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Then we derive the Edgeworth expansion of the distribution of $\theta_T$ up to third order, and prove its validity. By this Edgeworth expansion we can see that this minimum contrast estimator is always second-order asymptotically efficient in the class of second-order asymptotically median unbiased estimators. Also the third-order asymptotic comparisons among minimum contrast estimators will be discussed.
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Abstract

Let \( \{X_t\} \) be a Gaussian ARMA process with spectral density \( f_\theta(\lambda) \), where \( \theta \) is an unknown parameter. To estimate \( \theta \) we propose a minimum contrast estimation method which includes the maximum likelihood method and the quasi-maximum likelihood method as special cases. Let \( \hat{\theta}_T \) be the minimum contrast estimator of \( \theta \). Then we derive the Edgeworth expansion of the distribution of \( \hat{\theta}_T \) up to third order, and prove its validity. By this Edgeworth expansion we can see that this minimum contrast estimator is always second-order asymptotically efficient in the class of second-order asymptotically median unbiased estimators. Also the third-order asymptotic comparisons among minimum contrast estimators will be discussed.
1. Introduction

Recently some systematic studies of higher order asymptotic theory for stationary processes have been developed. In many cases such studies have used the formal Edgeworth expansions. Thus it has been required to prove their validities. Götze and Hipp (1983) showed that formal Edgeworth expansions are valid for sums of weakly dependent vectors. Durbin (1980) and Tani-guchi (1984) showed the validity of Edgeworth expansions of statistics derived from observations which are not necessarily independent and identically distributed. However their sufficient conditions for the validity are hard to check even in the fundamental statistics.

In this paper we propose a minimum contrast estimation method which includes the maximum likelihood method and the quasi-maximum likelihood method as special cases. Suppose that \( \{X_t\} \) is a Gaussian ARMA process with spectral density \( f_\theta(\lambda) \), where \( \theta \) is an unknown parameter. Let \( \hat{\theta}_T \) be the minimum contrast estimator of \( \theta \). Then we give the Edgeworth expansion of the distribution of \( \hat{\theta}_T \) up to third order, and prove its validity. That is, as special cases we get the valid Edgeworth expansions for the maximum likelihood estimator and the quasi-maximum likelihood estimator which is defined by the value minimizing \( \int_{-\pi}^{\pi} \left[ \log f_\theta(\lambda) + I_T(\lambda)/f_\theta(\lambda) \right] d\lambda \) with respect to \( \theta \), where \( I_T(\lambda) \) is the periodogram.

In Section 7 we consider the transformed statistic \( \hat{\theta}_m = \hat{\theta}_T + \frac{1}{T} m(\hat{\theta}_T) \), where \( m(\cdot) \) is a smooth function. Then we give the valid Edgeworth expansion for \( \hat{\theta}_m \). By this Edgeworth expansion
we can see that our minimum contrast estimator is always second-order asymptotically efficient in the class of second order asymptotically median unbiased estimators if efficiency is measured by the degree of concentration of the sampling distribution up to second order. Also the third-order asymptotic comparisons among minimum contrast estimators will be given.
2. Minimum Contrast Estimator

We propose a minimum contrast estimator which includes the maximum likelihood estimator and the quasi-maximum likelihood estimator as special cases.

Let $D_d$ and $D_{\text{ARMA}}^C$ be spaces of functions on $[-\pi, \pi]$ defined by

$$D_d = \{ f: f(\lambda) = \sum_{u=-\infty}^{\infty} a(u) \exp(-iu\lambda), a(u) = a(-u),$$

$$\sum_{u=-\infty}^{\infty} (1 + |u|)|a(u)| < d, \text{ for some } d < \infty \},$$

$$D_{\text{ARMA}}^C = \{ f: f(\lambda) = \frac{\sigma^2}{2\pi} \frac{\left| \sum_{j=0}^{q} a_j z^j \right|^2}{\left| \sum_{j=0}^{p} b_j z^j \right|^2}, (\sigma^2 > 0),$$

$$\sum_{u=-\infty}^{\infty} |\sum_{j=0}^{q} a_j e^{iu\lambda}|^2 \leq c, \text{ for } |z| \leq 1,$$

$$0 < c < c < \infty \}.$$

We set down the following assumptions.

Assumption 1. \( \{X_t\} \) is a Gaussian stationary process with the spectral density \( f_\theta(\lambda) \in D_{\text{ARMA}}^C, \theta_o \in C \subseteq \Theta \subseteq \mathbb{R}^1 \), and mean 0. Here \( \Theta \) is an open set of \( \mathbb{R}^1 \) and \( C \) is a compact subset of \( \Theta \).

Assumption 2. The spectral density \( f_\theta(\lambda) \) is continuously five times differentiable with respect to \( \theta \in \Theta \), and the derivatives
Assumption 3. There exists $d_1 > 0$ such that

$$I(\theta) = \frac{1}{4\pi} \int_{-\pi}^{\pi} \left( \frac{\partial}{\partial \theta} \log f_\theta(\lambda) \right)^2 d\lambda \geq d_1 > 0, \text{ for all } \theta \in \Theta.$$ 

Suppose that a stretch $X_T = (X_1, \ldots, X_T)$ of the series $\{X_t\}$ is available. Let $\Sigma_T = \Sigma_T(\theta_0)$ be the covariance matrix of $X_T$. The $(m,n)$th element of $\Sigma_T$ is given by

$$\int_{-\pi}^{\pi} \exp(i(m-n)\lambda) f_\theta(\lambda) d\lambda.$$ 

Let $A_T(\theta)$ and $B_T(\theta)$ be $T \times T$-Toeplitz matrices associated with harmonic functions $g_\theta(\lambda)$ and $h_\theta(\lambda)$, where $g_\theta \in D^C_{\text{ARMA}}$, $h_\theta \in D_d$ (i.e., the $(m,n)$th element of $A_T(\theta)$ and $B_T(\theta)$ are given by

$$\int_{-\pi}^{\pi} \exp(i(m-n)\lambda) g_\theta(\lambda) d\lambda \text{ and } \int_{-\pi}^{\pi} \exp(i(m-n)\lambda) h_\theta(\lambda) d\lambda,$$ 

respectively. We impose the following assumptions.

Assumption 4. The functions $g_\theta$ and $h_\theta$ are continuously four times differentiable with respect to $\theta \in \Theta$, and the derivatives $\partial g_\theta/\partial \theta$, $\partial^2 g_\theta/\partial \theta^2$, $\partial^3 g_\theta/\partial \theta^3$, $\partial^4 g_\theta/\partial \theta^4$, $\partial^5 g_\theta/\partial \theta^5$, $\partial^6 g_\theta/\partial \theta^6$, $\partial^7 g_\theta/\partial \theta^7$, $\partial^8 g_\theta/\partial \theta^8$, $\partial^9 g_\theta/\partial \theta^9$ belong to $D_d$. Also $g_\theta$ and $h_\theta$ satisfy

$$g_\theta(\lambda)^2 h_\theta(\lambda) = \frac{1}{2} f_\theta(\lambda)^2 \partial^2 \frac{\partial}{\partial \theta} f_\theta(\lambda). \quad (2.1)$$ 

Assumption 5. A function $b_T(\theta)$ is four times continuously differentiable with respect to $\theta$, and is written as

$$b_T(\theta) = \frac{1}{4\pi} \int_{-\pi}^{\pi} f_\theta(\lambda)^{-1} \partial^2 \frac{\partial}{\partial \theta} f_\theta(\lambda) d\lambda + O(T^{-1}).$$ 

Now consider the following equation;

$$\frac{1}{T} \Sigma_T A_T(\theta)^{-1} B_T(\theta) A_T(\theta)^{-1} \Sigma_T = b_T(\theta), \quad \theta \in \Theta. \quad (2.2)$$
A minimum contrast estimator $\hat{\theta}_T$ of $\theta_0$ is defined by a value of $\theta$ that satisfies the equation (2.2). This estimator $\hat{\theta}_T$ includes the following cases:

**Example 1.** Put $g_{\theta} = f_{\theta}$, $h_{\theta} = \frac{1}{2} \frac{\partial f_{\theta}}{\partial \theta}$ and $b_T(\theta) = \frac{1}{2T} \text{tr} \Sigma_T^{-1} \frac{\partial}{\partial \theta} (\Sigma_T)$, then by Theorem 1 in Taniguchi (1983)

$$b_T(\theta) = \frac{1}{4\pi} \int_{-\pi}^{\pi} f_{\theta}^{-1} \frac{\partial}{\partial \theta} f_{\theta} d\lambda + O(T^{-1}).$$

The estimator $\hat{\theta}_T$ becomes the maximum likelihood estimator (see Taniguchi (1983) or (1985)).

