ON THE CYCLABILITY OF
K-CONNECTED (K-1)-REGULAR GRAPHS

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k-CONNECTED (k+1)-REGULAR GRAPHS

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1. **Introduction**

In the past fifteen years or so, there have been quite a number of papers dealing with variations on the following general theme. Given a graph $G$ and a positive integer $m$, $m \leq |V(G)|$, find non-trivial conditions on $G$ which will guarantee that given a set $S = \{v_1, \ldots, v_m\} \subseteq V(G)$, there exists a cycle $C_S$ containing $S$. In the special case $m = |V(G)|$, we are dealing with conditions for the existence of Hamiltonian cycles, in itself a subject studied extensively by many graph theorists.

For the most recent survey of the subject for general $m$, the reader is directed to Holton [1983] and Plummer (1983). In particular, some interesting questions remain unsettled in the special case of regular graphs. Let $C(m)$ denote the class of all graphs which have the property that every set of $m$ points lie on some cycle. The largest $m$ for which $G \in C(m)$ is called the cyclability of $G$. Now suppose $k \geq 3$ and let $f(k)$ denote the largest integer $j$ such that in every $k$-connected $k$-regular graph every $j$ points lie on some cycle. It was proved by Holton (1982) and independently by Kelmans and Lomonosov (1982a) that $f(k) \geq k + 4$. This lower bound for $f(k)$ is not believed to be best possible. For example, Holton, McKay, Plummer and Thomassen (1984) proved that $f(3) = 9$. This result was also obtained by Kelmans and Lomonosov independently and announced without proof in (1982a). Meeth (1973) constructed an infinite family of graphs which show, among other things, that $F(k) \geq 10k - 11$. Thus a rather large gap in possible values for $f(k)$ remains at this writing. Recently, McQuaig and Rosenfeld (1984) have shown that for all even $k \geq 4$, there are infinite families of $k$-connected $k$-regular graphs with cyclabilities $6k - 4$ when $k \equiv 0 \pmod{4}$ and $8k - 5$ when $k \equiv 2 \pmod{4}$.

More recently, interest has been generated in the related question of cyclability of $k$-connected $r$-regular graphs for $r \geq k + 1$. First of all, Dirac (1960) proved that for any $k$-connected graph, regular or not, the cyclability is at least $k$. It is interesting to note that in the case of $k$-connected $(k+1)$-regular graphs having $k$ even, the Dirac bound cannot be improved. To see this, consider the complete bipartite graph $K_{k, k+1}$ where the bipartition sets $U$ and $W$ have $|U| = k$ and $|W| = k + 1$ respectively. The cyclability of $K_{k, k+1}$ is clearly $k$. We can easily modify $K_{k, k+1}$ to yield a graph $H_k$ which is $k$-connected and $(k+1)$-regular by replacing each point of $W$ by a copy of the graph obtained from $K_{k+2}$ by deleting a matching of cardinality $k/2$. (Figure 1.1 shows how this is done for $k = 4$.)
More generally, with \( k \) still even, but \( r \geq k + 1 \), Holton (1982) has constructed other graphs which are \( k \)-connected and \( r \)-regular, but which do not lie in \( C(\sqrt{r}) \) and hence have cyclability precisely \( k \).

Now suppose \( k \) is odd. If \( r \geq k + 2 \), Holton (1982) has constructed \( k \)-connected \( r \)-regular graphs which do not lie in \( C(\sqrt{r}) \). This, again together with Dirac's bound, shows that any \( k \)-connected \( r \)-regular graph has cyclability \( = k \), as long as \( k \) is odd and \( r \geq k + 2 \).

So in a sense, the only case left unsettled here is that of \( k \)-connected \( (k + 1) \)-regular graphs for \( k \geq 3 \) and \( k \) odd.

One can do a bit better than the Dirac bound here as was shown by Holton (1982), and independently by Kelmans and Lomonosov (1982b), via the following result.

**Theorem 1.1.** In any \( k \)-connected \( (k + 1) \)-regular graph with \( k \geq 3 \) and odd, any \( k + 2 \) points lie on a cycle.

Thus the cyclability of such graphs is bounded below by \( k + 2 \).

In fact, Kelmans and Lomonosov (1982b) claimed that the conclusion of Theorem 11 can be improved to \( k + 3 \), but this claim is false, at least for \( k = 3 \). For a counterexample due to the present authors, see Holton, (1983). Since Kelmans and Lomonosov did not publish the proof of the \( k + 3 \) bound, the situation...
for \( k \) odd and \( k \geq 5 \) is presently unknown, at least to the present authors. In his 1982 paper, Holton goes on to show that if \( k \) is odd and \( k \geq 3 \) and if \( h(k) \) is the largest positive integer \( m \) for which all \( k \)-connected \((k + 1)\)-regular graphs lie in \( C(m) \), then \( h(k) \leq 9k \).

