The purpose of this note is to give a direct proof of some recent important results of E.B. Dynkin and A. Mandelbaum [2]. This also provides immediately the results in [3] with a very simple proof. This is achieved by avoiding the use of Poisson process.

Let us set up some notation. Let \((L, E, \mu)\) be a probability space and \((X_k^k, \kappa_k, \mu^k)\) be the \(k\)-fold produce probability space. Let \(h_k(x_1, \ldots, x_k)\) be a symmetric function of \(k\)-variables. We call it canonical if \(\int h_k(x_1, \ldots, x_{k-1}, y) \, du = 0\) for all \(x_1, \ldots, x_{k-1}\). Let \(X_1, \ldots, X_n\) be a i.i.d. \(X\)-valued random variable on a probability space with distribution \(\mu\).
ON A LIMIT THEOREM AND INVARIANCE PRINCIPLE FOR SYMMETRIC STATISTICS

Approved for public release; distribution unlimited.

V. Mandrekar

Technical Report No. 142
July 1986
ON A LIMIT THEOREM AND INVARIANCE PRINCIPLE FOR SYMMETRIC STATISTICS

by

V. Mandrekar

Department of Statistics and Probability
Michigan State University
East Lansing, MI 68824

and

Center for Stochastic Processes
Department of Statistics
University of North Carolina
Chapel Hill, NC 27514

*This research supported by ONR N00014-85-K-0150 and the Air Force Office of Scientific Research Contract No. F49620 85C 0144.
0. Introduction: The purpose of this note is to give a direct proof of some recent important results of E.B. Dynkin and A. Mandelbaum [2]. This also provides immediately the results in [3] with a very simple proof. This is achieved by avoiding the use of Poisson process. Let us set up some notation. Let \((X, \Sigma, \mu)\) be a probability space and \((X^k, \Sigma^k, \mu^k)\) be the k-fold produce probability space. Let \(h_k(x_1, \ldots, x_k)\) be a symmetric function of \(k\)-variables. We call it canonical if \(\int h_k(x_1, \ldots, x_{k-1}, y) d\mu = 0\) for all \(x_1, \ldots, x_{k-1} \in X^{k-1}\). Let \(X_1, \ldots, X_n\) be a i.i.d. \(X\)-valued random variable on a probability space with distribution \(\nu\). As in [2], define
\[
\sigma^n_k(h_k) = \sum_{1 \leq s_1 < \ldots < s_k \leq n} h_k(X_{s_1}, \ldots, X_{s_k}), \text{ for } k \leq n
\]
\[
= 0 \quad \text{ for } k > n.
\]
Let \(H = \{ (h_0, h_1, \ldots) : h_k \text{ canonical and } \sum_{k=1}^{\infty} \frac{1}{k!} \| h_k \|_2^2 < \infty \}\) where \(h_0\) is a constant and \(\| \|_2\) is the norm in \(L^2(X^k, \Sigma^k, \mu^k)\). On \(H\) define
\[
\| h \|_2^2 = \sum_{k=0}^{\infty} \frac{\| h_k \|_2^2}{k!}.
\]
\(H\) is the so-called exponential (Foch) space of \(L^2(\Sigma, \mu)(\phi \in L^2(X, \Sigma, \mu) \text{ with } \mathbb{E}(\mathbb{E}(X) = 0))\). It is a Hilbert space under coordinate addition, scalar multiplication and \(\| \|\). For each \(\phi \in L^2_0(X, \Sigma, \mu)\),
\(h_\phi \in H\) with \(h_\phi = (\phi(x_1), \ldots, \phi(x_k))\). It can be easily seen that \(\text{sp}\{ h_\phi : \phi \in L^2_0(X, \Sigma, \mu) \}\) is dense in \(H\). Define for each \(h \in H\),
\[
(0.1) \quad Y_n(h) = \sum_{k=0}^{\infty} n^{-k/2} \sigma^n_k(h_k).
\]
Since \(\sigma^n_k(h_k) = 0\) for \(k > n\), this is a finite sum. Also, let
\[
(0.2) \quad Y_n^t(h) = \sum_{k=0}^{\infty} n^{-k/2} \sigma^n_{[nt]}(h_k).
\]
The main purpose is to show directly that \(Y_n(h) \overset{D}{\rightarrow} \sum_{k=0}^{\infty} \frac{I_k(h_k)}{k!}\) where \(\overset{D}{\rightarrow}\) denotes convergence in distribution and \(I_k(h_k)\) denotes Ito-Wiener multiple
integral of \( h_k \) with respect to Gaussian random measure \( W \) with
\[
EW(A)W(A') = \mu(A \cap A').
\]

