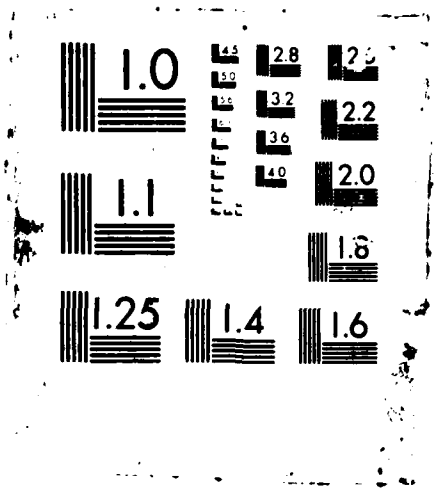
AD-A178 362 INFORMATION BASED NUMERICAL PRACTICE(U) MARYLAND UNIV COLLEGE PARK INST FOR PHYSICAL SCIENCE AND TECHNOLOGY I BABUSKA FEB 87 BN-1059 N00014-85-K-0169 1/1

UNCLASSIFIED F/G 12/1 NL

END
4-87
DTIC


1.0
2.8
2.2
3.2
3.6
4.0
2.0
1.1
1.8
1.25
1.4
1.6

AD-A178 362

(15)

MARYLAND

COLLEGE PARK CAMPUS

INSTITUTE FOR PHYSICAL SCIENCE
AND TECHNOLOGY

Technical Note BN-1059

INFORMATION BASED NUMERICAL PRACTICE

by

I. Babuška

DTIC
ELECTE
MAR 2 7 1987
S D

February 1987

DTIC FILE COPY

This document has been approved for public release and sale; its distribution is unlimited.

INSTITUTE FOR PHYSICAL SCIENCE
AND TECHNOLOGY

87 3 27 002

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER
Technical Note BN-1059

2. GOVT. ACCESSION NO.
AD-A183362

3. RECIPIENT'S CATALOG NUMBER

4. TITLE (and Subtitle)
Information Based Numerical Practice

5. TYPE OF REPORT & PERIOD COVERED
Final life of the contract

6. PERFORMING ORG. REPORT NUMBER

7. AUTHOR(s)
I. Babuska

8. CONTRACT OR GRANT NUMBER(s)
ONR N-00014-85-K-0169

9. PERFORMING ORGANIZATION NAME AND ADDRESS
Institute for Physical Science and Technology
University of Maryland
College Park, MD 20742

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS

11. CONTROLLING OFFICE NAME AND ADDRESS
Department of the Navy
Office of Naval Research
Arlington, VA 22217

12. REPORT DATE
February 1987

13. NUMBER OF PAGES
25

14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)

15. SECURITY CLASS. (of this report)

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release: distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The paper is a sruvey of some aspects of information based selection of numerical methods. Some theoretical ideas on how to deal with uncertain information are discussed and the example of an universal quadrature formula is introduced. Certain aspects of the selection of finite element method and adaptive mesh construction are discussed. Numerical examples illustrate the theoretical aspects.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE
S N 0102-LF-014-6601

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

INFORMATION BASED NUMERICAL PRACTICE

I. Babuška
Institute for Physical Science and Technology
University of Maryland, College Park, MD 20740

Partially supported by the Office of Naval Research under grant N-00014-85-K-0169.

Invited lecture at the Second Symposium on Complexity of Approximately solved Problems, April 20-24, 1987.

ABSTRACT

The paper is a survey of some aspects of information based selection of numerical methods. Some theoretical ideas on how to deal with uncertain information are discussed and the example of an universal quadrature formula is introduced. Certain aspects of the selection of finite element method and adaptive mesh construction are discussed. Numerical examples illustrate the theoretical aspects.

A1

1. INTRODUCTION

Any practical computation is aimed to obtain reliable results in an optimal way. The notion of the optimality is very complex in practice. It includes not only computer cost (for example, depending on various aspect of hardware), but includes also the manpower cost (salaries). Manpower cost makes today typically 90-95% of the total cost of an engineering project. The optimal choice-optimization of the method-strongly depends on the goal of the computation and on the available information. An essential part of the computation is also the assessment of the reliability of the obtained results. Let us refer to [Noor, Babuška (1987)] which gives a survey of the principles of the quality assessment of the finite element computation and major literature (about 200 references cited). This survey addresses many aspects inclusive adaptive approaches, optimal extraction of the desired information from the computed data (in the post processing phase), a-posteriori error analysis, etc.

In practice any computation is information based. Many numerical methods and codes for solving mathematically formulated problems are available today. A very important problem is to characterize the conditions under which a concrete method performs well. In engineering a large effort is spent to obtain indirectly such characterization by comparative computational studies of various benchmark problems. See e.g. [MacNeal, Harder (1985)], [Robinson, Blackham (1981)] any many others. Unfortunately, mathematical theory here is practically nonexistent.

Presently the research is also going on how to use the expert systems (in general, application of the principles of artificial intelligence) to select a numerical method, its part or other basic parameters of the computational analysis. See e.g. [Babuška, Rank (1987)], [Rogers, Barthelemy

(1985)], [Dym (1984)] and others.

Mathematically the optimal selection of numerical methods was for long time under consideration. See e.g. [Babuška, Sobolev (1965)], [Bachvalov (1968)], [Sobolev (1974)], [Traub, Wozniakowski (1985)], for basic ideas, results and literature. We will address some aspects of the optimization of numerical methods below.

