AN EFFICIENT WAY FOR EDGE-CONNECTIVITY AUGMENTATION

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AN EFFICIENT WAY FOR EDGE-CONNECTIVITY AUGMENTATION

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Keywords: Edge-connectivity augmentation problem; algorithm; computational complexity

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ABSTRACT

We present an algorithm for finding a minimum set of edges to be added so as to k-edge-connect a given graph $G = (V,E)$ with $k > 1$ and $|V| > 1$. The time complexity is $O(k^2|V|^3(k|V| + |E|))$ or $O(k^2(|V|^4 + k|V| + |E|))$ if we use Dinic's maximum flow algorithm or Malhotra, Kumar and Maheshwari's one, respectively, as a subroutine.
1. INTRODUCTION

The problem in which the object is to add a minimum weight set of edges to a graph \( G = (V,E) \) so as to satisfy a given vertex- or edge-connectivity condition is called the vertex- or edge-connectivity augmentation problem. This problem has a wide variety \([3,4,5,10,14-22]\).

Frank and Chou \([5]\) discussed the unweighted version of some edge-connectivity augmentation problem for graphs without edges, and showed that it is polynomially solvable. Eswaran and Tarjan \([3]\) considered the following problems:

(i) The strong connectivity augmentation problem for directed graphs.
(ii) The bridge-connectivity augmentation problem for undirected graphs.
(iii) The biconnectivity augmentation problem for undirected graphs.

They proved that the weighed versions of these three types are NP-complete and that each of the unweighted versions has an \( O(|V| + |E|) \) algorithm. Rosenthal and Goldner \([15]\) proposed an \( O(|V| + |E|) \) algorithm for the unweighted version of the biconnectivity augmentation problem. Frederickson and Ja'ja' \([6]\) discussed the NP-completeness of several restricted augmentation problems and showed \( O(|V|^2) \) approximation algorithms for above problems (i)-(iii).

We are interested in the k-edge connectivity augmentation problem for undirected graphs with \( k \geq 2 \), a generalization of (ii). The weighted version of the problem are easily shown to be NP-complete. The unweighted version, which had been one of open problems in graph theory \([2, p. 49]\), was solved by Watanabe and Nakamura \([20-22]\): it is shown that the cardinality of a minimum solution of the problem is equal to the k-augmentation number, \( EA_k(G) \), of a given graph \( G \) (the definition will be given later) and that a minimum solution can be obtained in \( O(k^2|V|+^{|V|}+|E|) \) time by using Dinic's maximum flow algorithm.

In this paper we consider an improvement of our previous algorithm, given in \([20-22]\), for k-edge-connectivity augmentation problems.
In section 2, graph-theory terminologies and technical terms used in this paper are given.

In section 3, we summarize our previous results on k-edge-connectivity augmentation problems.

In section 4, we describe an improvement of the algorithm mentioned in [20-22]. The previous algorithm repeats two procedures: the one constructs the data structure called the component tree by using a maximum flow algorithm, and the other searches for a pair of vertices, called an admissible pair, which are to be joined by a new edge. The most time-consuming part of the algorithm is the first procedure, which is repeated each time a new edge is added. Thus it is repeated \( EA_k(G) \) times, where \( EA_k(G) \leq k|V| \).

The definition of an admissible pair implies that any minimum solution \( Z \) of the problem is partitioned into some minimal sets \( Z(m+1), \ldots, Z(k) \) such that their addition in this order increases the edge connectivity one by one, where \( m \) is equal to the edge connectivity of \( G \).

We will describe how to determine such a minimal set of new edges whose addition to the current graph increases the edge connectivity by exactly one, without reconstructing the data structure. This will reduce the repetition of reconstructing the data structure to at most \( k-1 \) times, leading to a more efficient algorithm.

We consider the case where a given graph is disconnected or connected, respectively. In 4.1 or 4.2. In 4.3, we estimate the time complexity of the improved algorithm and show that it is

\[
O(k^2|V|^3(k|V| + |E|))
\]

if we use Dinic's maximum flow algorithm [4] or

\[
O(k^2(|V|^4 + k|V| + |E|))
\]

if we use the maximum flow algorithm proposed by Malhotra, Kumar and Maheshwari [4,13].
2. PRELIMINARIES

Many of graph-theory terminologies and technical terms used in this paper are more or less standard, and those not specified here can be identified in [1.4.8].

A graph \( G = (V,E) \) (or \( G=(V(G),E(G)) \)) is a finite set of vertices, \( V \), and a finite set of edges, \( E \). If \( E \) is a multiset, that is, if any edge may occur several times, then \( G \) is called a multigraph. Such edges are called multiple edges. Otherwise \( G \) is a simple graph. In this paper, the term "a graph" means an undirected multigraph unless otherwise stated.

Two vertices \( u,v \) which comprise an edge are said to be adjacent, and the edge is often denoted by \( (u,v) \), even if it is one of multiple edges, as long as no confusion arises. The edge \( (u,v) \) is incident to the vertices \( u,v \); \( u \) and \( v \) are incident to \( (u,v) \). The degree \( d_G(v) \) (or, simply \( d(v) \)) of a vertex \( v \) of \( G \) is the number of edges incident to it in \( G \). An edge \( (v,v) \), that is, an edge joining \( v \) to itself is referred to as a loop. \( G_1 = (V_1,E_1) \) is isomorphic to \( G_2 = (V_2,E_2) \) if \( |V_1| = |V_2|; |E_1| = |E_2| \) and there is a bijection \( \xi \) of \( V_1 \) onto \( V_2 \) such that \( (u,v) \in E_1 \) if and only if \( (\xi(u),\xi(v)) \in E_2 \).

A walk of \( G \) from \( v_1 \) to \( v_n \) (or a \( (v_1,v_n) \)-walk of \( G \)) is an alternating sequence of vertices and edges of \( G \), \( v_1,e_1,v_2,e_2,...,v_{n-1},e_{n-1},v_n(n \geq 1) \), such that \( e_i = (v_i,v_{i+1}), 1 \leq i \leq n-1 \). The length of this walk is \( n-1 \). A path (A trail, respectively) is a walk without any repeated vertices (edges) in it. For \( 1 \leq i < j \leq n \), the \( (v_i,v_j) \)-path consisting of edges \( (v_i,v_{i+1}),...,(v_{j-1},v_j) \) is referred to as the \( (v_i,v_j) \)-subpath of a \( (v_1,v_n) \)-path. If \( n > 2 \) then \( v_2,...,v_{n-1} \) are called the inner vertices of the path. If two paths have no edge in common, then they are said to be edge-disjoint (or simply, disjoint). Let \( M_{\odot}(u,v) \) (or simply, \( M(u,v) \)) denote the maximum number of pairwise edge-disjoint \( (u,v) \)-paths of \( G \).

G is connected if and only if every pair of vertices of \( G \) are joined by a path of \( G \). If \( G \) and \( H \) are two graphs such that \( V(H) \subseteq V(G) \) and \( E(H) \subseteq E(G) \), then \( H \) is a subgraph of \( G \). If \( H \) is a maximal connected subgraph of \( G \) (that is, if \( V(H) \neq V(G) \) then \( G \) is not connected) then \( H \) is called a connected component (or simply, component) of \( G \). Let \( Z \) be a set of edges such that \( Z \subseteq E(G) \) (\( Z \cap E(G) = \emptyset \) (empty), respectively), where any edge of \( Z \) joins two vertices of \( V(G) \). Then \( G-Z \)
(G+Z, respectively) denotes the graph obtained by deleting all edges of Z from G (by adding all edges of Z to G). If Z = {e} then it is denoted by G-e (G+e) for simplicity.

For two subsets S, S' ⊆ V(G), let E(S, S'; G) denote the set of all those edges of E(G) joining a vertex of S and one of S'. In particular, we denote E(S, V(G)-S; G) by K(S, G). If S = {v} then we write K(v; G). If S ≠ φ and S ⊆ V(G) (a proper subset) then K(S, G) is called a separator or a \(|K(S,G)|\)-separator of G. Clearly, if |V(G)| > 1 then G-K(S, G) is disconnected. Put
\[
d(S, G) = |K(S, G)|
\]
and we call it the degree of S (in G).

Let K be a separator of G, and suppose that K=K(T; G) for a nonempty subset T ⊆ V(G). A pair of disjoint subsets S, S' ⊆ V(G) (that is, S ∩ S' = φ) is said to be separated by K (or we say that K separates S from S') if S ⊆ T and S' ⊆ V(G)-T. K is referred to as an (S, S')-separator (of G). If S={u} and S'={v} then we simply call K a (u, v)-separator. An (S, S')-separator K with the minimum cardinality among all (S, S')-separators of G is referred to as an (S, S')-cut. A (u, v)-cut is defined similarly. Each component of G-K is called a K-block (of G). A K-block whose vertex set includes a subset S ⊆ V(G) is denoted by B(S, K; G) and is referred to as the (S, K)-block of G. (For simplicity, we often use the term "a K-block", meaning its vertex set. If S={u} then B({u}, K; G) is written by B(u, K; G).

Let m ⩽ k for a fixed integer k > 1. A subset S ⊆ V(G) is called an m-edge-component (or, simply, an m-component) of G if and only if the following (1), (2) hold:

(1) \( M_0(u, v) \geq m \) for any \( u, v \in S \).

(2) For any \( u' \in V(G)-S \), S has a vertex \( v' \) with \( M_0(u', v') < m \).

An m-edge-component that is not an (m-1)-edge-component is said to be critical. If S is an m-edge-component of G with \( 0 \leq d(S, G) < m \) then S is called an m- pendant. Clearly, a 1- pendant of G is identical to the vertex set of a component of G. Let \( P_m(G) \) denote the total number of m-pendants of G. An m-pendant S of G is referred to as an external m- pendant if K(S, G) is an (S, S')-cut of G for some m-edge-component S' (≠ S) of G.
The edge-connectivity $ec(G)$ of a graph $G$ is the minimum number of edges whose removal from $G$ disconnect it to more than one component or result in a single vertex:

$$ec(G) = \begin{cases} 
\min \{ |K| : K \text{ is a separator of } G \} & \text{if } |V(G)| > 1 \\
0 & \text{otherwise}
\end{cases}$$

$G$ is said to be $h$-edge-connected if $ec(G) \geq h$. Let $N_G(u,v)$ (or simply, $N(u,v)$) denote the cardinality of a $(u,v)$-cut. It is well known that

$$N_G(u,v) = M_G(u,v) \text{ for any } u,v \in V(G), u \neq v$$

and that

$$ec(G) = \min \{ M_G(u,v) : u,v \in V(G), u \neq v \} \text{ if } |V(G)| > 1.$$  

(See [4,8].)

Let $\lfloor x \rfloor$ ($\lceil x \rceil$, respectively) denote the minimum integer not less than $x$ (the maximum integer not greater than $x$).

For a subset $S \subseteq V(G)$, let $G[S]$ denote the graph defined by $V(G[S]) = S$ and $E(G[S]) = \{(u,v) \in E(G) : u,v \in S\}$. $G[S]$ is referred to as the subgraph induced by $S$ of $G$.

