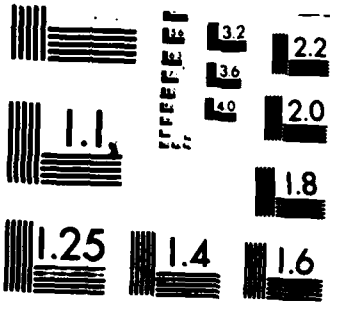
AD-A178 356 A SIMULATION STUDY OF RANDOM CAPS ON A SPHERE(U) 1/1
STANFORD UNIV CA DEPT OF STATISTICS H SOLOMON ET AL
17 MAR 87 TR-388 N00014-86-K-0156

UNCLASSIFIED F/G 12/1 NL



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

AD-A178 356

(12)

A SIMULATION STUDY OF RANDOM CAPS ON A SPHERE

BY

H. SOLOMON and C. SUTTON

TECHNICAL REPORT NO. 388
MARCH 17, 1987

PREPARED UNDER CONTRACT
N00014-86-K-0156 (NR-042-267)
FOR THE OFFICE OF NAVAL RESEARCH

Reproduction in Whole or in Part is Permitted
for any purpose of the United States Government

Approved for public release; distribution unlimited.

DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA

DTIC
ELECTE
MAR 3 0 1987
S D
E

DTIC FILE COPY

87 3 30 004

12

A SIMULATION STUDY OF RANDOM CAPS ON A SPHERE

BY

H. SOLOMON and C. SUTTON

DTIC COPY INSPECTED 6

Accession For	
NTIS GRA&I	☒
DTIC TAB	☐
Unannounced	☐
Justification	
By	
Distribution/	
Availability Codes	
Dist	Avail and/or Special
A-1	

TECHNICAL REPORT NO. 388

MARCH 17, 1987

Prepared Under Contract

N00014-86-K-0156 (NR-042-267)

For the Office of Naval Research

Herbert Solomon, Project Director

Reproduction in Whole or in Part is Permitted
for any purpose of the United States Government

Approved for public release; distribution unlimited.

DEPARTMENT OF STATISTICS

STANFORD UNIVERSITY

STANFORD, CALIFORNIA

DTIC ELECTE MAR 30 1987 S D E

A SIMULATION STUDY OF RANDOM CAPS ON A SPHERE

H. Solomon and C. Sutton

Abstract: This paper describes the computer simulation of a coverage problem in geometric probability, that of placing random caps on the surface of a sphere. The simulation results were compared with exact values where known and the differences were negligible. This suggested the use of simulation results to assess several approximation formulas in the literature.

1. Introduction

Consider a sphere of unit radius on which are placed N spherical caps subtending a half angle α at the center of the sphere ($0 < \alpha \leq \pi/2$). Assume that the centers of these caps are independently and uniformly distributed over the surface of the sphere. We seek to approximate the probability $\mathrm{P}(N)$ that the sphere is completely covered. An exact expression for $\mathrm{P}(N)$ is presently not known except for several special cases, see Solomon (1978).

An approximation of $\mathrm{P}(N)$ can be obtained via computer simulation as follows. For each of n trials we generate N random caps on the surface of a unit sphere. We let $m_{N,n}$ denote the number of trials in which the caps completely cover the sphere. Then $p_{N,n} = m_{N,n}/n$ is an approximate value of $\mathrm{P}(N)$. As n increases, the approximation should approach the exact value.

In order to determine whether or not the caps of a given trial completely cover a sphere, let us define a crossing as being a point of intersection of the circular boundaries of two overlapping caps. Then if there is an uncovered crossing, some area of the sphere outside the two overlapping caps must also be uncovered. Conversely, if at least two caps overlap, then the boundary of any uncovered area on the sphere must contain an uncovered crossing.

Thus we find that the sphere is completely covered if and only if there are at least two caps which overlap and every crossing is covered. This is the essential fact on which the simulation program is based.