Information about the class of problems (or set of their solutions) under consideration is essential for the optimal selection of a method. For theoretical aspects of the notion of an information, optimality and information based complexity, we refer to [Wozniakowski (1985)].

It is imperative to concentrate on information which is <u>practically available and realistic and not only mathematically elegant and convenient</u>. The tests on benchmark model problem reflecting the practice are necessary to keep the research in the prospective. The optimal method selection depends strongly on the set of solutions (information). Some mathematically elegant results could be practically misleading if taken out of context. For example [Smoljak (1966)] has shown, in the case of quadrature formulae, that for convex balanced set of integrated functions the linear algorithm is as good as nonlinear (adaptive) one. Nevertheless, he (see also Bachvalov [1968]) underlined that result of this type is very information dependent and has not to be overstated in general.

Heuristics is, and has to be, used directly or indirectly in the selection of an optimal method. This heuristics is either in the selection of the available information (although following mathematical theory can be rigorous) or in the optimization reasoning itself (and it is not mathematically rigorous). A very elegant analysis based on a probabilistic justification of practical heuristic arguments is in [Gao (1986)].

We will address here some problems and mathematical results and will present some basic ideas on sample examples. For simplicity we will discuss some aspects related to 3 directions.

a) Selection of nonadaptive methods.

b) Feedback and adaptive methods.

c) Reliability assessment.

2. SELECTION PRINCIPLES FOR NONADAPTIVE METHODS

2.1. THE QUADRATURE FORMULA

The simplest example studied in detail in the literature is the problem of the optimal quadrature formula. This example can also serve as a prototype of various approaches used in more complex setting.

The notion of the optimality is very broad and relates to different types of convergence. See, e.g. [Sobolev (1962a)], [Sobolev (1962b)] and others. For an extensive theory of optimal cubature formulae and the functional analytic prerequisites for the study of optimal formulae, we refer to the large monography (808 p) of [Sobolev (1974)].

Let us address some aspects of quadrative formula in its most elementary setting. Let

$$F(u) = \int_0^{2\pi} u(t)dt$$

be a functional defined on a certain space of continuous functions on $I = (0,2\pi)$. Assume that the values $u(t_i)$ are computable for any $0 < t_i < 2\pi$ (and only these values are computable). Let us be interested in the quadrature formula using N function values with the minimal error. This leads to the following problem. Given $N > 0$, integer, find $0 \le t_i \le 2\pi$, α_i, $i = 1,2,\ldots,N$ such that

$$(1) \qquad \left|F(u) - \sum_{i=1}^{N} u(t_i)\alpha_i\right| = E(u,N)$$

is minimal over a set S of functions u, i.e., let us be interested in

$$(2) \qquad \inf_{t_i,\alpha_i} \sup_{u\in S} E(u,N).$$

The formula which achieves (2) will be called optimal and denoted by $Q(S,N)$.

Essential here is the set S. It is typically selected as the unit ball in a Banach space. In (2) we have taken the infimum over all $0 \le t_i \le 2\pi$ and α_i without any constraints. We can be also interested in the case when t_i are constrained to uniform mesh, etc. For various results related to the optimality of this type, we refer to [Sobolev (1974)], Nikolski (1958)], [Traub, Wozniakowski (1980)] and others. The main problem for the application of the selection of a formula based on this (worst case) optimality principle is the selection of the set S. To illustrate this difficulty, let us consider one parametric family (scale) of spaces S_k

$$S_k = \{u \mid \|u\|_{H^k(I)} \le 1\}$$

where $H^k(I)$, $k > \frac{1}{2}$ is the standard Sobolev space. The optimal formula $Q(S_k, N)$ depends on k and N. Hence, the following natural question is: For a concrete u, what k to select? In this connection we can deal also with countably normed spaces, i.e. to assume that

$$\|u\|_{H^k(I)} \le \Phi(k), \qquad \tfrac{1}{2} < k < \infty,$$

with a-priori given $\Phi(k)$. For more we refer once more to [Sobolev (1974)] and also to [Babuška, Prager, Vitasek (1966)].

It seems to be, that the selection of the quadrature formula as an optimal one with respect to a special choice S_k, is practically ineffective because of the uncertainty in the selection of S_k.

Let us discuss this aspect on the problem when the function u is 2π-periodic. The problem of integration of a periodic function was analyzed in many papers. For example [Bachvalov (1964)] studied optimal lower and upper bounds for the errors in s-dimensional cases of classes $H_{s,p}^{(r_1,\ldots,r_s)}$ of

functions $u(x_1,\ldots,x_s)$ having period p in every direction and $\|f^{(r_1,\ldots,r_s)}\|_{L_p} \le 1$. The estimates were the best possible ones up to the term $\log N$. The upper estimates were obtained by number theoretical approaches of [Korobov (1963)] and the lower ones were studied by the theory presented in [Kolmogorov, Tichomirov [1959)]. Many other results are available.

Let us return now to the one dimensional case of periodic function and address the problem of <u>uncertainty</u> of the space selection. For a detailed theory, see [Babuška (1968)]. First we could ask: What is the intuitive content of the statement that a (complex) function u is a 2π-periodic, continuous one? We can formulate it so, that it belongs to a Hilbert space H which has the following property $\mathcal{P}$:

P_1: If $f \in H$, then f is continuous.