Let $Y$ be a nonempty subset of $V(G)$, and let $a \in Y$. Put

$$G - Y = G[V(G) - Y],$$

and let $G \langle a,Y \rangle$ be defined as follows:

$$V(G \langle a,Y \rangle) = (V(G) - Y) \cup \{a\}.$$  

$$E(G \langle a,Y \rangle) = E(G - Y) \cup \{(a,v) : (u,v) \in E(G), u \in Y, v \in V(G) - Y\}.$$  

It is said in [12] that $G \langle a,Y \rangle$ arises from $G$ by identification of $Y$ to $a$. Let

$$T(a,a';G) = \{X \subseteq V(G) : a \in X, a' \in V(G) - X, d(X,G) = M_G(a,a')\}.$$  

If $a \neq a'$ then $T(a,a';G)$ is nonempty.

**THEOREM 2.1.** [12].

In a graph $G$, let $Y \in T(a,a';G)$ for certain $a,a' \in V(G)$. Then, for any distinct vertices $u,v$ of $G \langle a,Y \rangle$. 

Let \( S \subseteq V(G) \), and let \( G \cdot S \) denote the graph defined by the following:

\[
V(G/S) = V(G) - S \cup \{v(S) \mid v(S) \notin V(G)\},
\]

and

\[
E(G/S) = E(V(G) - S, V(G) - S; G) \cup \{(u,v(S)) \mid (u,v) \in E(S,G), v \in S\},
\]

where \( v(s) \) is the new vertex corresponding to \( S \). This operation constructing \( G/S \) from \( G \) is called shrinking of \( S \) in \( G \). \( G/S \) is called the graph obtained by shrinking of \( S \) in \( G \). For simplicity, we also call \( G \cdot S \) shrinking of \( S \) if no confusion arises. Let \( \pi = \{S_1, \ldots, S_t\} \) (\( t \geq 2 \)) be a partition of \( V(G) \):

\[
V(G) = S_1 \cup \cdots \cup S_t, \quad S_i \cap S_j = \emptyset (i \neq j).
\]

Let \( |\pi| \) denote the total number of sets in \( \pi \). For each \( i, 1 \leq i \leq t \), put

\[
G_{\frac{i}{}} = G_{\frac{i-1}{}} / S_i,
\]

where \( G_{\frac{i}{}} = G \). Put

\[
G/\pi = G_{\frac{t}{}}
\]

and we call it the \( \pi \)-shrinking of \( G \). \( G/\pi \) is uniquely determined up to isomorphism, independently of the order of shrinking of the sets in \( \pi \). Since we identify two isomorphic graphs, we consider that \( G/\pi \) is unique for each \( \pi \).

Let \( \pi_0(m) \) denote the partition of \( V(G) \) into \( m \)-components of \( G \), where \( m > \text{ec}(G) \). Put

\[
G/m = G/\pi_0(m)
\]

for simplicity. \( G/m \) may have multiple edges. Let \( \rho_m \) denote a mapping

\[
\rho_m: V(G) \rightarrow V(G/m) = \{v(S_i) \mid S_i \in \pi(m)\}
\]

defined by \( \rho_m(v) = v(S_i) \in \pi(m) \) if and only if \( v \in S_i \). \( \rho_m \) is called the mapping of \( G \) induced by \( \pi_0(m) \). We fix \( \rho_m \) and denote \( \rho_m^{-1}(v(S_i)) = S_i \). For any set \( F \subseteq E(G) \), let

\[
\rho_m(F) = \{(\rho_m(u), \rho_m(v)) \mid (u,v) \in F\}.
\]
PROPOSITION 2.1.

Let $S_1, S_2 \in \pi(m)(S_1 \neq S_2)$ and $v_1, v_2 \in V(G/m)(v_1 \neq v_2)$. If $K_G$ is an $(S_1, S_2)$-cut of $G$ then $\rho_m(K_G)$ is a $(\rho_m(S_1), \rho_m(S_2))$-cut of $G/m$ with $|\rho_m(K_G)| = |K_G|$. Conversely, if $K$ is a $(v_1, v_2)$-cut of $G/m$ then $G$ has a $(\rho_m^{-1}(v_1), \rho_m^{-1}(v_2))$-cut $K_G$ with $|K_G| = |K|$.

PROOF.

It suffices to consider the case where $K_G \neq \emptyset, K \neq \emptyset$. Then $m > 1$. Let

$$B_i = \{ \rho_m(S): S \in \pi(m), S \subseteq B(S, K_G; G), i = 1, 2. \}
$$

It is easy to see that

$$\rho_m(K_G) = E(B_1 \cup B_2; G/m). \quad |\rho_m(K_G)| = |K_G|.
$$

If we have $S, S' \in B(S, K_G; G)(S \neq S')$ for either $i = 1$ or $i = 2$ then $G$ has a $(w, w')$-path $P$ for some pair $w \in S, w' \in S'$ such that

$$E(P) \cap K_G = \emptyset.
$$

This implies that $G/m$ has a $(\rho_m(S), \rho_m(S'))$-path $Q$ such that

$$E(Q) \cap \rho_m(K_G) = \emptyset.
$$

Hence it follows that $\rho_m(K_G)$ is a $(\rho_m(S_1), \rho_m(S_2))$-cut of $G/m$.

Conversely we prove the second part. For each $i, i = 1, 2$, put

$$B(i) = B(v_i, K; G/m),
$$

$$v_i = \bigcup_{x \in B(i)} \rho_m^{-1}(x).
$$

Let

$$e_i = (v_i, v_{i+1} \in K_G) = 1, \ldots, m \quad (m' = |K| < m),
$$

where

$$e_i \in B(i), \quad i = 1, 2.
$$

Then, for each $j, 1 \leq j \leq m'$, there is an edge
\[ f_j = (u_{j1}, u_{j2}) \in E(G). \ u_{jt} \in \rho_m^{-1}(v_{jt}). \ t = 1, 2. \]

Put

\[ K_G = \{ f_1, \ldots, f_m \}. \]

Then, clearly, we have

\[ K_G = E(V_1, V_2; G). \]

Similarly to the proof of the first part we can show that \( K_G \) is a \((\rho_m^{-1}(v_1), \rho_m^{-1}(v_2))\)-cut of \( G \).

Q.E.D.

Suppose that there is a pair \( u, v \in V(G)(u \neq v) \) such that \( G \) has exactly \( p \) edges

\[ e_1, \ldots, e_p \ (p \geq 1) \]

connecting them. Delete these \( p \) edges from \( G \) and add an edge \( e \)

\[ e = (u, v) \]

with weight

\[ c(e) = p. \]

We denote

\[ \sigma(e_i) = e_i. \ i = 1, \ldots, p. \]

Repeat such replacement until we obtain a simple graph \( G' \) with each edge having weight equal to the number of multiple edges of \( G \) connecting each pair of endvertices.

Construct a directed graph \( N(G') \) from \( G' \) by replacing each edge \( e = (u, v) \in E(G') \) by two directed edges

\[ e' = (u \rightarrow v), \ e'' = (v \rightarrow u) \]

both of which have weights equal to \( c(e) \)

\[ c(e') = c(e'') = c(e). \]

\( N(G) \) is called the network of \( G \).

Choose a pair of vertices \( s \neq t \) from \( V(N(G')) \) with \( M_0(s, t) > 0 \), and we call \( s \) and \( t \) a source and a sink respectively. Consider each \( e \in E \) to be the capacity of \( e \in E(N(G')) \), and any
existing algorithm for finding a maximum flow $f_m$ from a source to a sink can be applied to $N(G^*)$.

Let $\text{val}(f)$ denote the total flow of a flow $f$:

$$\text{val}(f) = \sum_{e \in \text{IN}(s)} f(e) - \sum_{e \in \text{OUT}(t)} f(e).$$

where $\text{IN}(w)$ ($\text{OUT}(w)$, respectively) denotes the set of all edges of $N(G^*)$ incoming to $w$ (outgoing from $w$). In particular, put

$$F(s \rightarrow t; G) = \text{val}(f_m) \quad \text{or} \quad F(s \rightarrow t) = \text{val}(f_m).$$

Let $S, S' \subseteq V(N(G^*))$ be nonempty disjoint sets, and let

$$E(S \rightarrow S') = \{e = (u \rightarrow v) : e \in E(N(G^*)), u \in S, v \in S'\},$$

$$(S, S') = E(S \rightarrow S') \cup E(S' \rightarrow S),$$

$$(S, S) = (S, V(N(G^*)) - S).$$

We call $(S, S)$ a cut of $N(G^*)$. Put

$$c(S, S) = \sum_{e \in S \rightarrow S} c(e),$$

which is called the capacity of a cut $(S, S)$. A minimum cut is a cut with the minimum capacity among all cuts of $N(G^*)$.

**PROPOSITION 2.2.**

$$F(s \rightarrow t) = M_G(s, t) \text{ if } M_G(s, t) > 0.$$

**PROOF.**

Let $K$ be any $(s, t)$-cut of $G$, where $|K| = M_G(s, t)$. Put

$$S = B(s, K; G), \quad S' = B(t, K; G).$$

Let

$$K' = \{\sigma(e) \in E(G^*), e \in K\}.$$  

Then clearly.
\[ \sum_{e \in \mathcal{E}(G')} e(e) = |k| \]

Hence for the cut \((S, \bar{S}) \subset \mathcal{E}(\mathcal{N}(G'))\) of \(\mathcal{N}(G')\), we have

\[ (S, \bar{S}) = F(S-S) \cup F(S-\bar{S}) \]

\[ F(S-S) = \{(u,v) : (u,v) \in \mathcal{E} K\} \]

\[ F(S-\bar{S}) = \{(u,v) : (u,v) \in \mathcal{E} K\} \]

Then

\[ c(K') = \sum_{e \in F(S-S)} c(e) = |k| \quad K' = (S, \bar{S}) \]

Since any flow \(f\) has

\[ \text{val}(f) \leq c(K') \]

we have

\[ \text{val}(f) \leq c(K') = |k| = \mathcal{M}_{\text{min}}(s, t) \]

Conversely suppose that we have a maximum flow \(f\). Then it is well-known that \(\mathcal{N}(G')\) has a minimum cut \(K'' = (S, \bar{S})\) such that

\[ s \in S \quad t \in \bar{S} \quad \forall e \in E \quad \Delta(K'') = F(S-\bar{S}) \]

\[ f(e) = \begin{cases} c(e) & \text{if } e \in F(S-S) \\ 0 & \text{if } e \in F(S-\bar{S}) \end{cases} \]

For each \(e = (u,v) \in F(S-S)\), there is \(e' = (v,u) \in F(S-S)\) and vice versa. Let

\[ K' = \{(u,v) : (u,v) \in F(S-S)\} \]

Then \(K'\) is an \(st\)-separator of \(G\) and
$$\sum_{e \in K} c(e) = c(K') = F(s-t).$$

Let

$$K = \{ e \in E(G) : c(e) \in K' \}.$$

Then $K$ is an $(s,t)$-separator of $G$ with

$$|K| = F(s,t).$$

We have

$$F(s,t) \geq M_{0}(s,t),$$

since

$$M_{0}(s,t) = N_{0}(s,t) \leq |K|.$$  

Q.E.D.

**COROLLARY 2.1.**

For any pair $u, v \in V(G)$ ($u \neq v$)

$$M_{0}(u, v) = F(u-v),$$

where we put $F(u-v) = 0$ if $M_{0}(u, v) = 0$.

