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A MODEL FOR THE EXTRACTION OF PERIODIC WAVEFORMS BY TIME DOMAIN AVERAGING

by

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SUMMARY

It is shown that the existing comb filter model of time domain
averaging does not correctly describe the extraction of periodic waveforms
from additive noise using a digital computer because it assumes a knowledge
of the signal over an infinite time and the result it produces is not
exactly periodic. A revised model is presented which requires only a
finite number of samples and which produces a result which is periodic. It
is demonstrated that the rejection of periodic noise of a known frequency
can be optimized by the selection of a suitable number of averages.
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1. INTRODUCTION

Time domain averaging is a signal processing technique which may be used to extract repetitive signals from additive noise. It requires either an accurate knowledge of the repetition frequency of the desired signal, if it is periodic, or else a second signal which is synchronous with the first but is free of noise. Using either the repetition frequency or the synchronous signal, successive blocks of the noisy signal may be sampled and ensemble averaged. When sufficient averages are taken, it is found that the noise in the ensemble average tends to cancel, leaving an improved estimate of the desired signal.

In mechanical engineering, there are many applications in which a periodic signal must be extracted from noise. One important example of this type of problem is in the analysis of the vibration of rotating machines, when it may be necessary to extract a periodic signal such as the tooth meshing vibration of a single gear from the total vibration of the machine, in order that the extracted waveform may be checked for signs of a malfunction. In practice, this averaging is commonly performed using a digital computer or other digital instrument.

Some understanding of the process of time domain averaging is advisable if the analyst is to appreciate the limitations of the technique and to optimize its performance for a particular application. Until now, time domain averaging has been modelled by the convolution of the noisy signal with a finite train of impulses in which the time between the impulses is equal to the period of the signal [1]. It has been shown that this process is equivalent in the frequency domain to the multiplication of the Fourier transform of the signal by a comb filter, thus passing only the fundamental and harmonic frequencies of the desired signal [1]. Increasing the number of averages taken narrows the teeth of the comb and reduces the level of the side lobes between the teeth, thus improving the estimate of the desired signal.

At first it would appear that the same comb filter model can be applied to the extraction of periodic waveforms [2], but it is shown in this memorandum that there are two features of the model which affect its performance in this context. Firstly, the model assumes a knowledge of the noisy signal over infinite time. Secondly, while the time domain average may approximate the desired periodic signal, it is not exactly periodic, and so it can only be represented completely by a signal which is defined over an infinite time. Clearly, it is not possible to meet these conditions in practice when using a digital computer to calculate the time domain average because a knowledge of the signal over infinite time is not available, nor could it be processed. Hence there is a need for a revised model for the time domain averaging of periodic signals which requires a knowledge of the signal over only a finite time, and produces a time domain average which is itself periodic and can therefore be represented completely by a single period.

This memorandum shows that such a model can be obtained from the existing comb filter model of time domain averaging by the application of a rectangular window to the noisy signal in the time domain and by the sampling of the Fourier transform of the signal in the frequency domain. These extra operations reveal some interesting features in the performance of the technique which are demonstrated using simple numerical examples.
Furthermore, it is demonstrated that the number of averages taken can be selected to optimize the rejection of periodic noise of a known frequency.

2. EXISTING MODEL

This section examines the application of the existing comb filter model of time domain averaging to the extraction of periodic signals and reveals the problems which arise.

Consider a signal \( y(t) \) which is composed of a periodic signal \( x(t) \) with known period \( T_R \) and an additive noise component \( e(t) \):

\[
y(t) = x(t) + e(t)
\]  

(1)

The convolution of signal \( y(t) \) with an ideal unit impulse \([3]\) located at \( t = -nT_R \), denoted by \( \delta(t+nT_R) \), causes \( y(t) \) to be shifted in the negative time direction by \( nT_R \), as shown in Figure 1. The shifted signal is described by [3]:

\[
y(t+nT_R) = y(t) \ast \delta(t+nT_R)
\]  

(2)

Now consider a function \( c(t) \) comprising a train of \( N \) ideal impulses of amplitude \( 1/N \) spaced at intervals \( T_R \):

\[
c(t) = \frac{1}{N} \sum_{n=0}^{N-1} \delta(t+nT_R)
\]  

(3)