Gilbert (1965) used the preceding idea to perform a similar simulation; however, his study was rather small and the results do not provide enough data with which to evaluate the accuracy of the various approximation formulas which have been suggested for $P(N)$. Prior results were also achieved by Moran and Fazekas de St. Groth (1962) by employing a small physical simulation. We desire to assess the accuracy of the suggested approximation formulas and have therefore performed a large computer simulation. Some of the main steps incorporated in our simulation program are described below.

2. Simulation

Random points on the surface of the sphere, which provide centers for the caps, are obtained as follows. For each point p_i we first generate $q_{i,1}$, $q_{i,2}$, and $q_{i,3}$ randomly, each according to a normal distribution with mean 0 and variance 1. We then perform a projection by letting

$$p_{i,1} = \frac{q_{i,1}}{||q||}$$
$$p_{i,2} = \frac{q_{i,2}}{||q||}$$
$$p_{i,3} = \frac{q_{i,3}}{||q||}$$

where

$$||q|| = \left(q_{i,1}^2 + q_{i,2}^2 + q_{i,3}^2\right)^{\frac{1}{2}}.$$

Due to the symmetry of the spherical normal distribution, the point $p_i = (p_{i,1}, p_{i,2}, p_{i,3})$ so obtained will be a random observation from a distribution which is uniformly distributed over the surface of the sphere.

A cap intersects another cap if its center falls within a great circle distance 2α from the center of the other cap. Letting $d(a,b)$ denote the straight line distance between two points a and b, we then have that the cap with center a intersects the cap with center b if and only if

$$\arcsin(d(a,b)/2) < \alpha.$$

Suppose it occurs that the cap with center $a = (a_1, a_2, a_3)$ intersects the cap with center $b = (b_1, b_2, b_3)$. Then two crossings are formed, one on each side of the great circle arc connecting points a and b. Denoting the midpoint of this arc by m, the crossings will lie on the great circle which forms a perpendicular intersection at the point m with the great circle passing through a and b. Using spherical trigonometry we obtain that the great circle distance between each crossing and the point m is given by

$$\arccos\left\{\frac{\cos\alpha}{\cos\left[\arcsin\frac{d(a,b)}{2}\right]}\right\} = \arccos\left\{\frac{\cos\alpha}{\left[1-\frac{d^2(a,b)}{4}\right]^{\frac{1}{2}}}\right\}.$$

Denote this distance by γ, and call the two crossings formed x and y.

We can determine the Cartesian coordinates (x_1, x_2, x_3) of x and (y_1, y_2, y_3) of y in the following manner. Set $v = (v_1, v_2, v_3)$ equal to $(b_1 - a_1, b_2 - a_2, b_3 - a_3)$. Let $||v||$ denote the length of the vector v. Let θ be the angle between the positive x-axis and the vector $(v_1, v_2, 0)$, measured in the direction from the positive x-axis towards the positive y-axis. Let $\phi = \arcsin(v_3/||v||)$. Consider the following series of rotation of axes. First rotate about the z-axis an angle θ in the direction of $(v_1, v_2, 0)$. Then rotate about the new y-axis an angle ϕ towards the positive z-axis. Next we rotate about the new x-axis an angle γ towards the positive y-axis. Now if we reverse the direction of the first two rotations, and rotate about the y-axis by ϕ and then about the z-axis by θ, the new coordinates of m will be the coordinates of the crossing x in the original system. This sequence of rotations of axes is equivalent to rotating the sphere through an angle γ about an axis that passes through the center of the sphere and which is parallel to the vector v. The coordinates of y are found in the same way, except that we rotate about the x-axis by an angle $-\gamma$.

A crossing formed by two caps will be covered if the great circle distance between the crossing and the center of any of the N-2 other caps is less than α. Equivalently, we can check whether or not the straight line distance is less than $2\sin(\alpha/2)$.