3. **THE K-EDGE-AUGMENTATION PROBLEM**

The $k$-edge-connectivity augmentation problem for any fixed $k \geq 1$ is defined by:

"Given a graph $G = (V, E)$ with $|V| > 1$, determine a minimum set $Z$ of edges joining two vertices $i, j \in V(G)$ such that $i \cap j = \emptyset$ and $G + Z$ is $k$-edge-connected."

We can assume that $G$ has no loop and that any added edge joins distinct vertices of $V(G)$. Suppose that $k \geq 1$. Let $R_{k}(G)$ denote the minimum number of edges whose addition to $G$ results in a $k$-edge-connected graph. If $k = 1$ then the problem is easy to solve. Therefore we assume that
in this paper.

3.1. THE DEMAND AND THE EDGE-AUGMENTATION NUMBER OF A GRAPH

We give definitions of edge demands of, demands of, component demands of m-components as well as the demand of a graph.

Assume that \( ec(G) \leq m \leq k \). Let \( S \) denote a nonempty subset of \( V(G) \). The edge demand of \( S \) (of \( G \)), \( ED_k(S,G) \), is defined by

\[
ED_k(S,G) = \begin{cases} 0 & \text{if } S = V(G) \text{ or } d(S,G) \geq k \\ k - d(S,G) & \text{otherwise} \end{cases}
\]

Here we denote a t-edge-component of \( G \) by \( S(t) \) for \( t > 0 \). If \( S = S(m) \) then the demand of \( S \) (of \( G \)), \( D_k(S,G) \), is defined recursively by the following (i), (ii):

(i) If \( S = S(k) \) then

\[ D_k(S,G) = ED_k(S,G). \]

(ii) If \( S = S(m) \) with \( m < k \) then

\[
D_k(S,G) = \begin{cases} \max(ED_k(S,G), \sum_{S(m+1) \subseteq S} D_k(S(m+1),G)) & \text{if there is an } S(m+1) \subseteq S \\ D_k(S(m+1),G) & \text{if } S = S(m+1) \end{cases}
\]

where \( \sum_{S(m+1) \subseteq S} D_k(S(m+1),G) \) denotes the total sum of demands \( D_k(S(m+1),G) \) of those \((m+1)\)-components \( S(m+1) \subseteq S \) of \( G \). (We note, as is in Corollary 3.1 of [20-22], that \( S \) is the disjoint union of some \((m+1)\)-components of \( G \) if \( S \) is a critical \( m \)-component of \( G \).)

We generalize the definition of \( D_k(S,G) \) to that for a subset \( S \subseteq V(G) \). Suppose that \( S \subseteq V(G) \) and \( S \neq S(m) \) for any \( m \leq k \), and let
Then we define the component demand of $S$ (of $G$), $CD_k(S,G)$, by the following:

$$CD_k(S,G) = \begin{cases} \sum_{S(h(S)) \subseteq S} D_k(S(h(S)),G) & \text{if } h(S) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

For any $S \subseteq V(G)$, the demand of $S$ (of $G$), $D_k(S,G)$, is defined by using this notation as follows:

$$D_k(S,G) = \begin{cases} D_k(S(m),G) & \text{if } S = S(m) \text{ for some } m \leq k \\ \max(ED_k(S(m),G),CD_k(S,G)) & \text{otherwise.} \end{cases}$$

Let $D_k(G)$, called the demand of $G$, denote the value determined by the following procedures $(1) - (3)$:

1. Compute the demand of $S(k)$, $D_k(S(k),G)$, for every $k$-component $S(k)$ of $G$.

2. If $k > \text{ec}(G) + 1$ then, for each $m$ with $m = k-1, \ldots, \text{ec}(G)$ in this order, compute recursively the demand of every $m$-component $S(m)$ of $G$:

$$D_k(S(m),G) = \begin{cases} \max(ED_k(S(m),G),CD_k(S(m),G)) & \text{if there is } S(m+1) \subseteq S(m) \\ D_k(S(m+1),G) & \text{if } S(m) = S(m+1). \end{cases}$$

3. Let $D_k(G) = D_k(V(G),G)$.

Put

$$EA_k(G) = \lfloor D_k(G)/2 \rfloor.$$

We call $EA_k(G)$ the $k$-edge-connectivity-augmentation number (or simply, the $k$-augmentation number).
number of $G$.

3.2. THE CHARACTERIZATION OF $\text{EA}_k(G)$. 

We summarize our previous results given in [20-22].

PROPOSITION 3.1. [20-22]

For any fixed $k \geq 2$, $G=(V,E)$ is $k$-edge-connected if and only if $\text{EA}_k(G)=0$.

LEMMA 3.1. [20-22]

$R_k(G) \geq \text{EA}_k(G)$ for any fixed $k \geq 2$.

We give the definitions of the edge condition and $m$-augmenting sets, which are necessary to prove the converse of Lemma 3.1:

$R_k(G) \leq \text{EA}_k(G)$ for any fixed $k > 1$.

Clearly, it suffices to consider the case with $\text{ec}(G) < k$. For any vertex $v$ and each $m$ with $\text{ec}(G) \leq m \leq k$, $G$ has exactly one $m$-component that contains $v$. Let $S(v,m;G)$ denote the $m$-component of $G$ that contains $v$.

Whenever $\text{ec}(G) < k$ we choose from $V(G)$ distinct vertices $u_1, u_2$ satisfying the following conditions (1) - (3) called the edge condition (for $G$):

(1) $S(u_i,m;G)$ is an external $m$-pendant of $G$ for $i = 1, 2$ and for any $m$ with $\text{ec}(G)+1 \leq m \leq k$.

(2) $S(u_i,m'+1;G) \subseteq S(u_i,m';G)$ for $i = 1, 2$ and for any $m'$ with $\text{ec}(G)+1 \leq m' < k$.

(3) $S(u_1,\text{ec}(G)+1;G) \neq S(u_2,\text{ec}(G)+1;G)$.

We note that, by Proposition 3.3-3 of [20-22], we can find a pair of vertices $u_1, u_2$ satisfying the edge condition for $G$. 


Put

\[ G' = G + (u_1, u_2) \]

(We use this notation throughout this section.) Then

\[ M_{G'}(u_1, u_2) = M_G(u_1, u_2) + 1 = e(G) + 1 \]

An m-component \( S \) of \( G' \) that is not an m-component of \( G \) is referred to as an \( m \)-augmenting set of \( G \) (with respect to the edge \((u_1, u_2)\)). Any m-component of \( G' \) is either an m-component of \( G \) or an m-augmenting set of \( G \). Since the addition of the edge \((u_1, u_2)\) to \( G \) can increase the number of pairwise edge-disjoint paths by at most one, each m-augmenting set of \( G \) is identical to the disjoint union of at least two m-components included in an \((m-1)\)-component of \( G \). Clearly, any m-augmenting set of \( G \) is a critical m-component of \( G' \).

A pair of distinct vertices \( u_1, u_2 \in V(G) \) is said to be admissible (with respect to \( G \)) if the following (1) and (2) hold:

1. The pair \( u_1, u_2 \) satisfy the edge condition for \( G \).
2. If \( e(G) = k - 1 \) and \( P^k(G) \geq 4 \) then \( S(u_1, k, G') = S(u_2, k, G') \) is not a \( k \)-pendant of \( G' \).

**LEMMA 3.2. [20-22]**

Suppose that \( 0 \leq e(G) < k \). Then we can find an admissible pair \( u_1, u_2 \) with respect to \( G \) such that

\[ E_A(G) - E_A(G') = 1. \]

We have proved in [20-22] the following theorem by induction on \( k \)-augmentation numbers of graphs.
THEOREM 3.1. [20-22]

For any graph $G$ with $|V(G)| > 1$ and for any fixed $k \geq 2$,

$$ R_k(G) = EA_k(G). $$

3.3. THE DATA STRUCTURE

We describe data structures used in an algorithm for finding a minimum solution $Z$ of the problem. We denote

$$ n_v = |V(G)|, \quad n_e = |E(G)| $$

for a given graph $G$.

3.3.1. THE DATA STRUCTURE FOR A GRAPH $G$

We assume here that $G$, a multigraph, is given by means of adjacency lists for $G'$, a weighted simple graph: it consists of a list head $G_{\omega}$ and some $v$nodes representing vertices of $G$, as in the following declarations, where $N = n_v$ (Figure 1).

```plaintext
type
  pnode = T vnode;
  G = array [1..N] of pnode;
  vnode = record
    VNAM, VAL, WT, integer;
    ITR pnode;
  end;
```

Vertices of $V(G)$ are integers $1,...,N$ and $VNAM$ maintains the corresponding integer. $VAL$ will be used to maintain current flow values in a maximum flow algorithm. $WT$ is set equal to the multiplicity of the corresponding edge in $G$. The adjacency lists for $G'$ can be considered as those
for $N(G^3)$, and are used also in a maximum flow algorithm which compute $F(u \rightarrow v) = M_G(u,v)$ for a pair $u,v \in V(G)$ on the network $N(G^3)$, where we put $F(u \rightarrow v) = 0$ if $M_G(u,v) = 0$. For any $Z' \subset Z$, $G + Z'$ will be maintained as adjacency lists for $(G + Z')^3$.

3.3.2. THE DATA STRUCTURE FOR A COMPONENT TREE

We use the following logical data structure, which is referred to as the component tree for $G$.

Let $S(m)$ ($S(m)'$, respectively) denote any $m$-component of $G$ (of $G'$).

The component tree $CT(G)$ is an undirected tree defined by the following (1) and (2):

1. $V(CT(G))$ consists of those vertices $u_G, u_{S(m)}, u_v$ representing, respectively, $V(G)$, each $S(m)$ $(1 \leq m \leq k)$, each vertex $v \in V(G)$. Vertices $u_G, u_{S(m)}, u_v$ are referred to as the root, an $m$-component vertex, a leaf, respectively.

2. For any distinct vertices $u, u' \in V(CT(G))$, there is an edge $(u, u') \in E(CT(G))$ if and only if one of the following (i)-(iii) holds:

   i) $u$ is the root and $u' = u_{S(1)}$.

   ii) $u = u_{S(i)}$ and $u' = u_{S(i+1)}$, such that $S(i+1) \subseteq S(i), 1 \leq i \leq k$.

   iii) $u = u_{S(k)}$ and $u' = u_v$, such that $v \in S(k)$.

If $k = 3$ and $G$ is as shown in Figure 2 then $CT(G)$ will be as in Figure 3, where $V_i, T_i, S_i$ are, respectively, 1-components, 2-components, 3-components of $G$.

We briefly describe the actual data structure of a component tree. The component tree $CT(G)$ consists of four kinds of data types, $linknode$, $LEVEL$, $CH$, and $cnode$ as in the following declarations (Figure 4):

```plaintext
type
    linknode = i cnode;
    LEVEL = array [1..k] of linknode;
    CH = array [1..N] of linknode;
```
cnode = record
  NAME: integer;
  NEXT, LLINK, RLINK, SON, TOP, F: linknode;
  DEG: integer;
end;

For each m, 0 ≤ m ≤ k, LEVEL[m] is the pointer to the first cnode of the list maintaining m-component vertices by means of NEXT. This list is called the m-level. For each i, 1 ≤ i ≤ N, CH[i] is the pointer to the cnode corresponding to the leaf i.