P_2: H is dense in C.

P_3: If c is real, $f \in H$, then also $g(x) = f(x+c) \in H$ and $\|f\|_H = \|g\|_H$.

P_4: There exists $K(H)$ such that

$$\|f\|_C \le K(H)\|f\|_H .$$

(We restrict ourself on the case of Hilbert spaces only).

We have

THEOREM 1. Let H be a Hilbert space having property $\mathcal{P}$. Then

(3a) 1) $e^{ikx} \in H$, $k = \ldots,-1,0,1,\ldots$

(3b) 2) functions e^{ikx}, $k = \ldots-1,0,1\ldots$ create orthogonal basis of H

(3c) 3) denoting $\eta_k = \|e^{ikx}\|_H$, then

$$\sum_{k=0}^{\infty} \eta_k^{-2} < \infty. \qquad \square$$

Space satisfying the condition P, respectively (3a), (3b) and (3c) will be called a periodic space. We will need also a stronger notion. The space H is called strongly periodic, if it is periodic and

(3d) $$\|e^{ikx}\|_H = \|e^{-ikx}\|_H$$

(3e) $$\text{If } |j| \geq |k|, \text{ then } \|e^{ijx}\|_H \geq \|e^{ikx}\|_H$$

(3f) $$\eta_{[\alpha n]}^2 \sum_{t=0}^{+\infty} \eta_{[\alpha n]+tn}^{-2} \leq D, \qquad 0 \leq \alpha \leq 2, \quad n \geq 1 \text{ integer}$$

where $[\alpha n]$ means the integral part of αn.

Let us now denote

(4) $$\omega(n,H) = \inf_{\alpha_j, t_j} \sup_{\|u\|_H \leq 1} \left|F(u) - \sum_{j=1}^{n} u(t_j)\alpha_j\right|$$

(5) $$\delta(n,H) = \inf_{\alpha_j} \sup_{\|u\|_H \leq 1} \left|F(u) - \sum_{j=1}^{n} \alpha_j u(\tfrac{2\pi}{n} j)\right|$$

(6) $$\Lambda(n,H) = \sup_{\|u\|_H \leq 1} |F(u) - T_n(u)|$$

where

(7) $$T_n(u) = \frac{2\pi}{n} \sum_{j=1}^{n} u(\tfrac{2\pi}{n} j)$$

is the trapezoid formula. Obviously ω is related to the optimal formula for all distributions of t_j, δ when we restrict ourself to uniform mesh and Λ is the error for the trapezoid formula.

Intuitively, one can expect that the trapezoid formula compares well with the optimal one. Nevertheless, we have

THEOREM 2. Let $\xi_j > 0$, $j = 1,2,\ldots$ arbitrary. Then there exists a periodic space H such that

$$\limsup_{n\to\infty} \frac{\Lambda(n,H)}{\omega(n,H)\xi_j} = \infty. \tag{8}$$ □

THEOREM 3. Let H be any strongly periodic space. Then

$$\limsup_{n\to\infty} \frac{\Lambda(n,H)}{\omega(n,H)\sqrt{n}} < \infty. \tag{9}$$ □

THEOREM 4. For every period space

$$\Lambda(n,H) > \delta(n,H). \tag{10}$$ □

THEOREM 5. Let H be a periodic space. Then

$$\lim_{n\to\infty} \frac{\Lambda(n,H)}{\delta(n,H)} = 1. \tag{11}$$

Only the trapezoid formula has the property that

$$\limsup_{n\to\infty} \frac{\Lambda(n,H)}{\delta(n,H)} < \infty \tag{12}$$

for all periodic spaces. □

These theorems show

1) Speaking about periodic function u, we likely mean that it belongs to a strongly periodic space.

2) Although the trapezoid formula is not optimal for any H, it is good for all periodic spaces, i.e. only the trapezoid formula is a robust formula.

The theorems show that the selection of an optimal formula for a particular space is likely a bad choice in practice.

Let us illustrate it in a concrete example. Let $\|u\|_H^2 = \int_0^{2\pi} (u^2 + u'^2)dx$, then the formula

$$C(n,H)T_n(u) \tag{13}$$

$$C^{-1}(n,H) = 1 + \frac{2}{n^2} \sum_{t=1}^{\infty} \frac{1}{(tA)^2 + (\frac{1}{n^2})} \tag{14}$$

is the optimal one, i.e.

$$\sup_{\|u\|_H \le 1} |F(u) - C(n,H)T_n(u)| = \rho(n,H).$$

Assume now that $u = e^{\alpha \sin x}$, $\alpha = 3,10$. Then

$$\frac{1}{2\pi} F(e^{3 \sin x}) = 4.88079258586502408$$

$$\frac{1}{2\pi} F(e^{10 \sin x}) = 2815.71662846625447$$

and the values obtained by the trapezoid and the optimal formula are given in Table 1.

TABLE 1

	TRAPEZOID FORMULA		OPTIMAL FORMULA	
n	$\alpha = 3$	$\alpha = 10$	$\alpha = 3$	$\alpha = 10$
8	4.88241999058958100	2047.90959481962441	4.64604604..	2900.35030..
16	4.88079258593666173	2815.77672896656761	4.81902223..	2780.14081..
24	4.88079258586502408	2815.71662897903758	4.85310536..	2799.74394..