Simulations were done using $\alpha = \pi/2$ and $N = 4, 5, \ldots, 13$. This value of α was selected since exact values for $P(N)$ have been determined in this case and hence can serve as anchors in our simulation study. J. G. Wendel (1962) showed that if N points are scattered at random on the surface of the unit sphere in n-space, the probability that all the points lie on some

hemisphere is given by

$$2^{-N+1}\sum_{k=0}^{n-1}\binom{N-1}{k}.$$

This expression yields, as a special case, the result that for spherical caps of angular radius $\alpha = \pi/2$

$$\text{(1)} \qquad \mathrm{P}(N) = 1 - 2^{-N}(N^2 - N + 2).$$

For each value of N 20,000 trials were accomplished. The number of covered spheres out of 20,000 trials, $m_{N,20000}$, is binomially distributed with parameters n = 20,000 and p = P(N). Hence the standard deviation of $p_{N,20000}$ ($= m_{N,20000}/20000$) is given by

$$SD_{N,20000} = \sqrt{\frac{\mathrm{P}(N)(1-\mathrm{P}(N))}{20000}}.$$

The following results give, for each value of N, the exact value of P(N) for $\alpha = \pi/2$, our approximation $p_{N,20000}$, and the value of $r_{N,20000} = (p_{N,20000} - \mathrm{P}(N))/SD_{N,20000}$.

N	P(N)	$p_{N,20000}$	$r_{N,20000}$
4	0.12500	0.12550	0.21
5	0.31250	0.31555	0.93
6	0.50000	0.50030	0.08
7	0.65625	0.66055	1.28
8	0.77344	0.77145	-0.67
9	0.85547	0.85720	0.70
10	0.91016	0.91100	0.42
11	0.94531	0.94430	-0.63
12	0.96729	0.96825	0.76
13	0.98071	0.98090	0.20

These results serve to validify our simulation program.

3. Moran and Fazekas de St. Groth results

We desire to investigate the various formulas that have been developed to approximate P(N). First the formulas will be compared for the case of $\alpha = 53.43°$. This value of α is chosen since it has been considered by previous investigators (because of a biological application). To serve as a basis for the comparisons we estimated P(N) using our simulation program for

$N = 10, 15, 20, \ldots, 60$ (and $\alpha = 53.43°$). Each estimate is based on 20,000 trials. We note that even for 20,000 trials the standard deviation of the estimator $p_{N,20000}$ can be as large as 0.0035 for some values of N. Hence the displayed simulation results in this paper may indicate more accuracy than is actually present.

Moran and Fazekas de St.Groth (1962) used a method of moments approach to derive two approximation formulas for $\mathrm{P}(N)$. Under the assumption that the uncovered region of the sphere consists of a single region (which will always be the case when $\alpha = \pi/2$ and should be nearly true for α close to $\pi/2$) they developed

$$\mathrm{P}(N) = 1 - \frac{1}{2}\pi^2[E(Y)]^2/E(Y^2), \tag{2}$$

where Y denotes the proportion of the surface not covered. It is found that

$$E(Y) = \cos^{2N}(\alpha/2)$$

and the calculation of $E(Y^2)$ may be accomplished via numerical integration using

$$E(Y^2) = \frac{1}{2}\int_0^\pi \left[1 - \frac{1}{4\pi}f(\phi)\right]^N \sin\phi \, d\phi \tag{3}$$

and

$$1 - \frac{f(\phi)}{4\pi} = \begin{cases} \left[1 - \frac{1}{\pi}\arccos\left(\frac{\tan(\phi/2)}{\tan\alpha}\right)\right]\cos\alpha + \frac{1}{\pi}\arccos\left(\frac{\sin(\phi/2)}{\sin\alpha}\right) & (0 \le \phi \le 2\alpha) \\ \cos\alpha & (2\alpha \le \phi \le \pi). \end{cases} \tag{4}$$

Under the hypothesis that the number of disjoint uncovered regions has a Poisson distribution and that the areas of the individual uncovered regions are distributed independently, Moran and Fazekas de St. Groth derived

$$\mathrm{P}(N) = \exp\left(-\frac{1}{2}\pi^2\{E(Y^2)[E(Y)]^{-2} - 1\}^{-1}\right). \tag{5}$$

They expected that the actual value of $\mathrm{P}(N)$ should be between the values given by (2) and (5).