**Example 2.** Put $g_{\theta} = \frac{1}{2\pi}$, $h_{\theta} = \frac{1}{8\pi^2} \frac{\partial f_{\theta}}{\partial \theta} f_{\theta}^{-2}$ and $b_T(\theta) = \frac{1}{4\pi} \int_{-\pi}^{\pi} f_{\theta}^{-1} \frac{\partial}{\partial \theta} f_{\theta} d\lambda$. Then (2.2) is written as

$$X_T \begin{pmatrix} n \\
\vdots \\
m \\
\end{pmatrix} = \frac{1}{4\pi} \int_{-\pi}^{\pi} f_{\theta}^{-1} \frac{\partial}{\partial \theta} f_{\theta} d\lambda.$$

(2.3)

We can see that the equation (2.3) is equivalent to

$$\frac{\partial}{\partial \theta} \int_{-\pi}^{\pi} \left[ \log f_{\theta}(\lambda) + \frac{I_T(\lambda)}{f_{\theta}(\lambda)} \right] d\lambda = 0,$$

where $I_T(\lambda) = \frac{1}{2\pi} \left| \sum_{t=1}^{T} X_t e^{it\lambda} \right|^2$. Thus the estimator $\hat{\theta}_T$ becomes the quasi-maximum likelihood estimator (see Dunsmuir and Hannan (1976), Hosoya and Taniguchi (1982)).

At first we present the following basic theorem which is
useful for the higher-order asymptotic theory up to third order in time series analysis.

**Theorem 1.** Assume that Assumptions 1-5 hold. Let $\alpha$ be an arbitrary fixed number such that $0 < \alpha < 3/8$.

1. There exists a statistic $\hat{\theta}_T$ which solves (2.2) such that for some $d_1 > 0$,

$$
P_{\theta_0}^T[|\hat{\theta}_T - \theta_0| < d_1 T^{\alpha - 1/2}] = 1 - o(T^{-1}),
$$

uniformly for $\theta_0 \in \mathcal{C}$.

2. For $[\hat{\theta}_T]$ satisfying (2.4),

$$
sup_{B \in \mathcal{A}_0} |P_{\theta_0}^T[[T(\theta_0)]^{1/2}(\hat{\theta}_T - \theta_0) \in B] - \int_B \varnothing(x)p^T_3(x)dx| = o(T^{-1}),
$$

uniformly for $\theta_0 \in \mathcal{C}$, where $\mathcal{A}_0$ is a class of Borel sets of $\mathbb{R}$, satisfying

$$
sup_{B \in \mathcal{A}_0} \int (\partial B) \varnothing(x)p^T_3(x)dx = o(C).
$$

Here $\varnothing(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, and $p^T_3(x) = 1 + \frac{q(x)}{\sqrt{T}} + \frac{r(x)}{T}$ where $q(x)$ and $r(x)$ are polynomials.

In Section 6 we shall give the coefficients of $q(x)$ and $r(x)$ by using the spectral density $f_{\theta}$. 
3. Stochastic expansion of minimum contrast estimator.

In this section we derive a stochastic expansion of $\hat{\theta}_T$. We set down

$$z_T(\theta) = X_T^t H_T(\theta) X_T - T b_T(\theta),$$

where $H_T(\theta) = A_T(\theta)^{-1} B_T(\theta) A_T(\theta)^{-1}$.

Let

$$Z_1(\theta) = \frac{1}{\sqrt{T}} (X_T^t H_T(\theta) X_T - T b_T(\theta)), \quad (3.1)$$

$$Z_2(\theta) = \frac{1}{\sqrt{T}} (X_T^t \ddot{H}_T(\theta) X_T - T \Sigma_T(\theta) \ddot{H}_T(\theta)), \quad (3.2)$$

$$Z_3(\theta) = \frac{1}{\sqrt{T}} (X_T^t \dddot{H}_T(\theta) X_T - T \Sigma_T(\theta) \dddot{H}_T(\theta)), \quad (3.3)$$

where $\ddot{H}_T(\theta) = \frac{\partial H_T(\theta)}{\partial \theta}$ and $\dddot{H}_T(\theta) = \frac{\partial^2}{\partial \theta^2} H_T(\theta)$. Henceforth, for simplicity, we sometimes use $A$, $B$, $H$, $\Sigma$, $Z_1$, $Z_2$ and $Z_3$ instead of $A_T(\theta)$, $B_T(\theta)$, $H_T(\theta)$, $\Sigma_T(\theta)$, $Z_1(\theta)$, $Z_2(\theta)$ and $Z_3(\theta)$, respectively. It is easy to show that

$$\ddot{H} = - A^{-1} \dot{A} A^{-1} - A^{-1} B A^{-1} A^{-1} + A^{-1} B A^{-1}, \quad (3.4)$$

$$\dddot{H} = A^{-1} \dot{A} A^{-1} - A^{-1} B A^{-1} A^{-1} - A^{-1} B A^{-1} A^{-1} + A^{-1} B A^{-1} A^{-1} A^{-1} - A^{-1} B A^{-1} A^{-1} + 2 A^{-1} B A^{-1} A^{-1} - A^{-1} B A^{-1} A^{-1} - A^{-1} B A^{-1} A^{-1} + A^{-1} B A^{-1} A^{-1} A^{-1}. \quad (3.5)$$
Since the minimum contrast estimator is approximated by simple functions of \(Z_1, Z_2\), and \(Z_3\). To give the asymptotic expansion, we must evaluate the asymptotic cumulants (moments) of \(Z_1, Z_2\), and \(Z_3\). The following lemma is useful to evaluate them (see Taniguchi (1983)).

**LEMMA 1.** Suppose that \(f_1(\lambda), \ldots, f_s(\lambda) \in D_d, g_1(\lambda), \ldots, g_s(\lambda) \in D_{ARMA}^C\). We define \(\Gamma_1, \ldots, \Gamma_s, \Lambda_1, \ldots, \Lambda_s\), the \(\mathcal{T} \times \mathcal{T}\)-Toeplitz type matrices associated with \(f_1(\lambda), \ldots, f_s(\lambda), g_1(\lambda), \ldots, g_s(\lambda)\), respectively. Then

\[
T^{-1} \text{tr } \Gamma_1 \Lambda_1^{-1} \Gamma_2 \Lambda_2^{-1} \cdots \Gamma_s \Lambda_s^{-1} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f_1(\lambda) \ldots f_s(\lambda) g_1(\lambda)^{-1} \cdots g_s(\lambda)^{-1} d\lambda + o(T^{-1}).
\]

We write

\[
E_{\theta}Z_1(\theta) = \frac{\mu(\theta)}{\sqrt{T}} + o(T^{-1}). \tag{3.6}
\]

Here \(\mu(\theta)\) will be evaluated explicitly for some cases in Section 7. Using Lemma 1 and (2.1), it is not difficult to show the following lemma.

