/2! + \ldots$$

Because the $\Delta\alpha$ - dependent part of g is a sum of trigonometric functions it is straightforward to demonstrate that the following properties of the infinite series hold: The odd ordered derivatives of g

ions it is straightforward to demonstrate that the following properties of the infinite series hold: The odd ordered derivatives of g vanish. The even ordered derivatives alternate in sign, with the exception of the zeroeth derivative which is identically zero :

$$g(0) = 0,\ g'(0) = 0,\ g''(0) > 0,\ g'''(0) = 0,\ g''''(0) < 0\ .\ .$$

Because g is an alternating infinite series, an upper bound can be found on the error caused by terminating the series after a finite number of terms. The error can be no larger in magnitude than the magnitude of the (n + 1) term of the series terminated after n terms.

One can as a result bound the integrand of I_1. The fact that the integral is strictly positive and nondecreasing means that the integral can be bounded as well. Write

$$\xi = \frac{\beta\omega^2}{2} - \frac{\beta^2\omega^4}{10} + \ .\ . \quad ; \quad \text{if } \xi = \frac{\beta\omega^2}{2} + \varepsilon \quad \text{then} \quad |\varepsilon| \le \frac{\beta^2\omega^4}{10}$$

so that $M^2S_1S_2/N^2/(1 + MS/N)\ \xi = R\omega^2 + R\varepsilon/(\beta/2)$. With $x = \omega R^{1/2}$,

$$|R\varepsilon/(\beta/2)| \le x^4\ R^{-1}\beta/5 \le W^4R\beta/5.$$

Therefore ξ and therefore the denominator of the integrand of (A4.2) can be upper and lower bounded :

$$1 + x^2 + \beta W^4 R/5 \ \ge\ 1 + x^2 + R^{-1}\beta x^4/5 + \ .\ .\ .\ \ge\ 1 + x^2 - \beta W^4 R/5$$

which yields bounds for the integral

$$\int_0^{WR^{1/2}} \frac{x^2dx}{1 + x^2 - \beta W^4R/5} \ \ge\ \int_0^{WR^{1/2}} \frac{x^2dx}{1 + x^2 - R^{-1}\beta x^4/5} \ \ge$$

$$\ge\ \int_0^{WR^{1/2}} \frac{x^2dx}{1 + x^2 + \beta W^4R/5}$$

It will be necessary to consider the magnitude of $\beta W^4 R/5$ relative to one.

βW^2	$W^2 R$	$\beta W^4 R/5$
.1	10	.2
.01	10	.02
.01	100	.2
.1	100	2

If $\alpha = \pi/2$ and $L/\lambda_{max} = 10$, then $\beta W^2 = \Delta\alpha^2 (100)(2\pi)^2/6$ so that if $\beta W^2 = .1$ $\Delta\alpha = .7$ deg.

Unless $W^2 R$ approaches 100, the product $\beta W^4 R/5$ will be smaller than one. In order for $W^2 R$ to be as large as 100, the signal to noise ratios must be as large as 30 db if βW^2 is to be smaller than one tenth. Based on these considerations we will assume that $\beta W^4 R/5 \ll 1$. Under these conditions the upper bound on the integral becomes:

$$\int_0^{WR^{1/2}} \frac{x^2}{1 + x^2 - W^4 R^{-1}/5} = WR^{1/2} - (1 - W^4 \beta R/5)^{1/2} \tan^{-1}(WR^{1/2})$$

The lower bound is given by :

$$\int_0^{WR^{1/2}} \frac{x^2 dx}{1 + x^2 + \beta W^2} = WR^{1/2} - (1 + \beta W^4 R/5)^{1/2} \tan^{-1}(WR^{1/2})$$

Substituting the above results into (A4.3) one obtains for the upper bound on I_1 :

$$I_1^{max} = \frac{TW}{2\pi} \frac{1}{1 + MS/N} R^{-1} \left\{ 1 - (WR^{1/2})^{-1} (1 - W^4 R\beta/5)^{1/2} \tan^{-1}(WR^{1/2}) \right\}$$

$$I_1^{min} = \frac{TW}{2\pi} \frac{1}{1 + MS/N} R^{-1} \left\{ 1 - (WR^{1/2})^{-1} (1 + W^4 R\beta/5)^{1/2} \tan^{-1}(WR^{1/2}) \right\}$$

For $WR^{1/2} \gg 1$

$$\tan^{-1}(WR^{1/2}) = \frac{\pi}{2} - (WR^{1/2})^{-1} + \ldots$$

and $W^4 R\beta/5 \ll 1$

$$(1 \pm W^4 R\beta/5)^{1/2} = 1 \pm \frac{W^4 R\beta}{10} \pm \frac{W^8 R^2 \beta^2}{40} + \ldots$$

so that

$$I_1^{max} \cong \frac{TW}{2\pi} \frac{1}{1 + MS/N} R^{-1} \left\{ 1 - (WR^{1/2})^{-1} \times \frac{\pi}{2} + (W^2 R)^{-1} + \right.$$

$$\left. + \frac{W^3 R^{1/2} \beta}{10} \frac{\pi}{2} - \frac{\beta W^2}{10} + \ldots \right\}$$

$$I_1^{min} \cong \frac{TW}{2\pi} \frac{1}{1 + MS/N} R^{-1} \left\{ 1 - (WR^{1/2})^{-1} \times \frac{\pi}{2} + (W^2 R)^{-1} + \right.$$

$$\left. \frac{\beta W^2}{10} - \frac{W^3 R^{1/2} \beta}{10} \frac{\pi}{2} + \ldots \right\}$$

The difference between the upper and lower bound has order smaller than $(WR^{1/2})^{-1}$. The Fisher matrix will be shown to be nonsingular to order $(WR^{1/2})^{-1}$ and therefore the upper and lower bounds will yield identical results. Therefore we write

$$I_1 \cong \frac{TW}{2\pi} \frac{1}{1 + MS/N} R^{-1} \left\{ 1 - (WR^{1/2})^{-1} \times \frac{\pi}{2} \right\} \tag{A4.3}$$

One can obtain analogous upper and lower bounds for I_2 and I_3 :

$$I_2^{max} \cong \frac{TW}{2\pi} \frac{1}{1 + MS/N} \frac{1}{3} R^{-1} \left(1 - 3 (W^2 R)^{-1} + \right.$$

$$\left. + \frac{3W^2 \beta}{5} \right)$$

$$I_2^{min} \cong \frac{TW}{2\pi} \frac{1}{1 + MS/N} \frac{1}{3} R^{-1} \left(1 - 3(W^2 R)^{-1} - \right.$$

$$\left. - \frac{3\beta W^2}{5} \right)$$

$$I_3^{max} = \frac{TW}{2\pi} \frac{1}{(1 + MS/N)^2} R^{-2} \left(1 - 3\pi/4 (WR^{1/2})^{-1} + \right.$$

$$+ \frac{3\pi\beta W^3 R^{1/2}}{40})$$

$$I_3^{min} = \frac{TW}{2\pi} \frac{1}{(1 + MS/N)^2} R^{-2}(1 - 3\pi/4(WR^{1/2})^{-1} -$$

$$- \frac{3\pi\beta W^3 R^{1/2}}{40})$$

The upper and lower bounds for I_2 and I_3 differ by $O(\beta W^2)$ and hence he upper and lower bounds nearly equivalent results.

$$I_2 \cong \frac{TW}{2\pi} \frac{1}{1 + MS/N} R^{-1} \frac{1}{3} \tag{A4.4}$$

$$I_3 \cong \frac{TW}{2\pi} \frac{1}{(1 + MS/N)^2} R^{-2}(1 - 3\pi/4 \ (WR^{1/2})^{-1}) \tag{A4.5}$$

Let $y \cong (W^2 R)^{-1/2}$. Substituting equations (A4.3) - (A4.5) into (33) - (35) one obtains :

$$J_{11} = \frac{TW}{2\pi} x_1^2 \frac{(MS_1/N)^2}{1 + MS/N} R^{-1} (1 + \frac{2S_2}{S_1} - y\pi/2 (1 + \frac{3S_1}{2S_2})) \tag{A4.6}$$

$$J_{21} = - \frac{TW}{2\pi} x_1 x_2 \frac{M^2 S_1 S_2/N^2}{1 + MS/N} R^{-1}(1 - y\pi) \tag{A4.7}$$

$$J_{22} = \frac{TW}{2\pi} x_2^2 \frac{(MS_2/N)^2}{1 + MS/N} R^{-1} (1 + \frac{2S_1}{S_2} - y\pi/2(1 + \frac{3S_2}{2S_1})) \tag{A4.8}$$

Dropping the $O(y)$ terms in (A4.6) - (A4.8) yields equations (48) - (50).