NAME has an integer greater than N if the cnode represents the root or an m-component vertex, and has one between 1 and N otherwise. Suppose that an m-component S is the disjoint union of (m+1)-components S1, ..., Sn. t ≥ 1. ec(G) ≤ m ≤ k, where we assume that a (k+1)-component means a leaf. Let R(S1), R(Sn), denote the cnodes representing S, Sn, respectively. Then R(S1), ..., R(Sn) are maintained as a doubly linked list by means of LLINK and RLINK. They are called sons of R(S). R(S1).SON is pointed to the first son of R(S), and R(Sn).F is pointed to R(S), i = 1, ..., t. R(S1).DEG is provided for dk(S,G) if R(S) represents a non-leaf. R(S1).TOP is set equal to R(S1).SON initially, and is used as the pointer to the first leaf in the sons of R(S) during the procedure is processing R(S) in the construction of CT(G). If R(S1).TOP = nil then the partitioning R(S) into R(S1), ..., R(Sn) has been finished. Figure 5 shows a part of the actual data structure corresponding CT(G) of Figure 2.

In the procedure which constructs CT(G), we compute M0(u,v), u,v ∈ V(G), by using a maximum flow algorithm as a subroutine with a modification such that it will terminate and return the value k whenever the current flow value exceeds k.

Let o(G) denote the time complexity of a maximum flow algorithm with such a modification to compute f(u,v,G) = M0(u,v). Then CT(G) can be constructed in O(n2o(G)) time. For example it is O(n2Tn2n2) if we use the Dinic's algorithm and is O(n2Tn2n2) if we use the algorithm pro-
posed by Malhotra, Kumar and Maheshwari (MKM, for short), where $T = \min\{k, n_v\}$. (For the
detail, see [4, 13].)

(In [9] it is shown that we can determine $F(u-v; G)(= M_G(u,v))$ for all pairs $u,v \in V(G)$ by
using a maximum flow algorithm only $O(n_v)$ times, instead of $O(n_v^2)$ times as described above. In
this paper, however, we do not take advantage of this result simply for ease of implementation.
The time complexity for constructing $CT(G)$ would be $O(n_v n_v^3)$ in Dinic's case and $O(n_v n_v^2)$ in
MKM's case.)

3.3.3. COMPUTATION ON $CT(G)$

Suppose that we have $CT(G)$. We describe the time complexity of the following computation

(1) on $CT(G)$

(1) $d_k(S(m), G)$, $D_k(G)$, $e_c(G)$ and $P^m(G)$.

The computation of the following (i) - (iv) can be done in $O(k(n_v + n_e))$ time:

(i) The degree $d_k(S(m), G)$ for every m-component $S(m)$ of $G$, $m = 1, \ldots, k$.

(ii) $D_k(G)$ (the demand of $G$).

(iii) $e_c(G)$ (the edge-connectivity of $G$).

(iv) $P^m(G)$ (the number of m-pendants of $G$) with $m = e_c(G) + 1$.

(2) Finding a pair $u_1, u_2$ satisfying the edge condition.

Let $S$ be an external m-pendant of $G$. Proposition 3.3 of [20-22] shows that $S$ includes at least
one external $(m+1)$-pendant of $G$. First, suppose that there are distinct $(m+1)$-components
$S_1, S_2 \subseteq S$ of $G$. Since

$M_G(v_i, v_j) = m$ for $v_i, v_j \in S_i$, $i = 1, 2$.

we have

$d_k(S_i, G) = k + |S_i, G| \geq m$, $i = 1, 2$. 

If $d_k(S_i, G) = m$ for either $i = 1$ or $i = 2$ then $K(S_i, G)$ is a $(S_1, S_2)$-cut of $G$, meaning that $S_i$ is an external $(m+1)$-pendant of $G$. If

$$d_k(S_i, G) > m$$

then $S_i$ is not an $(m+1)$-pendant of $G$. Suppose that $S$ is also an $(m+1)$-component of $G$. Then, clearly, $S$ is an external $(m+1)$-pendant of $G$.

Therefore if we specify a node of representing an external $m$-pendant $S(m)$ of $G$, we can find in $O(kn_v)$ time the node representing a leaf $u$ such that

$$u \in S(k) \subseteq \cdots \subseteq S(m+1) \subseteq S(m),$$

where each $S(t), m+1 \leq t \leq k$, is an external $t$-pendant of $G$. Hence we can find a pair $u_1, u_2$ satisfying the edge condition for $G$ in $O(kn_v)$ time.

(3) Constructing adjacency lists for $(G')^P$.

We can construct adjacency lists for $(G')^P$ in $O(n_v)$ time where $G' = G + (u_1, u_2)$.

(4) Finding an admissible pair.

Suppose that we have a pair $u_1, u_2$ satisfying the edge condition for $G$ and adjacency lists for $(G')^P$, where $G' = G + (u_1, u_2)$. It suffices to consider the case where $k = ec(G) + 1$ and $P(G') \geq 4$.

We choose a vertex $v_j$ from each $k$-component $S_i (\neq S(u_i, k, G); i=1,2)$ of $G$ and compute $M_{G'}(u_i, v_j)$. If $M_{G'}(u_i, v_j) \geq k$ then $DEG$ of every son of the node representing $S(v_j, k, G)$ is set to 1, which indicates $S(v_j, k, G) \subseteq S(u_1, k, G')$. Then we compute $d_k(S(u_1, k, G'), G')$ in $O(|E(G')|)$ time by counting the total weight of edges $(u, v) \in E(G') \cap K(S(u_1, k, G'), G')$.

If $d_k(S(u_1, k, G'), G') < k$ then $S(u_1, k, G')$ is a $k$-pendant of $G'$ and we choose another vertex $u_2'$ from a $k$-pendant not included in $S(u_1, k, G')$. This choice is done in $O(n_v)$ time. Lemma 3.2 of [20-22] shows that the pair $u_1, u_2'$ is an admissible pair. Thus we can find an admissible pair with respect to $G$ in $O(n_v \alpha(G') + |E(G)|)$ time if we compute $M_{G'}(u_1, v_j)$ by means of a maximum flow.
algorithm

(5) An algorithm proposed in [20-22].

The proof of Theorem 3.1 shows an algorithm for finding a minimum solution of the problem.

First construct the initial data structure and compute the initial data (i)-(iv):

(i) \( d_k(S(m), G) \) for every \( S(m), m = 1 \ldots k \).

(ii) \( D_k(G) \).

(iii) \( ec(G) \).

(iv) \( P^m(G) \) with \( m = ec(G) + 1 \).

Then repeat the following (a), (b) by \( EA_k(G) \) times.

(a) Finding an admissible pain \( u_1, u_2 \) with respect to \( G \) and the construction of \( CT(G') \),

\[ G' = G + (u_1, u_2) \]

(b) The computation of the following (i)-(iii) for \( G' \).

(i) \( d_k(S(m)', G') \) for every \( m \)-component \( S(m)' \) of \( G' \), \( m = 1 \ldots k \).

(ii) \( ec(G') \).

(iii) \( P^m(G') \) with \( m = ec(G') + 1 \).

Suppose that we use Dinic's maximum flow algorithm. Then the initial data structure and the initial data (i)-(iv) can be obtained in \( O(kn_0^2 n_v) \) time. (a), (b) for \( G' \) can be done in \( O(kn_0^2 |E(G')|) \) time. Since we have

\[ |Z| = FA_k(G) \leq kn_v \]

and

\[ |G| + \sum |G_i| + \sum |G_i + Z_i| = n_v + n_v + 1 + \ldots + (n_v + FA_k(G)) \leq (kn_v + i)(n_v + kn_v) \]

for any solution \( Z \), the total time is
If we use MAXW's maximum flow algorithm then the total time is

\[ O(k^2n^d(kn + n_d)). \]

The most time-consuming part of this algorithm is constructing a component tree, which is repeated each time a new edge is added. In the next section we will propose an improved algorithm in which constructing a component tree is repeated at most \( k-1 \) times instead of \( EA_k(G) \leq kn \) times mentioned above.

4. AN IMPROVEMENT OF THE ALGORITHM

We consider an improvement of the algorithm mentioned in 3.3.3-(5). The idea of the improvement is as follows:

The previous algorithm reconstructs the component tree each time we find only one edge to be added. We can expect a more efficient algorithm if there is an easy way to find as many edges to be added as possible before reconstructing component trees: it may reduce both time spent to find such edges and the number of times of reconstructing component trees. We will describe more precisely.

Suppose that

\[ ec(G)<k, \quad k \geq 2. \]

and let \( Z \) be a solution obtained by the algorithm mentioned in 3.3.3-(5). Since each edge of \( Z \) joins a pair of vertices satisfying the edge condition, \( Z \) has a partition

\[ Z = Z(ec(G) + 1) \cup \cdots \cup Z(k), \]

such that

\[ Z(i) \cap Z(j) = \emptyset (i \neq j), \quad Z(1) = \emptyset (ec(G) + 1 \leq i \leq k) \]

and

\[ ec(G_{i+1}) = ec(G_i) + 1 = i. \]
\[ ec(G, -e) = ec(G, -e) \] for \( \forall e \in Z(i) \).

where

\[ G_{ec} = G, G_i = G_{i-1} + Z(i), i = ec(G) + 1, \ldots, k. \]

We will consider two procedures, connect in 4.1 and add in 4.2: the first one determines \( Z(1) \) for \( G \) with \( ec(G) = 0 \), and the second one does \( Z(m+1) \), \( m = ec(G) \), for \( G \) with \( ec(G) > 0 \). The procedure \( add \) is used repeatedly to determine \( Z(m+1), \ldots, Z(k) \) in this order such that reconstructing the component tree will be done only between \( Z(j) \) and \( Z(j+1) \) for each \( j, ec(G) + 1 \leq j \leq k - 1 \). In 4.3 we describe the outline of an improved algorithm and estimate its time complexity.

**REMARK 4.1.**

(1) If \( ec(G) = 0 \) then \( |Z(1)| \geq P'(G) - 1 \) (Note that \( P'(G) \) is equal to the number of components of \( G \)).

(2) If \( ec(G) > 0 \) then \( |Z(i)| \geq \left[ P'(G_{i-1})/2 \right] \) for each \( i, ec(G) + 1 \leq i \leq k \).

Let \( m \) denote any fixed integer such that

\[ ec(G) \leq m \leq k - 1. \]

and put

\[ H = G_{m+1}/\pi(m+1). \]

where \( \pi(m+1) \) denotes the partition of \( V(G_{m+1}) \) into \( (m+1) \)-components of \( G_{m+1} \). Let \( \iota_{m+1} \) denote the mapping of \( G_m \) induced by \( \pi(m+1) \). We have

\[ d(u, H) \geq m, \quad M_H(u, H) = m \]

for any \( u \in V(H) \). A vertex \( u \) with \( d(u, H) = m \) is called an \( (m+1) \)-pendant or a pendant of \( H \), and \( d^{m+1}(H) \geq 2 \).
REMARK 4.2.