**LEMMA 2.** Under Assumptions 1-5, we have

\[
E_{\theta}(Z_1(\theta))^2 = I(\theta) + o(T^{-1}), \tag{3.7}
\]

\[
E_{\theta}(Z_1(\theta)Z_2(\theta)) = J(\theta) + o(T^{-1}), \tag{3.8}
\]

\[
E_{\theta}(Z_1(\theta))^3 = \frac{1}{\sqrt{T}} K(\theta) + \frac{3}{\sqrt{T}} T(\theta) \mu(\theta) + o(T^{-3/2}), \tag{3.9}
\]

\(\text{Here } u(\theta)\text{ will be evaluated explicitly for some cases in Section 7. Using Lemma 1 and (2.1), it is not difficult to show the following lemma.}\)
\[
E_\theta[Z_1(\theta)Z_3(\theta)] = L(\theta) + O(T^{-1}), \quad (3.10)
\]

\[
E_\theta[Z_2(\theta)]^2 = M(\theta) + O(T^{-1}), \quad (3.11)
\]

\[
E_\theta[Z_1(\theta)^2Z_2(\theta)] = \frac{1}{\sqrt{T}}N(\theta) + \frac{2}{\sqrt{T}}J(\theta)\mu(\theta) + O(T^{-3/2}), \quad (3.12)
\]

\[
\text{cum}_\theta[Z_1(\theta),Z_1(\theta),Z_1(\theta),Z_1(\theta)] = \frac{1}{T}H(\theta) + O(T^{-2}), \quad (3.13)
\]

\[
E_\theta[\frac{1}{T} \frac{\partial}{\partial \theta} \ell_T(\theta)] = - I(\theta) + O(T^{-1}), \quad (3.14)
\]

\[
E_\theta[\frac{1}{T} \frac{\partial^2}{\partial \theta^2} \ell_T(\theta)] = - 3J(\theta) - K(\theta) + O(T^{-1}), \quad (3.15)
\]

\[
E_\theta\left[\frac{1}{T} \frac{\partial^3}{\partial \theta^3} \ell_T(\theta)\right] = - 4L(\theta) - 3M(\theta) - 6N(\theta) - H(\theta) + O(T^{-1}), \quad (3.16)
\]

where

\[
J(\theta) = - \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[ \frac{\partial}{\partial \theta} f_\theta(\lambda) \right]^3 f_\theta(\lambda) - 3 d\lambda
\]

\[
+ \frac{1}{4\pi} \int_{-\pi}^{\pi} \left[ \frac{\partial}{\partial \theta} f_\theta(\lambda) \right] \left[ \frac{\partial^2}{\partial \theta^2} f_\theta(\lambda) \right] f_\theta(\lambda) - 2 d\lambda,
\]

\[
K(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[ \frac{\partial}{\partial \theta} f_\theta(\lambda) \right]^3 f_\theta(\lambda) - 3 d\lambda,
\]

\[
L(\theta) = \frac{3}{2\pi} \int_{-\pi}^{\pi} \left( \frac{\partial}{\partial \theta} f_\theta(\lambda) \right)^4 f_\theta(\lambda) - 4 d\lambda - \frac{3}{2\pi} \int_{-\pi}^{\pi} \left[ \frac{\partial}{\partial \theta} f_\theta(\lambda) \right]^2 \left( \frac{\partial^3}{\partial \theta^3} f_\theta(\lambda) \right) f_\theta(\lambda) - 3 d\lambda
\]

\[
+ \frac{1}{4\pi} \int_{-\pi}^{\pi} \left[ \frac{\partial}{\partial \theta} f_\theta(\lambda) \right] \left[ \frac{\partial^3}{\partial \theta^3} f_\theta(\lambda) \right] f_\theta(\lambda) - 2 d\lambda,
\]

\[
M(\theta) = \frac{1}{\pi} \int_{-\pi}^{\pi} \left[ \frac{\partial}{\partial \theta} f_\theta(\lambda) \right]^4 f_\theta(\lambda) - 4 d\lambda - \frac{1}{\pi} \int_{-\pi}^{\pi} \left[ \frac{\partial^3}{\partial \theta^3} f_\theta(\lambda) \right] f_\theta(\lambda) - 3 d\lambda.
\]
\[ N(\theta) = - \frac{1}{\pi} \int_{-\pi}^{\pi} \left\{ \frac{\partial^2}{\partial \theta^2} f_\theta(\lambda) \right\} f_\theta(\lambda)^{-4} d\lambda + \frac{1}{2\pi} \int_{-\pi}^{\pi} \left\{ \frac{\partial}{\partial \theta} f_\theta(\lambda) \right\} \left\{ \frac{\partial^2}{\partial \theta^2} f_\theta(\lambda) \right\} f_\theta(\lambda)^{-3} d\lambda, \]

\[ H(\theta) = \frac{3}{2\pi} \int_{-\pi}^{\pi} \left\{ \frac{\partial^2}{\partial \theta^2} f_\theta(\lambda) \right\} f_\theta(\lambda)^{-4} d\lambda. \]

Put \( BT = \Lambda_1^{-1} \Gamma_1 \Lambda_2^{-1} \ldots \Gamma_{S-1} \Lambda_s^{-1} \), where \( \Gamma_1, \ldots, \Gamma_{S-1}, \Lambda_1, \ldots, \Lambda_s \) are \( T \times T \)-Toeplitz type matrices associated with some harmonic functions \( u_{\theta}^{(1)}(\lambda) \in D_d, \ldots, u_{\theta}^{(S-1)}(\lambda) \in D_d, v_{\theta}^{(1)}(\lambda) \in D_{ARMA}, \ldots, v_{\theta}^{(S)}(\lambda) \in D_{ARMA} \), respectively.

**LEMMA 3.** Under Assumption 1, for every \( \delta > 0 \), and some \( d_2 > 0 \), we have

\[ P_{\theta}^T \left[ \frac{1}{\sqrt{T}} |X_T^B X_T - E_{\theta}(X_T^B X_T)| > d_2 T^\delta \right] = o(T^{-1}), \quad (3.17) \]

uniformly for \( \theta \in \Theta \).

**[PROOF]** Choose an integer \( \eta \geq 1 \) so that \( 2\eta \delta > 1 \). By Tchebychev's inequality, we have

\[ P_{\theta}^T \left[ \frac{1}{\sqrt{T}} |X_T^B X_T - E_{\theta}(X_T^B X_T)| > d_2 T^\delta \right] \]

\[ \leq E_{\theta} \left[ (X_T^B X_T - E_{\theta}(X_T^B X_T))^2 \right] / (d_2 T^\delta)^{2\eta}. \quad (3.18) \]

Since \( E_{\theta} \left[ (X_T^B X_T - E_{\theta}(X_T^B X_T))^2 \right] = o(1) \) (see Lemma 4 of Taniguchi (1985)), (3.18) implies (3.17).

**LEMMA 4.** Let \( Y_T \) be a random variable which has the stochastic
expansion

\[ Y_T = Y_T^{(3)} + T^{-3/2} \varepsilon_T, \quad (3.19) \]

where the distribution of \( Y_T^{(3)} \) has the following Edgeworth expansion:

\[ P\{Y_T^{(3)} \in B\} = \int_B \theta(x)p_T^3(x)dx + o(T^{-1}), \quad (3.20) \]

where \( B \) is a Borel set of \( R^n \) satisfying (2.6). Also \( \varepsilon_T \) satisfies

\[ P\{|\varepsilon_T| > \rho_T\sqrt{T}\} = o(T^{-1}), \quad (3.21) \]

where \( \rho_T \to 0, \rho_T T^{1/2} \to \infty \) as \( T \to \infty \). Then

\[ P\{Y_T \in B\} = \int_B \theta(x)p_T^3(x)dx + o(T^{-1}), \quad (3.22) \]

for \( B \in \mathcal{A}_0 \).

The above proof proceeds on a similar way to Chibisov (1972).