Table 1 suggests that in practice only for small n and small accuracy the "optimal" formula can be better, provided that the space H is properly selected (which will not likely to happen). For detailed theory and many additional aspects, we refer to [Babuška (1968)].

The observation about the universal property of the trapezoid formula and its analysis was made later in various settings by various authors. See, e.g., [Motornyj (1974)] and others. We refer here also to [Traub, Wozniakowski (1980)] where many papers about optimal formulae are cited.

Similar principles of (worst case) optimality can be applied in many more complex cases. We mentioned the case of optimal difference formula, the optimal choice of the trial functions in variational methods, etc., see, e.g. [Babuška, Prager, Vitasek (1966)], solution of initial value problem for ordinary differential equation, see, e.g. [Bachvalov (1963)], integral equations, see [Emeljanov, Iljin (1967)] and many others.

2.2. THE REGULARITY OF THE SOLUTION AS THE BASIC INFORMATION FOR FINITE ELEMENT SELECTION

Let us consider the model problem

$$-\Delta u = f \quad \text{on} \quad \Omega, \tag{15a}$$

$$u = g \quad \text{on} \quad \partial\Omega, \tag{15b}$$

where $\Omega \subset R^2$ is a bounded domain and $\partial\Omega$ is its boundary.

The finite element method consists (in the most simple case) in the partition of Ω into the set of triangle and quadrilateral elements τ_i (the mesh) and in the best approximation of u (in $H^1(\Omega)$) by piecewise polynomials of degree p_i on τ_i. The space of these piecewise polynomials is called trial space and its dimension N is called the number of degrees of freedom.

Typically it is assumed that the only information available about u is that it belongs to $H^k(\Omega)$, $k > 1$ and the optimal error (for a quasiuniform mesh) is

(16a) $$\|e\|_{H^1(\Omega)} \leq CN^{-\frac{\mu}{2}} \|u\|_{H^k(\Omega)}$$

where

(16.b) $$\mu = \min(p,k-1)$$

when $p_i = p$ for all elements τ_i. This error is the optimal one. Removing constraint $p_i = p$, the best estimate is still $O(N^{-\frac{k-1}{2}})$ which can be easily be proven by concept of the n-width.

If no other information than $u \in H^k(\Omega)$ is available, then the uniform partition is obviously preferable because it leads to the best possible estimate. For more details, see [Babuška, Suri (1987)].

The information that $u \in H^k(\Omega)$ is very far from the optimal one in practice (e.g. structural mechanics). Usually the data (i.e. boundary, f) are piecewise analytic. Then the solution of the problem (15a,b) belongs to a countably normed space. It is possible to prove that

(17a) $$\int_\Omega |D^k u|^2 \Phi^2_{k-2+\beta}(x)dx \leq Ck!d^k, \qquad k = 2,\ldots$$

where

(17b) $$\Phi_{k-2+\beta}(x) = \prod_i |x - x_i|^{k-2+\beta}$$

and x_i are typically the vertices of Ω.

For the proof, see [Babuška, Guo (1985)], [Babuska, Guo (1986)]. We have now

THEOREM 6. Let $u \in H^1(\Omega)$ satisfies (17). Then there exists sequence of meshes and elements degrees p_i on τ_i such that

$$\|e\|_{H^1(\Omega)} \leq Ce^{-\gamma\sqrt[3]{N}} \tag{18}$$

where $\gamma > 0$ depends on β and d in (17). □

The exponential rate (18) can be obtained in practical computations. For more, see [Guo, Babuška (1986a)], [Babuška, Guo [1986)]. Results of this type were implemented in the commercial finite element code PROBE (Noetic Tech., St Louis), [Szabo (1985)]. For the survey paper about the state of the art of the h-p version of the finite element method, we refer to [Babuska (1986)].

The decision in which space the solution u should be imbedded is crucial for selection of the finite element meshes and degrees. In Fig. 1 we show the accuracy of the computation of the elasticity problem of a L-shaped domain by various finite element methods (meshes, degrees) as function of N (which roughly expresses the cost) and which are optimal for various selection of the spaces of the solutions. This shows similar aspects we have addressed in previous section, namely very different performances of the method.

In one dimensional setting much more details are available. See [Gui, Babuška (1986)]. Consider the following sample problem: Let $u(x) = x^\alpha - x$, $\alpha > \frac{1}{2}$ be given on $I = (0,1)$. Consider the set of all partitions of Δ of I

$$\Delta := 0 = \Delta_0 < \cdots < \Delta_m = 1, \quad I_j' = (x_{j-1}, x_j).$$

Denote by $S(\Delta,\underline{p}) = \{u \in H_0^1(I) | u$ is polynomial of degree p_i on $I_i\}$.