Appendix 5

Derivation of equations (65) - (71)

In this appendix, the elements of matrices J_2 and J_3 are computed using the following expressions for arbitrary elements of the Fisher Information Matrix :

$$J_{ij} = \mathrm{Tr}\left\{ K^{-1} \frac{\partial K}{\partial \theta_i} K^{-1} \frac{\partial K}{\partial \theta_j} \right\} = \sum_{k=1}^{WT} \mathrm{Tr}\left\{ K_k^{-1} \frac{\partial K_k}{\partial \theta_i} K_k^{-1} \frac{\partial K_k}{\partial \theta_j} \right\}$$

$$= - \sum_{k=1}^{WT} \mathrm{Tr}\left\{ \frac{\partial}{\partial \theta_i} (K_k^{-1}) \frac{\partial K_k}{\partial \theta_j} \right\} \qquad (A5.1)$$

From section 2, K_k and K_k^{-1} are given by :

$$K_k = N_k I + S_1 \underline{v}_1 \underline{v}_1^{\dagger} + S_2 \underline{v}_2 \underline{v}_2^{\dagger}$$

$$K_k^{-1} = \frac{1}{N_k} \left[I - G_k \left\{ P_1 \underline{v}_1 \underline{v}_1^{\dagger} + P_2 \underline{v}_2 \underline{v}_2^{\dagger} - P_1 P_2 (\underline{v}_1 \underline{v}_1^{\dagger} \underline{v}_2 \underline{v}_2^{\dagger} + \underline{v}_2 \underline{v}_2^{\dagger} \underline{v}_1 \underline{v}_1^{\dagger}) \right\} \right],$$

where

$$P_1 = \frac{MS_1/N}{1 + MS_1/N} ; \quad P_2 = \frac{MS_2/N}{1 + MS_2/N} ; \quad G_k = \left[1 - P_1 P_2 |\underline{v}_1^{\dagger} \underline{v}_2|^2 \right]^{-1}.$$

Calculation of $\mathrm{Tr}\left\{ K_k^{-1} \frac{\partial K_k}{\partial S_i} K_k^{-1} \frac{\partial K_k}{\partial \alpha_j} \right\}$:

We have

$$\frac{\partial K_k}{\partial S_i} = \underline{v}_i \underline{v}_i^{\dagger} \quad ;$$

also let $\frac{\partial}{\partial \alpha_j} (\underline{v}_j \underline{v}_j^{\dagger}) \triangleq R_j$ and $\frac{\partial}{\partial \alpha_j} |\underline{v}_1^{\dagger} \underline{v}_2|^2 = \eta_j$. Then with $\frac{\partial G_k}{\partial \alpha_j} = G_k^2 P_i P_j \eta_j$, it follows that

$$\frac{\partial K_k}{\partial \alpha_j} = - \frac{G_k}{N_k} P_j \left\{ R_j - P_i (R_j \underline{v}_i \underline{v}_i^{\dagger} + \underline{v}_i \underline{v}_i^{\dagger} R_j) - P_i \eta_j G_k (\underline{v}_i \underline{v}_i^{\dagger} P_i + \underline{v}_j \underline{v}_j^{\dagger} P_j - P_i P_j (\underline{v}_i \underline{v}_i^{\dagger} \underline{v}_j \underline{v}_j^{\dagger} + \underline{v}_j \underline{v}_j^{\dagger} \underline{v}_i \underline{v}_i^{\dagger})) \right\}.$$

Then for $i \neq j$

$$- \mathrm{Tr}\left\{ \frac{\partial K_k}{\partial S_i} \frac{\partial K_k^{-1}}{\partial \alpha_i} \right\} = \frac{G_k P_j}{N_k} \mathrm{Tr} \left\{ \underline{v}_i \underline{v}_i^{+} R_j \left(1 - 2MP_i \right) - P_i \eta_j G_k \left(MP_i \underline{v}_i \underline{v}_i^{+} + P_j \underline{v}_i \underline{v}_i^{+} \underline{v}_j \underline{v}_j^{+} - P_i P_j \left(2 |\underline{v}_1^{+} \underline{v}_2|^2 \underline{v}_i \underline{v}_i^{+} \right) \right) \right\} \tag{A5.2}$$

It can be shown that $\eta_j = \mathrm{Tr}(\underline{v}_i \underline{v}_i^{+} R_j)$. One can then obtain for equation (A5.2)

$$= \frac{G_k P_j}{N_k} \eta_j \left(1 - 2MP_i - G_k P_i (M^2 P_i + P_j |\underline{v}_1^{+} \underline{v}_2|^2 - 2P_i P_j M |\underline{v}_1^{+} \underline{v}_2|^2 \right)$$

Substituting the definitions for P_i, P_j, ξ and H_k

$$\xi = 1 - \frac{|\underline{v}_1^{+} \underline{v}_2|^2}{M^2} \quad ;$$

$$H_k = \frac{G_k}{(1 + MS_1/N)(1 + MS_2/N)} = \left[1 + MS/N + M^2 S_1 S_2 / N^2 \xi \right]^{-1} \doteq H_k^{-1}$$

after a few steps of algebra

$$- \mathrm{Tr} \left\{ \frac{\partial K_k}{\partial S_i} \frac{\partial K_k^{-1}}{\partial \alpha_j} \right\} = \frac{M^2 S_i / N^2}{H_k^2} \frac{\eta_i}{M^2} \left(1 + MS_i/N \, \xi \right) \tag{A5.3}$$

For $i = j$, $\frac{\partial K_k}{\partial S_j} = \underline{v}_j \underline{v}_j^{+}$, (A5.2) becomes

$$- \mathrm{Tr} \left\{ \frac{\partial K_k}{\partial S_j} \frac{\partial K_k^{-1}}{\partial \alpha_j} \right\} = \frac{G_k P_j}{N_k} \mathrm{Tr} \left\{ \underline{v}_j \underline{v}_j^{+} R_j - P_i \left(\underline{v}_j \underline{v}_j^{+} R_j \underline{v}_i \underline{v}_i^{+} + \underline{v}_j \underline{v}_j^{+} \underline{v}_i \underline{v}_i^{+} R_j \right) + P_i \eta_j G_k \left(MP_i \underline{v}_j \underline{v}_j^{+} \underline{v}_i \underline{v}_i^{+} + MP_j \underline{v}_j \underline{v}_j^{+} - P_i P_j \underline{v}_j \underline{v}_j^{+} \left(\underline{v}_i \underline{v}_i^{+} \underline{v}_j \underline{v}_j^{+} + \underline{v}_j \underline{v}_j^{+} \underline{v}_i \underline{v}_i^{+} \right) \right) \right\} \tag{A5.4}$$

Since

$$\mathrm{Tr} \left\{ \underline{v}_j \underline{v}_j^{+} R_j \right\} = 0 \quad \text{and} \quad \mathrm{Tr} \left\{ \underline{v}_j \underline{v}_j^{+} (R_j \underline{v}_i \underline{v}_i^{+} + \underline{v}_i \underline{v}_i^{+} R_j) \right\} = M \eta_j$$

(A5.4) becomes

$$\frac{G_k P_j}{N_k} \eta_j \left\{ -MP_i + P_j G_k (MP_i |\underline{v}_1^{+} \underline{v}_2|^2 + M^2 P_j - 2MP_i P_j |\underline{v}_1^{+} \underline{v}_2|^2) \right\}$$

so that

$$-\mathrm{Tr} \left\{ \frac{\partial K_k}{\partial S_j} \frac{\partial}{\partial \alpha_j} \left(K_k^{-1} \right) \right\} = - \frac{M^3 S_1 S_2 / N^3}{H_k^2} \left(1 + MS_i/N \, \xi \right) \eta_j . \tag{A5.5}$$