The convolution of \( y(t) \) and \( c(t) \), illustrated in Figure 2, is given by:

\[
c(t) \ast y(t) = \frac{1}{N} \sum_{n=0}^{N-1} y(t+nT_R)
\]  

\[
= \frac{1}{N} [y(t)+y(t+T_R)+\ldots+y(t+(N-1)T_R)]
\]  

(4)

By definition, the time domain average \( a(t) \) is:

\[
a(t) = \frac{1}{N} \sum_{n=0}^{N-1} y(t+nT_R)
\]  

(5)

This equation has the same form as that proposed in the existing comb filter model [1], except that the waveform has been shifted to the right instead of to the left. Combining Equations 4 and 5 gives:

\[
a(t) = c(t) \ast y(t)
\]  

(6)

By the convolution theorem [3], the Fourier transform of \( a(t) \) is:

\[
A(f) = C(f) \cdot Y(f)
\]  

(7)
That is, the calculation of the time domain average is equivalent to the multiplication of the Fourier transform \( Y(f) \) of the signal by the function \( C(f) \). It can be shown [4] that \( C(f) \) is given by:

\[
C(f) = \frac{1}{N} \sum_{n=0}^{N-1} e^{i2\pi nT \cdot f} = \frac{1}{N} \frac{\sin(\pi mT f)}{\sin(\pi T f)}
\]  

Figure 3 illustrates the form of the amplitude spectrum \(|C(f)|\) for \( N=1,2,4,8,16 \) and 32. The spectrum takes the form of a comb with the teeth of the comb spaced at intervals \( f_R = 1/T \). Note that the peaks of the teeth of the comb have unit amplitude regardless of \( N \) because:

\[
\lim_{u \to 0} \frac{\sin(N(u+2m\pi))/\sin(u+2m\pi)}{\sin(u+2m\pi)} = N \text{ where } m=\text{integer}
\]

It can be seen from Equation 8 that increasing the number of averages \( N \) narrows the teeth of the comb and also reduces the amplitude of the side lobes between the teeth, so that for very large \( N \), only frequencies at exact multiples of the repetition frequency \( f_R \) will be passed. This is the form of the comb filter described by the existing model [1]. A detailed analysis of the properties of the filter when applied to the extraction of periodic waveforms has been made [2], including estimates of the equivalent noise bandwidth and the effects of jitter.

However, there are two features of the model which restrict its application to the extraction of periodic waveforms using a digital computer. Firstly, there are no bounds placed on the value of \( t \), as it will be observed from Figure 2 that the signal \( y(t) \), and hence the time domain average \( a(t) \), extend to infinite time. That is, the model assumes that \( y(t) \) is known for all \( t \) and therefore that the time domain average \( a(t) \) is defined for all \( t \), even though only a finite number of averages is calculated. In practice, the signal \( y(t) \) can only be examined for a finite time, which in effect applies a window to the signal. The existing model does not include the effects of this window and so cannot correctly predict the performance of the technique. Secondly, note that \( C(f) \) is continuous in \( f \), hence for general \( Y(f) \), \( A(f) \) will be continuous in \( f \) and so cannot be periodic in \( T_R \). Although the analyst may require an estimate of the periodic signal \( x(t) \), the result \( a(t) \) is not periodic, and can only be represented completely by a signal which extends over infinite time. Again, this cannot be achieved in practice. However, when a very large number of averages is taken, the result may begin to approximate a periodic signal and the effects of these restrictions will be reduced, but for fewer averages the existing model will not correctly describe the averaging process.

3. REVISED MODEL

In this section a revised model is developed to describe the extraction of periodic signals by time domain averaging. The revised model overcomes the problems with the existing model in that it requires a knowledge of only a finite block of the noisy signal and it produces a
result which is exactly periodic. Furthermore, the model includes the effects of the sampling of the signal at a frequency $f_s$.

