The theory and practice of the h-p version of finite element method

Ben Qi Guo and Ivo Babuška

BN-1062

April 1987
The theory and practice of the h-p version of finite element method

I. Babuska and B. Guo

Institute for Physical Science and Technology
University of Maryland
College Park, MD 20742

Department of the Navy
Office of Naval Research
Arlington, VA 22217

Approved for public release: distribution unlimited

The p and h-p version is a new development in finite element method in recent years. This paper is addressing some theoretical advances and presents numerical illustrations.
THE THEORY AND PRACTICE OF THE h-p VERSION
OF FINITE ELEMENT METHOD

Ben Qi Guo* Ivo Babuška**
Institute of Physical Science & Technology
and Department of Mathematics
University of Maryland
College Park, MD 20742
U.S.A.

Abstract: The p and h-p version is a new development in finite element method in recent years. This paper is addressing some theoretical advances and presents numerical illustrations.

1. Introduction

There are three versions of finite element method. The classical h-version achieves the accuracy by refining the mesh while using low degree p of elements, p = 1, 2 usually. The p-version keep the mesh fixed and the accuracy is achieved by increasing the degree p. The h-p version properly combines both approaches.

The h-p version is the new development of finite element method. It was first addressed by Babuška and Dorr [4]. The further analysis and computation for two dimensional problems were made by Guo, Babuška in [39]. The problem with non-homogeneous Dirichlet data was studied by Babuška, Guo, in [10]. The h-p version with p - 1 elements for the problem of 2nd order was discussed by Guo in [19]. The feedback and adaptive approach was developed by Gui, Babuška [41] and Babuška, Rank [42].

The p and h-p version for two dimensional problems were implemented in the commercial code PROBE by Noetic Tech., St. Louis. See [27,28]. The commercial program FIESTA (Istituto Sperimentale Modelli e Strutture) has limited p and h-p capabilities in three dimensions. The p and h-p versions of finite element method in three dimensions is being developed by Noetic Tech. and by Aeronautical Research Institute of Sweden.

The practical effectiveness of the p and h-p version is closely related to the problems in structural mechanics which are the problems of elliptic partial equations with piecewise analytic data (as domains, boundary conditions). The analysis of the interior regularity was given by Morrey [23], and the behavior of the solution in the neighborhood of corners and edges of the domain was given by Kondrat'ev, Oleinik [21,22] and Grisvard [16,19]. The characterization of the regularity of the solution on nonsmooth domains in the frame of countably normed spaces was given in the series of papers by Babuška, Guo in [5,6, 7,8] and [9].

The advances in the p-version were discussed in [26]. The computational aspects were addressed in [25].

* Supported by the National Science Foundation under Grant CNS-85-16191.
** Supported by the Office of Naval Research under Contract N00014-85-k-0169.

For a survey of the state of the art of the p and h-p versions we refer also to [2].

2. Finite Element Method and the Approximation

Let B(u,v) be a bilinear form defined on $H^1 \times H^1$, where $H^1_0 \subset H^2_0$. The problem with piecewise analytic data was made by Babuška, Guo in [19]. The problem with non-homogeneous Dirichlet data was studied by Babuška, Guo, in [10]. The h-p version with $p - 1$ elements for the problem of 2nd order was discussed by Guo, in [19]. The feedback and adaptive approach was developed by Gui, Babuška [41] and Babuška, Rank [42].

The practical effectiveness of the p and h-p version is closely related to the problems in structural mechanics which are the problems of elliptic partial equations with piecewise analytic data (as domains, boundary conditions). The analysis of the interior regularity was given by Morrey [23], and the behavior of the solution in the neighborhood of corners and edges of the domain was given by Kondrat'ev, Oleinik [21,22] and Grisvard [16,19]. The characterization of the regularity of the solution on nonsmooth domains in the frame of countably normed spaces was given in the series of papers by Babuška, Guo in [5,6, 7,8] and [9].

The advances in the p-version were discussed in [26]. The computational aspects were addressed in [25].