In the table below we give our simulation results for $\mathrm{P}(N)$ for $\alpha = 53.43°$ based on 20,000 trials for each value of N, as well as the values obtained from both of the formulas of Moran and Fazekas de St. Groth.

N	Simulation	(2),(3)&(4)	(5),(3)&(4)
10	0.0003	-2.6119	0.0000
15	0.0465	-1.3333	0.0120
20	0.2752	-0.2882	0.1750
25	0.5763	0.3643	0.4821
30	0.7846	0.7085	0.7336
35	0.8993	0.8738	0.8785
40	0.9556	0.9474	0.9482
45	0.9837	0.9787	0.9788
50	0.9929	0.9915	0.9916
55	0.9974	0.9967	0.9967
60	0.9988	0.9987	0.9987

It appears that both approximation formulas converge to a common value as N increases; however, they do not sandwich the simulation value as was anticipated by Moran and Fazekas de St. Groth. Rather, both formulas tend to underestimate P(N).

In their paper, Moran and Fazekas de St. Groth did not use (3) and (4) in their calculations of P(N). Instead they use a saddlepoint method to arrive at

$$\hat{m}_2 = \frac{1}{2}E(Y)\left[1 + \frac{N^2\tan^2(\alpha/2)}{\pi^2}\right]^{-1} \tag{6}$$

as an approximation of $E(Y^2)$. It is interesting to note that making use of this substitution leads to values of P(N) which better approximate these simulation values. A comparison is presented in the table below, where for each case it can be seen that the approximation (6) yields a better estimate.

N	Simulation value	Value from (2),(3)&(4)	Value from (2) and (6)	Value from (5),(3)&(4)	Value from (5) and (6)
10	0.0003	-2.6119	-2.6814	0.0000	0.0000
15	0.0465	-1.3333	-1.2616	0.0120	0.0154
20	0.2752	-0.2882	-0.2163	0.1750	0.1991
25	0.5763	0.3643	0.4050	0.4821	0.5084
30	0.7846	0.7085	0.7279	0.7336	0.7498
35	0.8993	0.8787	0.8815	0.8785	0.8857
40	0.9556	0.9474	0.9503	0.9482	0.9511
45	0.9837	0.9787	0.9798	0.9788	0.9799
50	0.9929	0.9915	0.9920	0.9916	0.9920
55	0.9974	0.9967	0.9969	0.9967	0.9969
60	0.9988	0.9987	0.9988	0.9987	0.9988

For $\alpha = 53.43°$ and $N = 10, 15, ..., 60$ we found that the combination of (5) and (6) constituted the best approximation for $P(N)$ produced by Moran and Fazekas de St. Groth.

4. Gilbert developement

E. N. Gilbert (1965) developed upper and lower bounds for $P(N)$. Let λ denote the probability that any specified point on the sphere will be covered by a randomly placed cap, so that

$$\lambda = \frac{1}{2}(1 - \cos\alpha) = \sin^2(\alpha/2).$$

The lower bound for $P(N)$ is given by

$$1 - \frac{4}{3}N(N-1)\lambda(1-\lambda)^{N-1}. \tag{7}$$

Gilbert also gives

$$1 - (1-\lambda)^N \tag{8}$$

as a general upper bound for $P(N)$. This is a weak upper bound. For the specific case of $\alpha = 53.43°$ a better upper bound can be found. The dominant terms of this closer upper bound are

$$1 - 6(1-\lambda)^N. \tag{9}$$

Note that (8) is just the probability that some fixed point V on the sphere is covered, and notice that the probability that V is covered exceeds the probability that the entire sphere is covered. To arrive at (9), consider points $V_1, V_2, \ldots, V_6$ which are points of intersection of the sphere and of a regular octahedron inscribed within the sphere. The method of inclusion and exclusion can be used to find an exact expression for the probability that the set of six vertices is covered by N . This probability will be an upper bound for $P(N)$. For large N, (9) gives the dominant terms of this expression. It can be seen from the table below that while the upper and lower bounds do indeed bracket the simulation values, neither provides a very good estimate of $P(N)$.