In the present paper, we will prove that, in fact, \( h(k) \leq 2k - 1 \). (This result was announced without proof by Holton (1983).) To accomplish this, we shall construct, given \( k \geq 3 \) and odd, a graph \( G_k \) which is \( k \)-connected and \((k + 1)\)-regular, but which has a set of \( 2k \) points which do not lie on a common cycle. The procedure will be as follows. First we construct a graph \( G'_k \) which is \( k \)-connected and which has all points \( k \) with degree either \( k \) or \( k + 1 \), but which has a set of \( 2k \) points not lying on any cycle. Then we modify \( G'_k \) first to obtain an intermediate graph \( G''_k \) and then, in turn, modify \( G''_k \) to obtain a \( k \)-connected \((k + 1)\)-regular graph \( G_k \) having a set of \( 2k \) points which lie on no common cycle.

The construction is done in two slightly different ways depending upon whether \( k \equiv 1 \pmod{4} \) or \( k \equiv 3 \pmod{4} \). The reader is encouraged to refer to graphs \( G'_5 \) and \( G'_7 \) to help understand the constructions in general. (See Figure 2.1.)

2. The Construction of \( G'_k \). Let \( k \geq 3 \) be an odd integer. In all cases \( G'_k \) will be a bipartite graph with bipartition \( (X \cup Y) \cup Z \cup Z' \) where

\[
X = \{x_0, x_1, \ldots, x_{k-1}\},
\]

\[
Y = \{y_0, \ldots, y_{k-2}\}/2, \quad Y' = \{y'_0, \ldots, y'_{k-2}\}/2,
\]

\[
Z = \{z_0, \ldots, z_{k-1}\}, \quad Z' = \{z'_0, \ldots, z'_{k-1}\}.
\]

The lines in \( G'_k \) are defined as follows. Every point of \( Y \) (respectively \( Y' \)) is adjacent to every point in \( Z \) (respectively \( Z' \)). For the remaining adjacencies we split the description into two cases. Suppose \( k \equiv 1 \pmod{4} \). For each \( i = 0, \ldots, k - 1 \), both \( z_i \) and \( z'_i \) are adjacent to \( x_i, x_{i+1}, x_{i-1}, \ldots, x_{i - \frac{k-1}{4}}, x_i + \frac{k-1}{4} \) where subscripts are taken modulo \( k \). In the case in which \( k \equiv 3 \pmod{4} \), for \( i = 0, \ldots, k - 1 \), both \( z_i \) and \( z'_i \) are adjacent to \( x_i, x_{i+1}, x_{i-1}, \ldots, x_{i - \frac{k-3}{4}}, x_i + \frac{k-1}{4} \), where again the subscripts are taken modulo \( k \).

The modulo \( k \) "circular symmetry" for adjacencies among the \( x_i \)'s, \( z_j \)'s and \( z'_k \)'s is important to bear in mind and will prove to drastically reduce the number of cases we will have to treat in order to prove that \( G'_k \) is \( k \)-connected.
Figure 2.1
Finally, we note that $G'_3$ is just the well-known Hershel graph.

3. The connectivity of $G'_k$.

Note that in $G'_k$ we have $\deg u = k$ for $u \in U \cup Z \cup Z'$ and $\deg u = k + 1$ for $u \in X$.

We now proceed to prove that $G'_k$ is $k$-connected. To this end, let $u$ and $v$ be two distinct points in $V(G)$. We must find $k$ openly disjoint paths joining $u$ and $v$. We shall refer to such a family of paths as openly disjoint $u \cdot v$ paths. Here openly disjoint (hereafter abbreviated as o.d.) means that the paths joining $u$ and $v$ are otherwise pairwise point disjoint. We shall often refer to a set of $k$ openly disjoint $u \cdot v$ paths as a $k$-skein joining $u$ and $v$ (or as a $u \cdot v$ $k$-skein).

1. First suppose $(u,v) \in Y$. Say $u = y_0$ and $v = y_1$. Then $y_0z_0y_1, y_0z_1y_1, \ldots, y_0z_{k-1}y_1$ suffices as the $u \cdot v$ $k$-skein. The $k$-connection between two points of $Y'$ follows by symmetry.

2. Suppose $u \in Y$ and $v \in Y'$. Without loss of generality, assume $u = y_0$ and $v = y_0'$. Then $(y_0z_0y_0z_0'y_0', \ldots, y_0z_{k-1}y_{k-1}'y_0')$ suffices.

For the rest of the cases, we will treat the congruence classes $k \equiv 1 \pmod{4}$ and $k \equiv 3 \pmod{4}$ separately.

First suppose $k \equiv 1 \pmod{4}$. (Thus $k \geq 5$.)

3a. Suppose $u \in Y$ and $v \in Z$, say $u = y_0$ and $v = z_{k-1}/2$. Note that for $i = 0, 1, \ldots, k-2$, we have $z_{k-1}/2$ adjacent to $x \cdot k-1/4 + i$. So let

$$P_i = y_0z_1x\cdot k-1/4+i\cdot z_{k-1}/2, \text{ for } i = 0, \ldots, k-3/2,$$

$$Q_i = y_0z_{k-1}/2+i\cdot y_1z_{k-1}/2, \text{ for } i = 1, \ldots, k-3/2, \text{ and let }$$

$$R_i = y_0z_{k-1}/2 \text{ and } S_i = y_0z_{k-1}/2z_{k-3}/4z_{k-1}/2.$$

Then $(P_0, \ldots, P_{k-3}/2, Q_1, \ldots, Q_{k-1}/2, R_1, S_1)$ is a $u \cdot v$ $k$-skein.
4a. Suppose $u \in Y$ and $v \in X$. Without loss of generality, suppose $u = y_0$ and $v = x \frac{k-1}{2}$.