**PROOF OF (1) IN THEOREM 1.**

In this proof we develop the discussion by using the argument similar to that of Bhattacharya and Ghosh (1978) and Taniguchi (1985). Consider the equation

\[ 0 = T^{-1} l_T(\theta_0) + T^{-1}(\theta - \theta_0) \frac{\partial}{\partial \theta} l_T(\theta_0) \]

\[ + (2T)^{-1}(\theta - \theta_0)^2 \frac{\partial^2}{\partial \theta^2} l_T(\theta_0) \]

\[ + (6T)^{-1}(\theta - \theta_0)^3 \frac{\partial^3}{\partial \theta^3} l_T(\theta_0) \]

\[ + R_T(\theta), \quad (3.23) \]
where $R_T(\theta)$ is the usual remainder in the Taylor expansion, for which it holds that

$$|R_T(\theta)| \leq \frac{1}{24T} |\theta - \theta_0|^4 \sup_{|\theta' - \theta| \leq |\theta - \theta_0|} \left| \frac{\partial^4}{\partial \theta^4} \ell_T(\theta') \right|. \quad (3.24)$$

In view of Lemma 3, we can see that for every $\alpha > 0$ there exist positive constants $d_3$ and $d_4$ such that

$$P_{\theta_0}^T \left[ |Z_1(\theta_0)| > d_3 T^\alpha \right] = o(T^{-1}), \quad (3.25)$$
$$P_{\theta_0}^T \left[ |Z_2(\theta_0)| > d_3 T^\alpha \right] = o(T^{-1}), \quad (3.26)$$
$$P_{\theta_0}^T \left[ |Z_3(\theta_0)| > d_3 T^\alpha \right] = o(T^{-1}), \quad (3.27)$$
$$P_{\theta_0}^T \left[ \frac{1}{\sqrt{T}} \frac{\partial^3}{\partial \theta^3} \ell_T(\theta_0) - E_{\theta_0} \frac{\partial^3}{\partial \theta^3} \ell_T(\theta_0) \right] > d_3 T^\alpha \right] = o(T^{-1}), \quad (3.28)$$
$$P_{\theta_0}^T \left[ |R_T(\theta)| > |\theta - \theta_0|^4 (d_4 T^\alpha) \right] = o(T^{-1}). \quad (3.29)$$

Therefore, on a set having $P_{\theta_0}^T$-probability at least $1 - o(T^{-1})$, for some constants $d_5 > 0$ and $d_6 > 0$ we can rewrite (3.23) as

$$\theta - \theta_0 = (I(\theta_0) + \eta_T)^{-1} \left[ \delta_T + (2T)^{-1} (\theta - \theta_0)^2 \frac{\partial^2}{\partial \theta^2} \ell_T(\theta_0) \right. \right.$$  

$$
+ \left. (6T)^{-1} (\theta - \theta_0)^3 \frac{\partial^3}{\partial \theta^3} \ell_T(\theta_0) + d_5 |\theta - \theta_0|^4 \zeta_T \right], \quad (3.30)$$

where $\eta_T$ and $\delta_T$ are random variables whose absolute values are less than $d_5 T^{-1/2 + \alpha}$ and $\zeta_T$ is a random variable whose absolute value is less than $d_4 T^\alpha$. There exist a sufficiently large $d_7 > 0$ and an integer $T_0$ such that if $T > T_0$ and $|\theta - \theta_0| \leq d_7 T^{-1/2 + \alpha}$, $(0 < \alpha < 3/8)$. 

the right-hand side of (3.30) is less than $d_T^{-1/2+\alpha}$. Applying the Brouwer fixed point theorem to the right-hand side of (3.30) we have proved (2.4).

Now we set down

$$V_T = \sqrt{T}(\hat{\theta}_T - \theta_0),$$

$$I_T(\theta) = -\frac{1}{T}E_\theta[\frac{3}{\theta^2}z_T(\theta)],$$

and

$$U_T(\theta) = \frac{Z_1(\theta)}{I_T(\theta)} + \frac{Z_1(\theta)Z_2(\theta)}{I(\theta)^2\sqrt{T}} - \frac{(3J(\theta)+K(\theta))}{2I(\theta)^3\sqrt{T}}Z_1(\theta)^2$$

$$+ \frac{1}{I(\theta)^3T}[Z_1(\theta)Z_2(\theta)^2 + \frac{1}{Z_1(\theta)^2}Z_3(\theta) + \frac{3(-3J(\theta)-K(\theta))}{2I(\theta)}Z_1(\theta)^2Z_2(\theta)$$

$$+ \frac{(3J(\theta)+K(\theta))^2}{2I(\theta)^2}Z_1(\theta)^3 - \frac{4I(\theta)+3M(\theta)+6N(\theta)+H(\theta)}{6I(\theta)}Z_1(\theta)^3].$$

**Lemma 5.** Under Assumptions 1-5, we have the following stochastic expansion

$$\sqrt{T}(\hat{\theta}_T - \theta_0) = U_T(\theta_0) + T^{-3/2}\zeta_T,$$

(3.31)

where $\zeta_T$ satisfies $P_{\theta_0}^T(|\zeta_T| > \rho_T\sqrt{T}) = o(T^{-1})$ for some sequence $\rho_T \to 0$, $\rho_T\sqrt{T} \to \infty$ as $T \to \infty$.

**Proof**

From the equation $\ell_T(\hat{\theta}_T) = 0$, we have

$$0 = Z_1(\theta_0) + T^{-1/2}Z_2(\theta_0)V_T - I_T(\theta_0)V_T$$

$$+ \frac{1}{2T}T^{-3/2}[\frac{\partial^2}{\partial \theta^2}\ell_T(\theta_0)]V_T^2 + \frac{1}{6T^2}[\frac{\partial^3}{\partial \theta^3}\ell_T(\theta_0)]V_T^3$$

$$+ \frac{1}{24T^{5/2}}[\frac{\partial^4}{\partial \theta^4}\ell_T(\theta_0^*)]V_T^4.$$

(3.32)
where $|\theta^*-\theta_0| \leq |\hat{\theta}_T-\theta_0|$. We rewrite (3.32) as

$$V_T = \frac{Z_1(\theta_0)}{I_T(\theta_0)} + \frac{Z_2(\theta_0)V_T}{\sqrt{I_T(\theta_0)}} + \frac{1}{2\sqrt{I_T(\theta_0)}}(\frac{1}{T} \frac{\partial^2}{\partial \theta^2} I_T(\theta_0))V_T$$

$$+ \frac{1}{6I_T(\theta_0)T}(\frac{1}{T} \frac{\partial^3}{\partial \theta^3} I_T(\theta_0))V_T^3 + \frac{1}{24I_T(\theta_0)T^2}(\frac{1}{T} \frac{\partial^4}{\partial \theta^4} I_T(\theta_0))V_T^4 \quad (3.33)$$

Noting (2.4), (3.25)-(3.29) with $0 < \alpha < 1/10$, we can write (3.33) as

$$V_T = \frac{Z_1}{I_T} + \frac{a_T(1)}{\sqrt{T}}, \quad (3.34)$$

where $P^T_{\theta_0} \{ |a_T(1)| > d_8 T^{2\alpha} \} = o(T^{-1})$, for some $d_8 > 0$. Substituting (3.34) for the right-hand side of (3.33), and noting (3.15) we have

$$V_T = \frac{Z_1}{I_T} + \frac{Z_2}{\sqrt{I_T}^2} - \frac{3J + K}{2\sqrt{I_T}^3} Z_1^2 + \frac{a_T(2)}{T} \quad (3.35)$$

where $P^T_{\theta_0} \{ |a_T(2)| > d_9 T^{3\alpha} \} = o(T^{-1})$, for some $d_9 > 0$. Again substituting (3.35) for the right-hand side of (3.33), and noting (3.16) we have

$$V_T = U_T(\theta_0) + \zeta_T/T^{3/2}, \quad (3.36)$$

where $P^T_{\theta_0} \{ |\zeta_T| > d_{10} T^{5\alpha} \} = o(T^{-1})$, for some $d_{10} > 0$. Since $0 < \alpha < 1/10$, we have the desired result.