Calculation of $\mathrm{Tr} \left\{ \frac{\partial K_k}{\partial S_i} \frac{\partial}{\partial S_j} (K_k^{-1}) \right\}$:

$$\frac{\partial}{\partial S_j}(K_k^{-1}) = -\frac{G_k}{N_k}\frac{\partial P_j}{\partial S_j}\left\{ \underline{v}_j\underline{v}_j^* - P_i(\underline{v}_i\underline{v}_i^*\underline{v}_j\underline{v}_j^* + \underline{v}_j\underline{v}_j^*\underline{v}_i\underline{v}_i^*)\right\}$$

$$-\frac{\partial G_k}{\partial S_j}\frac{1}{N_k}\left\{ P_i\underline{v}_i\underline{v}_i^* + P_j\underline{v}_j\underline{v}_j^* - P_iP_j(\underline{v}_i\underline{v}_i^*\underline{v}_j\underline{v}_j^* + \underline{v}_j\underline{v}_j^*\underline{v}_i\underline{v}_i^*)\right\}. \qquad \text{(A5.6)}$$

$$\frac{\partial P_j}{\partial S_j} = \frac{1}{(1 + MS_j/N)^2}\ ; \qquad \frac{\partial G_k}{\partial S_j} = \frac{\partial G_k}{\partial P_j}\frac{\partial P_j}{\partial S_j} = \frac{G_k^2|\underline{v}_1^*\underline{v}_2|^2P_i}{(1 + MS_j/N)^2}$$

$$= -\frac{G_k/N_k}{(1 + MS_j/N)^2}\left\{ \underline{v}_j\underline{v}_j^*(1 + P_jP_iG_k\,|\underline{v}_1^*\underline{v}_2|^2) + \underline{v}_i\underline{v}_i^*P_i^2|\underline{v}_1^*\underline{v}_2|^2G_k\right.$$

$$\left. -P_i(\underline{v}_i\underline{v}_i^*\underline{v}_j\underline{v}_j^* + \underline{v}_j\underline{v}_j^*\underline{v}_i\underline{v}_i^*)(1 + G_kP_iP_j|\underline{v}_1^*\underline{v}_2|^2)\right\}$$

With $1 + P_jP_iG_k|\underline{v}_1^*\underline{v}_2|^2 = G_k$, (A5.6) becomes

$$= \frac{G_k^2/N_k}{(1 + MS_j/N)^2}\left\{ \underline{v}_j\underline{v}_j^* + \underline{v}_i\underline{v}_i^*P_i^2|\underline{v}_1^*\underline{v}_2|^2 - P_i(\underline{v}_i\underline{v}_i^*\underline{v}_j\underline{v}_j^* + \underline{v}_j\underline{v}_j^*\underline{v}_i\underline{v}_i^*)\right\}$$

It follows that $\mathrm{Tr}\left\{\frac{\partial K_k}{\partial S_i}\frac{\partial}{\partial S_j}(K_k^{-1})\right\} = \frac{(1/N)^2}{H_k^2}\,|\underline{v}_1^*\underline{v}_2|^2$ (A5.7)

and

$$\mathrm{Tr}\left\{\frac{\partial K_k}{\partial S_j}\frac{\partial}{\partial S_j}(K_k^{-1})\right\} = \frac{(1/N)^2}{H_k^2}\,(1 + MS_i/N\ \xi\ )^2 \qquad \text{(A5.8)}$$

Equations (65) - (72) can be obtained from equations (A5.3), (A5.5), (A5.7) and (A5.8).

Appendix 6

Derivation of equations (65) - (68)

The elements of J_2 and J_3 can be expressed in terms of integrals I_3 - I_5 as indicated earlier. The integrands are functions of the infinite series $\xi = \beta\omega^2/2\ (\ 1 - \beta\omega^2/5 + \ .\ .\ .\ )$:

$$I_4 = \frac{T}{2\pi} \int_0^W \frac{\omega^2 d\omega}{(\ 1 + MS/N + M^2 S_1 S_2/N^2\ \xi\ )^2}$$

A second order approximation to ξ is as follows :

$$\cong \frac{T}{2\pi} \frac{1}{(\ 1 + MS/N\ )^2}\ R^{-3/2} \int_0^{WR^{1/2}} \frac{x^2 dx}{1 + x^2 + R^{-1}\beta x^4/5}$$

The ratio of first to second terms in the series at any value of $x = \omega R^{1/2}/$ is :

$$\frac{5x^2}{R^{-1}\beta x^4} = 5\ (\beta\omega^2)^{-1}$$

According to the definition of closely spaced sources, the above ratio is guaranteed to be a large number, provided the angular spacing (appearing in the function β) is sufficiently small. It will be assumed to be the case that βW^2 is much smaller than one so that a first order approximation for ξ will yield a useful approximation. One can then work with the following simpler integral:

$$I_4 = \frac{T}{2\pi} \frac{1}{(1 + MS/N)^2} R^{-3/2} \int_0^{WR^{1/2}} \frac{x^2 dx}{(1 + x^2)^2}$$

After direct integration

$$I_4 = \frac{TW}{2\pi} \frac{W^2}{(1 + MS/N)^2} (2RW^2)^{-1} \left\{ - \frac{1}{1 + W^2R} + (W^2R)^{-1/2} \times \tan^{-1}(W R^{1/2}) \right\}$$

$$I_5 = \frac{TW}{2\pi} \frac{1}{2(1 + MS/N)^2} \left\{ \frac{1}{1 + W^2R} + (W^2R)^{-1/2} \times \tan^{-1}(W R^{1/2}) \right\}$$

From Appendix 4

$$I_3 = \frac{TW}{2\pi} \frac{1}{(1 + MS/N)^2} R^{-2} \left\{ 1 - \frac{3}{2} (W^2R)^{-1/2} \times \tan^{-1}(WR^{1/2}) + \frac{1}{2} \frac{1}{1 + W^2R} \right\}$$

For $RW^2 \ll 1$ one obtains the Taylor expansions :

$$I_3 \cong \frac{TW}{2\pi} \frac{W^4}{5(1 + MS/N)^2} \left(1 - \frac{10}{7} W^2R + \ldots \right) \quad \text{(A6.1)}$$

$$I_4 \cong \frac{TW}{2\pi} \frac{W^2}{3(1 + MS/N)^2} \left(1 - \frac{6}{5} W^2R + \ldots \right) \quad \text{(A6.2)}$$

$$I_5 \cong \frac{TW}{2\pi} \frac{1}{(1 + MS/N)^2} \left(1 - \frac{2}{3} W^2R + \frac{3}{5} W^4R^2 - \ldots \right) \quad \text{(A6.3)}$$

Substituting (A6.1) - (A6.3) into (65) - (71) :

$$J_{13} = -\frac{TW}{2\pi}\ \frac{M^2S_1S_2/N^2}{(1+MS/N)^2}\ \frac{W^2}{3}\ x_1\ \beta^{1/2}\ \frac{M}{N}\left\{1 + \frac{3\beta W^2}{10}\left[MS_2/N - \frac{2M^2S_1S_2/N^2}{1+MS/N}\right] - \frac{6\beta W^2}{25}\right\} \tag{A6.4}$$

$$J_{14} = \frac{TW}{2\pi}\ \frac{MS_1/N\ (1+MS_1/N)}{(1+MS/N)^2}\ \frac{W^2}{3}\ x_1\ \beta^{1/2}\ \frac{M}{N}\left\{1 - \frac{6}{25}\beta W^2 - \frac{3\beta W^2}{5}\ \frac{M^2S_1S_2/N^2}{1+MS/N}\right\} \tag{A6.5}$$

$$J_{23} = -\frac{TW}{2\pi}\ \frac{MS_2/N\ (1+MS_2/N)}{(1+MS/N)^2}\ \frac{W^2}{3}\ x_2\ \beta^{1/2}\ \frac{M}{N}\left\{1 - \frac{6}{25}\beta W^2 - \frac{3\beta W^2}{5}\ \frac{M^2S_1S_2/N^2}{1+MS/N}\right\} \tag{A6.6}$$

$$J_{24} = \frac{TW^2}{2\pi}\ \frac{M^2S_1S_2/N^2}{(1+MS/N)^2}\ \frac{W^2}{3}\ x_2\ \beta^{1/2}\ \frac{M}{N}\left\{1 + \frac{3\beta W^2}{10}\left[MS_1/N - \frac{2M^2S_1S_2/N^2}{1+MS/N}\right] - \frac{6\beta W^2}{25}\right\} \tag{A6.7}$$