If \( u \) is a pendant of \( H \) then \( K\{u, H\} \) is a (\( u, H \))-cut of \( H \) for any \( u \in V(H) \) and \( \xi_{m+1}(u) \in \pi(m+1) \) is an external (\( m+1 \))-pendant of \( G_m \). Any \( (m+1) \)-pendant \( S \) of \( G_m \) is external and \( \xi_{m-1}(S) \in V(H) \) is a pendant of \( H \).

Let

\[
Y = \{y_1, \ldots, y_q\} \quad (q = p_{m+1}(H))
\]
denote the set of all pendants of \( H \), and put

\[
r = \lfloor q/2 \rfloor.
\]

Put

\[
V(e) = \{u, v\} \quad \text{for an edge } e=(u, v),
\]

and

\[
V(E) = \bigcup_{e \in E} V(e) \quad \text{for a set } E \text{ of edges}
\]

Let \( E \) be a set of edges. We call \( E \) an attachment for \( H \) if and only if the following (1)-(4) hold:

1. \( V(E) \subseteq Y \).
2. \( E \cap E(H) = \emptyset \).
3. \( V(e) = V(e') \) for all \( e, e' \in E \), \( e \neq e' \).
4. There is at most one pair \( e, e' \in E \) such that \( |V(e) \cap V(e')| = 1 \).

4.1. THE PROCEDURE \textit{CONNECT}(G, e(G)=0)

We consider the procedure \textit{connect}, which determines \( Z \) and constructs an adjacency list \( \mathcal{L} \) for \((G^* + Z, F)\) with \( e(G^*) = 0 \). Let

\[
m = e(G^*) = 0.
\]
and consider

\[ H = G / \pi(1) \]

Then

\[ V(H) = \mathbb{N} q \geq 2 \quad E(H) = \phi \]

Define a set of edges

\[ Z_H = \{ e_i = (y_i, y_{i+1}) \mid i = 1, \ldots, q-1 \} \]

where

\[ Z_H \cap E(H) = \phi \]

Clearly:

\[ e_0(H + Z_H) = 1, \quad e_0(H + Z_H') = 0 \text{ for } Z_H' \subseteq Z_H \]

Now we will show how to choose a pair of vertices satisfying the edge condition. For each \( i, 1 \leq i \leq q \), put

\[ S_i = \xi_1^{-1}(y_i) \in \pi(1) \]

There are two cases concerning each \( S_i \):

1. \( S_i \) is a \( k \)-component of \( G \)
2. \( S_i \) is a critical \( t \)-component of \( G \), where \( 1 \leq t < k \)

Proposition 3 of [20-22] shows that, in both (1) and (2), there is a sequence

\[ S(k) \subseteq S \subseteq S(t+1) \subseteq S \]

- in each \( i = 1, 2 \), where \( S(t) \) denotes an external \( t \)-pendant of \( G(t) = t+1, \ldots, k \).

\[ S = S \quad \text{ if } S \text{ is a } k \text{-component.} \]

\[ S = t \times S_{t+1} = \phi \quad \text{if } S \text{ is a critical } t \text{-component } t < k \]
Choose vertices $u_{xj}$, $x=1,...,q-1$; $j=1,2,$ as follows:

(i) If $S_x$ ($S_{x+1}$, respectively) satisfies (1) then 
   $u_{x1} \in S_x, u_{x2} \in S_{x+1}$.
(ii) If $S_x$ ($S_{x+1}$) satisfies (2) then 
   $u_{x1} \in S_{x1}(k), u_{x2} \in S_{x+12}(k))$.

Put 

$$e_x = (u_{x1}, u_{x2}), x=1,...,q-1.$$ 

where 

$$e_x \not\in E(G).$$

We consider the case where $x=1$:

$$G', G = G + e_1.$$ 

Clearly, the pair $u_{11}, u_{12}$ satisfies the edge condition for $G$. We will show that if $q \geq 3$ then the pair $u_{21}, u_{22}$ satisfies the edge condition for $G'$. If this is shown then the discussion for $x=1$ can be applied to the general case:

$$G + \{e_1, \ldots, e_{x-1}\}, G + \{e_1, \ldots, e_{x-1}, e_x\}, x \geq 3.$$ 

Proposition 3.6 and Lemma 3.4 show the following (a)-(c):

(a) $S(u_{11}, 1; G') = S(u_{12}, 1; G') = S_1 \cup S_2$, and it is the only 1-augmenting set of $G$ with respect to $e_1$.

(b) Any $m'$-component $S'$ of $G'$ is also an $m'$-component of $G$ if $m' > 1$ or if $S' \neq S(u_{11}, 1; G')$ with $m' = 1$.

(c) For any $m'$-component $S$ of $G$, $ec(G) \leq m' \leq k$. 

$$D_k(S, G) = 2 \text{ if } V(e_1) \subset S \text{ and either } P^{m'}(G) \neq 3 \text{ or } m' \neq k.$$ 

$$D_k(S, G) = 1 \text{ if } |V(e_1) \cap S| = 1, \text{ or if } V(e_1) \subset S \text{ and } P^{m'}(G) = 3 \text{ with } m' = k.$$ 

0 otherwise.
Hence we have (d), (e):

(d) If \( S_2 = S_2(k) \) then \( S_2 \) is also a \( k \)-component of \( G' \) with \( D_k(S_2, G') = D_k(S_2, G) - 1 \) (\( = k - 1 \))

(e) If \( S_2 \) is a critical \( t \)-component of \( G, 1 \leq t < k \), then (i), (ii) hold.

(i) \( S_{2i}(t') \) is an external \( t' \)-pendant of \( G' \) with \( D_k(S_{2i}(t'); G') = D_k(S_{2i}(t'), G) \) for each \( t' \), \( t + 1 \leq t' \leq k \).

(ii) If \( t \geq 2 \) then \( S_2 \) is an external \( t'' \)-pendant of \( G' \) for each \( t'', 2 \leq t'' \leq t \).

It follows that there is a sequence of external pendants of \( G' \)

\[ S_{21}(k) \subseteq \cdots \subseteq S_{2i}(t+1) \subseteq S_1 \cup S_2 \]

such that if \( S_2 \) is a critical \( t \)-component of \( G \) with \( 2 \leq t < k \) then \( S_2 \) is an external \( t'' \)-pendant of \( G' \) and

\[ S_{2i}(t+1) \subseteq S_2 \subseteq S_1 \cup S_2 \]

for each \( t'', 2 \leq t'' \leq t \). Thus the pair \( u_{21}, u_{22} \) satisfies the edge condition for \( G' \), and we obtain the following proposition.

**PROPOSITION 4.1.**

Let

\[ E_1 = \{ e_1, \ldots, e_{q-1} \}, \quad e_i = (u_{i1}, u_{i2}), 1 \leq i \leq q-1. \]

Then we can set

\[ Z(1) = E_1. \]

The procedure `connect` repeats two procedures: finding a pair \( u_{i1}, u_{i2} \) satisfying the edge condition for the current graph \( G \) in \( O(kn_v) \) time and then constructing adjacency lists for \( (G + (u_{i1}, u_{i2})) \) in \( O(n_v) \) time. Thus the procedure `connect` finds \( Z(1) \) and constructs adjacency lists for \( (G + Z(1)) \) in \( O(kn_v^2) \) time.

\( CT(G + E_1) \) is easily obtained from \( CT(G) \): coalescing all \( 1 \)-component vertices of \( CT(G) \) into one \( 1 \)-component vertex, merging corresponding sons lists into one list, and changing degrees of
corresponding component vertices in $O(kn_c)$ time (by means of (c)).

4.2. THE PROCEDURE F/N/D ($ec(G) > 0$)

We consider the procedure $fnd$, which determines $Z(m-1)$ and constructs adjacency lists for $(G + Z(m+1))^p$ with $m = ec(G) > 0$. In 4.2.1 and 4.2.2, we consider the procedure $h/fnd$, which determines a minimum attachment $Z_H$ for $H = G/\pi(m+1)$ such that $ec(H + Z_H) = m + 1$. In 4.2.3, we consider how to determine an edge $(u', v') \in Z(m+1)$ from each edge $(u, v) \in Z_H$. In 4.2.4 we describe the procedure $fnd$, a modified version of the procedure $h/fnd$.

Let $E$ be any attachment for $H$. For each edge $e = (u, v) \in E$, $H + E$ has a new $(m+1)$-component, denoted by $A(e, H + E)$, containing $V(e)$, since

$$M_{H+E}(u, v) = M_H(u, v) + 1 = m + 1 \leq M_H(u, v).$$

$A(e, H + E)$ is referred to as the $(m+1)$-augmenting set for $e$ (with respect to $E$).

First we show the following proposition for an attachment $E$ for $H$.

**PROPOSITION 4.2.**

Suppose that an attachment $F$ for $H$ satisfies the following (1), (2):

1. $V(F) = Y$
2. $H + F$ has an $(m+1)$-component $A$ such that $V(E) \subseteq A$.

Then

$$A = V(H).$$

**PROOF.**

Assume that

$$A \subseteq V(H).$$

Corollary 3.1 of [20-22] shows that $H - I$ has an $(m+1)$-component $S$ such that

$$S \subseteq V(H) - A.$$
Proposition 3.1 of [20-22] shows that $H + E$ has an $(A.S)$-cut $K$ with $|K| = m$. Proposition 3.3 of [20-22] shows that $B(S.K,H+E)$ contains an $(m+1)$-pendant $S'$ of $H + E$. We can show, by using Theorem 3.1 of [20-22], $S'$ is also an $(m-1)$-pendant of $H$. Hence $S' = \{v\}$ for some vertex $v \in V(H) - A$. It follows that

$$(V(H) - A) \cap Y \neq \emptyset.$$ 

a contradiction.

Q.E.D.

We will show, in 4.2.1 and 4.2.2, that we can find an attachment $E$. 

$$F = \{e_1, \ldots, e_r\} \quad (r = \lfloor q/2 \rfloor),$$ 

satisfying the following (i) - (iii):

(i) For each $i$, $1 \leq i \leq r-1$,

$$A(e_i, H_i) \cap A(e_{i+1}, H_{i+1}) \neq \emptyset \quad \text{if } r \geq 2$$

where

$$H_0 = H, H_r = H_{r-1} + e_r, \quad j = 1, \ldots, r$$

(ii) $V(E) = Y$.

(iii) $V(e_i) \cap V(e_j) \neq \emptyset$ if and only if $q$ is odd, $i = r-1$ and $j = r$.

If such $F$ exists then $A(e_r, H_r)$ is an $(m-1)$-component of $H + E$ and

$$V(H) \subseteq A(e_r, H_r).$$

Proposition 4.2 shows that

$$A(e_r, H_r) = V(H)$$
4.2.1. THE CASE WHERE $q = 2$

If $q = 2$ then let

$$Z_H = \{e_1 = (y_1, y_2) \mid e_1 \in E(H)\}.$$ 

Clearly $Z_H$ is an attachment for $H$. It is easy to see that $Z_H$ and $Z(e_1, H_1)$ satisfy Proposition 4.2-(1), (2), showing that

$$A(e_1, H_1) = V(H).$$

Thus we obtain the following.

PROPOSITION 4.3.

If $q = 2$ then

$$A(e_1, H_1) = V(H)$$

for $e_1 = (y_1, y_2) \in E(H)$.