**REMARK.** By Lemma 4, the Edgeworth expansion for $\sqrt{T}(\hat{\theta}_T-\theta_0)$ (up to order $T^{-1}$) is equal to that for $U_T(\theta_0)$ on $B \in \mathcal{A}_0$. Thus we have only to derive the Edgeworth expansion for $U_T(\theta_0)$. □

As we saw in the previous section we have to seek the Edgeworth expansion for $U_T(\theta_0)$. To do so we have to derive the Edgeworth expansion for $Z = (Z_1(\theta), Z_2(\theta), Z_3(\theta))'$. Thus, in this section, we give an asymptotic expansion of the characteristic function of $Z$.

Put

$$\tau(\xi) = E_\theta(e^{i\xi'Z}),$$

where $\xi = (t_1, t_2, t_3)'$. Then it is easy to show

$$\tau(\xi) = \text{det}(I(T\xi T)) - \frac{2i}{\sqrt{T}} \sum_{j=1}^3 (t_1 H + t_2 H + t_3 H)^{1/2} \log \left( 1 - \frac{2i}{\sqrt{T}} \rho_j \right)$$

$$- \left. \frac{1}{\sqrt{T}} (t_1 T_b T(\theta) + t_2 \text{tr}H\Sigma + t_3 \text{tr}H\Sigma) \right),$$

(4.1)

where $I(T\xi T)$ is the $T\xi T$-identity matrix. Let $\rho_j$ be the jth latent root of $S = \sum_{j=1}^3 (t_1 S_1 + t_2 S_2 + t_3 S_3)^{1/2}$ ($\rho_1 \geq \ldots \geq \rho_T \geq 0$).

Of course each $\rho_j$ is a real number. Then we have

$$\log \tau(\xi) = \frac{1}{2} \sum_{j=1}^3 \log(1 - \frac{2i}{\sqrt{T}} \rho_j)$$

$$- \frac{1}{\sqrt{T}} (t_1 T_b T(\theta) + t_2 \text{tr}H\Sigma + t_3 \text{tr}H\Sigma).$$

(4.2)

Notice the relation

$$\log(1-ih) = -ih + \frac{h^2}{2} + \frac{ih^3}{3} - \frac{h^4}{4} - \frac{ih^5}{5}$$

$$+ h^6 \int_0^1 (1 - v)^5 \frac{dv}{(1 - ivh)^5},$$

(4.3)
where
\[ \left| \int_0^1 (1 - v)^5 \frac{dv}{(1 - ivh)^6} \right| \leq 1 \]

(e.g., Bhattacharya and Rao (1976, p.57)). By (4.3), the relation (4.2) is

\[ \log \tau(t) = -\frac{1}{2} \sum_{j=1}^T \left[ -\frac{2i \rho_j}{\sqrt{T}} + \frac{4i \rho_j^2}{2T} + \frac{8i \rho_j^3}{3T^{3/2}} \right. \\
- \left. \frac{16 \rho_j^4}{4T^2} - \frac{2^{5/2} \rho_j^5}{5T^{3/2}} + \frac{2^4 \rho_j^6}{T^3} \right] \\
- \frac{1}{\sqrt{T}}(t_1 T_b(t) + t_2 \text{tr} H \Sigma + t_3 \text{tr} H^2 \Sigma), \quad (4.4) \]

where \( |\gamma_j| \leq 1 \). Remembering (3.6), we have

\[ \log \tau(t) = it_1 \left( \mu(t) + o(T^{-1}) \right) + \frac{i^2}{T} \text{tr} S^2 + \frac{4i^3}{3T^{3/2}} \text{tr} S^3 \\
+ \frac{2^4 i}{T^2} \text{tr} S^4 + \frac{16 i^5}{5T^{3/2}} \text{tr} S^5 + R_6, \quad (4.5) \]

where \( |R_6| \leq \frac{2^5}{T^3} \text{tr} S^6 \). Using Lemma 1 we can rewrite as

\[ \frac{2i^2}{T} \text{tr} S^2 = \sum_{j,k=1}^3 \left( A_{jk} + \frac{B_{jk}}{T} + o(T^{-3/2}) \right)(it_j)(it_k), \quad (4.6) \]

\[ \frac{8i^3}{T} \text{tr} S^3 = \sum_{j,k,l=1}^3 \left( A_{jkl} + o(T^{-1}) \right)(it_j)(it_k')(it_l'), \quad (4.7) \]
\[
\frac{481}{T} \text{tr} S^4 = \sum_{j,k,l,m=1}^{3} [A_{jklm} + o(T^{-1})]
\]
\[
x (it_j)(it_k)(it_l)(it_m),
\]  
(4.8)

\[
\frac{3841}{T} \text{tr} S^5 = \sum_{j,k,l,m,n=1}^{3} [A_{jklmn} + o(T^{-1})]
\]
\[
x (it_j)(it_k)(it_l)(it_m)(it_n),
\]  
(4.9)

\[
\frac{16}{T} \text{tr} S^6 = \sum_{j,k,l,m,n,p=1}^{3} [A_{jklmp} + o(T^{-1})]
\]
\[
x (it_j)(it_k)(it_l)(it_m)(it_n)(it_p),
\]  
(4.10)

For examples we can see that \( A_{11} = I(\theta), A_{12} = J(\theta), A_{13} = L(\theta), \)
\( A_{22} = M(\theta), A_{111} = K(\theta), A_{112} = N(\theta), A_{1111} = H(\theta), \) e.t.c...

Thus (4.5) is written as

\[
\log r(t) = \text{tr} \left[ \frac{\mu(\theta)}{\sqrt{T}} + o(T^{-1}) \right]
\]

\[
+ \frac{1}{2} \sum_{j,k=1}^{3} [A_{jk} + B_{jk}/T + o(T^{-3/2})](it_j)(it_k)
\]

\[
+ \frac{1}{6\sqrt{T}} \sum_{j,k,l=1}^{3} [A_{jkl} + o(T^{-1})](it_j)(it_k)(it_l)
\]

\[
+ \frac{1}{24T} \sum_{j,l,m=1}^{3} [A_{jkm} + o(T^{-1})](it_j)(it_k)(it_m)
\]

\[
+ \frac{1}{120T^{3/2}} \sum_{j,k,l,m,n=1}^{3} [A_{jklmn} + o(T^{-1})](it_j)(it_k)(it_l)(it_m)(it_n)
\]

\[+ R_5. \]  
(4.11)
We set down \( \Omega = \{A_{jk}\} \), 3x3-matrix, and \( \|t\|=\sqrt{t_1^2+t_2^2+t_3^2} \). If \( \Omega \) is singular it is not difficult to show that

\[
Z_1(\theta) = c_1(\theta)Z_2(\theta) + d_1(\theta) = c_2(\theta)Z_3(\theta) + d_2(\theta), \text{ a.s.}
\]

for some constants \( c_1(\theta), d_1(\theta), (i = 1, 2) \),

which implies that the joint distribution of \( z \) is reduced to that of \( Z_1 \). Thus, without loss of generality, henceforth we consider the case when \( \Omega \) is nonsingular.