Let us be interested now on what can be said about

$$(18) \qquad Q = \inf_{\substack{\omega \in S(\Delta,\underline{p}) \\ \dim S(\Delta,\underline{p}) = N}} \|u - \omega\|_{H^1(I)} .$$

We have then

THEOREM 7.

$$(19) \qquad C_1(\alpha) \frac{1}{\sqrt{N^{\alpha - \frac{1}{2}}}} g_0^{\sqrt{(\alpha-\frac{1}{2})N}} \leq Q \leq C_2(\alpha) g_0^{\sqrt{(\alpha-\frac{1}{2})N}}$$

where $g_0 = (\sqrt{2} - 1)^2$. □

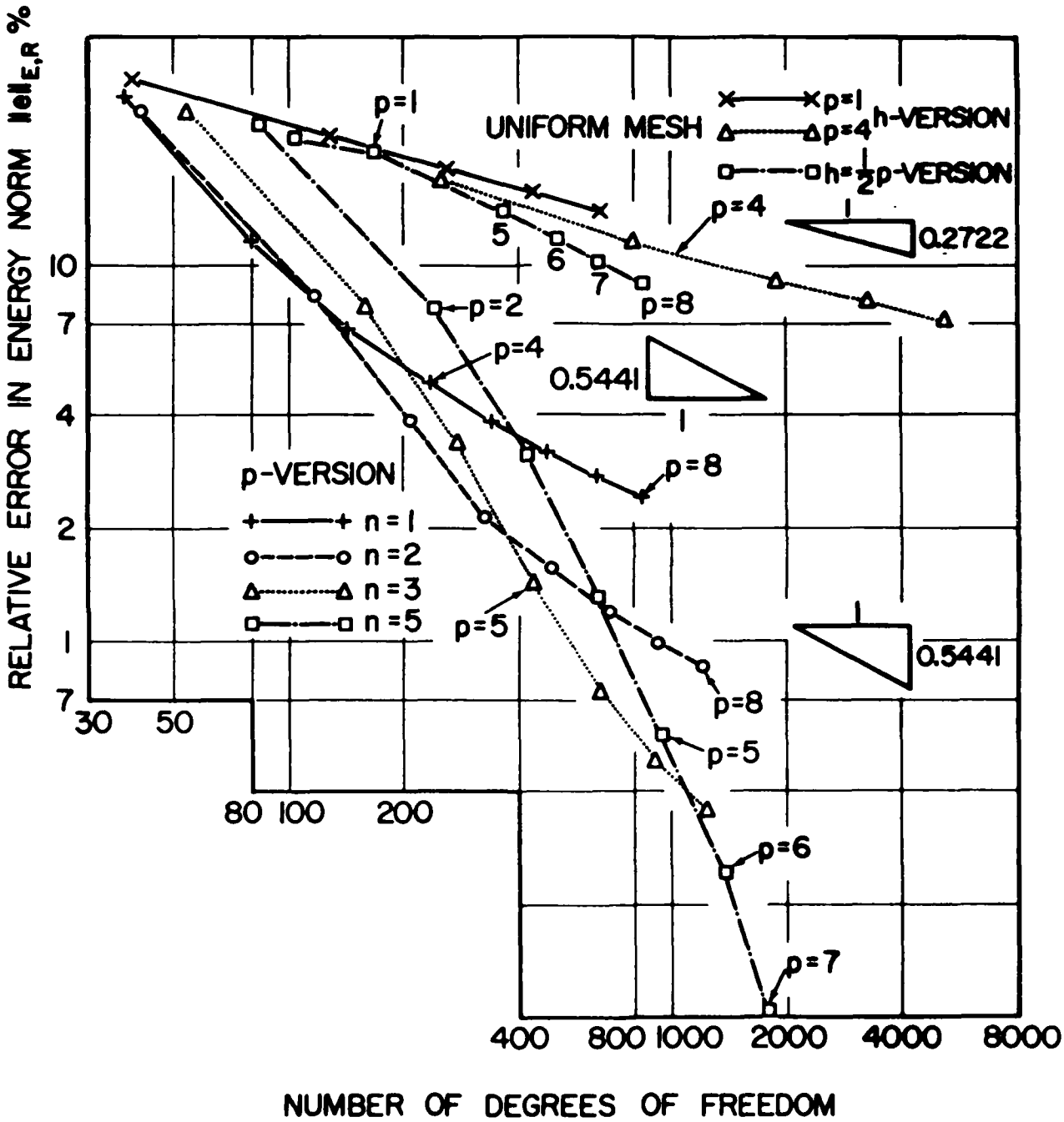


Fig. 1. The error in the energy norm in dependence on N for various meshes.

3. THE ADAPTIVE METHODS

It is worthwhile to distinguish between a feedback method and an adaptive method. A feedback method is any method which utilizes the computed values to steer its direction. An adaptive method is a feedback method which is optimal in a precise sense. For more elaboration, we refer to [Rheinboldt (1983)], [Babuska (1986b)].

3.1. THE FINITE ELEMENT METHOD

Let us explain the main ideas on the example of an adaptive finite element solver for one dimensional boundary value problem

$$-(au')' + bu = f \qquad x \in I = (0,1) \tag{20a}$$

$$u(0) = u(1) = 0. \tag{20b}$$

The finite element method consists of the mesh

$$\Delta := 0 = x_0 < x_1 < \cdots < x_N = 1$$

of the cardinality $N = \kappa(\Delta)$. Denote by $I_j = (x_{j-1}, x_j)$ the element. In the most simple case the finite element solution $u(\Delta) \subset H_0^1(I)$ is a continuous function on I, linear on I_j, $j = 1,\ldots,N$. Given the mesh Δ, the finite element solution $u(\Delta)$ is uniquely defined. Denoting by u_0 the (exact) solution of (20), the error is $e(\Delta) = u(\Delta) - u_0$. Assume that the error is measured in the energy norm $\|e\|_E$,

$$\|e\|_E^2 = \int_0^1 (ae'^2 + be^2)dx.$$

The feedback method now consists of the construction of a sequence Δ_j, $j = 1,2,\ldots,$ so that Δ_j depends on Δ_ℓ and $u(\Delta_\ell)$, $\ell < j$. The operator $\mathcal{T}$ giving Δ_j is called transition operator and defines the

feedback. Sequence $\{\Delta_i\}$ is called the trajectory. A feedback method is called adaptive if it creates an optimal trajectory. Various optimality definitions can be considered. See [Babuska (1986b)]