$$J_{33} = \frac{TW}{2\pi}\ \frac{(M/N)^2}{(1+MS/N)^2}\left\{1 + \frac{\beta W^2}{3}\ \frac{MS_2/N\ (1+MS_2/N)}{1+MS/N} + \frac{(MS_2/N)^2(\beta W^2)}{20}\ \frac{(1+MS_2/N)(1+MS_2/N-2MS_1/N)}{(1+MS/N)^2}\right\}$$

$$J_{34} = \frac{TW}{2\pi}\ \frac{(M/N)^2}{(1+MS/N)^2}\left\{1 - \frac{\beta W^2}{6}\left(1 + \frac{2M^2S_1S_2/N^2}{1+MS/N}\right) + \frac{(\beta W^2)^2}{20}\left(\frac{2}{5} + \frac{3(M^2S_1S_2/N^2)^2}{(1+MS/N)^2} - \frac{2M^2S_1S_2/N^2}{1+MS/N}\right)\right\} \tag{A6.9}$$

$$J_{44} = \frac{TW}{2\pi}\ \frac{(M/N)^2}{(1+MS/N)^2}\left\{1 + \frac{\beta W^2}{3}\ \frac{MS_1/N\ (1+MS_1/N)}{1+MS/N} + (MS_1/N)^2\ \frac{(\beta W^2)^2}{20}\left(\frac{(1+MS_1/N)(1+MS_1/N-2MS_2/N)}{(1+MS/N)^2}\right)\right\} \tag{A6.10}$$

Computing the determinant of J_3 :

$$\det(J_3) \cong \left[\frac{TW}{2\pi} \right]^2 \frac{(M/N)^4}{(1 + MS/N)^4} \left\{ \frac{\beta W^2}{3} (1 + MS/N) - \frac{61(\beta W^2)^2}{900} \right.$$

$$\left. + \frac{(\beta W^2)^2}{20} \left[\frac{(MS/N)^2(1 + MS/N)^2 + 2M^2S_1S_2/N^2(1 - 2MS/N - 3(MS/N)^2}{(1 + MS/N)^2} \right] \right.$$

(A6.11)

Calculation of X_{11} :

$$X_{11} = \frac{J_{13}^2 J_{44} - 2J_{13}J_{14}J_{34} + J_{14}^2 J_{33}}{\det(J_3)}$$

We consider the case in which $MS_i/N \ll (\beta W^2)^{-1}$; $i = 1,2$. After a few steps of algebra, one can show that

$$J_{13}^2 J_{44} \cong \left[\frac{TW}{2\pi} \right]^2 \left[\frac{M}{N} \right]^2 \frac{(M^2S_1S_2/N^2)^2}{(1 + MS/N)^6} x_1^2 \beta \frac{W^4}{9} \left\{ 1 + \frac{3}{5} \beta W^2 \times \right.$$

$$\left. \left[MS_2/N - \frac{2M^2S_1S_2/N^2)}{1 + MS/N} \right] - \frac{12}{25} \beta W^2 + \frac{\beta W^2}{3} \frac{(MS_1/N)(1 + MS_1/N)}{1 + MS/N} \right\}$$

$$2J_{13}J_{14}J_{34} = - \left[\frac{TW}{2\pi} \right]^2 \left[\frac{M}{N} \right]^2 \frac{2(MS_1/N)^2 MS_2/N (1 + MS_1/N)}{(1 + MS/N)^6} \times$$

$$x_1^2 \beta \frac{W^4}{9} \left\{ 1 + \frac{3}{10} \beta W^2 \left[MS_2/N - \frac{4M^2S_1S_2/N^2}{1 + MS/N} \right] - \frac{12}{25} \beta W^2 + \right.$$

$$\left. - \frac{\beta W^2}{6} \left[1 + \frac{2M^2S_1S_2/N^2}{1 + MS/N} \right] \right\}$$

$$J_{14}^2 J_{33} = \left[\frac{TW}{2\pi} \right]^3 \left[\frac{M}{N} \right]^4 \frac{(MS_1/N)^2 (1 + MS_1/N)^2}{(1 + MS/N)^6} x_1^2 \beta \frac{W^4}{9} \times$$

$$\left\{ 1 - \frac{12}{25} \beta W^2 + \frac{\beta W^2}{3} \frac{MS_2/N (1 + MS_2/N)}{1 + MS/N} - \frac{6\beta W^2}{5} \frac{M^2S_1S_2/N^2}{1 + MS/N} \right\}$$

After considerable algebra one can show that

$$J_{13}^2 J_{44} - 2J_{13}J_{14}J_{34} + J_{14}^2 J_{33} \cong \left[\frac{TW}{2\pi} \right]^3 \frac{(M/N)^4}{(1 + MS/N)^4} x_1^2 \beta \frac{W^2}{9} \times$$

$$(MS_1/N)^2 \left\{ 1 - \frac{12}{25} \beta W^2 + \frac{3\beta W^2}{5} \frac{(MS_2/N)(MS_2/N - 2MS_1/N)}{(1 + MS/N)} \right\}$$

so that

$$X_{11} = \frac{TW}{2\pi} \frac{(MS_1/N)^2}{(1 + MS/N)} x_1^2 \frac{W^2}{3} \left\{ 1 - \frac{12}{25} \beta W^2 + \frac{61 \beta W^2}{300(1 + MS/N)} \right.$$

$$+ \frac{3\beta W^2}{5} \frac{(MS_2/N)(MS_2/N - 2MS_1/N)}{(1 + MS/N)} - \frac{3\beta W^2}{20} \left[\frac{(MS/N)^2}{1 + MS/N} \right]$$

$$\left. - \frac{3\beta W^2 M^2 S_1 S_2/N^2}{10(1 + MS/N)^2} (3MS/N - 1) \right\} \tag{A6.12}$$

Entirely analogous computations can be carried out to determine X_{12} and X_{22}. The results are given by :

$$X_{12} = \frac{TW}{2\pi} x_1 x_2 \frac{M^2 S_1 S_2/N^2}{1 + MS/N} \frac{W^2}{3} \left\{ 1 - \frac{12}{25} \beta W^2 - \frac{11 \beta W^2}{300(1 + MS/N)} - \right.$$

$$- \frac{3\beta W^2}{20} \left[\frac{(MS/N)^2}{1 + MS/N} + \frac{2M^2 S_1 S_2/N^2 (3MS/N - 1)}{(1 + MS/N)^2} \right]$$

$$\left. + \frac{3\beta W^2}{10} \frac{(MS_2/N)^2 + (MS_1/N)^2}{1 + MS/N} \right\}$$

$$X_{22} = \frac{TW}{2\pi} x_2^2 \frac{(MS_2/N)^2}{(1 + MS/N)} \frac{W^2}{3} \left\{ 1 - \frac{12}{25} \beta W^2 + \frac{61 \beta W^2}{300(1 + MS/N)} \right.$$

$$+ \frac{3\beta W^2}{5} \frac{(MS_1/N)^2 (1 + MS_1/N)}{(1 + MS/N)^2} - \frac{3\beta W^2}{20} \left[\frac{(MS/N)^2}{1 + MS/N} - \right.$$

$$\left. \left. + \frac{2M^2 S_1 S_2/N^2 (3MS/N - 1)}{(1 + MS/N)^2} \right] \right\} \tag{A6.13}$$

Appendix 7

Derivation of equations (94), (95), (97), (98)

In this appendix the form of the matrix X is computed for the case :

$$MS_1/N \ll (\beta W^2)^{-1} \quad \text{and} \quad MS_2/N \gg (\beta W^2)^{-1} \quad \text{so that } R\beta W^2 \ll 1.$$

According to the above, $MS_2/N \gg 1$ and $S_2 \gg S_1$. Based on the results of Appendix 5, the following expressions are approximations for elements of the J_2 and J_3 matrices under the specified conditions.