**Lemma 6.** If we take \( T \) sufficiently large, then for a \( \delta_1 > 0 \) and for all \( \xi \) satisfying \( \|\xi\| \leq \delta_1 \sqrt{T} \), there exists a positive definite matrix \( Q_0 \) and polynomial functions \( F_1(\cdot) \) and \( F_2(\cdot) \) such that

\[
|\tau(\xi) - A(\xi; 3)| = \exp[-\frac{1}{2} \xi' Q \xi] \times F_1(\|\xi\|) \cdot O(T^{-3/2})
\]

\[
+ \exp[-\xi' Q_0 \xi] \times F_2(\|\xi\|) \cdot O(T^{-3/2}),
\]

where

\[
A(\xi; 3) = \exp[-\frac{1}{2} \xi' \Omega \xi] \times [1 + \frac{1}{\sqrt{T}} \mu_1]
\]

\[
+ \frac{1}{\delta \sqrt{T}} \sum_{j, k, \ell = 1}^{3} A_{jk \ell} \xi_j \xi_k \xi_\ell
\]

\[
+ \frac{1}{2 \sqrt{T}} \sum_{j, k = 1}^{3} B_{jk} \xi_j \xi_k + \frac{\mu^2(\mu_1)^2}{2T}
\]

\[
+ \frac{\mu(\mu_1)}{6T} \sum_{j, k, \ell = 1}^{3} A_{jk \ell} \xi_j \xi_k \xi_\ell
\]
\[ + \frac{1}{24T} \sum_{j,k,\ell,m=1}^{3} A_{jk\ell m}(it_j)(it_k)(it_\ell)(it_m) \]

\[ + \frac{1}{72T} \sum_{j,k,\ell,j',k',\ell'=1}^{3} A_{jk\ell}A_{j'k'\ell'}(it_j)(it_k)(it_\ell)(it_{j'})(it_{k'})(it_{\ell'}) \]
\[ + \frac{1}{120T^{3/2}} \sum_{j,k,\ell,m,n} (A_{jkmn} + O(T^{-1}))(it_j)(it_k)(it_\ell)(it_m)(it_n) + R_6(t), \]  

where \( F_4(\cdot) \) is a polynomial function. Let \( w > 0 \) be the smallest eigen value of \( \Omega \). Then for sufficiently large \( T \), we can choose \( \delta_1 > 0 \) so that

\[
\delta_1 - \frac{\delta_1}{2} \sum_{j,k,\ell} |A_{jkl\ell}| - \frac{\delta_2}{24} \sum_{j,k,\ell,m} |A_{jklm}| - \frac{\delta_3}{120} \sum_{j,k,\ell,m,n} |A_{jklmn}| - 2^{5\delta_1} \sum_{j,k,\ell,m,n,p} |A_{jklmnp}| > 0. \tag{4.17}
\]

Thus the last exponential term in (4.16) is dominated by

\[
\exp[\delta_1(u + o(T^{-1/2}))]\exp[\|z\|^2(\frac{w}{4} + O(T^{-1}))],
\]

for \( \|z\| \leq \delta_1/\sqrt{T} \). \tag{4.18}

which implies the existence of \( Q_0 \) in (4.13).

We also have the following lemma.

**Lemma 7.** Under Assumptions 1-5, for every \( r > 0 \), there exists \( \delta_2 > 0 \) such that

\[
|\tau(z)| \leq (1 + 4\delta_2 r^2 q^2(T)/u). \tag{4.19}
\]

for all \( z \) satisfying \( \|z\| \geq r/\sqrt{T} \), where \( q(T) = \lfloor cT \rfloor \), for some constant \( c \).
[PROOF]

Notice that \( p_1^2 = \max \underline{e}^\prime S^2 \underline{e} \), where \( \underline{e} = (e_1, \ldots, e_T)' \in \mathbb{R}^T \)
and \( \underline{e}^\prime \underline{e} = 1 \). Also we have

\[
\underline{e}^\prime S^2 \underline{e} = \underline{e}^\prime (\Sigma^{1/2} (t_1 \Sigma \Sigma^{1/2} t_2 \Sigma + t_3 \Sigma^{1/2} t_4 \Sigma^{1/2})^2 \underline{e} 
\leq 2t_1^2 \underline{e}^\prime \Sigma^{1/2} \Sigma \Sigma^{1/2} \Sigma \Sigma^{1/2} \underline{e} + 2t_2^2 \underline{e}^\prime \Sigma^{1/2} \Sigma \Sigma^{1/2} \Sigma \Sigma^{1/2} \underline{e} 
+ 2t_3^2 \underline{e}^\prime \Sigma^{1/2} \Sigma \Sigma^{1/2} \Sigma \Sigma^{1/2} \underline{e}.
\] (4.20)

It is not difficult to show that

\[
\underline{e}^\prime \Sigma^{1/2} \Sigma \Sigma^{1/2} \underline{e} \leq c_1,
\] (4.21)

\[
\underline{e}^\prime \Sigma^{1/2} \Sigma \Sigma^{1/2} \Sigma \Sigma^{1/2} \underline{e} \leq c_2,
\] (4.22)

\[
\underline{e}^\prime \Sigma^{1/2} \Sigma \Sigma^{1/2} \Sigma \Sigma^{1/2} \underline{e} \leq c_3,
\] (4.23)

where \( c_1, c_2 \) and \( c_3 \) are some positive constants. For exposition we prove (4.21). Since \( f_\theta(\lambda), h_\theta(\lambda) \in D_d \) and \( \underline{e} \in D_{\text{ARMA}} \), we can set

\[
f_1 = \max_{\lambda} f_\theta(\lambda) < \infty,
\]

\[
h_1 = \max_{\lambda} |h_\theta(\lambda)| < \infty,
\]

\[
\underline{e}_1 = \min_{\lambda} \underline{e}_\theta(\lambda) > 0.
\]

Thus, using discussions of Anderson (1971, p.573-4) we have

\[
\underline{e}^\prime \Sigma^{1/2} \Sigma \Sigma^{1/2} \underline{e} = \underline{e}^\prime A^{-1} \Sigma A^{-1} \Sigma A^{-1} \Sigma A^{-1} \Sigma^{1/2} \underline{e} 
\leq \underline{e}^\prime A^{-1} \Sigma A^{-1} \left( \begin{array}{cc}
2f_1 & 0 \\
0 & 2f_1
\end{array} \right) A^{-1} \Sigma A^{-1} \Sigma^{1/2} \underline{e}.
\]
Thus we have proved (4.21). The proofs of (4.22) and (4.23) are similar. From (4.20) we have

\[
\rho_T^2 \leq \ldots \leq \rho_1^2 \leq \|\mathbf{t}\|^2 \cdot d_{ll}
\]  

(4.25)

for any \( \mathbf{t} \), where \( d_{ll} \) is a positive constant. While by Lemma 1, we get

\[
T^{-1} \sum_{j=1}^{\infty} \rho_j^2 = T^{-1} tr S^2 
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( t_1 A(\lambda) + t_2 B(\lambda) + t_3 C(\lambda) \right)^2 d\lambda + \|\mathbf{t}\|^2 O(T^{-1}) 
\]

\[
= \frac{\|\mathbf{t}\|^2}{2\pi} \int_{-\pi}^{\pi} \left( \frac{t_1 A(\lambda) + t_2 B(\lambda) + t_3 C(\lambda)}{\|\mathbf{t}\|} \right)^2 d\lambda + \|\mathbf{t}\|^2 O(T^{-1}), 
\]  

(4.26)
where
\[ A(\lambda) = \frac{1}{2} \frac{\delta}{\delta \theta} f_\theta(\lambda) \cdot f_\theta(\lambda)^{-1}, \]
\[ B(\lambda) = \frac{1}{2} \left[ -2 f_\theta(\lambda)^{-2} \left( \frac{\delta}{\delta \theta} \frac{\theta}{\delta \theta} \right) + f_\theta(\lambda) \frac{\delta^2}{\delta \theta^2} f_\theta(\lambda) \right], \]
and
\[ C(\lambda) = \left( 6 f_\theta(\lambda)^{-3} \left( \frac{\delta}{\delta \theta} \frac{\theta}{\delta \theta} \right)^3 \right) - \frac{2}{3} f_\theta(\lambda)^{-2} \frac{\delta}{\delta \theta} \frac{\theta}{\delta \theta} \frac{\delta^2}{\delta \theta^2} f_\theta(\lambda) + f_\theta(\lambda) \frac{\delta^3}{\delta \theta^3} f_\theta(\lambda) / 2. \]