4.2.2. THE CASE WHERE $q \geq 3$

Let $E$ be any attachment for $H$ such that there are vertices

$$v_{ij} \in V - V(E), \ i, j = 1, 2,$$

where $v_{11}, v_{12}, v_{21}$ are pairwise distinct, and if $v_{22}$ is equal to one of the rest then we assume, without loss of generality, that

$$v_{22} = v_{12}.$$

Put

$$L = H + E, \ e = (v_{11}, v_{12}), \ e' = (v_{21}, v_{22}),$$

where

$$e, e' \in E(L).$$

Put

$$A(e) = A(e, L + \{e, e'\}), \ A(e') = A(e', L + \{e, e'\}),$$

$$\lambda e = A(e, L + e), \ \lambda e' = A(e', L + e').$$

Clearly...
\(A(e) \subseteq A(e').\) 

\(A(e) \cup A(e') \subseteq A(e')\) if \(A(e) \cap A(e') \neq \emptyset.\)

In the following we first consider the case I where

\(A(e) \cap A(e') = \emptyset,\)

i.e., \(L + \{e,e'\}\) has an \((A(e),A(e'))-cut\) consisting of \(m\) edges. Then we proceed to another case II where

\(A(e) \cap A(e') \neq \emptyset.\)

**CASE I.** \(A(e) \cap A(e') = \emptyset.\) (Then \(V(e) \cap V(e') = \emptyset.\))

Let \(K\) be any fixed \((A(e),A(e'))-cut\) of \(L + \{e,e'\}\), and let \(B_i, i = 1, 2\), denote the \(K\)-block of \(L + \{e,e'\}\) such that

\[V(e) \subseteq A(e) \subseteq A(B_1), \quad V(e') \subseteq A(e') \subseteq V(B_2).\]

(In the following discussion, we write \(B_i\) instead of \(V(B_i)\) for simplicity.) We note that \(K\) is also an \((A(e),A(e'))-cut\) of \(L\) of \(L + e\) or of \(L + e'\).

Let \(f, f'\) be two edges defined by either

\[f = (v_{11}, v_{21}), \quad f' = (v_{12}, v_{22})\]

or

\[f = (v_{11}, v_{22}), \quad f' = (v_{12}, v_{21})\]

where

\[\{f,f'\} \cap E(L) = \emptyset.\]

Let \(h\) be any fixed vertex from \(\{v_{21}, v_{22}\}\) and put

\[L = (L + e) < h, B_2 >.\]

We note that

\[V(1) = B_1 \cup \{h\}, \quad 1 = (1 + e,e') < h, B_2 >.\] (See Fig. 6.)
PROPOSITION 4.4.

Suppose that $A(e) \cap A(e') = \phi$. Then

$$M_{L'}(u,u') = M_l(u,u')$$

for any $u,u' \in V(1)$, where $L' = L + e$ or $L' = L + \{e,e'\}$.

PROOF.

Since

$$M_{L+e}(v_{11},v_{21}) = M_{L+e'}(v_{11},v_{21}) = m = |K|,$$

Theorem 3.1 of [20-22] shows that

$$M_{L'}(u,u') = M_l(u,u').$$

Q.E.D.

PROPOSITION 4.5.

Suppose that $A(e) \cap A(e') = \phi$. Then

$$M_{L+e}(v,v') \leq M_{L+e'}(v,v')$$

for any $v,v' \in B_1$, where $L' = L + e$ or $L' = L + \{e,e'\}$.

PROOF.

It suffices to say that

$$M_{L+e}(v,v') \leq M_{L+e'}(v,v').$$

Since 1-K has just two components whose sets of vertices are $B_1$ and $B_2$, 1-K has a $(v_{11},v_{12})$-path $Q_j$ such that

$$E(Q_j \cap K) = \phi, \quad j = 1, 2.$$

Hence $L + \{f, f'\}$ has a circuit $C$ such that

$$E(C) = E(Q_1) \cup E(Q_2) \cup \{f, f'\}.$$

Let $P_j$ denote the $(v_{11},v_{12})$-subpath of $C$ defined by
\[ E(P_i) = E(C) - E(Q_1). \]

Let \( P_1, \ldots, P_n \) denote any fixed set of \((v,v')\)-paths of \( L + e \), where \( t = M_{L+e}(v,v') \). If none of them passes through \( e \) then all of them are paths of \( L + \{f,f'\} \) and the proposition follows. Suppose that \( e \in E(P_i) \). Let \( P_{ij} \), \( j = 1, 2 \), denote the two subpaths of \( P_i \) except \( e \). Then we may have

\[ E(P_{11}) = \phi \quad \text{or} \quad E(P_{12}) = \phi. \]

If none of them passes through \( e \) then all of them are paths of \( L + \{f,f'\} \) and the proposition follows. Suppose that \( e \in E(P_i) \). Let \( P_{ij} \), \( j = 1, 2 \), denote the two subpaths of \( P_i \) except \( e \). Then we may have

\[ E(P_{11}) = \phi \quad \text{or} \quad E(P_{12}) = \phi. \]

That is,

\[ M_{L+e}(v,v') \leq M_{L+\{f,f'\}}(v,v'). \]

Q.E.D.

**COROLLARY 4.1.**

Suppose that \( A(e) \cap A(e') = \phi \). Then \( L + \{f,f'\} \) has the \((m+1)\)-component \( A \) such that \( A \subseteq A(e) \subseteq A \).

Put

\[ f_1 = (v_{11},v_{21}), \quad f_3 = (v_{11},v_{22}), \quad f_2 = (v_{12},v_{22}), \quad f_4 = (v_{12},v_{21}). \]

Then

\[ A(f_i) = \begin{cases} A(f_i,L + \{f_1,f_2\}) & \text{if } 1 \leq i \leq 2, \\ A(f_i,L + \{f_3,f_4\}) & \text{if } 3 \leq i \leq 4. \end{cases} \]

**PROPOSITION 4.6.**

Suppose that we have

\[ A(e) \cap A(e') = \phi \quad \text{and} \quad A(f_1) \cap A(f_2) = \phi. \]

Then
PROOF.

$L + \{f_1, f_2\}$ has an $(A(f_1), A(f_2))$-cut consisting of $m$ edges. Let $K'$ denote any fixed $(A(f_1), A(f_2))$-cut of $L + \{f_1, f_2\}$. Then $K' \neq K$. $K'$ is also an $(A(f_1), A(f_2))$-cut of $L$ of $L + f_1$ or of $L + f_2$. Let $B_i'$ denote the $K'$-block of $L + \{f_1, f_2\}$ containing $V(f_i)$, $i = 1, 2$. We have

$$v_{i,j} \in B_i \cap B_j', \ i, j = 1, 2.$$

Suppose that

$$V(K) \cap B_1 = \{v\}.$$ 

Then any $(v_{1,1}, v_{2,2})$-path, $j=1,2$, of $L$ and of $L + \{e, e'\}$ passes through $v$. Each of $m$ edge-disjoint $(v_{1,1}, v_{2,2})$-paths is decomposed into two subpaths: the $(v_{1,1}, v)$-subpath and the $(v, v_{2,2})$-subpath. Showing that

$$M_{1+i(f_1f_2), (v_{i,j}, v)} = m + 1, \ i, j = 1, 2.$$

That is,

$$v \in A(f_1) \cap A(f_2),$$

a contradiction. Similarly we can show that

$$|V(K) \cap B_i| \geq 2, \ i = 1, 2, \ |V(K') \cap B_i'| \geq 2, \ j = 1, 2.$$

Let

$$K = \{e_i: 1 \leq i \leq m\}, \ K' = \{e'_i: 1 \leq i \leq m\},$$

and let $P_{31}, \ldots, P_{3m}$ denote any set of $m$ edge-disjoint $(v_{11}, v_{22})$-paths of $L$. Then

$$|E(P_{3i}) \cap K| = |E(P_{3i}) \cap K'| = 1, \ i = 1, \ldots, m.$$ 

If we assume that there is $e \in K \cup K'$ such that

$$V(e) \cap (B_i \cap B_i') \neq \phi, \ V(e) \cap (B_1 \cap B_2') \neq \phi$$

then some $P_{3i}$ passes through $e$, showing a contradiction that

$$|E(P_{3i}) \cap K| \geq 2 \ or \ |E(P_{3i}) \cap K'| \geq 2.$$ 

Hence no such $e \in K \cup K'$ exists. Let $P_{41}, \ldots, P_{4m}$ denote any set of $m$ edge-disjoint $(v_{21}, v_{12})$-paths of $L$. Then we can similarly show that there is no $e \in K \cup K'$ such that
For each $i$, $i = 1, 2$, put

$$k_i = \{e_j \in K : V(e_j) \subseteq B_i^{'\prime}\}. \quad k_i' = \{e_j' \in K^{'\prime} : V(e_j') \subseteq B_i\}.$$

Clearly,

$$K = K_1 \cup K_2. \quad K' = K_1' \cup K_2'. \quad K_1 \cap K_2 = K_2' \cap K_2' = \emptyset.$$

We also have

$$|K_i| = |K_2| = |K_i'| = |K_2'|.$$

showing that $m$ is even. Put

$$x = m/2.$$

For each $P_{3i}$, $i = 1, \ldots, m$, put

$$P_{3i}^{1i} = P_{3i}[B_1 \cap B_i^{'\prime}]. \quad i = 1, 2.$$

Similarly, for each $P_{4i}$, $i = 1, \ldots, m$, put

$$P_{4i}^{1i} = P_{4i}[B_2 \cap B_i^{'\prime}]. \quad P_{4i}^{2i} = P_{4i}[B_1 \cap B_2^{'\prime}].$$

We note that these $4m$ subpaths are pairwise edge-disjoint. It follows that $L$ has pairwise edge-disjoint $(v_1, v_2)$-paths $Q_i$, $i = 1, \ldots, m$, defined by these $4m$ subpaths and $K \cup K'$. (See Figure 7).

Let $Q_{m1}$, $Q_{m2}$, respectively denote the $(v_11, v_12)$-subpath (the $(v_22, v_23)$-subpath) of $Q_m$, and let $R_{m1}$, $R_{m2}$, respectively denote the $(v_11, v_21)$-path of $L + \{f_3, f_4\}$ defined by joining

$$Q_{m1}, f_4 \quad (Q_{m2}, f_3).$$

The paths $Q_1, \ldots, Q_m$, $Q_{m1}, Q_{m2}$ are pairwise edge-disjoint $(v_11, v_21)$-paths of $L + \{f_3, f_4\}$, showing that

$$\Lambda(f_3) \cap \Lambda(f_4) \neq \emptyset.$$

Q.E.D.

The next corollary follows from Corollary 4.1 and Proposition 4.6.
COROLLARY 4.2.

Suppose that \( A(e) \cap A(e') = \emptyset \). Then

\[
\begin{align*}
A(e) & \subseteq A(e) \subseteq A(f_1) = A(f_2) \quad \text{if } A(f_1) \cap A(f_2) \neq \emptyset, \\
A(e) & \subseteq A(e) \subseteq A(f_3) = A(f_4) \quad \text{otherwise.}
\end{align*}
\]

CASE II. \( A(e) \cap A(e') \neq \emptyset \).