In the next section we discuss the convergence of \( Y_n(h) \). We observe that for \( \phi \in L^2_0(\Sigma, \mu) \)
\[
Y_n(h^\phi) = \sum_{k=0}^{n} \frac{\phi(X_{s_1}) \cdots \phi(X_{s_k})}{\sqrt{n}^{s_k}} \cdots \frac{\phi(X_{s_1}) \cdots \phi(X_{s_k})}{\sqrt{n}^{s_k}}
\]
\[
= \prod_{1}^{n} (1 + \frac{\phi(X)}{\sqrt{n}}).
\]

Let us observe that for any \( \epsilon > 0 \),
\[
\sum_{j} P\left( |\phi(X_j)| > \sqrt{\epsilon} \right) = \sum_{j} P\left( |\phi(X_j)|^2 > \epsilon \right) \leq \|\phi\|^2 < \infty.
\]

Hence by Borel-Cantelli lemma, a.s. (for \( j \leq n \))
\[
|\phi(X_j)| \leq \sqrt{\epsilon j} \leq \sqrt{\epsilon n} \quad \text{for} \quad j \geq \text{some } N(\omega) \quad (N(\omega) < \infty).
\]

But
\[
\prod_{1}^{n} (1 + \frac{\phi(X)}{\sqrt{n}}) = \prod_{1}^{n} (1 + \frac{\phi(X)}{\sqrt{n}}) \quad \text{giving for a.s. } \omega, \text{ so}
\]
\[
l_1 \lim Y_n(h^\phi) = \lim_{n} \prod_{1}^{n} (1 + \frac{\phi(X)}{\sqrt{n}}). \quad \text{Thus } WLOG, \text{ we can assume for } n \text{ large}
\]
\[
\phi(X_{j}) \frac{1}{\sqrt{n}} < 1 \quad \text{a.s. for all } j \leq n \text{ and } Y_n(h^\phi) = \prod_{1}^{n} (1 + \frac{\phi(X)}{\sqrt{n}}). \quad \text{Taking log on both sides and expanding log}(1+x) \text{ we have}
\]
\[
\log \prod_{1}^{n} (1 + \frac{\phi(X)}{\sqrt{n}}) = \frac{\phi(X)}{\sqrt{n}} - \frac{1}{2} \sum_{1}^{n} \frac{\phi(X)^2}{\sqrt{n}} + \epsilon_n(\phi)
\]

where \( \epsilon_n(\phi) \overset{P}{\rightarrow} 0 \) by the WLLN and since \( \max_{1}^{n} |\phi(X)\| \overset{P}{\rightarrow} 0 \) by Chebychev's Inequality,
i.e. the $(Y_n(h^\phi)) \overset{D}{=} \exp[I_1(\zeta) - \frac{1}{2}||\zeta||_2^2]$. Using Cramér-Wold device and the above argument we get

**0.3 Lemma:** For any finite subset $\{\phi_1, \ldots, \phi_k\} \subseteq L^2(\mathcal{X}, \mathcal{I}, \mu)$

$(Y_n(h^{\phi_1}), \ldots, Y_n(h^{\phi_k})) \overset{D}{=} (\exp(I_1(t_1)) - \frac{1}{2}||t_1||_2^2, \ldots, \exp(I_1(t_k)) - \frac{1}{2}||t_k||_2^2)$.

As a consequence, we get for $\{\phi_i, i \in I\}$ a finite subset of $L^2(\mathcal{X}, \mathcal{I}, \mu)$ and $\{c_i, i \in I\} \subseteq \mathbb{R}$,

$(0.3)' \quad Y_n(\sum_{i \in I} c_i h^{\phi_i}) \overset{D}{=} \sum_{k=0}^{\infty} \frac{I_k(\sum_{i \in I} c_i h^{\phi_i})}{k!}$.

We now observe that for $h, h' \in H$,

$\sum_{k=0}^{\infty} \frac{I_k(h)}{k!} h' = \sum_{k=0}^{\infty} \frac{I_k(h')}{k!} h$ by (2), p. 744. Also,

$(0.4) \quad E[Y_n(h) - Y_n(h')]^2 = \sum_{k=0}^{\infty} \frac{I_k(\sum_{i \in I} c_i h^{\phi_i})}{k!} h_k - h'_k = E||h - h'||^2$.