We will call the trajectory $\{\Delta_i\}$ optimal with respect to the convergence if

$$\|e(\Delta_j)\|_E \to \infty \quad \text{as} \quad j \to \infty. \tag{21}$$

Define now

$$\Phi_u(N) = \inf_{\kappa(\Delta)=N} \|e(\Delta)\|_E \tag{22}$$

i.e. $\Phi_u(N)$ is the smallest error which can be obtained by the mesh with the cardinality N.

We will call the trajectory $\{\Delta_i\}$ optimal with respect to convergence rate if

$$\limsup_{i\to\infty} \frac{\|e(\Delta_i)\|_E}{\Phi_u(\kappa(\Delta_i))} < \infty. \tag{23}$$

For a transition operator $\mathcal{A}$ defining the feedback method, we define $S_c(u_0,\Delta_1,\mathcal{A})$ (respectively $S_R(u_0,\Delta_1,\mathcal{A})$) = $\{(u_0,\Delta_1,\mathcal{A})|$ the trajectory is optimal with respect to the convergence (convergence rate)$\}$. The goal is to design $\mathcal{A}$ so that S_c and S_R will be so large that they include all cases which are important in practice. In [Babuška, Vogelius (1984)] we considered some feedback methods (the details are technical and cannot be given here) which have the following properties:

1) $S_c(u_0,\Delta_1,\mathcal{A}) = H_0^1(I)$, i.e. that the feedback leads always to the convergence.

2) $S_R(u_0,\Delta_1,\mathcal{A}) \neq H_0^1(I)$. Sufficient condition for $u_0 \in S_R$ are given. These conditions are satisfied for solutions which are important in

application as, e.g., $u_0 = x^\alpha - x$, $\alpha > \frac{1}{2}$ and many other functions. Functions $u \in H_0^1(I)$, $u \notin S_R(u_0, \Delta_1, \mathfrak{a})$ were also constructed. For mathematical details see [Babuška, Vogelius (1984)].

The selection of the method depends very much on the information about the solution. This information is usually not too reliable and hence robust methods have to be preferred; in our case, robustness is directly related to the size of S_R. The decision about the available information will always be left to the user. Likely, the physical-engineering arguments are the most reliable way to characterize available information. Probability approach does not avoid this problem because the information about the probability field is not available and is usually made on very arbitrary basis.

We mentioned one dimensional problem. The ideas were extended to more dimensional problems. See, e.g. [Babuška, Miller (1987)]. An adaptive code FEARS (for elements of degree $p = 1$) has been written and experimented with. Although in more dimensions such a detailed and complete analysis, as in one dimension, is not available, the numerical experiments indicate that the coded method is adaptive with respect to the convergence rate for a very broad set of solution of engineering interest. Fig. 2 shows the finite element mesh for the elasticity problem of a cracked panel. The exact solution has singularity at the tip of the crack and uniform mesh would give the rate $O(N^{-1/4})$ while the adaptive solver leads to the rate $O(N^{-1/2})$ (which is the same as for the smooth solution). As could be seen from Table 2, the rate $O(N^{-1/2})$ was practically achieved. Table 2 shows also the effectivity index θ of the estimator as defined in the next Section.

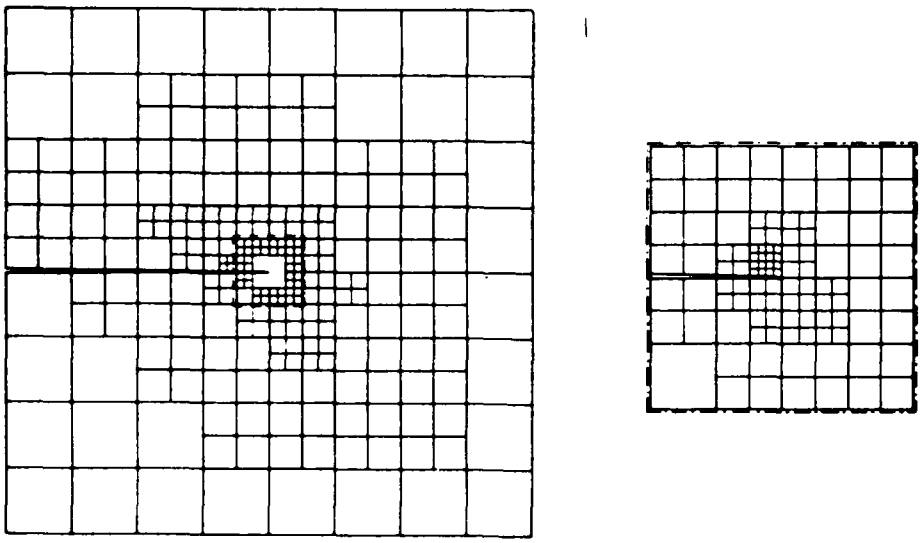


Figure 2. The mesh constructed by FEARS for N = 617.

