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An Overview

New developments in computer architecture (parallel computers and multi-processors) and a renewed interest in 3-dimensional problems and higher-order problems have revived interest in iterative methods for elliptic and parabolic finite-difference/finite-element equations.

Consider the linear system

\[ Ax = y \]

which arises from the discretization of an elliptic boundary-value problem. A direct (linear stationary) iterative method arises from the splitting

\[ A = M - N \]

and takes the form

\[ Mx^{k+1} = Nx^k + y. \]

The study of the rates of convergence of such an iterative scheme is essentially the study of the spectral radius \( \rho \) of the iteration matrix \( K = M^{-1}N \).

Parter has been studying these problems for over 20 years. Since 1977 he has been working with several researchers at the Los Alamos National Laboratory, studying methods which are easily implemented on the CRAY. This work has led to several reports and publications. Most of this work has been concerned with finite-difference equations. However, in [1] Parter and Steuerwalt developed the theory for finite-element equations. In a certain sense this paper was the culmination of this line of research. The survey papers [2] and [7] describe the general approach, the basic results and the significance of these results in applications.

This theory yields results of the form

\[ \rho = 1 - Ch^p + o(h^p) \]

\[ 1 > \rho \geq 1 - Ch^p + o(h^p) \]
where $C > 0$, $p > 0$ are constants determined by the problem, and $h$ is the discretization parameter which is going to zero. Unfortunately, in any practical situation, $h$ is small and $p$ is near “1”. Thus, while the results are useful and properly describe the rates of convergence of many of the methods currently in use, the results also imply that it is imperative that we consider other solution schemes.

In recent years there have been two new approaches to this problem: (1) Multigrid methods and (2) (wisely) preconditioned conjugate gradient methods. In both of these approaches, which are intrinsically related, one obtains error estimates of the form

\[(6) \quad \|e^{(k+1)}\|_A \leq \rho \|e^{(k)}\|_A\]

where $e^k$ and $e^{k+1}$ are the errors at the $k^{th}$ and $(k+1)^{st}$ iteration, $\| \|_A$ denotes the operator norm and $\rho$ is a constant, independent of $h$, $0 < \rho < 1$.

However it seems we are just beginning to understand these powerful methods. In particular, there are the questions: (1) in preconditioning methods, how does one choose an effective preconditioner, (2) in multigrid methods, how do we choose the interpolation and projection operators? The smoothing operators? And, how can we obtain sharp, numerical estimates on the quantity $\rho$?

Working with David Kamowitz, a graduate student in the Computer Sciences Department who has just completed his phd., Parter has been involved in several multigrid projects on both a theoretical and an experimental level.

In [3] they studied difference equations for boundary value problems for ordinary differential equations of the form

\[(7a) \quad -(pu')' + bu' + qu = f, \quad 0 \leq x \leq 1\]

\[(7b) \quad u(0) = u(1) = 0.\]

This paper contains both a theoretical multigrid analysis and a computational study. In [13] Kamowitz extends these ideas to study singular perturbation problems with and without turning points. Thus, [13] is concerned with problems of the form

\[(8a) \quad -\epsilon u'' + bu' = f, \quad 0 \leq x \leq 1\]
\[ u(0) = u(1) = 0. \]


\( \nabla \cdot a(x,y) \nabla u = f, \quad \Omega, \)

\( u = 0, \quad \partial \Omega. \)

Earlier studies were limited to the Poisson equation.


\[ \psi(t) = \frac{7}{2} \int_{0}^{1} E_1(|z - z'|) \psi(z') dz' + h(z), \quad 0 \leq z \leq 1. \]

The lengthy report [12] by Faber, Manteuffel and Parter develops some basic ideas of "norm equivalence" as well as establishing some new results on effective preconditioning methods for the important "indefinite case". Earlier studies of preconditioning methods have been based on the concept of "spectral equivalence". Unfortunately, this approach limits one to the case where the symmetric part of the operator is positive definite.
PUBLICATIONS


Contacts with other Research Groups

1. Professor Parter is a member of the SCIENCE COUNCIL FOR ICASE (Institute for Computer Applications in Science and Engineering, NASA Langley Research Center, Hampton, Virginia). In this capacity he visits ICASE one or two times each year. During these visits he spends some time discussing scientific matters with the staff.

2. Professor Parter is a member of the Panel for Applied Mathematics for the Center for Applied Mathematics at the National Bureau of Standards. In this capacity he visits the Center for Applied Mathematics at least once a year. During these visits he spends some time discussing scientific matters with the staff.

3. Professor Parter is a regular visitor to the Los Alamos National Laboratory where he collaborates with the staff of the numerical analysis group C-3. Some of this work is closely related to the work under this contract.
Lectures and Visits

I. The following is a partial list of Professor Parter's lectures and visits.


2. International Multigrid Conference, April 6-8, 1983, Copper Mountain, Colorado.

3. Conference on Large Scale Scientific Computation, May 17-19, 1983, Madison, WI. Professor Parter was the chairman of the organizing committee and the editor of the published proceedings.


9. Workshop on Computational Mechanics, September 24-25, 1984. NASA Lewis Research Center, Cleveland, Ohio. Professor Parter was the chairman of the organizing committee.


II. The following is a list of David Kamowitz's lectures and visits.


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