Consider a rectangular window $v(t)$ of unit amplitude and width $T_R$ centred at $t=0$, as shown in Figure 4:

$$v(t) = \begin{cases} 
1.0 & |t| < T_R/2 \\
0.5 & |t| = T_R/2 \\
0 & \text{elsewhere}
\end{cases}$$

(10)

The Fourier transform $V(f)$, also shown in Figure 4, is given by [3]:

$$V(f) = \frac{T_R \sin(\pi T_R f)}{(\pi T_R f)}$$

(11)

Now shift the window in the positive time direction by an amount $(T_R/2)-(T_s/2)=(T_R-T_s)/2$ where $T_s=1/f_s$ is the period between the samples of the input signal. The shifted window $w(t)$ is defined by:

$$w(t) = v(t-(T_R-T_s)/2)$$

(12)

Its Fourier transform $W(f)$ is obtained by the time shifting theorem [3]:

$$W(f) = \frac{e^{-j\pi(T_R-T_s)}}{\pi T_R f} \frac{V(f)}{e^{j\pi(T_R-T_s)/2}}$$

(13)

Both $w(t)$ and $W(f)$ are illustrated in Figure 5. The window is defined in this manner with the edges of the window located midway between the sampling impulses to avoid the problem of having an impulse occurring at the edge of the window.

Now consider an infinite train $s(t)$ of ideal unit impulses with the spacing between the impulses given by $T_s$:

$$s(t) = \sum_{j=-\infty}^{\infty} \delta(t-jT_s)$$

(14)

Its Fourier transform $S(f)$ is given by [3]:

$$S(f) = f_s \sum_{j=-\infty}^{\infty} \delta(f-jf_s)$$

(15)

Both $s(t)$ and $S(f)$ are illustrated in Figure 6.

Consider now the sampling of the signal $y(t)$ at the frequency $f_s$ over the window of duration $T_R$, as illustrated in Figure 7. The window, defined by $w(t-nT_R)$, consists of the window $v(t)$ shifted by $t=nT_R$. The sampling of the signal is produced by the multiplication of $y(t), v(t)$ and $s(t)$ to give the result shown in Figure 7. This result is then convolved with the unit impulse $\delta(t+nT_R)$ located at $t=-nT_R$, as illustrated in Figure 8, thus performing a shift of the sampled signal by $nT_R$ in the negative time direction. The result is given by:

$$\delta(t+nT_R) * [y(t) * v(t-nT_R) * s(t)] = y(t+nT_R) * v(t) * s(t+nT_R)$$

(16)
Now the function \( s(t) \) is periodic in \( T_s \). If \( T_s \) is chosen such that an integral number \( M \) of samples occurs per repetition period \( T_R \) then:

\[
T_R = MT_s
\]  

(17)

Therefore \( s(t) \) will also be periodic in \( T_R \), so that:

\[
s(t) = s(t+nT_R)
\]  

(18)

It is then possible to replace \( s(t+nT_R) \) in Equation 16 by \( s(t) \) to give:

\[
\delta(t+nT_R)[y(t).w(t-nT_R).s(t)] = y(t+nT_R).w(t).s(t)
\]  

(19)

The same result could have been achieved by shifting the signal \( y(t) \) and then sampling and windowing, but the above approach is instructive as it demonstrates that a knowledge of the signal over infinite time is not necessary.

An estimate of the revised time domain average \( g(t) \) is given by:

\[
g(t) = \frac{1}{N} \sum_{n=0}^{N-1} y(t+nT_R).s(t).w(t)
\]  

(20)

But \( s(t) \) and \( w(t) \) are both independent of \( n \) and so can be moved outside the summation giving:

\[
g(t) = \frac{s(t).w(t)}{N} \sum_{n=0}^{N-1} y(t+nT_R)
\]  

(21)

Substituting Equation 5 into the above gives:

\[
g(t) = s(t).w(t).a(t)
\]  

(22)

Note that although \( a(t) \) is not bounded in time, \( g(t) \) is bounded in time because of the effect of the window \( w(t) \). The revised time domain average therefore satisfies the requirement that a knowledge of the signal over only a finite time be required.