Since $Ee^{\sum_{k=0}^{\infty} \frac{I_k(h)}{k!} h_k} = (\sum_{k=0}^{\infty} \frac{I_k(h)}{k!} h_k)$ by (2), p. 744). Also,

$(0.5) \quad E(\sum_{k=0}^{\infty} \frac{I_k(h)}{k!} h_k)^2 = ||h - h'||^2$.

Thus we get

$(0.6) \quad \textbf{Theorem:} \text{ For any } h \in H,$

$Y_n(h) \overset{D}{=} W(h) = \sum_{k=0}^{\infty} \frac{I_k(h)}{k!}$

**Proof:** Let $h \in H$ and $\varepsilon > 0$. Choose $h' = \sum_{i \in I} c_i h_i$ such that $||h - h'||^2 < \varepsilon/2$.

Now consider for $t \in \mathbb{R}$

$|e^{itY_n(h)} - e^{itY_n(h')}| + |E(e^{itY_n(h)} - e^{itY_n(h')})| + |E(e^{itY_n(h)} - e^{it\mathcal{W}(h')})|$

$+ |E(e^{it\mathcal{W}(h')} - e^{it\mathcal{W}(h)})|$.

Using Schwartz's Inequality and the fact $|e^{ix} - 1| \leq |x|$ we get that the first
and third term of the above inequality are dominated by $t^2 E\|h-h'\|^2$ using (0.4) and (0.5). Hence by (0.3)',

$$\lim_{n \to \infty} Ee^{i t Y_n(h)} - Ee^{i t W(h)} = \varepsilon/2.$$ 

As $\varepsilon$ is arbitrary we get the result.

Finally, we make some observations to be used later.

(0.7) \[ Y^t_n(h) = \sum_{k=0}^{\lfloor t \rfloor} \frac{n^{-k/2}}{k!} \sum_{1 \leq s_1 < \cdots < s_k \leq \lfloor t \rfloor} \phi(X_{s_1}) \cdots \phi(X_{s_k}) = \sum_{k=0}^{\lfloor t \rfloor} \frac{\phi(X_{s_k})}{k! \sqrt{n}}. \]

Also, $\min(t,s) \cup (A \cup A')$ is a covariance on $[0,\infty) \times \Omega$ giving that there exists a centered Gaussian process $W(t,A)$ with $E(W(t,A)W(s,A')) = \min(t,s) \cup (A \cup A')$. Let for $T < \infty$

$$\mathcal{H}_T = \{(h_0, h_1, \ldots) \in H : \sum_{k=0}^T \frac{\|h_k\|^2}{k!} < \infty\}.$$ 

1. **Invariance Principle:** Let $D[0,T]$, $(T \leq \infty)$ be the space of right continuous functions on $[0,T] ([0,\infty))$ with left limits at each $t \leq T$. The space $D[0,T]$ is endowed with Skorohod topology [1]. The topology in $D[0,\infty)$ is the one described in Whitt [4]. We note that

$$X_{\lfloor nt \rfloor} = \sum_{k=1}^{\lfloor nt \rfloor} \frac{\phi^2(X_k) - E\phi^2}{n}$$

has stationary independent increments. So for $\varepsilon > 0$

$$P\left( \sup_{0 \leq t \leq T} |X_{\lfloor nt \rfloor}| > \varepsilon \right) \leq C.P\left( |X_{\lfloor nt \rfloor}| \geq \varepsilon \right) \to 0$$

by the weak law of large numbers. Using this, the arguments preceding Lemma 0.3, invariance principle and Cramér-Wold device we get the following analogue of Lemma 0.3.

**Lemma 1.1:** \( (Y^t_n(h_1), \ldots, Y^t_n(h_k)) \xrightarrow{D_{k,T}} (\exp(1^t(\phi_j) - \frac{1}{2} t \|\phi_j\|^2), j = 1, \ldots, k) \)
where \( I^t(\cdot_\cdot_\cdot) = \int_{[0,t]} (u) \cdot \phi_j(x) W_k(du, dx) \). Here \( D_{k,T} \rightarrow \) denotes convergence in \( D^k[0,T] \) with respect to product topology.