Let $P_i = y_0 z \frac{k+1}{4} + i x \frac{k+1}{2}$, for $i = 0, ..., \frac{k+1}{2}$.

Now let $Q_i = y_0 x \frac{k+1}{4} + i z \frac{k+1}{2} + i x \frac{k+1}{2}$, for $i = 0, ..., \frac{k-5}{2}$

and let the "mirror images" of the $Q_i$'s about the axis $z \frac{k+1}{2} x \frac{k+1}{2} z' \frac{k-1}{2}$ be

$P_i = x \frac{k+1}{2} z_1 \frac{k+1}{4} z_2 \frac{k+1}{4} + i y_0$, for $i = k+1, ..., 3k-3$.

We then have a total of $k+1 + k-1 + 3k-3 - k \frac{1}{2} = k \text{ o.d. } u - v \text{ paths as desired.}$

5a. Suppose $u \in Y$ and $u \in Z$. Without loss of generality, let $u = y_0$ and $v = z' \frac{k-1}{2}$. Then let

$P_i = y_0 z \frac{k+1}{4} + i z \frac{k+1}{4} + i z' \frac{k+1}{2}$, for $i = 0, ..., \frac{k+1}{2}$.

$Q_i = y_0 x \frac{k+1}{4} + i x \frac{k+1}{2} + i z \frac{k+1}{4} + i y_0$, for $i = 0, ..., \frac{k-5}{2}$.

Now let $P_i = y_0 z \frac{3k+1}{4} + i x \frac{3k+1}{4} + i z' \frac{3k+1}{4} + i y_0$, for $i = 0, ..., \frac{k-5}{4}$.

We then have a total of $k+1 + k-1 + k \frac{1}{4} = k \text{ o.d. } u - v \text{ paths as sought.}$

6a. Suppose $u$ and $v$ are both in $Z$.

First note that any pair of $z_i$'s have at least one common neighbour in $X$ (and in fact, there are pairs of $z_i$'s which have exactly one common neighbour). For example, (and for the sake of symmetry when working with the drawing in this case) let $u = z \frac{k-1}{4}$ and let $v = z' \frac{3k-3}{4}$.

Now let

$P_i = z \frac{k+1}{4} y_i \frac{z k+3}{4}$, for $i = 0, ..., \frac{k-3}{2}$ and let

$P = \frac{k+1}{2} = z \frac{k+1}{4} x \frac{k+1}{4} z \frac{3k+3}{4}$. Next let

$Q_i = z \frac{k+1}{4} x \frac{k+1}{4} y_i \frac{z k+3}{4} + i x \frac{k+1}{2} + i z \frac{3k+3}{4}$ for $i = 0, ..., \frac{k-3}{2}$. 
Then we have a total of \( \frac{k-1}{2} + \frac{k-1}{2} = k \) o.d. \( u - v \) paths as desired.

Now if the two \( z_i \)'s chosen for \( u \) and \( v \) have \( r \geq 2 \) common neighbours in \( X \), then in addition to the \( \frac{k+1}{2} \) paths of type \( P_i \) above we get \( r-1 \) more of the form \( u x_j v \). Taking these together with \( \frac{k+1}{2} \cdot (r-1) \), type \( Q_i \) above, we get a total of \( \frac{k+1}{2} + r-1 + \frac{k+1}{2} \cdot (r-1) = k \) o.d. \( u - v \) paths as desired.

7a. Suppose \( u \in Z \) and \( v \in X \). Without loss of generality, assume \( u = z \). There are now two cases to consider.

First suppose that \( v \in \Gamma(u) \cap X = \Gamma(z) \cap X \). (Here and throughout the rest of this paper \( \Gamma(u) \) denotes the neighbourhood of \( u \).) Let \( M \) denote the "vertical" matching of \( \Gamma(z) \) into all lines of which are of the form \( x z_i \). Then \( |M| = \frac{k+1}{2} \), and we can find \( \frac{k+1}{2} \) o.d. \( u - v \) paths using \( M \), \( r \) of them of length 3 where \( r \) is the number of neighbours of \( v \) in \( Z \) which are covered by \( M \) and \( \frac{k+1}{2} \cdot r \) of length 5 which are of the form \( v z_j y_k z' m x_m z \). where \( z_m x_m \in M \), but \( z' m \notin \Gamma(v) \cap Z \). On the other hand, \( v \) always has at least \( \frac{k+1}{2} - 1 = \frac{k-1}{2} \) neighbours in \( Z \) which are not equal to \( z \) and these can be used to form an additional \( \frac{k-1}{2} \) o.d. \( u - v \) paths. Again, then, we get \( \frac{k+1}{2} - \frac{k-1}{2} = k \) o.d. \( u - v \) paths as required.