$$J_{13} \cong -\frac{TW}{2\pi} x_1 \frac{W^2}{3} \frac{M}{N} MS_1/N \frac{3\beta}{10} \left(1 - 2\frac{S_1}{S_2} + \frac{10}{3} (MS_2/N\beta W^2)^{-1} + \right.$$
$$\left. + \frac{4}{5} (MS_2/N)^{-1} \right)$$

$$J_{14} \cong \frac{TW}{2\pi} \frac{M^2 S_1/N^2 (1 + MS_1/N)}{(MS_2/N)^2} \beta^{1/2} x_1 \frac{W^2}{3} \left(1 - \frac{6}{25}\beta W^2 - \frac{3}{5} MS_1/N\, \beta W^2 \right)$$

$$J_{23} \cong \frac{TW}{2\pi} \frac{M}{N} \frac{W^2}{3} x_2 \beta^{1/2} \left(1 - \frac{6}{25}\beta W^2 - \frac{3}{5} MS_1/N\beta W^2 \right)$$

$$J_{24} \cong -\frac{TW}{2\pi} \frac{M}{N} \frac{S_1}{S_2} x_2 \frac{W^2}{3} \beta^{1/2} \left(1 - \frac{3}{10} MS_1/N\, \beta W^2 - \frac{6}{25}\beta W^2 \right)$$

$$J_{33} \cong \frac{TW}{2\pi} \frac{M^2}{N^2} \frac{(\beta W^2)^2}{20} \left(1 + \frac{20}{3} (MS_2/N\, \beta W^2)^{-1} \right)$$

$$J_{34} \cong \frac{TW}{2\pi} \frac{M^2}{N^2} (MS_2/N)^{-2} \cong J_{44}$$

It follows from the above that $\det J_3 \cong J_{33}J_{44}$, with $J_{33} \gg J_{44}$ when

$$MS_2/N \gg \frac{(\beta W^2)^{-1}}{(20)^{1/2}} \cong \frac{(\beta W^2)^{-1}}{4.5}$$

We assume that the above statement holds, that $MS_2/N\beta W^2$ may be arbitrarily large. From section 3 :

$$X_{11} = \frac{J_{13}^2 J_{44} - 2J_{13}J_{14}J_{34} + J_{14}^2 J_{33}}{\det J_3} .$$

Based on the earlier comments, one obtains the simplified expression :

$$\cong \frac{J_{13}^2}{J_{33}} - \frac{2J_{13}J_{14}}{J_{33}} + \frac{J_{14}^2}{J_{44}}$$

Inserting the above approximations yields the following result.

$$\cong \frac{TW}{2\pi} (MS_1/N)^2 \frac{3\beta W^2}{5} \frac{W^2}{3} x_1^2 (1 - 4 \frac{S_1}{S_2} + \frac{10}{3}(MS_2/N\beta W^2)^{-1} - \frac{4}{5} (MS_2/N)^{-1} + \ldots)$$

For X_{12} :

$$X_{12} = \frac{J_{13}J_{44}J_{23} - J_{14}J_{34}J_{23} - J_{34}J_{24}J_{13} + J_{14}J_{24}J_{33}}{\det J_3}$$

$$\cong \frac{J_{13}J_{23}}{J_{33}} - \frac{J_{14}J_{23}}{J_{33}} - \frac{J_{24}J_{13}}{J_{33}} + \frac{J_{14}J_{24}}{J_{44}}$$

$$\cong \frac{TW}{2\pi} \; 2MS_1/N \; x_1x_2 \frac{W^2}{3} \; (1 - \frac{6}{25} \beta W^2 - \frac{3}{5} MS_1/N \; \beta W^2 - 4 \frac{S_1}{S_2} + \ldots)$$

$$X_{22} = \frac{J_{23}^2J_{44} - 2J_{23}J_{24}J_{34} + J_{24}^2J_{33}}{\det J_3}$$

$$\cong \frac{J_{23}^2}{J_{33}} - 2 \frac{J_{23}J_{24}}{J_{33}} + \frac{J_{24}^2}{J_{44}}$$

$$\cong \frac{TW}{2\pi} \frac{20}{3} \; (\beta W^2)^{-1} \; x_2^2 \frac{W^2}{3} \; (1 - \frac{12}{25} \beta W^2 - \frac{6}{5} MS_1/N \; \beta W^2 + \ldots)$$

Section 2, equations (38) -(40) give the elements of J_1. For $MS_2/N >> (\beta W^2)^{-1}$ and $MS_1/N << (\beta W^2)^{-1}$ these become :

$$J_{11} \cong \frac{TW}{2\pi} \frac{(MS_1/N)^2}{MS_2/N} x_1^2 \frac{W^2}{3} \; (1 + \frac{3}{5} MS_2/N \; \beta W^2 - \frac{3}{10} MS_1/N \; \beta W^2)$$

$$J_{12} \cong \frac{TW}{2\pi} MS_1/N \; x_1x_2^2 \frac{W}{3} \; (1 - \frac{6}{25} \beta W^2 - \frac{9}{10} MS_1/N \; \beta W^2)$$

$$J_{22} \cong \frac{TW}{2\pi} MS_2/N \; x_2^2 \frac{W^2}{3} \; (1 + \frac{3}{5} (MS_1/N)^2 \beta W^2 - \frac{3}{10} MS_1/N \; \beta W^2)$$

Neglecting terms of inverse powers of MS_2/N and $MS_2/N\beta W^2$ one obtains :

$$J_{11} - X_{11} \cong \frac{TW}{2\pi} x_1^2 \frac{W^2}{3} \frac{(MS_1/N)^2}{MS_2/N} \; (1 + \frac{9}{10} MS_1/N \; \beta W^2)$$

$$J_{12} - X_{12} \cong -\frac{TW}{2\pi} x_1 x_2 \frac{W^2}{3} MS_1/N \left(1 - \frac{6}{25} \beta W^2 + \frac{3}{10} MS_1/N \, \beta W^2 \right)$$

$$J_{22} - X_{22} \cong \frac{TW}{2\pi} x_2^2 \frac{W^2}{3} MS_2/N \left(1 - \frac{3}{10} \beta W^2 MS_1/N \right)$$

At low signal to noise ratios for source 1, i.e., $MS_1/N \ll 1$, the variance of bearing errors are given by :

$$Var(\hat{\alpha}_1) \geq \left\{ \frac{TW}{2\pi} x_1^2 \frac{12}{75} W^2 \beta W^2 (MS_1/N) \frac{S_1}{S_2} \right\}^{-1} \tag{A7.1}$$

$$Var(\hat{\alpha}_2) \geq \left\{ \frac{TW}{2\pi} x_2^2 \frac{12}{75} W^2 \beta W^2 MS_2/N \right\}^{-1} \tag{A7.2}$$

At large signal to noise ratios where $MS_1/N \gg 1$:

$$Var(\hat{\alpha}_1) \geq \left\{ \frac{TW}{2\pi} x_1^2 W^2 \frac{2\beta W^2}{5} (MS_1/N)^2 \frac{S_1}{S_2} \right\}^{-1} \tag{A7.3}$$

$$Var(\hat{\alpha}_2) \geq \left\{ \frac{TW}{2\pi} x_2^2 W^2 \frac{2\beta W^2}{5} (M^2 S_1 S_2/N^2) \right\}^{-1} \tag{A7.4}$$

Equations (A7.1) - (A7.4) are equations (94), (95), (97), (98)

Appendix 3

Derivation of equations (100) and (101)

Since the matrix elements of J_2 and J_3 can be expressed in terms of integrals I_3 - I_5 it will be necessary to obtain large W^2R approximations for the integrals.

A method for obtaining upper and lower bounds on integrals I_1 - I_3 for the large W^2R mode was given in Appendix 4. The method involves bounding the truncated infinite series for ξ which leads to a bound on the integrals I_1 - I_3. The approach is taken here to evaluate integrals I_4 and I_5.

After simple computations, one can show that the integrals must lie between the values :

$$I_4 = \frac{TW}{2\pi} \frac{1}{(1 + MS/N)^2} R^{-1} \left\{ WR^{1/2} \left(1 \pm \frac{\beta W^4 R}{10} \right) \pi/4 \right\}$$

$$I_5 = \frac{TW}{2\pi} \frac{1}{(1 + MS/N)^2} \pi/4 \left\{ WR^{1/2} \left(1 \pm \frac{\beta W^4 R}{10} \right) \right\}$$

The upper and lower bounds are approximately equal because we assume on the basis of physical arguments that

$$\beta W^2 \overset{\text{singularity}}{<<} WR^{1/2} .$$

Appendix 4 contains details of the argument.