Supported by the National Science Foundation under Grant CNS-85-16191.

**Supported by the Office of Naval Research under Contract N00014-85-k-0169.
Let $Q \subset \mathbb{R}^2$ be a bounded domain (curvilinear polygon) shown in Figure 3.1 with vertices $A_1, 1 \leq i \leq M$, the boundary $\partial Q$ be a piecewise analytic curve $\Gamma = \bigcup \Gamma_i$ where $\Gamma_i$'s are open arcs with endpoint $A_i$ and $A_{i+1}$. We denote the internal angle by $\omega_i, 0 < \omega_i \leq 2\pi, 1 \leq i \leq M.$

\[\text{Fig. 3.1. The domain with piecewise analytic boundary.}\]

Let $\gamma_0 = \bigcup \Gamma_i$ and $\gamma^1 = \gamma - \gamma_0$ be the Dirichlet and Neumann boundary respectively. For simplicity we consider only the problem

\[-Au + u = f \text{ in } \gamma_0 \quad (3.1a)\]

\[u = g^0 \text{ on } \gamma^0 \quad (3.1b)\]

\[\frac{2u}{\omega_i} = g^1 \text{ on } \gamma^1 \quad (3.1c)\]

If $f \in L^2(\gamma), g^1 \in L^2(\gamma^1), g^0 \in H^1(\gamma^1), i \in D,$ $g^0$ is continuous on $\gamma^0$, the problem (3.1) has a unique solution $u \in H^1(\Omega)$ if for $\nu \in \mathbb{R}^k$ with $\nu \neq 0$ the singularities appear at the corners of the domain. Hence the standard Sobolev spaces are not a powerful tool for this type of problem, and various weighted Sobolev norms were introduced.

Let $\beta = (\beta_1, \ldots, \beta_M)$ be an $M$-tuple of real numbers $0 < \beta_i < 1, 1 \leq i \leq M$. For any integer $k \geq 0$ we shall write $\beta + k = (\beta_1, k, \ldots, \beta_M + k)$. By $x_i(x)$ we denote the Euclidean distance between $x \in \Omega$ and the vertex $A_i, 1 \leq i \leq M$. We denote then $\Phi_{\beta+k}(x) = \int_{x_1}^{\beta_1+k} \frac{1}{x_1} dx_1 \ldots \frac{1}{x_M} dx_M.$

Let $S = (s_1, \ldots, s_M)$ be an $M$-tuple of real numbers. Define for $k \geq 0$, $V^{s,k}(\Omega) = \{u \in H^k(\Omega) \text{ s.t. } \beta + k - 1 \leq \Phi_{\beta+k}(x) \leq \beta + k + 1 \text{ for } x \in \Omega\}.$

Theorem 3.1. Let $\Omega$ be a polygon, $f \in C^0(\Omega), g^1 \in H^1(\gamma^1), g^0 \in H^k(\gamma^1)$, $0 < \beta_1 < 1, \beta_i > 1 - \frac{E}{\omega_i}$ (resp. $\beta_i = 1 - \frac{E}{\omega_i}$ if different types of boundary condition are imposed on $\Gamma_i$ and $\Gamma_{i+1}$, $1 \leq i \leq M$, then the problem (3.1) has a unique solution $u \in H^1(\Omega)$ and $u \in C^0(\Omega)$ for proof see [5].

Remark 3.1. Since $B^{s,k}(\Omega) \subset C^0(\Omega), k \geq 0$ for arbitrary $\epsilon > 0$ the result of Theorem 3.2 is weaker than that of Theorem 3.1. Nevertheless, it will not affect the asymptotic rate of convergence for the $h-p$ version.

Remark 3.2. Theorems 3.1 and 3.2 are also valid for generally strongly elliptic equation and system with analytic coefficients satisfying inf-sup condition [5, 6]. The interface problems with piecewise analytic interfaces and the eigenvalue problems have the same properties too, see [8]. The solution of elliptic problem of $2m$ order on polygonal domain belongs to $W^{2m}(\Omega)$ [7]. Hence for this class of problems including many structural problems the solution set $K = B^{s,k}(\Omega)$ or $C^0(\Omega), k \geq 2$.