N	lower bound (7)	simulation	upper bound (9)
10	-2.1790	0.0003	0.3725
15	-1.3989	0.0465	0.7970
20	-0.4039	0.2752	0.9344
25	0.2831	0.5763	0.9788
30	0.6638	0.7846	0.9931
35	0.8513	0.8993	0.9978
40	0.9370	0.9556	0.9993
45	0.9741	0.9837	0.9998
50	0.9896	0.9929	0.9999
55	0.9959	0.9974	1.0000
60	0.9984	0.9988	1.0000

5. Comparison of approximations for $\alpha = 53.43°$

R. E. Miles (1969) provides yet another way of approximating $P(N)$. Theorem 3 of his paper yields, as a special case, the approximation

$$1 - 2^{-N} N(N-1) \sin^2 \alpha \, (1 + \cos \alpha)^{N-2} \tag{10}$$

for $P(N)$. The table below compares values for $\alpha = 53.43°$ obtained from (10) with our simulation results. The best estimates of Moran and Fazekas de St. Groth, using (5) and (6), and the best estimates of Gilbert, using his lower bound (7), are included for comparison.

N	simulation result	Miles	Moran and Fazekas de St. Groth	Gilbert
10	0.0003	-1.3842	0.0000	-2.1790
15	0.0465	-0.7992	0.0154	-1.3989
20	0.2752	-0.0529	0.1991	-0.4039
25	0.5763	0.4623	0.5084	0.2831
30	0.7846	0.7479	0.7498	0.6638
35	0.8993	0.8885	0.8857	0.8513
40	0.9556	0.9527	0.9511	0.9370
45	0.9837	0.9806	0.9799	0.9741
50	0.9929	0.9922	0.9920	0.9896
55	0.9974	0.9970	0.9969	0.9959
60	0.9988	0.9988	0.9988	0.9984

We can see that although Moran and Fazekas de St. Groth's candidate does the best for the smaller values of N, Miles' formula seems to be the most accurate for the higher values of N. We note that each of the formulas appear to underestimate $P(N)$.

6. Comparison of approximations for $\alpha = \pi/2$

In order to provide another basis of comparison for the various approximation schemes, we have also calculated each method's estimate of $P(N)$ for $\alpha = \pi/2$ and $N = 4, 5, 6, \ldots, 25$. We can judge the accuracy of the estimates by comparing them with the exact value of $P(N)$ obtained from (1).

Let us first examine the four candidates from the paper of Moran and Fazekas de St. Groth. Recall that they derived (2) and (5) from different hypotheses, and that for each formula we can either use the exact value of $E(Y^2)$ obtained by numerical integration or we can make use of the approximation (6). For $N = 5, 10, 15, 20, 25$ (and $\alpha = \pi/2$) the estimates of $P(N)$ from these four methods, as well as the known values of $P(N)$, are displayed in the table below.

N	$P(N)$	Value from (2),(3)&(4)	Value from (2) and (6)	Value from (5),(3)&(4)	Value from (5) and (6)
5	0.3125	-0.5009	-0.0897	0.1157	0.2470
10	0.9102	0.8640	0.8927	0.8695	0.8961
15	0.9935	0.9915	0.9928	0.9915	0.9928
20	0.9996	0.9996	0.9996	0.9996	0.9996
25	1.0000	1.0000	1.0000	1.0000	1.0000

One can see that for $\alpha = \pi/2$, the best estimate for $P(N)$ comes from using (5) and (6). This was also the case for $\alpha = 53.43°$. Also note that each approximation formula underestimates $P(N)$, as was the case previously. Because there is only one uncovered region when $\alpha = \pi/2$, it is somewhat surprising that (5) approximates $P(N)$ better than does (2), since (2) was derived under the hypothesis that there was only one vacant region.