We note that \( W(t,A) \) is a Brownian motion for each \( A \in \Sigma \). Thus we can choose \( I^t(\cdot_\cdot_\cdot) \) continuous for each \( \phi \) and a martingale in \( t \) as \( I^t(\cdot_\cdot_\cdot) = \int \phi(x) W(t, dx) \). We get for \( \{c_1, \ldots, c_k\} \subseteq \mathbb{R} \), \( (k \text{ finite}) \),

\[
Y^t(\sum_{j=1}^k c_j \phi_j) = \sum_{j=1}^k c_j \exp(I^t(\phi_j) - \frac{1}{2} t \|\tau_j\|^2).
\]

Let \( \phi \in L^2(X, \Sigma, \mu) \), \( \|\tau\| = 1 \), and denote

\[
(c^k)^t = \phi(x_1) \ldots \phi(x_k) I_{[0,t]}(u_1) \ldots I_{[0,t]}(u_k).
\]

Define \( I^t(\cdot_\cdot_\cdot) = k! H_k(t, I(\cdot_\cdot_\cdot)) \) where \( H_k \) is Hermite polynomial, i.e.

\[
\sum_{k=0}^\infty H_k(t,x) = \exp(yx - \frac{1}{2} t^2).
\]

For \( \phi \in L^2(X, \Sigma, \mu) \), \( \|\tau\| = 1 \), we define for \( (h^+)^t = (1, \phi^t, (\phi^2)^t, \ldots) \),

\[
W(h^')^t = \sum_{k=0}^\infty \frac{I^t_{k}(\cdot_\cdot_\cdot)}{k!} ,
\]

and extend it linearly to \( (\sum_j \phi_j^j)^t \). It is a martingale. Let \( h \in H_T \), \( \{h(n)\} \) a sequence in \( \text{sp} \{(h^+)^t \} \), \( \phi \) in CONS in \( L^2_0(X, \Sigma, \mu) \) in \( H_T \), then

\[
P(\sup_{t \leq T} |W^t(h(n) - h(m))| > \varepsilon) \leq E|W^T(h(m) - h(n))|^2
\]

\[
= \sum_{k=0}^\infty \frac{\|h_k(m) - h_k(n)\|^2}{k!}
\]

using Doob's inequality and argument as in (0.5). Define for \( h \in H_T \),

\[
W^t(h) = -\lim_{n \to \infty} W^t(h_n) \quad \text{where the limit is uniform on compact for } h_n \to h.
\]

Then \( W^t(h) \) is right continuous martingale and has the same distribution as

\[
\sum_k I^t_{k}(h_k)/k!.
\]

Now we derive the main theorem of [3].

**Theorem 1.2:** \( Y^t_n(h) \to W^t(h) \) in \( D[0,T] \) for \( h \in H_T \) for each \( T < \infty \).
Proof: Let $h \in H$ and $\varepsilon > 0$, choose $h'_{nk} \in \text{sp}\{h' : \phi \in L^2_0(X, \mathbb{E}, \mathbb{P}) \} \ni h_k \rightarrow h$. Now define

$$X_{nk}^* = Y^*_n(h'_k), Z_n^* = Y^*_n(h), X_k^* = W^*_k(h'_k) \text{ and } X = W^*(h).$$

Then $X_{nk}, h_k \in D[0,T]$ for each $T < \infty$ by Lemma 1.1. Also $X_k^* \overset{D}{\rightarrow} X$ as $n \rightarrow \infty$ in $D[0,T]$. In addition,

$$P(\sup_{0 \leq t \leq T} |X_{nk}^* - Z_n^*| > \varepsilon) \leq E|Y_n^*(h - h'_k)|^2 \leq T||h - h'_k||$$

giving $\lim_{n,k \rightarrow \infty} P(\rho(X_{nk}, Z_n) > \varepsilon) = 0$ with $\rho$ being the Skorohod metric on $D[0,T]$. This implies by ([1], Thm 4.2, p. 25) that $Z_n^* \overset{D}{\rightarrow} W^*(h)$ in $D[0,T]$ ($T < \infty$) giving the result.

Remark: In the above arguments we may use an interpolated version of $Y_n^*(h)$ from the beginning and use appropriate version of Donsker's Invariance Principle to conclude above convergence occurs in $D[0,T]$ in sup norm giving $W^*(h)$ continuous.

References


END

4-1-81

DTIC