So we may suppose that \( v \in \Gamma(u) \cap X = \Gamma(z) \cap X \). This time we have \( \frac{k-1}{2} \) o.d. \( u - v \) paths of length 3 of form \( z \frac{k-1}{2} y z_j v \), the line \( z \frac{k-1}{2} \) and \( \frac{k-1}{2} \) additional paths of length 3 or 5 obtained as follows. Consider the matching \( M' \) of \( \Gamma(u) \cap X \) "vertically" into \( Z \); that is, all lines of \( M' \) are of the form \( x i z_i \). Delete from \( M' \) the line covering \( v \) and denote by \( M'' \) the resulting matching of size \( \frac{k-1}{2} \). Now if \( M'' \) covers a neighbour of \( v \) we get a path of length 3, while if a line \( e \) of \( M'' \) does not cover a point of \( \Gamma(v) \cap Z \), we can find a \( u - v \) path of length 5 using \( e \) by detouring through \( Y' \).

Now if \( u \) and \( v \) have \( r \geq 2 \) common neighbours, then it is easy to see that there is still a set of \( \frac{k-1}{2} \) \( u - v \) paths of length 2 or 4 where the paths of length 4 are of the form \( z \frac{k-1}{2} \ y_j z_m x_m z_i \), where \( \frac{k-1}{2} = u \) and \( z_i = v \). Then one can find an additional \( \frac{k+1}{2} \) \( u - v \) paths of length 4 of the form \( z \frac{k+1}{2} \ x_m z' m y_j z_i \) and the remaining \( \left\lfloor r + \frac{k+1}{2} - r + \frac{k+1}{2} \right\rfloor = r \cdot 1 \) paths of length 6 having the form...
8a. Suppose $u \in Z$ and $v \in Z'$. Without loss of generality, let $u = z \frac{k-1}{2}$. Now note that regardless of where $v$ is in set $Z'$, $r = |(\Gamma(u) \cap X) \cap (\Gamma(v) \cap X)| > 0$. So let $s_1 = |(\Gamma(u) \cap X) - (\Gamma(v) \cap X)|$, let

$s_2 = |(\Gamma(u) \cap X) - (\Gamma(v))|$, and $s_3 = |X - (\Gamma(u) \cup \Gamma(v))|$. Then clearly $r + s_1 + s_2 + s_3 = k$. Those members of $X$ counted by $r$ give rise to o.d. $u \cdot v$ paths of length 2. For each point $x_i$ of $X$ counted by $s_1$ take line $xi, x'_i$, for each counted by $s_2$ take line $xi, z_i$ and for each counted by $s_3$ take the path $z_i, x_i, z'_i$. The lines $xi, x'_i$ counted by $s_1$ give rise to $u \cdot v$ paths of length 4 all of form $u x_i z'_i y_j v$, those counted by $s_2$ yield $u \cdot v$ paths of length 4 of form $u y_j z_i x_i v$ and those counted by $s_3$ yield $u \cdot v$ paths of length 6 all of the form $u y_j z_i x_i z'_i y_j v$. Altogether, these form a collection of $k$ o.d. $u \cdot v$ paths.

9a. Finally, suppose both $u$ and $v \in X$. Let $u = x_i$ and $v = x_j$. Consider $\Gamma(x_i) \cap Z = N_Z(x_i)$.

If $z_m \in N_Z(x_j)$ then if it is also in $\Gamma(x_j) \cap Z = N_Z(x_j)$ we have a path of length 2 - namely $x_i z_m x_j$ joining $x_i$ and $x_j$. On the other hand, if $z_m \notin N_Z(x_i)$ then we have a path of length 4 - namely $x_i z_m y_n z_s x_j$ joining $x_i$ and $x_j$. This yields a total of $\frac{k+1}{2}$ o.d. $u \cdot v$ paths and they all lie within $G_k[X \cup Z \cup Y]$. But clearly there is a second set of $\frac{k+1}{2}$ o.d. $u \cdot v$ paths (the reflections of the first set of paths in the $X$ axis) which, together with the first set yields a total of $k + 1$ o.d. $u \cdot v$ paths as sought.

Now let us suppose $k \equiv 3 \pmod{4}$.

3b. Suppose $u \in Y$ and $v \in Z$. Without loss of generality, suppose $u = y_0$ and $v = z \frac{k-1}{2}$.

Let $P_i = y_0 z \frac{k-1}{2} + i z \frac{k-1}{2}$, for $i = 0, \ldots, k-3$.

$Q_i = y_0 z \frac{k-1}{2} + i y_j z \frac{k-1}{2}$, for $i = 1, \ldots, k-3$.

$R_i = y_0 z \frac{k-1}{2}$ and $S_i = y_0 z \frac{k-1}{2}$.

Then $\{P_0, \ldots, P_k, Q_0, \ldots, Q_k, R, S\}$ is a $k$-skein joining $u$ and $v$. 
4b. Suppose \( u \in Y \) and \( v \in X \). Without loss of generality, suppose \( u = y_0 \) and \( v = x_{k-1} \frac{1}{2} \).