For $\alpha = \pi/2$ we can again use (7) to obtain a lower bound for $P(N)$, and we can use (8) to obtain an upper bound for $P(N)$. These bounds, both by Gilbert, and the exact value of $P(N)$ are given below for various values of N.

N	lower bound (7)	known value (1)	upper bound (8)
5	0.1667	0.3125	0.9688
10	0.8828	0.9102	0.9990
15	0.9915	0.9935	1.0000
20	0.9995	0.9996	1.0000
25	1.0000	1.0000	1.0000

In the table below we display the approximated values of $P(N)$ derived from the work of Miles, equation (10). Also shown are the best approximations of Moran and Fazekas de St. Groth, (5) used with (6), and Gilbert's lower bound (7). The known values of $P(N)$ from equation (1) are also given.

N	known value	Miles	Moran and Fazekas de St. Groth	Gilbert
4	0.1250	0.2500	0.0903	0.0000
6	0.5000	0.5313	0.4324	0.3750
8	0.7734	0.7813	0.7360	0.7083
10	0.9102	0.9121	0.8961	0.8828
12	0.9673	0.9678	0.9629	0.9570
14	0.9888	0.9889	0.9875	0.9852
16	0.9963	0.9963	0.9959	0.9951
18	0.9988	0.9988	0.9987	0.9984
20	0.9996	0.9996	0.9996	0.9995
22	0.9999	0.9999	0.9999	0.9999
24	1.0000	1.0000	1.0000	1.0000

It can be seen that the formula of Miles provides the most accurate estimates of $P(N)$, and also that Moran and Fazekas de St. Groth's candidate is better than Gilbert's. Note that Miles' formula overestimates $P(N)$ for this value of α, whereas before, with $\alpha = 53.43°$, it underestimated $P(N)$. It is interesting to note that for $\alpha = \pi/2$ Miles' formula reduces to

$$P(N) \approx 1 - 2^{-N}(N^2 - N)$$

which is very similar to the exact result,

$$P(N) = 1 - 2^{-N}(N^2 - N + 2).$$

References

GILBERT, E. N. (1965). *The probability of covering a sphere with N circular caps.* Biometrika 52, 323-30.

MILES, R. E. (1969). *The asymptotic values of certain coverage probabilities.* Biometrika 56, 661-80.

MORAN, P. A. P. & FAZEKAS DE ST GROTH, S. (1962). *Random circles on a sphere.*

Biometrika 49, 389-96.

SOLOMON, H. (1978). *Geometric Probability.*

CBMS - NSF Regional Conference Series in Applied Mathematics, No. 28.

SIAM Publication, Philadelphia, Pa.

WENDEL, J. G. (1962). *A problem in geometric probability.*

Math. Scandinavica 11, 109-11.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 388	2. GOVT ACCESSION NO. A178356	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) A Simulation Study Of Random Caps On A Sphere		5. TYPE OF REPORT & PERIOD COVERED TECHNICAL REPORT
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) H. Solomon and C. Sutton		8. CONTRACT OR GRANT NUMBER(s) N00014-86-K-0156
9. PERFORMING ORGANIZATION NAME AND ADDRESS Department of Statistics Stanford University Stanford, CA 94305		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS NR-042-267
11. CONTROLLING OFFICE NAME AND ADDRESS Office of Naval Research Statistics & Probability Program Code 411SP		12. REPORT DATE March 17, 1987
		13. NUMBER OF PAGES 13
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

APPROVED FOR PUBLIC RELEASE: DISTRIBUTION UNLIMITED.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Random spherical caps, Coverage, Simulation.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This paper describes the computer simulation of a coverage problem in geometric probability, that of placing random caps on the surface of a sphere. The simulation results were compared with exact values where known and the differences were negligible. This suggested the use of simulation results to assess several approximation formulas in the literature.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE
S/N 0102-014-6601

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

END

4-87

DTIC