TABLE 2

MESH	N	$\frac{\lvert e \rvert_E}{\lvert u0 \rvert_E}$ %	θ
1	101	26.38	.885
2	143	21.35	.991
3	221	16.79	1.058
4	301	13.61	1.116
5	617	9.63	1.088

As in one dimension, the set $S_R \neq H^1(\Omega)$ and hence there exists cases where the method does not lead to an optimal (convergence rate) trajectory. Nevertheless, these cases are very likely without any practical importance. For the principles of adaptive method, we refer also to [Babuška, Gui (1986)] and [Noor, Babuška (1987)].

3.2. THE ADAPTIVE ODE's SOLVERS

The modern standard ODE's solvers are of feedback type. They are usually based on "per unit step" or "per step" tolerance criterion. The tolerance τ (which is an input) has in principle two purposes: the accuracy control or the feedback control. We will discuss here briefly the feedback aspects. Let

$$\dot{x} = f(t,x), \quad x(0) = x_0, \qquad 0 < t < T \tag{23}$$

be the problem under consideration. Then approximate solution $\xi(t)$ satisfies

$$\dot{\xi}(t) = f(t,\xi) + \eta(t), \qquad \xi(0) = x_0, \quad 0 < t < T$$

and we can judge the quality of $\xi(t)$ by $\eta(t)$ and define $\mathrm{dist}(x(t),\xi(t)) = \|\eta\|_{L_p((0,T))}$. Now we have

THEOREM 8. The "per unit step" (respectively "per step") approach is adaptive with respect to the rate of convergence for $\|\eta\|_{L_\infty((0,T))}$, respectively $\|\eta\|_{L_1((0,T))}$ measure, for a large set of f.

In practice it is not necessary to find exactly the optimal mesh because the effectivity of the solution is not too sensitive to the perturbances of optimal mesh.

4. A POSTERIORI ASSESSMENT OF THE ACCURACY

The a-posteriori assessment of the accuracy of the computation is essential but very delicate. For various approaches we refer once more to [Noor, Babuška (1987)] and literature cited there. In general, for the a-posteriori error estimate we desire that the effectivity index Θ of the estimator E

$$\Theta = \frac{\text{estimate}}{\text{true error}}$$

has the following properties:

a) $$K_1 < \Theta < K_2$$

where K_1 and K_2 are independent of any detailed characteristics of the mesh, solution, etc.

b) $$\Theta \to 1 \quad \text{as} \quad E \to 0$$

and for reasonable accuracies $.9 < \Theta < 1{,}1$ (say), for reasonable problems and meshes.

It is practically important that Θ is close to one for practical reasons. Table 2 has shown this effectivity index for the cracked panel problem. As before, we wish to characterize the set of the solutions and the meshes for which the above properties hold (i.e. we characterize analogs to S_c and S_R).

5. CONCLUSIONS

Any numerical analysis is information based. Hence, mathematical analysis is very important, provided that it addresses the pertinent problems of the computational analysis. Nevertheless, various mathematical results could be practically misleading if their assumptions are not confronted with the circumstances of practical computations. Hence, it is imperative to apply the obtained theoretical results in the environment of practical computations, made comparative numerical studies and test the applicability of the theoretical conclusion. Otherwise the results of information based complexity would have no impact on computational analysis. In the paper we did try to address some theoretical aspects in the relation with computational practice in a most simple setting.

REFERENCES

Babuška, I. (1968). The universally optimal quadrature formulae. Apl. Mat. 13, pp. 304-338, 388-403 (in German).

Babuška, I (1986a). The p and h-p versions of the finite element method. The state of the art, University of Maryland, Institute for Physical Science and Technology, Tech. Note BN-1156. To appear in Proceedings of Finite Element Workshop, R. Voigt, ed., Springer, 1987.

Babuška, I. (1986b), Feedback. Adaptivity and A-Posteriori Estimates in Finite Elements: Aims, Theory and Experience in Accuracy Estimates and Adaptivity Refinements in Finite Elements Computations, I. Babuška, O.C. Zienkiewicz, J. Gago, E. R. de A. Oliveira, eds., J. Wiley, New York.

Babuška, I., Gui, W. (1986). Basic principles of feedback and adaptive approaches in the finite element method, Computer Methods in Appl. Mech. and Engrg. 55, pp. 27-42.

Babuška, I. Guo, B. (1985). Regularity of the solutions of elliptic problem with piecewise analytic data, Institute for Physical Science and Technology Tech. Note BN-1047, University of Maryland.

Babuška I., Guo, B. (1986). The h-p version of the finite element method for domains with curved boundaries, Institute for Physical Science and Technology Tech. Note BN-1057, University of Maryland.

Babuška, I., Miller, A. (1987). A Feedback Finite Element Method with a-posteriori error estimation, Part I: The finite element method and some basic properties of the a-posteriori error estimator. Computer Methods in Appl. Mech. and Engrg.

Babuška, I., Prager, M., Vitasek, E. [1966). Numerical Processes in Differential Equations, John Wiley, New York.

Babuška, I., Rank, E. (1987), An expert-system-like feedback approach in the

h-p version of the finite element method. To appear in Finite Elements in Analysis and Design.

Babuška, I., Sobolev, S. L. [1965), Optimalization of numerical methods, Apl. Mat. 10, pp. 96-129 (in Russian).

Babuška, I., Suri, M. [1987), The h-p version of the finite element method with quasiuniform meshes. To appear in RAIRO.