Let
\[
P_i = y_0 z \frac{k-3}{4} + i x \frac{k-1}{2}, \quad i = 0, \ldots, k-1.
\]
Now if \( k = 3 \), let
\[
Q_i = y_0 z \frac{k-3}{4} + i x \frac{k-1}{2} + i \frac{k-3}{4}, \quad i = 0, \ldots, k-1
\]
and if \( k = 3 \), let \( Q_0 = 0 \).

Also let
\[
P_i = x \frac{k-1}{2} z_1 z_{i+1} z \frac{k+1}{4} + i y_0, \quad \text{for } i = 0, \ldots, 3k-5.
\]
We then have a total of \( \frac{k+1}{2} + \frac{k+1}{4} + 3k-5 - \frac{k-3}{4} = k \) o.d. \( u \) \(-\) \( v \) lines in all cases.

5b. Suppose \( u \in Y \) and \( v \in Z \). Without loss of generality, let \( u = y_0 \) and \( v = z' \frac{k-1}{2} \).

Then let
\[
P_i = y_0 z \frac{k+1}{4} + i x \frac{k+1}{4} + i z' \frac{k+1}{2}, \quad \text{for } i = 0, \ldots, k-1
\]
\[
Q_i = y_0 z \frac{k+1}{4} + i x \frac{k+1}{4} + i z' \frac{k+1}{2}, \quad \text{for } i = 0, \ldots, k-1, \text{ and}
\]
and let
\[
P_i = y_0 z \frac{2k-3}{4} + i x \frac{3k-3}{4} + i z' \frac{3k-3}{4} + i y' \frac{k+1}{4} + i z' \frac{k-1}{2}, \quad \text{for } i = 0, \ldots, k-1, \quad \text{when } k \geq 7.
\]
(For \( k = 3 \), let \( R_3 = \emptyset \).

We then have a total of \( \frac{k+1}{2} + \frac{k+1}{4} + \frac{k-3}{4} = k \) o.d. \( u \) \(-\) \( v \) paths as desired.

6b. Suppose \( u \) and \( v \) are both in \( Z \). Again, as in 6a, note that every pair of \( x_i \)'s have at least one common neighbour in \( X \) and in fact there are pairs with exactly common neighbour. Let \( u = z \frac{k-3}{4} \) and \( v = z \frac{2k-5}{4} \), for example.

First suppose \( k \geq 7 \).

Now let
\[
P_i = z \frac{k-3}{4} y_1 z \frac{2k-5}{4}, \quad \text{for } i = 0, \ldots, k-3
\]
and let
\[
R_{k-1} = z \frac{k-3}{4} x \frac{k+1}{2} z \frac{2k-3}{4}.
\]
Next let
\[
P_i = z \frac{k-3}{4} x \frac{k+1}{2} z \frac{k+1}{2} + i z \frac{2k-5}{4}, \quad \text{for } i = 0, \ldots, k-3.
\]
Thus we obtain a total of \( k^2 \cdot \frac{1}{2} + k^2 \cdot \frac{1}{2} = k \) o.d. u-v paths as desired. If \( k = 3 \), then 3 o.d. u-v paths are obvious.

Now if \( k \geq 7 \) and the 2 \( z_i \)'s chosen for u and v have \( r \geq 2 \) common neighbours in \( X \), then in addition to the \( \frac{k^2\cdot 1}{2} \) paths of type \( P_i \) above, we get \( r - 1 \) more of the form \( u \cdot x_i \cdot v \). Taking these together with \( \frac{k^2\cdot r - 1}{2} \) of type \( Q_i \) above, we get a total of \( \frac{k^2 + r - 1 + k^2 - (r - 1)}{2} = k \) o.d. u-v paths as desired.

The proofs of Cases 7a (\( u \in Z \), \( v \in X \)) and 9b (\( u, v \in Z \)) are identical to those for Cases 7a, 8a and 9a respectively.

This completes the proof that \( G'k \) is \( k \)-connected.

4. The Construction of \( G_k \).

Recall that in graph \( G'k \), each point in \( Y \cup Y' \cup Z \cup Z' \) has degree \( k \), while each point in \( X \) has degree \( k + 1 \). We now proceed to construct a \((k + 1)\)-regular graph \( G_k \) from \( G'k \) as follows.

First consider each line joining some \( y_i \in Y \) to a \( x_j \in Z \). Insert a new "midpoint" on this line and call it \( \alpha_i \). Similarly, insert a midpoint \( \beta_i \) on each line joining a \( z_i \) to an \( x_j \). Midpoints are similarly inserted on lines joining a \( y_i \) to a \( z_i \) and on lines joining a \( z_i \) to an \( x_j \). They are called \( \alpha_i \) and \( \beta_i \) respectively.

Now we replace each point of \( Y \cup Y' \cup Z \cup Z' \) with a set of points as follows.