On the basis of these approximations and the expression for I_3 also from Appendix 4 one obtains the following :

$$I_3 = \frac{TW}{2\pi} \frac{1}{(1 + MS/N)^2} R^{-2} \left\{ 1 - \frac{3y\pi}{4} \left(1 \pm \frac{\beta W^4 R}{10} \right) \right\}$$

$$I_3 \cong \frac{TW}{2\pi} \frac{1}{(MS/N)^2} R^{-2} \left\{ 1 - \frac{3y\pi}{4} \right\}$$

$$J_{13} = - \frac{TW}{2\pi} \beta^{-1/2} x_1 \frac{M}{N} \left\{ \frac{2}{MS_1/N} + y \frac{\pi}{2} \left(\frac{1}{MS/N} - \frac{3}{MS_1/N} \right) \right\}$$

$$J_{14} = \frac{TW}{2\pi} \beta^{-1/2} x_1 \frac{M}{N} \left\{ y \frac{\pi}{2} \frac{S_1}{S_2} \frac{1}{MS/N} \right\}$$

$$J_{23} = - \frac{TW}{2\pi} \beta^{-1/2} x_2 \frac{M}{N} \left\{ y \frac{\pi}{2} \frac{S_2}{S_1} \frac{1}{MS/N} \right\}$$

$$J_{24} = \frac{TW}{2\pi} \beta^{-1/2} x_2 \frac{M}{N} \left\{ \frac{2}{MS_2/N} + y \frac{\pi}{2} \left(\frac{1}{MS/N} - \frac{3}{MS_1/N} \right) \right\}$$

$$J_{33} = \frac{TW}{2\pi} \frac{(M/N)^2}{(MS_1/N)^2} \left\{ 1 - \frac{y\pi}{4(MS/N)^2} MS_2/N \left(4MS_1/N + 3\, MS_2/N \right) \right\}$$

$$J_{34} = \frac{TW}{2\pi} \frac{(M/N)^2}{(MS/N)^2} \frac{y\pi}{4}$$

$$J_{44} = \frac{TW}{2\pi} \frac{(M/N)^2}{(MS_2/N)^2} \left\{ 1 - \frac{y\pi}{4(MS/N)^2} MS_1/N \left(4MS_2/N + 3\, MS_1/N \right) \right\}$$

By inspection of the above approximations for the elements of J_3 :

$\mathrm{Det}\,(J_3) = J_{33}J_{44} + O(y^2)$

so that

$$X_{11} = \frac{J_{13}^2}{J_{33}} - \frac{2J_{13}J_{14}J_{34}}{J_{33}J_{44}} + \frac{J_{14}^2}{J_{44}}$$

$$X_{11} = \left[\frac{TW}{2\pi} \beta^{-1} x_1^2 \right] \times \left[4 - y\pi \left(3 + \frac{S_1^2}{S^2} \right) \right] \tag{A8.1}$$

$$X_{12} = \frac{J_{13}J_{23}}{J_{33}} - \frac{J_{14}J_{34}J_{23}}{J_{33}J_{44}} - \frac{J_{34}J_{24}J_{13}}{J_{33}J_{44}} + \frac{J_{14}J_{24}}{J_{44}}$$

$$= \left[\frac{TW}{2\pi} \beta^{-1} x_1 x_2 \right] \times \left[y\pi \left(1 + \frac{S_1 S_2}{S^2} \right) \right] \tag{A8.2}$$

$$X_{22} = \left[\frac{TW}{2\pi} \beta^{-1} x_2^2 \right] \times \left[4 - y\pi \left(3 + \frac{S_2^2}{S^2} \right) \right] \tag{A8.3}$$

Equations (A8.1) - (A8.3) are equations (91) - (93).

Appendix 9

Derivation of equations (107) and (108)

$\mu(s)$ is given by :

$$\mu(s) = \sum_{k=1}^{TW} \ln \left[\frac{\det(K_0)}{\det(K_1)} \right]^s - \ln \left[\det(K_0) \det(sK_1^{-1} + (1-s)K_0^{-1}) \right]$$

Consider $\det(K_0)$. K_0 has the form of the identity matrix plus an outer product matrix. The matrix

$$N_k I + S\underline{v}_0\underline{v}_0^*$$

has (M-1) eigenvalues of N_k and one remaining eigenvalue of $N_k + \lambda$, where λ is the single nonzero eigenvalue of $S\underline{v}_0\underline{v}_0^*$. This nonzero eigenvalue is equal to $S\underline{v}_0^*\underline{v}_0 = MS$. Therefore, the determinant of K_0 is

$$\det(K_0) = N_k^M (1 + MS_k/N_k)$$

K_1 has the form of the identity matrix plus two outer product matrices:

$$K_1 = N_k I + S_1\underline{v}_1\underline{v}_1^* + S_2\underline{v}_2\underline{v}_2^*$$

The matrix K_1 has (M-2) eigenvalues of N_k and two others which are of the form (the k dependence of N_k is hereafter suppressed) :

$$N + \lambda_1 \quad , \quad N + \lambda_2$$

where λ_1 and λ_2 are the two nonzero eigenvalues of the matrix $S_1\underline{v}_1\underline{v}_1^* + S_2\underline{v}_2\underline{v}_2^*$. $\lambda 1$ and λ_2 can be computed from the following eigenvalue equation. Eigenvectors associated with nonzero eigenvalues of the matrix $\underline{S}_1\underline{v}_1\underline{v}_1^* + S_2\underline{v}_2\underline{v}_2^*$ have the form

$$a\underline{v}_1 + a\underline{v}_2$$

so that

$$(S_1\underline{v}_1\underline{v}_1^* + S_2\underline{v}_2\underline{v}_2^*)(a\underline{v}_1 + b\underline{v}_2) = \lambda(a\underline{v}_1 + b\underline{v}_2)$$

where the above solutions for λ are λ_1 and λ_2. Using the above eigenvalue equation one can solve for the roots :

$$\det \begin{bmatrix} MS_1 - \lambda & S_1 \ (\underline{v}_1^*\underline{v}_2) \\ S_2 \ (\underline{v}_2^*\underline{v}_1) & MS_2 - \lambda \end{bmatrix} = 0$$

With $\xi_{12} \triangleq 1 - \frac{1}{M^2} |\underline{v}_1^* \underline{v}_2|^2$, one obtains

$$\lambda^2 - \lambda MS + M^2 S_1 S_2 \xi_{12} = 0$$

The following properties of the solutions to quadratic equations yield

$$\lambda_1 + \lambda_2 = MS, \qquad \lambda_1 \lambda_2 = M^2 S_1 S_2 \xi_{12}$$

$$\text{so that } \det(K_0) = N^{M-2} (N + \lambda_1)(N + \lambda_2) = N^{M-2} (1 + MS + M^2 S_1 S_2 \xi_{12})$$

$$= N^M H_k \quad ; \quad H_k = 1 + MS/N + M^2 S_1 S_2 / N^2$$

One can use an identical approach to compute the determinant of the matrix

$$sK_1^{-1} + (1-s)K_0^{-1} = \frac{1}{N_k} \Big[I - s\, P_0 \underline{v}_0 \underline{v}_0^* - (1 - s) G_k (P_1 \underline{v}_1 \underline{v}_1^* + P_2 \underline{v}_2 \underline{v}_2^* - P_1 P_2 (\underline{v}_1 \underline{v}_1^* \underline{v}_2 \underline{v}_2^* + \underline{v}_2 \underline{v}_2^* \underline{v}_1 \underline{v}_1^*) \Big]$$

$$\text{where } P_0 \triangleq \frac{S/N}{1 + MS/N}.$$

The above matrix in brackets has the form $I - A$. The matrix A has three nonzero eigenvalues which correspond to eigenvectors of the form

$$a\underline{v}_0 + b\underline{v}_1 + c\underline{v}_2.$$

The matrix $I - A$ has $(M-3)$ eigenvalues of 1 and three others of the form

$$1 - \lambda_i \; ; \; i = 1,2,3$$

where the λ_i are the nonzero eigenvalues of A.