Put

\[
e = (v, w), \quad e' = (v', w'), \quad L' = L + e.
\]

Suppose that there are distinct vertices

\[
v'', w'' \in Y - (V(E) \cup V(e) \cup V(e'))
\]

such that

\[
A(e', L' + \{e, e''\}) \cap A(e'', L' + \{e', e''\}) = \emptyset,
\]

where

\[
e'' = (v'', w'') \in E(L).
\]

Corollary 4.2 shows that we can find a pair of edges \( f, f' \) such that

\[
A(f, L' + \{f, f'\}) \cap A(f', L' + \{f, f'\}) = \emptyset,
\]

where

\[
V(f) \cup V(f') = V(e') \cup V(e'').
\]

\[
V(f) \cap V(f') = \emptyset.
\]

We assume, without loss of generality, that

\[
f = (v'', w''), \quad f' = (v'', w').
\]

PROPOSITION 4.7.

If

\[
A(e, L') \cap A\{L' + 1\} = \emptyset
\]

then
PROOF.

First we note that

\[ A(e,L') \cap A(f,L + f') \neq \emptyset. \]

\[ A(e,L') \subseteq A(e,e' + e') \subseteq A(f,f' + f') = A(f',f' + f') \] (by Corollary 42).

\[ A(e,L') \subseteq A(e,e + f) \subset A(f,f + f) \]

\[ A(e,L') \subseteq A(e,e' + f) \subset A(f,f + f) \]

\[ A(e,L') \subseteq A(e,e' + f) \subset A(f,f + f) \]

\[ L' + f \] has an \((A(e,L' + f), A(f,f + f))\)-cut \(K\) with \(|K| = m\). Let \(B_e, B_f\) denote the \(K\)-blocks of \(L' + f\) such that

\[ A(e,L') \subseteq B_e, A(f,f + f) \subset B_f. \]

We can assume, without loss of generality, that

\[ w' \in B_e, v'' \in B_f. \] (See Fig. 8).

Put

\[ k_{e'} = k \cup \{e'\}, k_{f'} = k \cup \{f'\}. \]

Since

\[ M_{L'+e'}(w,w') = M_{L'+e'}(v'',w') = m + 1. \]

\(k_{e'}\) or \(k_{f'}\) is a \((w',w')\)-cut of \(L' + e'\) or a \((v'',w')\)-cut of \(L' + f'\), respectively. \(B_e, B_f\) are the \(k_{e'}\)-blocks of \(L' + e'\) and the \(k_{f'}\)-blocks of \(L' + f'\). Let \(v_i\) be any fixed vertex of \(B_i\). Then

\(L'' = (L' + e') \cup B_i\) is isomorphic to \((L' + f') \cup v_i B_i\). Put

\[ L'' = (L' + e') \cup v_i B_i. \]

Then, by Theorem 3.1 of [20-22],

\[ M_{L'+e''}(u,u') = M_{L'+e''}(u,u') = M_{L'+e''}(u,u') \]

for any \(u, u' \in \partial L'' = B_i \cup \{v_i\}\). This shows that

\[ M_{L'+e''}(w,w') \geq m + 1. \]

Since
\[ M_{L'}(w, w') \geq m + 1 \]

That is,

\[ A(e, L') \setminus A(f, L' + f) \neq \emptyset. \]

Q.E.D.

**COROLLARY 4.3.**

\[ A(e, L' + f, f') = A(f, L' + f, f') = A(f, L' + f, f'). \]

We will show, by using Propositions 4.3, 4.6, and 4.7, that if \( ec(G) > 0 \) then we can find a sequence of edges

\[ e_1, \ldots, e_r = [q, 2] \]

such that

\[ A(e, H) \subset A(e, H_{r-1}), \quad i = 1, \ldots, r-1, \]

where

\[ H_i = H \cup H_{r-1} = H + e_{r-1}. \]

We first describe the procedure **hand** where we put for each \( i \leq r - 1 \)

\[ A(e_i) = A(e_i, H_{i+1}) \]

\[ A(e_{r}) = A(e_{r}, H_{r-1}) \]

for the edges \( e, f \) such that

\[ M \cup V : \mathcal{G} \setminus M \neq \emptyset \].
procedure hand

begin

i = 1

while i < n do begin

j = i + 1

if \( j \) is odd and \( j = r \) then \( A = \{ A_{\text{even}} \} \) else \( A = \{ A_{\text{odd}} \} \)

if \( A \cap A_{\text{even}} \neq \emptyset \) then begin

\( U = \{ A \} \)

if \( A \cap A_{\text{odd}} \neq \emptyset \) then begin

\( U = \{ A \} \)

end

end

else if \( i = S - 1 \) then

if \( A \in H \cap \bigcap_{k=1}^{S} H_{k} \neq \emptyset \) then begin

\( U = \{ A \} \)

end

else

\( U = \{ A \} \)

end

end
The part from line 4 through line 22 is called the $i$-phase for each $i, 1 \leq i \leq r-1$.

Let $f_1', f_{i+1}$ denote the two edges $(v,w), (v',w')$, respectively, that we have at the beginning of
the $i$-phase, and let $e_1, f_{i+1}'$ denote those edges obtained at the end of the $i$-phase, where

$$f_1' = (y_1, y_2), f_r' = e_r.$$

Then

$$V(f_1') \cup V(f_{i+1}) = V(e_r) \cup V(f_{i+1}').$$

For each $j, 1 \leq j \leq r$, put

$$Y_j = \begin{cases} Y & \text{if } q \text{ is odd and } j = r, \\ \{y_1, y_2, \ldots, y_{2j-1}, y_{2j}\} & \text{otherwise.} \end{cases}$$

**PROPOSITION 4.8.**

$A(e_1, H_i) \subseteq A(e_{i+1}, H_{i+1})$ for each $i, 1 \leq i \leq r-1$.

**PROOF.**

Proposition 4.6 shows that, at the end of the $1$-phase, we have two edges

$$e_1, f_2'$$

such that

$$A(e_1, H_1) \subseteq A(e_1, H_1 + f_2') = A(f_2', H_1 + f_2').$$

Suppose that, at the end of the $(i-1)$-phase, $2 \leq i \leq r-1$, we have two edges

$$e_{i-1}, f_i'$$

such that

$$A(e_{i-1}, H_{i-1}) \subseteq A(e_{i-1}, H_{i-1} + f_i') = A(f_i', H_{i-1} + f_i').$$

We consider the $i$-phase. At the beginning of the phase we have two edges

$$f_1', f_{i+1}.'$$

Put

$$H_1 = H_{i-1}, H_1 = H_{i-1} + f_i'.$$

and suppose that
Then Proposition 4.6 assures that, at line 12, we obtain two edges \( f, f' \) such that

\[
V(f) \cup V(f') = V(f_1') \cup V(f_{i+1}).
\]

\[
V(f) \neq V(f')
\]

\[
A(f, L' + \{f, f'\}) = A(f', L' + \{f, f'\}).
\]

Proposition 4.7 also assures that we have

\[
A(e_{i-1}, L') \cap A(f, L' + f) \neq \phi \text{ or } A(e_{i-1}, L') \cap A(f', L' + f') \neq \phi.
\]

Hence, at line 20, we obtain two edges

\[
e_i, f_{i+1}
\]

such that

\[
A(e_{i-1}, L') \subseteq A(e_{i-1}, L' + e_i), \text{ and } e_i + e_{i-1} = H_i
\]

At the end of the \((r-1)\)-phase, we obtain two edges

\[
e_{r-1}, f_r' = e_r
\]

such that

\[
A(e_{r-1}, H_{r-1}) \subseteq A(e_{r-1}, H_{r-1} + f_r') = A(f_r', H_{r-1} + f_r') = A(e_r, H_r).
\]

Q.E.D.

**REMARK 4.3.**

(1) For each \( 1 \leq i \leq r-1 \),

\[
|V(e_i) \cap V(e_{i+1})| = 1 \quad \text{if } q = \text{odd and } i = r-1.
\]

\[
V(e_i) \cap V(e_{i+1}) = \phi \quad \text{otherwise.}
\]

\[
A(e_r, H_r + f_{r-1}') \cap Y = Y_{r-1},
\]

\[
A(e_r, H_r) \cap Y = V(e_r) \cup \ldots \cup V(e_i) \subseteq Y_{i-1}.
\]

(2) \( Y \subseteq A(e_r, H_r) \).
Thus we obtain the following proposition by Propositions 4.8, 4.3 and Remark 4.3.

**PROPOSITION 4.9.**

\[ A(e_i, H) = \emptyset. \]

**COROLLARY 4.4.**

\[ Z_H = \{e_1, \ldots, e_r\} \] is a minimum solution to the \((m+1)\)-edge-connectivity augmentation of a graph \(H\).

**PROPOSITION 4.10.**

Suppose that \(q \geq 3\). Then, for each \(i, 1 \leq i \leq r-1\), \(A(e_i, H)\) is an \((m+1)\)-pendant of \(H\) if and only if \(q\) is odd and \(i = r-1\).

**PROOF.**

Put

\[ A = A(e_i, H). \]

First suppose that \(q\) is odd and \(i = r-1\). Since \(|Y - A| = 1\), \(H_{r-1}\) has an \(m\)-cut \(K = K(\{y\}, H_{r-1})\), where \(Y - A = \{y\}\). The \(K\)-block \(B(A, K; H_{r-1})\) contains at least one \((m+1)\)-pendant of \(H_{r-1}\) (by Proposition 3.3 of [20-22]). Hence if we assume that \(A\) is not an \((m+1)\)-pendant of \(H_{r-1}\) then we have an \((m+1)\)-pendant \(y'\) of \(H_{r-1}\) such that

\[ y' \in B(A, K; H_{r-1}) - A, \]

meaning that

\[ y' \notin Y - A, \quad y' \neq y, \]

a contradiction. Thus \(A\) is an \((m+1)\)-pendant of \(H_{r-1}\).

Conversely suppose that \(A\) is an \((m+1)\)-pendant of \(H\). Then \(A\) is external. Put

\[ K = K(A, H), \]

where \(|K| = m\). Let \(B_A, B\) be the two \(K\)-blocks of \(H\) such that
\[ B_A = A, \quad B = V(H) - A \]

Assume that either \( q \) is even or \( q \) is odd and \( 1 < r - 1 \). Then there is an edge \( e_{i+1} \) for which

\[ V(e_{i+1}) \subseteq B \text{ (by Remark 4.3-(1)).} \]

Hence \( K \) is also a \((B_A, B)\)-cut of \( H_{i+1} = H_{i} + e_{i+1} \), meaning that

\[ A \cap A(e_{i+1}, H_{i+1}) = \emptyset \]

This contradicts Proposition 4.8.

**Q.E.D.**

**COROLLARY 4.5.**

For each \( i, 1 \leq i \leq r \), the pair of vertices of \( V(e_i) \) is an admissible pair with respect to \( H_{i-1} \).