First suppose \( k \equiv 1 \mod 4 \). For each \( j \in \{0, \ldots, k-2\} \), replace \( y_j \) by a set \( \Delta_j \) of \( 2k \) new points joined two by two to midpoints \( \alpha_0, \alpha_1, \ldots, \alpha_{k-1} \), respectively. Now replace \( y_{k-2} \) by a set \( \Delta' \), consisting of \( 2k \) points, \( k \) of them joined one at a time to each \( \alpha_{k-2} \) for \( j = 0, \ldots, k-1 \) and the remaining \( k \) joined to yet another new point \( b \). Replace the \( y_i \)'s with sets \( \Delta_i \) and \( \Delta'_i \), in a symmetric manner.
Next, replace each $z_j \in Z$ with a set $C_j$ of points as follows. For each line of the form $\alpha_j z_j$ for $r = \frac{k+1}{2}$, insert $k-1$ new points into $C_j$ and join each to $\alpha_j$. Also replace $\alpha_j z_j$ with an additional $k$ new points $C_j$. Furthermore, for each line of the form $\beta_j x_j$, insert $k$ new points into $C_j$ and join each to $\beta_j$. See Figure 4.1a.) Thus altogether, $C_j$ contains $(k-3)(k-1) + k + (k+1)k = \frac{2k^2 + k + 3}{2}$ points, which since $k \equiv 1 \pmod{4}$ is an even number.

Thus when $k \equiv 1 \pmod{4}$, all of the sets $A_i$, $B_j$, and $B_{k-1}$ contain an even number of points.

"Mirror image" sets $A_i'$, $B_j'$, $C_j$, and point $b'$ are constructed analogously.

Now since each of the sets $A_1$, $A_i$, $B_j$, $B_{k-3}$, $B_{k-3}^i$, $C_j$, $C_j$ have more than $k$ points and each is even, we may invoke Lemma 4a of Wang and Kleitman (1973) to conclude that there exists a $k$-regular graph on each of these sets of points. Insert such a $k$-regular graph on each such point set. Finally, join points $b$ and $b'$. Clearly, the resulting graph $G_k$ is $(k+1)$-regular.

Now suppose $k \equiv 3 \pmod{4}$. In this case, we can construct a $k$-regular $G_k$ which is even smaller than that built for the case $k \equiv 1 \pmod{4}$ in that no "special" replacement for $\frac{k+1}{2}$ is necessary.

In $G_k$, insert midpoints $\alpha_i$ and $\beta_{ij}$ as before. For each $i = 1, \ldots, \frac{k+3}{2}$, replace $\alpha_i$ by a set $A_i$ of $2k$ points joined two by two to each $\alpha_i$. Replace each $z_j$ by a set $C_j$ consisting of $\frac{k+1}{2}$ points each joined twice to each of the $k+1$ different $\alpha_i$'s. Also add $\frac{k+1}{2}$ additional points to $C_j$ joined to all the midpoints of the different $\beta_{ij}$'s. (See Figure 4.b.)

Once again construct the "mirror image" sets $A_i'$ and $C_j'$ analogously.

Now each $A_i$ and $A_i'$ contains $2k$ points while each $C_j$ and $C_j'$ contains $\frac{k+1}{2} + \frac{k+1}{2} = k+1$ points which is also an even number since $k \equiv 3 \pmod{4}$. Thus again by the Wang and Kleitman result we can construct $k$-connected $k$-regular graphs on each of these sets and hence obtain our $(k+1)$-regular graph $G_k$. 
$k = 1 \pmod{4}$:

![Diagram](image-url)
Figure 41 (b)
5. **The Connectivity of $G_k$.**

To prove that $G_k$ is $k$-connected we proceed in two steps. First we consider an intermediate graph $G^*_k$ obtained from $G_k$ by inserting only the $C_i$'s. (From this point on we shall denote the subgraphs guaranteed by the Wang and Kleitman result on $A_i$ by $<A_i>$, on $B_{k-1}$ by $<B_{k-1}>$, etc.)

We now proceed to show $G^*_k$ to be $k$-connected. Let $u$ and $v$ be two distinct points in $G^*_k$.

Suppose first that neither $u$ nor $v$ is a midpoint.

1. If $u$ and $v$ lie in the same $C_i$ then there exist $k$ o.d. $u-v$ paths in $<C_i>$, since $<C_i>$ is $k$-connected. The analogous result holds when $u$ and $v$ lie in the same $C_j$.

2. If $u$ and $v$ lie in two different $C_i$'s, $C_j$'s or one in a $C_i$ and the other in a $C_j$, then there exist $k$ o.d. $u-v$ paths since such a set of paths exists in $G^*_k$. More precisely, suppose $u \in C_U$ and $v \in C_V$. In $C_U$ for each midpoint adjacent to $C_U$ choose a point in $C_U$ different from $u$. (Henceforth we shall refer to such a point as a *foot* of this midpoint in $C_U$.) This is possible because each midpoint has at least $k-1>2$ such feet in $C_U$. So the feet selected in this way form a set of $k$ distinct points in $C_U$ different from $u$.

Now since $<C_U>$ is $k$-connected by a well-known variation of Menger's Theorem, there exists a fan of paths in $<C_U>$ from $u$ to each of the $k$ feet chosen.

Repeat this procedure in $<C_V>$ and use these two $k$-fans, together with suitable pieces of the $k$ o.d. paths in $G^*_k$ joining $<C_U>$ contracted to a point to $<C_V>$ contracted to a point.

This argument is also valid if $u$ and $v$ are in the same $C_i$, the same $C_j$ or one is in a $C_i$ and the other in a $C_j$.