Since we are now assuming that \( \Omega \) in (4.11) is nonsingular, the functions \( A(\lambda), B(\lambda) \), and \( C(\lambda) \) are linearly independent in the \( L^2 \)-norm (\( \int_{-\pi}^{\pi} \cdot |^2 d\lambda \)). So we can show that for sufficiently large \( T \), there exists \( d_{12} > 0 \) such that
\[ T^{-1} \sum_{j=1}^{T} \rho_j^2 \geq \| \tilde{z} \|_{12}^2, \tag{4.27} \]
for any \( \tilde{z} \). The relations (4.25) and (4.27) imply that there exist \( \delta_2 > 0 \) and \( q(t) = [cT] \) such that
\[ \rho_1^2 \geq \cdots \geq \rho_{q(T)}^2 \geq \delta_2 \| \tilde{z} \|_{12}^2. \]

Noting that
\[ | \tau(t) | = \prod_{j=1}^{T} \left( 1 + \frac{4}{T} \rho_j^2 \right)^{-1/4} \]
\[ \leq \prod_{j=1}^{q(T)} \left( 1 + \frac{4 \delta_2}{T} \| \tilde{z} \|_{12}^2 \right)^{-1/4} \]
\[ \leq (1 + 4 \delta_2 \tau)^{-q(T)/4}, \quad \text{for} \quad \| \tilde{z} \| \geq \frac{n}{T}, \]
the proof is completed.
5. Edgeworth expansion for $Z$

In this section we shall give the Edgeworth expansion for $Z$. We set down $B(x,r) = \{ z \in \mathbb{R}^p : \| z - x \| \leq r, x \in \mathbb{R}^p \}$. For a probability measure $P$, we denote the variation norm of $P$ by $\| P \|$. The following lemma is known as a smoothing lemma (see Bhattacharya and Rao (1976, p.97-98 and p.113)).

**Lemma 8.** Let $P$ and $Q$ be probability measures on $\mathbb{R}^p$ and $\mathfrak{m}^p$ the class of all Borel subsets of $\mathbb{R}^p$. Let $\varepsilon$ be a positive number. Then there exists a kernel probability measure $K_\varepsilon$ such that

$$\sup_{B \in \mathfrak{m}^p} |P(B) - Q(B)| \\
\leq \frac{2}{3} \| P - Q \| + \frac{4}{3} \sup_{B \in \mathfrak{m}^p} Q((\mathfrak{m}B)^{2\varepsilon}), \quad (5.1)$$

where $K_\varepsilon$ satisfies

$$K_\varepsilon(B(q,r)^c) = O((\frac{\varepsilon}{r})^3), \quad (5.2)$$

and the Fourier transform $\hat{K}_\varepsilon$ satisfies

$$\hat{K}_\varepsilon(\tau) = 0 \text{ for } \| \tau \| > 8p^{4/3}/\pi^{1/3}\varepsilon. \quad \square$$

For $B \in \mathfrak{m}^3$, define

$$Q_3(B) = \mathbf{1}_{B} N(\mathbf{z}; \Omega)[1 + \frac{\mathbf{u}}{\sqrt{T}} H_1(\mathbf{z})]$$

$$+ \frac{1}{6T} \sum_{j,k,l=1}^{3} A_{jk} A_{jl} H_{jk} H_{jl}(\mathbf{z}) + \frac{\mathbf{u}^2}{2T} H_{11}(\mathbf{z})$$
+ \frac{1}{2T} \sum_{j,k=1}^{3} B_{jk} H_{jk}(z) + \frac{1}{6T} \sum_{j,k,l=1}^{3} \mu_{jkll} H_{jkll}(z)

+ \frac{1}{24T} \sum_{j,k,l,m=1}^{3} A_{jkll} H_{jkll}(z)

+ \frac{1}{72T} \sum_{j,k,l,k',l'=1}^{3} A_{jkll} A_{j'k'l'l'} H_{jkll'k'l'}(z),

(5.3)

where \( z = (z_1, z_2, z_3)' \),

\[
N(z; \Omega) = (2\pi)^{-3/2} |\Omega|^{-1/2} \exp -\frac{1}{2} z' \Omega^{-1} z,
\]

\[
H_{j_1 \ldots j_s}(z) = \frac{(-1)^s}{N(z; \Omega)} \frac{\delta^s}{\partial z_{j_1} \ldots \partial z_{j_s}} N(z; \Omega).
\]

This measure \( Q_2^{(3)}(\cdot) \) corresponds to the characteristic function \( A(z; 3) \) in Lemma 6. Then we have the following lemma.

**LEMMA 9.** Suppose that Assumptions 1-5 are satisfied. Then

\[
\sup_{B \in \mathbb{R}^3} |P^T_\theta(z \in B) - Q_2^{(3)}(B)| = o(T^{-1})
\]

\[
+ \frac{4}{3} \sup_{B \in \mathbb{R}^3} Q_2^{(3)}((\partial B)^{2\varepsilon}),
\]

(5.4)

uniformly for \( \theta \in \Theta \), where \( \varepsilon = T^{-1-\rho}, 0 < \rho < 1/2 \).

**[PROOF]** Substituting \( P^T_\theta(z \in B) \) and \( Q_2^{(3)}(\cdot) \) for \( P(z) \) and \( Q_2(z) \) in Lemma 8, respectively, we get
\[ \sup_{B \in \mathcal{B}} |P^T_\theta(Z \in B) - Q^{(3)}_Z(B)| \]

\[ \leq \frac{2}{3} \| (P^T_\theta - Q^{(3)}_Z) \ast K_\varepsilon \| \]

\[ + \frac{1}{3} \sup_{B \in \mathcal{B}} Q^{(3)}_Z((\delta B)^{2\varepsilon}). \tag{5.5} \]

Notice that

\[ \| (P^T_\theta - Q^{(3)}_Z) \ast K_\varepsilon \| \]

\[ = 2 \sup \{|(P^T_\theta - Q^{(3)}_Z) \ast K_\varepsilon(B)|; B \in \mathcal{B}^3 \} \]

\[ \leq 2 \sup \{|(P^T_\theta - Q^{(3)}_Z) \ast K_\varepsilon(B)|; B \in \mathcal{B}^3 \text{ and } B \subseteq B(Q,r_T) \} \]

\[ + 2 \sup \{|(P^T_\theta - Q^{(3)}_Z) \ast K_\varepsilon(B)|; B \in \mathcal{B}^3 \text{ and } B \subseteq B(Q,r_T) \} \]. \tag{5.6} \]

where \( r_T = T^T, 0 < r < 1/6 \). Here we put \( \varepsilon = T^{-1-\rho}, 0 < \rho < 1/2 \).

Then, for \( B \subseteq B(Q,r_T) \) we have

\[ |(P^T_\theta - Q^{(3)}_Z) \ast K_\varepsilon(B)| \leq |P^T_\theta \ast K_\varepsilon(B)| + |Q^{(3)}_Z \ast K_\varepsilon(B)| \]

\[ \leq P^T_\theta(\|Z\| \geq r_T/2) + K_\varepsilon(B(Q,r_T/2)) \]

\[ + Q^{(3)}_Z(B(Q,r_T/2)) + K_\varepsilon(B(Q,r_T/2)). \tag{5.7} \]

It is easy to check

\[ Q^{(3)}_Z(B(Q,r_T/2)) = o(T^{-1}). \tag{5.8} \]
The relations (3.25), (3.26) and (3.27) imply

$$P_T^*(\|Z\| \geq r_T/2) = o(T^{-1}). \quad (5.9)$$

While (5.2) implies

$$K_e[B(Q, r_T/2)^C] = o(T^{-3-3\rho-3\tau}) = o(T^{-1}). \quad (5.10)$$

Thus we have

$$|(P_T^* - Q_Z^{(3)})*K_e(B)| = o(T^{-1}), \quad (5.11)$$

for $B \in B(Q, r_T)^C$. Now we have only to evaluate

$$\sup[|(P_T^* - Q_Z^{(3)})*K_e(B)| ; B \in B(Q, r_T)].$$

By Fourier inversion we have

$$|(P_T^* - Q_Z^{(3)})*K_e(B)|$$

$$\leq (2\pi)^{-3} \frac{\pi^{3/2}}{3^{\rho/2}} \frac{r_T^3}{3^{\rho/2}} \int |(P_T^* - \hat{Q}_Z^{(3)})(t)\hat{K}_e(t)| dt. \quad (5.12)$$