The definition of $B^{s,k-1/2}(\gamma)$ does not give the structure of the space, and it is often difficult to verify in general whether $q$ belongs to this space. Hence further characterization of the structure of $B^{s,k-1/2}(\gamma)$ is important for application.

Let $1 = (a, b) \subset \mathbb{R}$. Analogously as before we shall define the spaces $H^{s,k-1}(\Omega)$ and $B^{s,k}(\Omega).$ Let $\rho_1 = \|x - a\|, \rho_2 = \|x - b\|$ and $y = (y_1, y_2)$ be the 2-tuple of real numbers. $0 < y_i < 1, i = 1, 2$. For any integer $k \geq 0$ we shall write $y + k = (y_1 + k, y_2 + k), k.$
\[ V_{T}^{k} = \prod_{i=1}^{N} V_{T_{i}}^{k} = \prod_{i=1}^{N} \mathbb{P}_{k} \quad \text{for } k \geq 1 \leq N. \]

For \( k \geq 1 \geq 0 \), we define \( h_{k}^{n}(I) = (u \in H^{k-1}(I), \| u - u_{h_{k}^{n}}^{(m)} \|_{L^{2}(I)}, \leq m \leq k) \)
\( (\text{if } \ell = 0, \text{the condition that } u \in H^{k-1}(I) \text{ is absent}) \)
and \( h_{k}^{n}(I) = (u \in H_{Y}^{k}(I), \| u - u_{h_{k}^{n}}^{(m)} \|_{L^{2}(I)} \leq C \delta h_{k}^{n}(k-1)) \)
\( k = 1, 1, \ldots, C \) and \( d \) independent of \( k \).

Analogously we define spaces \( B_{k}^{n}((i)) \) on \( T_{i} \)
with \( Y_{k} \) instead of \( Y \). We have the following theorem.

**Theorem 3.1.** Let \( \Omega \) be a polygon, further

1. **If** \( g_{0} \) is continuous on \( T_{i}^{0} \), \( g_{1} = g_{0} \mid T_{i}^{1} \)

   \( B_{1}^{1}((i)), i \in D \), with \( Y_{i} = (Y_{1}, Y_{2}), 0 < Y_{1}, Y_{2} < 1 / 2 \), \( i \leq 1,2 \), then there is a function \( g_{1}^{1} \in B_{1}^{1}(D) \)
   such that \( g_{1} = g_{1}^{1} \) and \( g_{1}^{1} = \max(\gamma_{1}, \gamma_{2}) + 1/2 \) for \( i,i-1 \in D, 1/2 < \gamma_{i} < 1 \) for \( i,i \in D, 0 < \gamma_{i} < 1 \) are arbitrary for \( i,i \in D \).

2. **If** \( g_{1} = g_{1}^{1} \mid T_{i}^{1} \) \( \neq B_{1}^{0}((i)) \), with \( Y_{i} = (Y_{1}, Y_{2}), 0 < Y_{1}, Y_{2} < 1 / 2, i \in D, \ell = 1,2 \), then

   there is a function \( g_{1}^{2} \in B_{1}^{2}(D) \) such that \( g_{1} = g_{1}^{2} \) and \( g_{1}^{2} = \max(\gamma_{1}, \gamma_{2}) + 1/2 \) for \( i,i-1 \in D, 1/2 < \gamma_{i} < 1 \) for \( i,i \in D, 0 < \gamma_{i} < 1 \) are arbitrary for \( i,i \in D \).