3. Suppose $u \in Y \cup Y' \cup X$ and $v \in C_i$ or $C_j$. Without loss of generality, suppose $v \in C_j$. Since $G^*_k$ is $k$-connected, there exist $k$ o.d. $u-v$ paths in $G^*_k$ and using the argument of Case 2, we can find $k$ o.d. $u-v$ paths in $G^*_k$.

4. If $(u,v) \subseteq Y \cup Y' \cup X$, then $k$ o.d. $u-v$ paths are found using the $k$-connectedness of $G^*_k$ and the fact that all $<C_j>$'s are themselves $k$-connected.
So it remains to treat the cases when at least one of \( u \) and \( v \) is a midpoint. Note that in \( G^*_k \), the midpoints have degree \( k \) if they lie between \( Y \) and \( Z \) or between \( Y' \) and \( Z' \), and they have degree \( k + 1 \) if they lie between \( X \) and \( Z \) or between \( X' \) and \( Z' \).

First suppose both \( u \) and \( v \) are midpoints in \( G^*_k \).

Let us now first consider the case when \( u \) and \( v \) are adjacent to the same \( \langle C_i \rangle \) (or \( \langle C_j' \rangle \)). Then \( u \) and \( v \) are adjacent to at least \( k-1 \) different points of \( C_i \) respectively. By Menger's Theorem there are at least \( k-1 \) o.d. \( u \cdot v \) paths in the subgraph \( \langle C_i \cup \{u,v\} \rangle \) of \( G^*_k \). Call them \( P_1, ..., P_{k-1} \). Also since \( G'_k \) is 2-connected, there is a cycle \( N \) in \( G'_k \) containing the lines \( L_u \) and \( L_v \) (whose midpoints are \( u \) and \( v \)) and hence \( N \cdot L_u \cdot L_v \) is a path which may be used to construct a path \( Q \) joining \( u \) and \( v \) which is openly disjoint from all the \( P_i \)'s. Thus \( \{P_1, ..., P_{k-1}, Q\} \) is the desired \( u \cdot v \) \( k \)-skein.

Now suppose \( u \) and \( v \) are adjacent (as midpoints) to different \( \langle C_j \rangle \)'s, say \( \langle C_u \rangle \) and \( \langle C_v \rangle \) respectively. Now in \( G'_k \) the 2 points corresponding to the contractions of \( \langle C_u \rangle \) and \( \langle C_v \rangle \) are joined by \( k \) o.d. paths. Call them \( P_1, ..., P_k \). One of these - say \( P_1 \) - uses line \( L_u \). Choose \( k-1 \) distinct feet of \( u \) in \( C_u \). Call this set \( U_1 \). Also for each path \( P_i, i = 1, ..., k \), choose exactly one foot in \( C_u \). Call this set \( U_2 \). We then have \( U_1 \cup U_2 = C_u \), \( U_1 \cap U_2 = \emptyset \) and \( |U_1| = |U_2| = k-1 \). Since \( \langle C_u \rangle \) is \((k-1)\)-connected, by Menger's Theorem there exist \( k-1 \) totally disjoint paths in \( \langle C_u \rangle \) joining the points of \( U_1 \) to those of \( U_2 \). A similar argument applies to \( \langle C_v \rangle \). Using these paths within \( \langle C_u \rangle \) and \( \langle C_v \rangle \) as well as paths \( P_1, ..., P_k \), we can construct \( k \) o.d. \( u \cdot v \) paths in \( G^*_k \).

Finally, suppose \( u \) is a midpoint in \( G^*_k \) but \( v \) is not. Suppose \( u \) is adjacent to \( \langle C_u \rangle \). But this is even simpler than the preceding case. In \( G'_k \), let \( P_1, ..., P_k \) be \( k \) o.d. paths joining the contraction to a point of \( \langle C_u \rangle \) with point \( v \). As before, let \( U_1 \) be a set of \( k-1 \) feet of \( u \) in \( C_u \) and choose \( U_2 \) so that it contains precisely one foot of each of the rest of the midpoints adjacent to \( \langle C_u \rangle \). Then \( |U_1| = |U_2| = k-1 \), \( U_1 \cap U_2 = \emptyset \) and since \( \langle C_u \rangle \) is \((k-1)\)-connected we can proceed as before to get \( k \) o.d. \( u \cdot v \) paths.

This completes the proof that \( G^*_k \) is \( k \)-connected.
Now insert the $A_i$’s, $A_i$’s, $(\text{and } B_{k \cdot 3} \frac{k}{2} \text{ and } B'_{k \cdot 3} \frac{k}{2})$ if $k \equiv 1 \pmod{4}$ into $G_k'$. Also insert points $b$ and $b'$ together with their respective $k$-fans to $B_{k \cdot 3} \frac{k}{2}$ and $B'_{k \cdot 3} \frac{k}{2}$. But do not join $b$ and $b'$ yet.