Babuška, I., Vogelius, M. (1984), Feedback and adaptive finite element solution on one-dimensional boundary value problems, Num. Math. 44, pp. 75-107.

Bachvalov, N. S. (1963), Optimal properties of the numerical integration of Adams and Gregory type. Proc. Problems of computational mathematics and computational technique, MASGIZ, pp. 5-22.

Bachvalov, N. S. (1964), The optimal estimates of the convergence of quadratures and methods of Monte-Carlo type for classes of functions. Proceedings numerical method for solving differential and integral equations and quadrature formulae, Nauka, Moscow, pp. 5-63 (in Russian).

Bachvalov, N. S. [1968), Optimal methods for problems solving. Apl. Mat 13, pp. 27-38.

Dym, C. (1984), Expert Systems: New approaches to computer aided engineering, Proceedings of 25th AIAA-ACME-ASCE-ANS, Structures, Structural Dynamics and Material Conference, Palm Spring, CA, pp. 99-115.

Gui, W., Babuška, I. (1986), The h, p and h-p versions of the finite element method in 1 dimension.

Part I: The error analysis of the p-version, Numer. Math. 49, pp. 577-612.

Part II: The error analysis of the h and h-p versions, Numer. Math 49, pp. 613-657.

Part III: The adaptive h-p version, Numer. Math. 49, pp. 659-683.

Guo, B., Babuška, I. (1986), The h-p version of the finite element method. Part I: The basic approximation results, Comp. Mech. 1, pp. 21-41. Part II: General results and applications, Comp. Mech. 1, pp. 203-220.

Kolmogorov, A. N., Tichomirov, B. M. (1959), ε-entropy and ε-capacity of the sets in functional spaces, Uspehi Mat. Nauk, 14 86, pp. 3-80 (in Russian).

Korobov, N. M. (1963), Number theoretical methods in the approximation analysis, Moscow, Fizmatgiz (in Russian).

MacNeal, H., Harder, R. L. (1985), A proposed standard set of problems to test finite element accuracy, Finite Elements in Analysis and Design 1, pp. 3-20.

Motornyj, V. P. (1974), On the best quadrature formulae of the form $\sum_{k=1}^{n} p_k f(x_k)$ for some classes of periodic differentiable functions, Ivz. Akad. Nauk USSR Ser. Mat. 38, pp. 583-614 (in Russian).

Nikolskij, S. M. (1958), Quadrature formulae, Moscow, Fizmatgiz (in Russian).

Noor, A. K., Babuška, I. (1987), Quality Assessment and Control of Finite Element Solutions. Finite Elements in Analysis and Design.

Smoljak, C. A (1966), And Optimal Restoration of Functions and Functionals of them. Candidate Dissertation, Moscow State University (in Russian).

Rheinboldt, W. C. (1983), Feedback Systems and Adaptivity for Numerical Computations, in Adaptive Computational Methods for Partial Differential Equations, edited by I. Babuška, J. Chandra, J. Flaherty, SIAM, pp. 3-19.

Robinson, J. Blackham, S. (1981), An evaluation of plate bending elements NSC/NASTRN, ASAS, PAFEC, ANSYS and SAP4, Robinson and Associates, Dorset, England.

Rogers, J. L., Barthelemy, J. F. (1985), An expert system for choosing the

best combination of options in general-purpose program for automated design synthesis. NASA Technical Memorandum 86320.

Sobolev, S. L. (1962a), Various types of convergence for cubature and quadrature formulae, Doklady Academy of Sciences, USSR 146, 1, pp. 41-42 (in Russian).

Sobolev, S. L. (1962b), Cubature formulae on the sphere, which are invariant with respect to the finite group of rotations. Doklady Academy of Sciences, USSR 146, 4, pp. 770-773 (in Russian).

Sobolev, S. L. (1974), Introduction in the theory of cubature formulae, Nauka, Moscow (in Russian).

Szabo, B. A. (1985), PROBE - Theoretical Manual, NOETIC Teach. Corp. St. Louis.

Traub, J. F., Wozniakowski, H. (1980), A General Theory of Optimal Algorithms, Acad. Press, New York.

Wozniakowski, H. (1985), A Survey of Information-Based Complexity, Journal of Complexity 1, pp. 11-44.

The Laboratory for Numerical analysis is an integral part of the Institute for Physical Science and Technology of the University of Maryland, under the general administration of the Director, Institute for Physical Science and Technology. It has the following goals:

- To conduct research in the mathematical theory and computational implementation of numerical analysis and related topics, with emphasis on the numerical treatment of linear and nonlinear differential equations and problems in linear and nonlinear algebra.

- To help bridge gaps between computational directions in engineering, physics, etc., and those in the mathematical community.

- To provide a limited consulting service in all areas of numerical mathematics to the University as a whole, and also to government agencies and industries in the State of Maryland and the Washington Metropolitan area.

- To assist with the education of numerical analysts, especially at the postdoctoral level, in conjunction with the Interdisciplinary Applied Mathematics Program and the programs of the Mathematics and Computer Science Departments. This includes active collaboration with government agencies such as the National Bureau of Standards.

- To be an international center of study and research for foreign students in numerical mathematics who are supported by foreign governments or exchange agencies (Fulbright, etc.)

Further information may be obtained from Professor I. Babuška, Chairman, Laboratory for Numerical Analysis, Institute for Physical Science and Technology, University of Maryland, College Park, Maryland 20742.

END

4-87

DTIC