By the convolution theorem [3], the Fourier transform of \( g(t) \) is:

\[
G(f) = S(f) * W(f) * A(f)
\]  

(23)

\( A(f) \) and \( W(f) \) are both continuous functions, so that \( W(f) * A(f) \) will also be continuous and therefore so will \( G(f) \), in which case \( g(t) \) cannot be periodic. However a periodic function can be created from \( g(t) \) by sampling \( G(f) \) in the frequency domain. This may be performed by multiplying \( G(f) \) by an infinite train of ideal impulses, denoted by \( R(f) \), with the impulses being spaced at multiples of the repetition frequency \( f_R \). The function \( r(t) \) is defined by:

\[
r(t) = \sum_{k=-\infty}^{\infty} \delta(t-kT_R)
\]  

(24)

Its Fourier transform \( R(f) \) is given by:
\[ R(f) = \sum_{k=-\infty}^{\infty} \delta(f-kf_R) \]  

(25)

The resulting Fourier transform is denoted by \( H(f) \), and the corresponding signal \( h(t) \) in the time domain will have a period \( T_R \). The functions \( h(t) \) and \( H(f) \) are given by:

\[ h(t) = g(t) * r(t) \]  

(26)

\[ H(f) = G(f) \cdot R(f) \]  

(27)

Although \( h(t) \) apparently extends to infinite time, it is periodic in \( T_R \) and so can be represented completely on a digital computer by the samples over a single period \( T_R \). Hence \( h(t) \) is periodic and can be described by a finite number of samples. The result thus satisfies both criteria.

Now examine the function \( H(f) \). Combining Equations 23 and 27 gives:

\[ H(f) = R(f) \cdot [S(f) \cdot W(f) \cdot A(f)] \]  

(28)

But \( A(f) \) is the time domain average as defined by the original model.

Combining Equations 28 and 7 gives:

\[ H(f) = R(f) \cdot [S(f) \cdot W(f) \cdot [C(f) \cdot Y(f)]] \]  

(29)

Recall that \( C(f) \), defined in Equation 8, is the comb filter of the original model, and that \( C(f) \cdot Y(f) \) was the result predicted by the original model of time domain averaging. Clearly, the convolution of \( C(f) \cdot Y(f) \) in the above equation with \( W(f) \) and subsequent operations with \( R(f) \) and \( S(f) \) will change the result markedly from that produced by the original model. For general \( y(t) \) and \( Y(f) \), this equation cannot be simplified, and so it is not possible to define a simple filter like the comb filter to describe the characteristics of time domain averaging.

4. EXAMPLES

In this section some of the properties of the time domain average which are not represented in the simple comb filter model, but are implicit in the revised model, will be demonstrated using simple numerical examples synthesized on a DEC LSI11 digital computer with the number of samples per repetition period constant at \( M=128 \). All examples were calculated using single precision sine functions of unit amplitude but with varying numbers of averages and varying frequency and relative phase of the sine waves. Results are presented in the form of the amplitude spectrum of the time domain average on a logarithmic scale of 0 to -100 dB with 0 dB being equal to unit amplitude. Only small numbers of averages have been calculated for convenience, but the results are equally applicable to the greater numbers of averages necessary in practice.

Consider first a signal consisting of sine waves at 16 and 32.5 cycles per revolution period, referred to hereafter as orders. Figure 9 shows the spectrum which is obtained after two averages. Because the component at 16
orders coincides with the centre of a main lobe of the comb filter of Figure 3b, convolution with $W(f)$ does not alter the spectrum. The component at 32.5 orders coincides with the node midway between orders 32 and 33 in the comb filter and is thus completely suppressed. The result is therefore a single spectral line at 16 orders. The components with amplitudes below -80 dB in this spectrum are errors due to the finite word length of the computer.

Figure 10 shows the spectrum which is obtained after three averages of the same signal. The component at 16 orders still coincides with a main lobe of the comb filter and so is passed with an amplitude of 0 dB, while the component at 32.5 orders now coincides with the centre of a side lobe of the comb filter lying between 32 and 33 orders, and the result is then convolved with the function $W(f)$ to produce a pattern of sidebands. Note that these sidebands are not predicted by the original model. Increasing the number of averages changes the comb filter $C(f)$ but does not change $W(f)$. Thus the amplitude passed by the comb filter is reduced by a known magnitude, but while that component does not coincide with the centre of a main lobe nor with a node of $C(f)$, $H(f)$ will contain a side lobe pattern which on a logarithmic scale will have an identical shape but a lower amplitude.

