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On Color Polynomials of Fibonacci Graphs

by

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Graph Theory
Fibonacci Graphs
Color Polynomials
King Polyomino Graphs

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On Color Polynomials of Fibonacci Graphs

Sherif El-Basil*

Chemistry Department, University of Georgia
Athens, GA 30602 U.S.A.

Abstract

A recursion exists among the coefficients of the color polynomials of some of the families of graphs considered in recent work of Balasubramanian and Ramaraj. Such families of graphs have been called Fibonacci graphs. Application to king patterns of lattices is given. The method described here applies only to the so called Fibonacci graphs.

Key words

Graph Theory
Fibonacci Graphs
Color Polynomials

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1. Introduction

Recently Balasubramanian and Ramaraj\textsuperscript{1} wrote an interesting paper on a newly defined color polynomial of certain graphs. They related their work to the pioneering work of Motoyama and Hosoya\textsuperscript{2} on king polynomials. Their paper has its merits in both the areas of statistical mechanics and "chemical" graph theory.

The purpose of this communication is to cite an observation on a recursive relation occurring among the coefficients of the color polynomials of some of the families of graphs and their corresponding king patterns which they considered. The observation may be of value from both the computational and graph-theoretical viewpoints. The method which will be described here applies only to the so-called Fibonacci graphs\textsuperscript{3}.

2. Definition of Fibonacci Graphs\textsuperscript{3}

In a homologous series of graphs the set \( \{G_n, G_{n+1}, G_{n+2}, \ldots\} \) where the number of vertices, \( n \), may or may not be finite, has been called a set of Fibonacci graphs\textsuperscript{3} if the following recursion is satisfied:

\[
\theta(G_{n+2}, k+1) = \theta(G_{n+1}, k+1) + \theta(G_n, k)
\]  

(1)

where \( \theta(G,k) \) is some graph-theoretical invariant of \( G \) which may include the following:

i) The number of \( k \)-matchings\textsuperscript{4} in a graph

ii) The number of \( k \) mutually resonant but nonadjacent sextets when \( G=B \), a benzenoid system

iii) The number of \( k \) independent sets of vertices when \( G=C \), the so-called Clar graph\textsuperscript{5,6}. 

Inter-relations among these invariants have been recently published\textsuperscript{7}. Hosoya\textsuperscript{8} seems to be the first who observed recursive relations of the type of eqn. 1 but only for the paths and the cycles when $\theta(G, k)$ becomes the number of matchings and $G$ is either a path or a cycle. Recently this author\textsuperscript{3} and Gutman\textsuperscript{9} generalized the concept to other types of graphs which obey eqn. (1) and to several graph invariants.

3. **Construction of Fibonacci Graphs**

The (finite or infinite) set \{G\(_n\), G\(_{n+1}\), ... G\(_{n+s}\)\}, n \geq 0, s > n+1 is called a set of Fibonacci graphs. Further, if either $v_0$ or $v_1$ is of degree one, then also \{G\(_{-1}\), G\(_0\), ..., G\(_n\)\} is a set of Fibonacci graphs. Such a set must possess at least three elements. The above construction is illustrated in Fig. 1 on the molecular graph of the benzyl radical. There are two modes of graph growth leading to Fibonacci graphs, i.e. "Fibonacci growth", viz., (a) external graph growth (path growth) and (b) internal graph growth (cycle growth).

4. **Application to Color Polynomials\textsuperscript{1} and king Patterns\textsuperscript{1,2}**

First we observe that the color polynomials given in ref. 1 are equivalent to the independence polynomials\textsuperscript{5,6} introduced earlier. Thus $O(G, k)$ is defined\textsuperscript{5,6} to be the number of selections of $k$ independent vertices from $G$. This is precisely the number of ways of coloring $k$ vertices black so that no two black vertices are adjacent. Table VI\textsuperscript{1} of ref. 1 lists color polynomials of some cycles. Of course a homologous series of rings form a set of Fibonacci graphs and thus should conform to eqn. 1 where $I(G, k) = O(C; k)$, $C = c; c \leq e$. The coefficients (i.e. $O(C; k)$'s)

\*Balasubramanian and Ramaraj\textsuperscript{1} have shown that the coefficients of the color polynomials of the paths are the **Fibonacci numbers** while those of the cycles are **manage numbers**.
are reproduced here to demonstrate the validity of eqn. 1.

\[
\begin{array}{cccc}
\alpha & \beta & \gamma & \delta \\
2 & 1 & 2 & 0 \\
3 & 1 & 3 & 0 \\
4 & 1 & 4 & 2 & 0 \\
5 & 1 & 5 & 5 & 0 \\
6 & 1 & 6 & 9 & 2 & 1 \\
7 & 1 & 7 & 14 & 7 & 0 \\
\end{array}
\]

As a further application of the concept of Fibonacci graphs we calculate the color polynomial of \( G_{10,14} \); a graph on 25 vertices.

\[
G_{10,14}
\]

There are a number of routes for the homologation to \( G_{10,14} \) from smaller graphs.

One such route is indicated below

Homologation to \( G_{2,14} \) is shown in Table 1. To obtain \( G_{10,14} \) from \( G_{2,14} \) we need the color polynomial of \( G_{3,14} \) which is calculated using recursion 27
where $C(G;x)$ is the cycle polynomial\textsuperscript{1,6,7} of $G$ and other symbols have their usual meanings. If one chooses the tetravalent vertex the polynomial is obtained in terms of (the known) path polynomials:

$$C(G_3;14;x) = 1 + 18X + 134X^2 + 535X^3 + 1243X^4 + 1708X^5 + 1352X^6 + 575X^7 + 115X^8 + 8X^9$$

Then $G_{2,14}$ and $G_{2,15}$ are the first two leading Fibonacci graphs for the second internal growth in ring B (Table 2).

Obviously $G_{10,14}$ corresponds to the lattice in Fig. 2.

5. **Conclusion**

Recursive relations of form 1 are very helpful in construction of counting polynomials of potentially very large graphs. Such a buildup from very small units is conceptually similar to expanding the secular determinant of a graph by pruning it down to smaller fragments\textsuperscript{10}. The identification of a particular family of a Fibonacci graph is certainly of topological and computational importance and is probably equivalent to a botanical identification of a plant family.

**Acknowledgments**

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References

Fig. Legends

Fig. 1
The two types of Fibonacci growths of graphs:
(a) External subdivision and (b) Internal subdivision.

Fig. 2
The lattice graph corresponding to $G_{10,14}$. There are 34362 king patterns generated when 6 kings assume nontaking positions. (c.f. Tables 1 and 2). Observe that $G_{10,14}$ is the dualist graph of the above lattice.
Homologation from $G_{2,2}$ to $G_{2,14}$. Numbers are coefficients of color polynomials.

Relation 1 is observed throughout. The computation involves 12 "Fibonacci-growths".

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**Table 2**

Homolgation $G_{2,14} + G_{10,14}$ via 8 internal Fibonacci growths. Numbers are coefficients of color polynomials.

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$1, 25, 274, 1732, 6989, 18822, 34362, 42344,$

$34438, 17689, 5320, 819, 48, 0.$
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