### 4.2.3. DETERMINING \( Z(m+1) \) FROM \( Z_H \)

We describe how to determine \( Z(m+1) \) from \( Z_H \) obtained by the procedure \( h\text{find} \). For each \( i, 1 \leq i \leq r \), put

\[ e_i = (v_{i1}, v_{i2}) \]

\[ V_i = \bigcup_{v \in A(e_i, H_i)} P_{m+1}^{-1}(v) \]

Each \( P_{m+1}^{-1}(v) \) is an external \((m+1)\)-pendant of \( G \), and Proposition 3.3 of [20-22] shows that there are two sequences

\[ S_j(k) \subseteq \cdots \subseteq S_j(m+2) \subseteq P_{m+1}(v_{i1}), \quad j = 1, 2. \]

where \( S_j(t') \) denotes an external \( t' \)-pendant of \( G, t' = m+2, \ldots, k \). For each \( i, 1 \leq i \leq r \), choose two vertices

\[ w_{ij} \in S_j(k), \quad j = 1, 2. \]

and put
Let $G, G_i = G_{i-1} + g_i$, $i = 1, \ldots, r$. 

**PROPOSITION 4.11.**

For each $i$, $1 \leq i \leq r$, the following (1) - (3) hold:

1. $V_i$ is an $(m+1)$-component of $G_i$, and if $S$ is any $(m+1)$-component of $G_i$ such that $\bigcap V_i = \emptyset$ then $S \in \pi(m+1)$.
2. $d(V_i, G_i) = d(V_i, G) = d(A(e_i, H_i), H_i) = d(A(e_i, H_i), H)$.
3. $V_i$ is an external $(m+1)$-pendant of $G_i$ if and only if $A(e_i, H_i)$ is an external $(m+1)$-pendant of $H$.

**PROOF.**

First we prove (1), where it suffices to consider the case where $V_i \subseteq V(G)$, or $i < r$.

Let $S_1, S_2 \in \pi(m+1)$, $S_1 \neq S_2$, and let $K_G$ be any $(S_1, S_2)$-cut of $G$, where $|K_G| = m$. There are distinct vertices $u \in V(H)$ such that

$$S_j = \rho_{m-1}^{-1}(u_j), j = 1, 2.$$

Proposition 2.1 shows that $H$ has a $(u_1, u_2)$-cut $K_H = \rho_{m-1}(K_G)$, for which

$$B(u, K_H, H) = \{ \rho_{m-1}(S) : S \in \pi(m+1), S \subseteq B(S, K_G, G), j = 1, 2 \}.$$

Put

$$A = A(e_i, H_i), B(u) = B(u, K_H, H), B(S_j) = B(S, K_G, G).$$

Suppose that

$$S \subseteq V, j = 1, 2.$$

Then
meaning that $K_H$ is no longer a $(u_1, u_2)$-separator of $H$. There is some $e_i = (v_{i1}, v_{i2}) \in E_H$, $1 \leq i \leq i$, such that

$v_{ij} \in B(u_j)$, $j = 1, 2$.

For the corresponding edge $g_i = (w_{i1}, w_{i2})$,

$w_{ij} \in \rho_{m-1}(v_i) \subseteq B(S_i)$, $j = 1, 2$.

Therefore any $(S_1, S_2)$-cut of $G$ cannot be an $(S_1, S_2)$-separator of $G$.

Next suppose that

$S_1 \subseteq V, S_2 \subseteq V(G) - V$.

Then

$u_j \in A, u_2 \in V(H_1) - A$.

$H_i$ has an $(A, \{u_2\})$-cut $K'_H$ with $|K'_H| = m$. Since

$V(e_i) \subseteq A, j = 1, \ldots, i$.

$K'_H$ is also an $(A, \{u_2\})$-cut of $H$, and $\{u_2\}$ is an $(m+1)$-component of $H$ and of $H$. Proposition 2.1 shows that $G$ has an $(S_1, S_2)$-cut $K'_G$ with $|K'_G| = |K'_H|$. Since

$V(g_j) \subseteq V, \subseteq B(S_1, K'_G, G)$, $j = 1, \ldots, i$.

$K'_G$ is an $(V, S_2)$-cut of $G$, and $S_2 \subseteq (m+1)$. Thus (1) follows.

If

$g \in K(V, G)$, $e \in K(A, H)$

then

$g \in K(V, G)$, $e \in K(A, H)$.
respectively, and (2) follows. Clearly, any (m+1)-pendant of $G_i$ (of $H_i$) is external, and (3) follows from (2).

Q.E.D.

PROPOSITION 4.12.

For each $i$, $1 \leq i \leq r$, the pair $w_{i1},w_{i2} \in V(G_i)$ is an admissible pair with respect to $G_{i-1}$.

PROOF.

We will prove the proposition by induction on $i$.

Inductive basis ($i=1$): $\rho_{m+1}^{-1}(v_{ij})$, $j=1,2$, are distinct external (m+1)-pendants of $G_{i-1}$, and therefore, the pair $w_{i1},w_{i2}$ satisfies the edge condition for $G_{i-1}$. If $q \geq 4$ then, by Propositions 4.10, 4.11, $V_i$ is not an (m+1)-pendant of $G_i$. Hence the pair $w_{i1},w_{i2}$ is an admissible pair with respect to $G_{i-1}$.

Inductive hypothesis ($i \geq 2$): For any $t$, $1 \leq t < i$, the pair $w_{t1},w_{t2}$ is an admissible pair with respect to $G_{t-1}$.

Inductive step ($i \geq 2$): If $q$ is odd and $i=r$ then, by Propositions 4.10, 4.11, $\rho_{m+1}^{-1}(v_{ij}) = V_{i-1}$, which is an (m+1)-pendant of $G_{i-1}$. (Note that we have assumed that $v_{r-1,1} \neq v_{r-1,2}$ if $q$ is odd). Hence, regardless of $q$ or $i$, our choice shows that $\rho_{m+1}^{-1}(v_{ij})$, $j=1,2$, are always distinct external (m+1)-pendants of $G_{i-1}$.

Suppose that $G_{i-1}$ has at least four external (m+1)-pendants. Since

$$
\begin{align*}
  &i-1 < r-1 \quad \text{if } q \text{ is even}, \\
  &i-1 < r-2 \quad \text{if } q \text{ is odd}
\end{align*}
$$

Proposition 4.10 shows that $V_i$ is not an (m+1)-pendant of $G_i$. Thus the pair $w_{i1},w_{i2}$ is an admissible pair with respect to $G_{i-1}$.

Q.E.D.
COROLLARY 4.6.

We can set

\[ Z(m+1) = \{g_1, \ldots, g_r \}. \]

4.2.4. THE PROCEDURE FIND AND ITS TIME COMPLEXITY

The procedure \textit{find} is a modified version of the procedure \textit{hfind}: we find edges \( g_i \in Z(m+1) \), add them to \( G' \), and constructs adjacency lists for \((G' + Z(m+1))^e\) without handling \( H \). In the procedure \textit{hfind}, the index \( i \), \( 1 \leq i \leq r-1 \), is used, where \( r = \lfloor q/2 \rfloor \), \( q = |Y| \). In the procedure \textit{find}, we search the \((m+1)\)-level for a pair of \((m+1)\)-pendants not yet processed. Concerning vertices, say \( v \) or \( w \), and edges, say \( t = (v, w) \), appearing in the procedure \textit{hfind}, we choose vertices \( a_v, a_w \) from corresponding \((m+1)\)-pendants of \( G \) and maintain adjacency lists for \((G + (a_v, a_w))^e\) if the edge \((a_v, a_w)\) is added to \( G \). Accordingly, for example, determining if \( A(f) \cap A(f') = \emptyset \) at line 7 of the procedure \textit{hfind} is done by finding vertices \( a_v, a_w, a_v', a_w' \) from corresponding \((m+1)\)-pendants of \( G \) and by computing \( M_G \) values by means of adjacency lists for \( G = (G' + \{(a_v, a_w), (a_v', a_w')\})^e \).

If \( e = H_{i-1} + e_i \) is maintained as adjacency lists for \((G_i)^e = ((G_{i-1})^e + g_i)^e\).

One edge \( g_i \) can be found in \( O(kn_c + \alpha(G_{i-1}''')) \) time, where \( G_{i-1}''' = G_{i-1} + \{(a_v, a_w), (a_v', a_w')\} \).

If we use Dinic's maximum flow algorithm then the total time of the procedure \textit{find} is

\[ O(\left| kn_c + \sum_{i=1}^{r} (m+1)n_c(n_c + i+1) \right|). \]

or

\[ O(kn_c^2(n_c + n_c)). \]

If we use MKM's maximum flow algorithm then the total time is

\[ O(kn_c^3). \]
4.3. THE IMPROVED ALGORITHM

The improved algorithm repeats the following three steps (1) - (3) at most $k-1$ times:

(1) The procedure `comptree`, which constructs $CT(G)$ for the current graph $G$.

(2) Computing (i) - (iv) or (i), (iii) (iv) mentioned in 3.3.3 - (1).

(3) \textbf{if} $ec(G) = 0$ \textbf{then} the procedure `connect`, which find $Z(1)$ and constructs adjacency lists for $(G^1 + Z(1))$; \textbf{else} the procedure `find`, which finds $Z(m+1)$, $m = ec(G)$, and constructs adjacency lists for $(G^1 + Z(m+1))$.

Let

$$n_i = |E(G_i)|, \quad i = 0, \ldots, k.$$ 

Then the time complexity of each step is as follows:

(1) $O(kn^i_n n_i)$ (Dinic) or $O(kn^i_n)$ (MKM)

(2) $O(k(n_e + n_i))$

(3) The procedure `connect` $O(kn^i_e)$. The procedure `find` $O(kn^i_e(n_e + n_i))$ (Dinic) or $O(kn^i_e)$ (MKM).

Since

$$n_e \leq n_e + kn_e,$$

the total time is

$$O(k^2n^i_e(n_e + n_i)) \text{ (Dinic)}$$

or

$$O(k^2(n^i_e + kn_e + n_i)) \text{ (MKM)}.$$

We note that space complexity is $O(kn_e + n_i)$ plus space required by a maximum flow algorithm.
5. CONCLUDING REMARKS

We have proposed an improved version of an algorithm for finding a minimum solution to the k-edge-connectivity augmentation problem. Taking advantage of the results in [9] to reduce time complexity in constructing component trees, as mentioned at the end of 3.3.2, will lead to a more efficient algorithm. We can also expect that a maximum flow algorithm will spend less time on H than on G. If we actually construct H and use the procedure hfind then we may be able to obtain a more efficient algorithm with the increase in space complexity.

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REFERENCES


21. T. Watanabe and A. Nakamura, On a smallest augmentation to k-edge-connect a graph
Technical Report No. C-20, Department of Applied Math., Faculty of Engineering Hiroshima
University, Higashi-Hiroshima, Japan (1984-07; revised 1986-07).

22. T. Watanabe and A. Nakamura, Edge-connectivity augmentation problems, to appear in J
Comput System Sci.
Figure 1. \( G \) and \( v_{node} \)

Figure 2. A graph \( G \)

Figure 3. The component tree \( C(T(G)) \)

Figure 4. \( LEVEL \), \( CH \) and \( v_{node} \)

Figure 5. A part of the actual data structure of \( C(T(G)) \), where \( TOP, DEG \), or \( nil \) are not written.

Figure 6. \( I - \{ e, e' \} \) and \( (1 + \{ e, e' \}) < h, B_1 \)>

Figure 7. The situation of \( K_1, K_1', i=1,2 \)

Figure 8. The situation of \( G_{\alpha}(\alpha \notin \omega) \).
Fig. 1.
Fig. 2.
Fig. 3.
Fig. 5.
Fig. 6.
Fig. 7.
Fig. 8.
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