**Remark 3.3.** **Theorem 3.3** also holds for curvilinear polygon with piecewise analytic boundary (see [10]).

**Remark 3.4.** **Theorem 3.3** allows us to verify the boundary conditions mentioned in Theorems 3.1 and 3.2.

### 4. The Mesh and Finite Element Spaces

Mesh design is very crucial to the accuracy of method and depends very much on the solution set \( K = B_{0}^{2}(D) \) and \( G_{2}^{2}(D) \). We assume for simplicity that \( \Omega \)

the polygon contained in a unit disc centered at origin which coincides with the vertex \( A_{1} \) of \( \Omega \), and \( \Omega = B_{0}^{2}(D) \) with \( \delta \), \( \delta \), \( i.e., \), assume that the singularity appears only at one vertex of \( \Omega \).

Mesh typically used in the \( h-p \) version is such that domain is divided into several layers by geometric progression. The \( j \)-th layer, \( 1 \leq j \leq n+1 \), consists of cells \( Q_{i,j} \), \( 1 \leq i \in I(j) \). In addition to the usual conditions in the theory of finite element method, the main characterization of the \( (n_{g}) \) mesh \( \Omega = \{ Q_{i,j}, 1 \leq i \in I(j), 1 \leq j \leq n+1 \} \) is following:

1. **Let** \( \text{mesh factor } \sigma \) be an arbitrary number, \( \sigma < 1 \), and let \( d_{i,j} \) be the distance between

   origin and \( O_{i,j} \) and \( h_{i,j} \) be the maximum and minimum of length of edges of \( Q_{i,j} \) then \( d_{i,j} \) \( h_{i,j} \) satisfy

   \[ d_{i,j}^{n+2-j} \leq d_{i,j} \leq d_{i,j} \]

   and

   \[ h_{i,j}^{n+2-j} \leq h_{i,j} \leq h_{i,j} \]

   for \( 1 \leq i \leq i(j), 1 < j \leq n+1 \) and

   \[ d_{i,j} = 0 \]

   \[ h_{i,j}^{n+1} \leq h_{i,j} \leq h_{i,j} \leq d_{i,j} \]

   for \( 1 \leq i \leq i(j), 1 < j \leq n+1 \).

   **(C2)** Let \( M = (N_{i,j}, 1 \leq i \in I(j), 1 \leq j \leq n+1) \), \( N_{i,j} \) is a one-to-one mapping of standard square \( S \)

   (resp. standard triangle \( T \)) onto \( Q_{i,j} \). Let \( \Phi_{i} \) and \( \Psi_{i} \) denote the vertex and side of \( Q_{i,j} \), then \( \omega_{i-1}^{1}(T) \)

   \( \omega_{i-1}^{1}(S) \) are the vertex and side of \( S \) (resp. \( T \)), \( 1 \leq i \leq 4 \) (resp. \( 1 \leq i \leq 3 \)). Moreover, if \( M_{i,j} \) and \( M_{i,j,k} \) map \( S \) (resp. \( T \)) onto element \( O_{i,j} \) and \( M_{i,j,k} \) with common side \( \gamma_{i} = A_{1}A_{2} \), then

   \[ \text{dist}(N_{i-1}^{1}(A), N_{i-1}^{1}(A_{2})) = \text{dist}(N_{i-1}^{1}(A), N_{i-1}^{1}(A_{2})) \]

   for any \( A \in Y_{i}, i = 1,2 \). We assume each side \( Y_{i} \) of \( Q_{i,j} \) is analytic curve, \( 1 \leq k \leq 4 \) (resp. \( 1 \leq k \leq 3 \)),

   \[ x = h_{i,j} y_{i,j}^{1}(x) \]

   \[ y = h_{i,j} y_{i,j}^{2}(x) \]

   \[ x \in I = (0,1) \]

   and

   \[ |\psi_{i,j}^{(k)}| \leq C_{1} k \]

   where \( C_{1} \) and \( L \) are independent of \( i, j \). Accordingly, the mapping \( M_{i,j} \) of \( S \) (resp. \( T \)) onto \( O_{i,j} \)

   is analytic on \( S^{*} \) (resp. \( T^{*} \)) and can be extended to \( \bar{S}^{*} \). Let \( J_{i,j} \) be the Jacobian of \( M_{i,j} \). We shall assume that

   \[ C_{1} h_{i,j} \leq J_{i,j} \leq C_{2} h_{i,j} \]

   with constants \( C_{1}, C_{2} \) independent of \( i,j \).