Actually, we will now show that $G_k - b - b'$ is $k$-connected. So suppose $(u,v) \in V(G_k) \cdot \{b,b'\}$

1. Suppose $(u,v) \cap (A_i \cup A_i') = \emptyset$, for all $i$ and $(u,v) \cap (B_{k \cdot 3} \frac{k}{2} \cup B'_{k \cdot 3} \frac{k}{2}) = \emptyset$ when $k \equiv 1 \pmod{4}$. Then since $G_k'$ is $k$-connected there are $k$ o.d. $u$-$v$ paths $P_1, \ldots, P_k$ in $G_k'$. Since all $<A_i>$’s, $<A_i'>$’s, $B_{k \cdot 3} \frac{k}{2}$ and $B'_{k \cdot 3} \frac{k}{2}$ are connected, paths $P_1, \ldots, P_k$ give rise to $k$ o.d. paths $G_1, \ldots, G_k$ joining $u$ and $v$ in $G_k$.

Before proceeding to the next case, we state and prove the following statement.

**Claim** (a) If $y_i \in Y$ corresponds to inserted subgraph $A_i$ (respectively $B_{k \cdot 3} \frac{k}{2}$) in $G_k$ and $L_k$ are the $k$ lines incident with $y_i$ in $G_k'$, then given any point $u \in A_i$ (respectively $B_{k \cdot 3} \frac{k}{2}$) there exists a $k$-fan in $A_i$ (respectively $B_{k \cdot 3} \frac{k}{2}$) which can be extended to a $k$-fan joining $u$ to the midpoints of $L_k$.

(b) Analogous statements hold for $y'_i \in Y'$ with respect to $A_i$’s (respectively $B'_{k \cdot 3} \frac{k}{2}$)

**Proof of Claim.** We prove only part (a) as (b) is proved in just the same way. Suppose $y_i$ corresponds to $A_i$. Choose any point $u \in A_i$. Then $u$ is one of exactly two feet in $A_i$ of some midpoint $x_i$. Suppose $w$ is the other of these two feet. Form a set $U$ of $k$ points by including $w$ and exactly one of the two feet of all the other $k-1$ midpoints adjacent to $A_i$. Then since $u \in U$ and $<A_i>$ is $k$-connected, there exists a fan of points from $u$ to each of the $k$ points in $U$ which in turn leads to the $k$-fan sought.

Now suppose $y_i$ corresponds to $B_{k \cdot 3} \frac{k}{2}$. That is, $y_i = y_{k \cdot 3} \frac{k}{2}$. Let $u \in B_{k \cdot 3} \frac{k}{2}$. There are two cases to consider.
First suppose $u$ is the foot of new point $b$. Then since $u$ is not a foot of any of the midpoints of $L_1, \ldots, L_k$ and since $B_{\frac{k-3}{2}}$ is $k$-connected, there is a $k$-fan of paths from $u$ to the (unique) foot of each $L_i$ in $B_{\frac{k-3}{2}}$. There are $k$ such feet and this fan clearly extends to one from $u$ to each of the $k$ midpoints of $L_1, \ldots, L_k$.

So suppose $u$ is the foot of some $L_j$ in $B_{\frac{k-3}{2}}$. Without loss of generality, suppose $u$ is the foot of some $L_j$. Then since $B_{\frac{k-3}{2}}$ is $k$-connected, there is a fan at $u$ to the feet in $B_{\frac{k-3}{2}}$ of each of the $k - 1$ lines $L_2, \ldots, L_k$. These $k - 1$ paths, together with the line from the foot of $L_j$ to the midpoint of $L_j$, clearly extends to a fan from $u$ to the midpoint of each of $L_1, \ldots, L_k$ as desired. This completes the proof of the Claim.

2. Now suppose at least one of $u, v$ lies in an $A_i, A'_i, B_{\frac{k-3}{2}}$, or $B'_{\frac{k-3}{2}}$, but that $u$ and $v$ do not both lie in the same one of these sets. Since $G_k = G_{\frac{k-2}{2}}$, there are $k$ odd $u - v$ paths in $G_k$, which together with the fans guaranteed by the above Claim, where necessary, yield $k$ odd $u - v$ paths in $G_k$.

3. If both $u$ and $v$ lie in the same $A_i$, $A'_i$, $B_{\frac{k-3}{2}}$ or $B'_{\frac{k-3}{2}}$, then since all of these subgraphs are $k$-connected, there exist $k$ odd $u - v$ paths as desired.

Thus $G_k \cdot b \cdot b'$ is $k$-connected. It remains now to add points $b$ and $b'$, join them to $s$ points each in $B_{\frac{k-3}{2}}$ and $B'_{\frac{k-3}{2}}$, as described earlier. But if we join $b$ to its $s$ points, the resulting graph $G_k \cdot b$ is $k$-connected by Menger's Theorem and then joining $b'$ to its $s$ neighbours, the resulting graph $G_k \cdot b \cdot b'$ is $k$-connected by the same reasoning. But then adding line $bb'$ we obtain $G_k$ which must be $k$-connected. Clearly $G_k$ is $k - 1$-regular.

Finally we note that trivially the $2k$ points of $Z \cup Z'$ lie on no common cycle in $G_k$ since $Z \cup Z'$ is an independent set and $\forall (G_k) : Z \cup Z' = Y \cup Y' \cup X_1 = k - 1$.
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