Increasing the number of averages will in general increase the rejection of a discrete noise component, but the restriction of the number of averages to a power of two will not necessarily produce the best result. There will be many instances in practice where a strong interfering signal of known frequency occurs which is not a harmonic of the repetition frequency. It may be possible to choose the number of averages to be calculated so that the interfering signal coincides with a node of the comb filter and so is strongly attenuated. For example, consider a signal at 32.05 orders. Figure 11 shows the spectrum obtained after 32 averages, consisting of a large spectral line at 32 orders and a pattern of sidebands. Yet if 20 averages are performed, the signal at 32.05 orders will coincide with the node lying between the main lobe of the comb filter at 32 orders and the first side lobe, producing the spectrum shown in Figure 12 which illustrates the attenuation of the interfering signal by more than 100 dB, even though fewer averages have been taken.

Another important factor is the frequency of the interfering signal. Figure 10 showed the spectrum obtained for a signal at 32.5 orders after three averages, and as the signal is located near the centre of the spectrum, the spectrum is nearly symmetrical. Figure 13 shows the spectrum obtained for a signal of 8.5 orders after three averages. Note the asymmetry of the pattern with the higher amplitude components appearing on the low order side of the peak. This occurs because the time domain average is a real signal and so has an amplitude spectrum which is symmetrical about $f=0$, although by convention only frequencies on the positive side of $f=0$ are shown. Nevertheless, components do exist on the negative frequency side and sidebands can extend from the negative side into the positive side and vice versa, and can either reinforce or cancel sidebands on the positive side and so produce asymmetry of the components there [3].

Now consider a signal at 55.5 orders. Figure 14 shows the spectrum obtained after three averages. The pattern is again symmetrical but this time with the higher amplitude components occurring on the high frequency side of the peak. This happens because the signal is sampled at 128 orders
by the sampling pulse train $s(t)$, so that in the frequency domain the spectrum of the signal is convolved with $S(f)$ producing replication of the spectrum about $f=0$ and about $f=f_s=128$ orders and its harmonics. Hence side lobes from the replicated spectrum will extend below 64 orders and either reinforce or cancel the side lobes there producing asymmetry.

The relative phase of signals can also affect the spectrum which is produced. Figure 15 shows the spectrum obtained after five averages of a signal with components at 20.5 and 43.5 orders which are in phase. In this instance the sidebands lying between the peaks cancel because both the signals are sinusoidal and so the upper and lower sidebands the produce are of opposite phase. Figure 16 shows the spectrum obtained after five averages for signals at the same frequencies but 180 degrees out of phase. Observe that the sidebands between the peaks reinforce. For cosinusoidal signals the opposite behaviour would be observed.

From the examples given in this section it can be seen that the extraction of periodic waveforms by time domain averaging is more complicated than the simple model of the comb filter would suggest. The rejection of noise depends on the frequency of the noise signal and on the number of averages taken, and this number need not be restricted to a power of two. In general, increasing the number of averages improves the rejection of noise, but for pure tone noise of a known frequency it may be possible to optimize the rejection of that noise by selecting the number of averages so that a node in the comb filter coincides with the frequency of the noise. However, in most instances this will not be possible, and a pattern of sidebands will be produced in the spectrum of the time domain average. Considerable asymmetry may occur in the pattern of sidebands depending upon the frequency of the noise signal. Where several pure tone signals occur, sidebands may tend either to cancel or to reinforce depending on the relative phase of the signals.

5. CONCLUSION

It has been shown that the existing simple model of a comb filter does not correctly describe the extraction of periodic waveforms from additive noise by time domain averaging because it assumes a knowledge of the noisy signal over an infinite time and the result it produces is not exactly periodic. For the model to correctly describe time domain averaging by digital computer, it should require only a finite number of samples of the signal and produce a result which is exactly periodic. It has been shown that this can be achieved by application of a rectangular window to the noisy signal in the time domain and by the sampling of the Fourier transform of the signal in the frequency domain.

The effect of varying the number of averages has been examined and it has been demonstrated that the rejection of periodic noise of a known frequency can be optimized by the selection of the number of averages so that a node in the comb filter coincides with the frequency of the noise. Thus the common practice of selecting a number of averages which is a power of two will not in general give optimal rejection.
REFERENCES


FIG. 2
FIG. 3

(a) $N = 1$

(b) $N = 2$

(c) $N = 4$
\[ y(t) \]

\[ w(t - nT_R) \]

\[ s(t) \]

\[ y(t) \cdot w(t - nT_R) \cdot s(t) \]
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