**Remark 4.1.** Figure 6.3 is an example of the geometric mesh for the problem with singularity at one corner, but the mesh can be analogously generalized for problems with singularity at every corner.

**Remark 4.2.** If mesh \( O \) contains triangular elements some additional assumptions have to be imposed. In the practice these assumptions can easily be satisfied, see [9].

Let \( P = (p_{i,j}, 1 \leq i \in I(j), 1 \leq j \leq n+1) \) and \( Q = (q_{i,j}, 1 \leq i \in I(j), 1 \leq j \leq n+1) \) be the degree vector with integer \( p_{i,j} \) and \( q_{i,j} = 0 \).

We define the finite element spaces
5. Basic Approximation Theorems of the h-p Version

We will list some basic approximation results in the case that $H_1 = H_2 = H^1(\Omega), \Omega = B^2(0)$ or $\mathbb{R}^2$ and $S_1 = S_2 = S^P = \mathbb{R}^d$, i.e., we seek the estimates of $Z(u,H^1(\Omega), S^P) = Z(\Omega)$ for $u \in K$.

**Theorem 5.1.** Let $\Omega$ be a polygon and $u \in \mathbb{R}^2(\Omega) \cap H^2$ be a curve, then for any $u \in (0,1)$, $P - Q, v \in P, u \leq u_0, 0 \leq v \leq u < = 1$, we have

$$Z(u,H^1(\Omega), S^P) \leq C u^{-1/3}$$

where $b$ and $C$ are independent of $N = \text{dim}(S^P)$, the number of degree of freedom. For proof, see [9].

**Theorem 5.2.** If $u \in \mathbb{R}^2(\Omega) \cap H^2$, $\Omega$ is a curvilinear polygon, the boundary of domain is piecewise analytic, then the result of the previous theorem holds. For proof, see [9].

**Remark 5.1.** Mesh factor $\sigma$ can be any number $\in (0,1)$ the computation shows that $\sigma = 0.15$ is the optimal value. In [18] it has been proved that $\sigma = (\sqrt{2} - 1)^2 = 0.17$ is the optimal mesh factor in one dimensional setting. The value $\sigma = 0.15$ in two dimensional problems reflects the fact the solutions in the neighborhood have essentially one dimensional character.

**Remark 5.2.** If $z^0 = g|_{\Gamma_1} \in B^1(\Gamma_1), \Gamma_1 = (8_1 - 1/2, 8_{i+1} - 1/2)$ are non-homogeneous Dirichlet boundary conditions, theorems above hold provided $z^0$ is properly projected on the trace of finite element space $S^P$.

**Remark 5.3.** For problems of order $2m$, the theorems hold when geometric mesh contains only parallelogram and triangular elements. For details, see [22].

6. Numerical Results

We will present some numerical results for the solution of a plane strain elasticity problem. We selected the model of crack panel loaded by traction that the exact solution is the first (symmetric) and second (antisymmetric) mode of stress intensity factor solution. This problem was selected because it characterizes the usual difficulties of engineering computation. Due to the symmetry and antisymmetry we need only to solve the problem in the upper half of the plane shown in Figure 6.1. The solution has singular behavior at the tip of the crack, i.e., the displacement $U = (u,v)$ has the expression $(r^{1/2}, \theta)$ near the origin. Obviously $u,v \in K$ and $u,v \in B^2(\Omega)$ for $B > 1/2$.