Rearranging the terms in A:

$$A = P_0 \underline{v}_0 \underline{v}_0^* + G_k P_1 \underline{v}_1 (\underline{v}_1^* - P_2 (\underline{v}_1^* \underline{v}_2) \underline{v}_2^*) + G_k \underline{v}_2 (\underline{v}_2^* - P_1 (\underline{v}_2^* \underline{v}_1) \underline{v}_1^*)$$

and writing the matrix equation $\det(A - \lambda I) = 0$ one obtains the following expression. For brevity of notation, define

$$\rho_{ij} \triangleq \underline{v}_i^* \underline{v}_j \text{ for } i,j = 0,1,2.$$

det

$$\begin{bmatrix} sPM - \lambda & sP_0\rho_{01} & sP_0\rho_{02} \\ (1-s)P_1G_k(\rho_{10}-P_2\rho_{12}\rho_{20}) & (1-s)P_1G_k(M-P_2|\rho_{12}|^2)-\lambda & (1-s)P_1G_k\rho_{12}(1-MP_2) \\ (1-s)P_2G_k(\rho_{20}-P_1\rho_{21}\rho_{10}) & (1-s)P_2G_k\rho_{21}(1-MP_1) & (1-s)P_2G_k(M-P_1|\rho_{12}|^2)-\lambda \end{bmatrix}$$

$= 0.$

After considerable algebra one can show that the above equation results in the following cubic equation in λ :

$$\lambda^3 - \lambda^2 \frac{1}{H_k(1 + MS/N)} \Big[(1 + MS/N)(MS/N + 2M^2S_1S_2/N^2\xi_{12}) - sM^2S_1S_2/N^2 \times (2 + MS/N) \Big] + \frac{\lambda\,(1 - s)}{H_k(1 + NMS/N)} \Big[(1 - s)(1 + MS/N)M^2S_1S_2/N^2\xi_{12} + sMS/N\,(MS_1/N\,\xi_{01} + MS_2/N\,\xi_{02} + M^2S_1S_2/N^2(2\,\xi_{12} + \xi_{01} + \xi_{02} - 2\chi\,)\,)\Big] - \frac{s(1 - s)^2}{H_k(1 + MS/N)}\, MS/N\, M^2S_1S_2/N^2 \Big[\xi_{12} + \xi_{01} + \xi_{02} - 2\,\chi \Big] \qquad (A9.1)$$

The following notation is used in (A9.1) :

$$\xi_{ij} \doteq 1 - \frac{1}{M^2}\,|\rho_{ij}|^2 \quad \text{and} \quad \chi = 1 - \frac{1}{M^3}\,\mathrm{Re}\,(\rho_{01}\rho_{12}\rho_{20})$$

Since only the sums and products of the λ_i enter into the determinant it is not necessary to solve for the individual eigenvalues. Write the above equation as follows :

$$\lambda^3 + a_0\,\lambda^2 + a_1\,\lambda + a_2 = 0.$$

Then

$$\lambda_1 + \lambda_2 + \lambda_3 = -\,a_0;\ \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = a_1;\ \ \lambda_1\lambda_2\lambda_3 = -\,a_2$$

After a few steps of algebra one obtains :

$$\det(sK_1^{-1} + (1-s)K_0^{-1}) = \frac{1}{(1 + MS/N)\ H_k} \Big[1 + MS/N + s^2 M^2 S_1 S_2/N^2\ \xi_{12}$$

$$+ s(1-s)\ MS/N\ (\ MS_1/N\ \xi_{01} + MS_2/N\ \xi_{02}\ ) + s^2(1-s)(\ \xi_{12} + \xi_{01}$$

$$+ \xi_{02} - 2\ \chi\ \Big]$$

Combining with the previous results for $\det(K_1)$ and $\det(K_0)$, the expression for $\mu(s)$ becomes :

$$\mu(s) = \sum_{k=1}^{TW} s \ln \Big[1 + \bar{R}\ \xi_{12} \Big] - \Big[1 + s^2 \xi_{12} + s(1-s)\ (\ \frac{S}{S_2}\ \xi_{01} +$$

$$+ \frac{S}{S_1}\ \xi_{02}\ )\ \bar{R} + s^2(1-s)\ MS/N\ \bar{R}\ (\xi_{12} + \xi_{02} + \xi_{01} - 2\ \chi) \Big] \quad \text{(A9.2)}$$

which is equation (107).

Now consider equation (A9.2) with the specialization of $\underline{v}_0$:

$$\underline{v}_0 = [\ 1,\ e^{-j\omega_k \nu_1},\ e^{-j\omega_k \nu_2},\ \ldots,\ e^{-j\omega_k \nu_{M-1}}\ ]^T$$

$$\nu_i = d\ \sin(\ \frac{S_1\alpha_1 + S_2\alpha_2}{S}\ )/c$$

From the results of section 2 :

$$\xi_{12} = 1 - \frac{1}{M^2} \sum_{i,j=-\frac{M-1}{2}}^{\frac{M-1}{2}} \cos(\omega_k d(i-j)(\cos\alpha_1 - \cos\alpha_2)/c)$$

With

$$\alpha \triangleq \frac{\alpha_1 + \alpha_2}{2} \quad ; \quad \Delta \triangleq \frac{\alpha_1 - \alpha_2}{2} \quad ; \quad \Delta\alpha = 2\Delta$$

$$\alpha_1 = \alpha + \Delta \qquad \alpha_2 = \alpha - \Delta$$

For $|\alpha_1 - \alpha_2| \ll 2\pi$,

$$\cos\alpha_1 - \cos\alpha_2 \cong \Delta\alpha \sin(\alpha)$$

If in addition, $WL\sin(\alpha)\Delta\alpha/c \ll 2\pi$, one obtains

$$\xi_{12} = \sum_{i,j, = \frac{M-1}{2}}^{\frac{M-1}{2}} \frac{1}{2} (\omega_k{}^2 d^2 \sin^2(\alpha)\ \Delta\alpha^2)(i-j)^2 - \ldots$$

$$\cong L^2 \omega_k^2 \sin^2(\alpha)\ \Delta\alpha^2/6c^2$$

$$\xi_{01} = 1 - \frac{1}{M^2} \sum_{i,j = \frac{M-1}{2}}^{\frac{M-1}{2}} \cos(\omega_k d(\cos\alpha_0 - \cos\alpha_1)(i-j)/c)$$

Using the previous definitions, $\alpha_0 = \alpha + \Delta \frac{S_1 - S_2}{S}$ and it therefore follows that

$$\cos\alpha_0 - \cos\alpha_1 = \cos(\alpha + \Delta \frac{S_1 - S_2}{S}) - \cos(\alpha + \Delta) \cong$$

$$\cong \Delta(1 - \frac{S_1 - S_2}{S})\sin\alpha = 2\Delta \frac{S_2}{S} \sin\alpha = \Delta\alpha \frac{S_2}{S}$$

After a few steps of algebra, it follows that

$$\xi_{01} \cong \omega_k^2 L^2 \sin^2\alpha\ \Delta\alpha^2 (\frac{S_2}{S})^2 /6c^2$$

Similarly $\xi_{02} \cong \omega_k^2 L^2 \sin^2\alpha\ \Delta\alpha^2 (\frac{S_1}{S})^2 /6c^2$

Let $\beta \cong L^2 \sin^2\alpha\ \Delta\alpha^2/6c^2$. Then

$$\xi_{12} = \omega_k^2 \beta \qquad \xi_{01} = \omega_k^2 \beta\, S_2^2/S^2 \qquad \xi_{02} = \beta\omega_k^2\, S_1^2/S^2 \tag{A9.3}$$

Finally for χ:

$$\chi = 1 - \frac{1}{M^2}\,\mathrm{Re}(\rho_{01}\rho_{12}\rho_{20}) =$$

$$= 1 - \frac{1}{M^2}\,\mathrm{Re} \sum_{i,j,l = \frac{M-1}{2}}^{\frac{M-1}{2}} \exp\left(-j\omega_k \frac{d}{c} \left\{ i(\cos\alpha_0 - \cos\alpha_1) + j(\cos\alpha_1 - \cos\alpha_2) + l(\cos\alpha_2 - \cos\alpha_0) \right\} \right)$$

$$= 1 - \frac{1}{M^3} \sum_{i,j,l = \frac{M-1}{2}}^{\frac{M-1}{2}} \cos\left(\omega_k \frac{d}{c}\, \Delta\alpha \sin\alpha \left\{ i\,\frac{S_2}{S} + j - l\,\frac{S_1}{S} \right\} \right)$$

Under conditions in which $LW\, \Delta\alpha \sin\alpha / c \ll 2\pi$, one can expand the cosine function in a power series resulting in :

$$\chi = \frac{1}{2}\left(\omega_k^2 d^2 \Delta\alpha^2 \sin^2\alpha \,/\, M^3 c^2\right) \sum_{i,j,l = \frac{M-1}{2}}^{\frac{M-1}{2}} \left(i(S_2/S) + j - l(S_1/S)\right)^2 + \ldots$$

Because the i,j,l sums are carried over the symmetric interval, only terms with even powers contribute the sums. Using the definition for β it can be shown that

$$\chi \cong \frac{1}{2}\,\omega_k^2 \beta \left(1 + \frac{S_1^2}{S} + \frac{S_2^2}{S} \right) \tag{A9.4}$$

Returning to equation (A9.2), by inserting the results of (A9.3) and (A9.4) one obtains the simplification :

$$\xi_{12} + \xi_{01} + \xi_{02} - 2\chi = 0$$

$$\frac{S}{S_2}\,\xi_{01} + \frac{S}{S_1}\,\xi_{02} = \beta\omega_k^2$$

and it therefore follows that

$$\mu(s) = \sum_{k=1}^{WT} \ln\left(1 + \bar{R}\beta\omega_k^2\right)^s - \ln\left(1 + \bar{R}\beta\omega_k^2 s\right)$$

which is equation (108).