The energy of $U$ is defined as

$$G(U) = \frac{E}{2(1-\nu)(1+\nu)} \int_{\Omega} [(2u_x)^2 + (2v_y)^2] + 2u_x 2v_y + (1-2\nu)\int_{\Omega} [(2u_x)^2 + (2v_y)^2] \text{d}x \text{d}y = \|U\|^2_E$$

where $E$ and $\nu$ are the Young’s modulus of elasticity and Poisson ratio. The error $e = U - U_{FE}$ is measured in energy norm, and by (5.1)

$$\|e\|_E \leq C e^{-bn^{1/3}}.$$

The relative error is defined as

$$\|e\|_{E,R} = \|e\|_E/\|U\|_{E} < 10\%.$$
Table 6.1. Relationship between $\| e \|_{E,R}^N$, $n$, $p$, $b$ and $C$ for the $h$-$p$ version. The symmetric problem $(\varepsilon = 1, \nu = 0.3)$ on mesh $A_i$, $1 \leq i \leq 6$, $\alpha = 0.15$, $p = n$.

<table>
<thead>
<tr>
<th>Mesh $A_i$</th>
<th>$p$</th>
<th>$n$</th>
<th>$N$</th>
<th>$k_{\text{ls}}$</th>
<th>$b$</th>
<th>$C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>1</td>
<td>9</td>
<td>6</td>
<td>60.92</td>
<td>0.764</td>
<td>1.455</td>
</tr>
<tr>
<td>$A_2$</td>
<td>2</td>
<td>48</td>
<td>6</td>
<td>20.23</td>
<td>0.740</td>
<td>2.303</td>
</tr>
<tr>
<td>$A_3$</td>
<td>3</td>
<td>121</td>
<td>6</td>
<td>7.61</td>
<td>0.776</td>
<td>2.098</td>
</tr>
<tr>
<td>$A_4$</td>
<td>4</td>
<td>256</td>
<td>6</td>
<td>2.57</td>
<td>0.720</td>
<td>1.810</td>
</tr>
<tr>
<td>$A_5$</td>
<td>5</td>
<td>477</td>
<td>6</td>
<td>0.90</td>
<td>0.670</td>
<td>1.683</td>
</tr>
<tr>
<td>$A_6$</td>
<td>6</td>
<td>808</td>
<td>6</td>
<td>0.33</td>
<td>0.670</td>
<td>1.688</td>
</tr>
</tbody>
</table>

In Figure 6.4 we show the dependence of the error on $\alpha$. We see the best value of $\alpha$ is close to $(V^2 - 1)^2 = 0.17$ which is the theoretical optimal value in one dimension.

Figure 6.2. Mesh

The table 6.1 shows the relationship between $N$, $n$, $p$ and $\| e \|_{E,R}$, where $n$ is the number of layers, $p$ is the element degree and $N$ is the number of degrees of freedom. The relationship is plotted on a $\ln \| e \|_{E,R}^N$ vs. $1/N^3$ scale and shown in Figure 6.3. The curve characterizing the convergence of the $h$-$p$ version is the envelope of six curves of the $p$-version on mesh $A_i$, $1 \leq i \leq 6$ and is nearly a straight line. The slope of the line is $b$ in (6.1) and is numerically 0.87.

Figure 6.3. Relative error in energy norm vs. number of degree of freedom. The symmetric problem $(\varepsilon = 1, \nu = 0.3)$ on mesh $A_i$, $1 \leq i \leq 6$, $\alpha = 0.15$, $p = n$.

Figure 6.4. Dependence of Relative error of the $h$-$p$ version in energy on the mesh factor $N$. The anti-symmetric problem $(\varepsilon = 1, \nu = 0.3)$ on mesh $A_i$, $1 \leq i \leq 6$.

Figure 6.5 shows that the $h$-$p$ version is insensitive to change of Poisson ratio. The slope of the curves of the $h$-$p$ version for $\nu = 0.3$ and 0.49 are almost the same. The locking problem never occurred.

Figure 6.5. Insensitivity of Relative error of the $h$-$p$ version in energy norm to change of Poisson ratio. The anti-symmetric problem $(\varepsilon = 1, \nu = 0.3)$ on mesh $A_i$, $1 \leq i \leq 6$. 

---

In Figure 6.4 we show the dependence of the error on $\alpha$. We see the best value of $\alpha$ is close to $(V^2 - 1)^2 = 0.17$ which is the theoretical optimal value in one dimension.
Table 6.2. Estimated error of the h-p version. The symmetric problem \((K = 1, v = 0.3)\) on Mesh \(A_n, 1 \leq n \leq 6, \sigma = 0.15, p = n.\)

<table>
<thead>
<tr>
<th>(n)</th>
<th>(|e|_{L^2})</th>
<th>(|e|_H^1)</th>
<th>(|e|_{H^1, a})</th>
<th>(|e|_{H^1, a})</th>
<th>(|e|_{H^1, a})</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.996E+1</td>
<td>2.966E+1</td>
<td>60.83</td>
<td>60.92</td>
<td>0.2189</td>
</tr>
<tr>
<td>3</td>
<td>3.703E-2</td>
<td>3.705E-2</td>
<td>7.606</td>
<td>7.611</td>
<td>0.0606</td>
</tr>
<tr>
<td>4</td>
<td>1.248E-2</td>
<td>1.250E-2</td>
<td>2.563</td>
<td>2.567</td>
<td>0.0926</td>
</tr>
<tr>
<td>5</td>
<td>4.339E-3</td>
<td>4.369E-3</td>
<td>0.891</td>
<td>0.897</td>
<td>0.6689</td>
</tr>
</tbody>
</table>

In Figure 6.6 we compare the performance of the h,p and h-p versions of \(\|e\|_{L^2, a}\) for a range of meshes. We see that the accuracy 0.5% - 1% is very expensive and probably not achievable for all the p-version and h version with \(p = 1\). The h-p version allows us to use a relatively small number of elements to obtain high accuracy.

Figure 6.6. Relative error in energy norm vs. number of degree of freedom for the h,p,h-p versions. The symmetric problem on which mesh factor \(\sigma = 0.15\).

Since the exponential rate of convergence can be achieved for low \(p\) and \(n\) as well as high \(p\) and \(n\), we can use the computational results from three successive computations to obtain estimation of error in energy norm. Table 6.2 shows that the estimated error is very reliable with relative error less than 1%. For the formulation of estimated error, see [20].

7. Conclusions

From theoretic analysis and computation we can conclude the following:

1. The asymptotic theory reflects accurately the computational practice in the entire range of engineering accuracy. The exponential rate of convergence is achieved in computation in industrial practice.

2. The optimal geometric factor of mesh refinement is close to 0.15.

3. The performance of the p and h-p versions is not influenced when the Poisson ratio \(v \leq 0.5\) (i.e., when the material is nearly incompressible).

4. Industrial experience with the method (by program PROBE) indicates high effectiveness and advantages of the h-p version, see [15].

5. Preliminary computation and theoretical analysis show that in the three dimensions the p and h-p have superior qualities in practical computation of problems in structural mechanics.

References


The Laboratory for Numerical analysis is an integral part of the Institute for Physical Science and Technology of the University of Maryland, under the general administration of the Director, Institute for Physical Science and Technology. It has the following goals:

- To conduct research in the mathematical theory and computational implementation of numerical analysis and related topics, with emphasis on the numerical treatment of linear and nonlinear differential equations and problems in linear and nonlinear algebra.

- To help bridge gaps between computational directions in engineering, physics, etc., and those in the mathematical community.

- To provide a limited consulting service in all areas of numerical mathematics to the University as a whole, and also to government agencies and industries in the State of Maryland and the Washington Metropolitan area.

- To assist with the education of numerical analysts, especially at the postdoctoral level, in conjunction with the Interdisciplinary Applied Mathematics Program and the programs of the Mathematics and Computer Science Departments. This includes active collaboration with government agencies such as the National Bureau of Standards.

- To be an international center of study and research for foreign students in numerical mathematics who are supported by foreign governments or exchange agencies (Fulbright, etc.)

Further information may be obtained from Professor I. Babuška, Chairman, Laboratory for Numerical Analysis, Institute for Physical Science and Technology, University of Maryland, College Park, Maryland 20742.
END
9-87
DTIC