Appendix 10

Derivation of the solution $\dot{\mu}(s) = 0$

From equation (107) $\dot{\mu}(s)$ is given by :

$$\dot{\mu}(s) = \sum_{k=1}^{TW} \ln(1 + \bar{R}\beta\omega_k^2) - \frac{\bar{R}\beta\omega_k^2}{1 + \bar{R}\beta\omega_k^2 s}$$

For large TW products, one can approximate the k-sums by integrals.

$$\dot{\mu}(s) = \frac{T}{2\pi} \int_0^W \left[\ln(1 + \bar{R}\beta\omega^2) - \frac{\bar{R}\beta\omega^2}{1 + \bar{R}\beta\omega^2 s} \right] d\omega$$

Changing variables :

$$\dot{\mu}(s) = \frac{T}{2\pi} \left[(\bar{R}\beta)^{-1/2} \int_0^{W(\bar{R}\beta)^{1/2}} \ln(1 + x^2)\, dx - (\bar{R}\beta s^3)^{-1/2} \int_0^{W(\bar{R}\beta s)^{1/2}} \frac{x^2\, dx}{1 + x^2} \right]$$

$$= \frac{TW}{2\pi} \left[\ln(1 + \bar{R}\beta W^2) - 2(1 - (\bar{R}\beta W^2)^{-1/2} \tan^{-1}\left[(\bar{R}\beta W^2)^{1/2} \right]) \right.$$

$$\left. - s^{-1} (1 - (\bar{R}\beta W^2 s)^{1/2} \tan^{-1}\left[(\bar{R}\beta W^2 s)^{1/2} \right]) \right]$$

For $\bar{R}\beta W^2 \ll 1$, power series expansions of the $\ln(*)$ and $\tan^{-1}(*)$ yield the following polynomial in s :

$$\dot{\mu}(s) = \frac{TW}{2\pi} \left[- \frac{(\bar{R}\beta W^2)^2}{10} + \frac{(\bar{R}\beta W^2)^2}{5} s + \frac{(\bar{R}\beta W^2)^3}{21} - \frac{(\bar{R}\beta W^2)^3}{7} s^2 + \ldots \right]$$

One can compute the root, s_m, for any order of the above polynomial. However, it can be shown by straighforward calculations that the nth term in the s polynomial is proportional to

$$(\bar{R}\beta W^2)^{n+2}$$

so that corrections to the location of the root by including higher and higher order terms will tend to vanish. The first order

approximation obtained by retaining terms of order $(\bar{R}\beta W^2)^2$ and larger only yields

$$s_m = 1/2.$$

A second order approximation yields

$$s_m = 1/2 - (5/21)\ \bar{R}\beta W^2 .$$

According to the assumption that the sources are closely spaced, $\bar{R}\beta W^2 \ll 1$ so that $s_m \cong 1/2$.

Appendix 11

Derivation of equation (109)

$$\mu(1/2) = \sum_{k=1}^{WT} \ln\left(1 + \bar{R}\beta W^2\right)^{1/2} - \ln\left(1 + (1/2)\bar{R}\beta W^2\right)$$

For large TW products, the following integral approximations are alid :

$$\mu(1/2) = \frac{T}{2\pi}\left[\frac{1}{2}(\bar{R}\beta)^{-1/2}\int_0^{(W^2\bar{R}\beta)^{1/2}} \ln\left(1 + x^2\right) dx - (\bar{R}\beta/2)^{1/2} \times \int_0^{(W^2\bar{R}\beta/2)^{1/2}} \ln\left(1 + x^2\right) dx\right]$$

$$= -\frac{TW}{2\pi}\frac{1}{20}(\bar{R}\beta W^2)^2\left[1 - \frac{5}{7}\bar{R}\beta W^2\right]$$

$$\ddot{\mu}(1/2) = \sum_{k=1}^{TW}\left[\frac{\bar{R}\beta\omega_k^2}{1 + \bar{R}\beta\omega_k^2/2}\right]^2 \cong \frac{T}{2\pi}\int_0^W \frac{(\bar{R}\beta\omega^2)^2}{(1 + \bar{R}\beta\omega^2/2)^2} d\omega$$

$$= \frac{T}{2\pi}(\bar{R}\beta/2)^{-1/2} 4\int_0^{(W^2\bar{R}\beta/2)^{1/2}} \frac{x^2\, dx}{(1 + x^2)^2}$$

$$= \frac{TW}{2\pi}\frac{(\bar{R}\beta W^2)^2}{5}\left(1 - \frac{10}{7}\bar{R}\beta W^2 + \ldots\right)$$

Hence $\mu(s_m) + \frac{s_m^2}{2}\ddot{\mu}(s_m) \cong -\frac{TW}{2\pi}\frac{(\bar{R}\beta W^2)^3}{40}$

and $s_m(\ddot{\mu}(s_m))^{1/2} \cong \left[\frac{TW}{20\pi}\right]^{1/2}\bar{R}\beta W^2$

The false alarm and miss probabilities are equal to the each other and to the probability of error :

$$Pr(\epsilon) = \mathrm{erfc}_*\left[\bar{R}\beta W^2\left(\frac{TW}{20\pi}\right)^{1/2}\right] \times \exp\left[-\frac{TW}{80\pi}(\bar{R}\beta W^2)^2\right]$$

which is equation (109).

References

1. M. Wax and T. Kailath, "Optimum localization of multiple sources by passive arrays," IEEE Trans. on Acoustics, Speech and Signal Processing, vol. ASSP-31, no. 5, pp. 1210-1217, October 1983.

2. T.J. Shan and T. Kailath, "Adaptive beamforming for coherent signals and interference," IEEE Trans. on Acoustics, Speech and Signal Processing, vol. ASSP-33, no. 3, pp. 527-536, June 1985.

3. S.R. DeGraaf and D.H. Johnson, "Capability of array processing algorithms to estimate source bearings," IEEE Trans. on Acoustics, Speech and Signal Processing, vol. ASSP-33, no. 3, pp. 1368-1379, June 1985.

4. G. Su and M. Morf, "Modal decomposition signal subspace algorithms," IEEE Trans. on Acoustics, Speech and Signal Processing, vol. ASSP-34, no. 3, pp. 585-602, June 1986.

5. R.D. Schmidt, "Multiple emitter location and signal parameter estimation," IEEE Trans. on Antennas and Propagation, vol. AP-34, pp. 276-280, March 1986.

6. B. Heilweil, Direction Estimation for Multiple Remote Sources, Ph.D. dissertation, Yale University, 1983.

7. H. Messer and P.M. Schultheiss, "Parameter estimation for two sources with overlapping spectra using an arbitrary M sensor array," Conference on Spectra Estimation, October 1985.

8. L.C. Ng, "Optimum multisensor, multitarget time delay estimation," NUSC Technical Report 6757, April 1983.

9. U. Sandkühler and J.F. Böhme, "Properties of maximum likelihood estimates for the multiple source location problem," to be submitted to the IEEE.

10. H.L. Van Trees, Detection, Estimation and Modulation Theory, Part I, John Wiley and Sons, New York, 1968.

END

10-87

DTIC