By Lemma 6 and noting $\hat{Q}_Z^{(3)}(t) = A(t; 3)$, the right-hand side of (5.12) is dominated by

$$o(T^{3\tau-3/2}) \int_{\|t\| \leq \delta \sqrt{T}} |\exp\{-\frac{t}{T}, Q_0 t\} \times F_1(\|t\|)$$

$$+ \exp\{-t', Q_0 t\} \times F_2(\|t\|) \|\hat{K}_e(t)\| dt$$

$$+ o(T^{3\tau}) \int_{\delta \sqrt{T} \leq \|t\| \leq 83^{1/3}} \delta_1^{1/3} - 3^{1/3} \|t\|^{1+\rho/3-1/3} |(P_T^* - \hat{Q}_Z^{(3)})(t)\hat{K}_e(t)| dt. \quad (5.13)$$
Evidently the first term of the above is of order $O(T^{-4/3})$. Also by Lemma 7, we have

$$O(T^3)\int_{\delta_1 \sqrt{T} < \|2\| < 8(3)^{4/3}T^{1+\rho}/m^{1/3}} |(P_{\theta} \cdot Q_{z}^{(3)})\hat{f}(\xi)K_{\epsilon}(\xi)| dt$$

$$\leq O(T^3)\int_{\delta_1 \sqrt{T} < \|2\| < d_{14}T^{1+\rho}d_{13}^2(1 + 4\delta_2\delta_1)^{-q(T)/4}dt + o(T^{-1}), \quad (5.14)$$

where $d_{13}$ and $d_{14}$ are appropriate positive constants. The above (5.14) is dominated by

$$O(T^{3+3+3\rho})(1 + 4\delta_2\delta_1)^{-q(T)/4}dt + o(T^{-1}) = o(T^{-1}). \quad (5.15)$$

Therefore we have proved

$$\sup\{|(P_{\theta} \cdot Q_{z}^{(3)})\hat{f}(\xi)|; B \subset B(2, r_T)\} = o(T^{-1}),$$

which completes the proof. \qedsymbol
6. Proof for (2) of Theorem 1.

Consider the following transformation

\[ W_1(\theta) = Z_1(\theta) \]
\[ W_2(\theta) = Z_2(\theta) - J(\theta)I(\theta)^{-1}Z_1(\theta) \]
\[ W_3(\theta) = Z_3(\theta) - L(\theta)I(\theta)^{-1}Z_1(\theta) \]  \hspace{1cm} (6.1)

Henceforth, for simplicity we sometimes use \( W_1, W_2 \) and \( W_3 \), instead of \( W_1(\theta), W_2(\theta) \) and \( W_3(\theta) \), respectively. Evidently (6.1) is a continuous bijective transformation. We denote (6.1) by \( \mathbf{W}_X(\mathbf{Z}) \), where \( \mathbf{W}_X = (W_1, W_2, W_3)' \). By Lemma 9, we have

\[
\sup_{B \in \Theta^3} |P_\theta^T[Z \in X^{-1}(B)] \cdot Q_\mathbf{Z}^{(3)}[X^{-1}(B)]| \\
= \frac{4}{3} \sup_{B \in \Theta^3} Q_\mathbf{Z}^{(3)}[(aX^{-1}(B))^2] + o(T^{-1}). \]  \hspace{1cm} (6.2)

Here we put \( Q_\mathbf{Z}^{(3)}(B) = Q_\mathbf{Z}^{(3)}[X^{-1}(B)] \). Then it is not difficult to show

\[
Q_\mathbf{Z}^{(3)}(B) = \int_B N(w_1:1)N(w_2, w_3: \Omega_2)[1 + \frac{3}{\sqrt{T}} \sum_{j=1}^{3} c_j^{(1)} H_j(\mathbf{W}) + \frac{1}{6\sqrt{T}} \sum_{j,k,l=1}^{3} c_j^{(1)} H_{jkl}(\mathbf{W}) + \frac{1}{2T} \sum_{j,k=1}^{3} (c_j^{(3)} + c_j^{(1)} c_k^{(1)}) H_{jk}(\mathbf{W}) + \frac{1}{6T} \sum_{j,k,l,m=1}^{3} c_j^{(1)} c_k^{(1)} H_{jklm}(\mathbf{W})].
\]
\[ + \frac{1}{24T} \sum_{j,k,l,m=1}^{3} c^{(1)}_{jklm} H_{jklm}(z) \]

\[ + \frac{1}{72T} \sum_{j,k,l,j',k',l'=1}^{3} c^{(1)}_{jkl} c^{(1)}_{j'k'l'} H_{jklj'k'l'}(z) \]

\[ = \int_B q_T(w) dW, \text{ say,} \quad (6.3) \]

where \( W = (w_1, w_2, w_3)' \), \( N(w_1; I) = (2\pi)^{-1/2} I^{-1/2} \exp -\frac{w^2}{2I} \)

and \( N(w_2, w_3; \Omega_2) = (2\pi)^{-1} |\Omega_2|^{-1/2} \exp -\frac{1}{2}(w_2, w_3) \Omega_2^{-1}(w_2, w_3) \)

\[ \Omega_2 = \begin{pmatrix} \Omega_{22} & \Omega_{23} \\ \Omega_{32} & \Omega_{33} \end{pmatrix}, \text{ 2x2-matrix.} \]

For examples we can see

\[ c^{(1)}_1 = u(\theta), \quad c^{(1)}_2 = -J(\theta)u(\theta)/I(\theta), \quad c^{(1)}_3 = -L(\theta)u(\theta)/I(\theta), \]

\[ \Omega_{22} = M(\theta) - J(\theta)^2 I(\theta)^{-1}, \quad c^{(1)}_{12} = N(\theta) - J(\theta)K(\theta)/I(\theta), \]

\[ c^{(1)}_{1111} = H(\theta), \text{ e.t.c..} \]

Since \( \chi \) is continuous, we have

\[ \partial \chi^{-1}(B) \subset \chi^{-1}(\partial B), \]

\[ (\partial \chi^{-1}(B))^{2\epsilon} \subset (\chi^{-1}(\partial B))^{2\epsilon}. \quad (6.4) \]

By the continuity of \( \chi \), there exists \( a > 0 \) such that

\[ (\chi^{-1}(\partial B))^{2\epsilon} \subset \chi^{-1}(\partial B) a^{2\epsilon}. \]

(6.5)

Thus we have
**Lemma 10.** Under Assumptions 1-5

\[
\sup_{B \in \mathcal{B}^3} |P_T^\mathcal{B}(\mathcal{W} \in B) - q_\mathcal{W}^{(3)}(B)|
\]

\[
= \frac{4}{3} \sup_{B \in \mathcal{B}^3} q_\mathcal{W}^{(3)}[(\lambda B) \alpha^e] + o(T^{-1}),
\]

(6.6)

uniformly for \( \theta \in \Theta \), where \( \alpha \) is a positive constant and \( \epsilon = T^{-1-\rho} \), \( 0 < \rho < 1/2 \).

Now we rewrite \( U_T(\theta) \) in Lemma 5 as

\[
U_T(\mathcal{W}) = \frac{W_1}{I_T} + \frac{W_2}{I^{2\sqrt{T}}} - \frac{(J + K)w_2^2}{2T^{3/4}}
\]

\[
+ \frac{1}{I^3} \left[ w_1^2 + \left(-\frac{5J - 3K}{2I}w_2^2 + \frac{1}{2}w_1^2w_3 \right)
\]

\[
+ \left( \frac{2J^2 + 3KJ + K^2}{2I^2} \right)w_3^2 - \frac{L + 3M + 6N + H}{6I}w_3 \right].
\]

(6.7)

Consider the following transformation

\[
\begin{cases}
S_1 = U_T(\mathcal{W}) \\
S_2 = W_2 \\
S_3 = W_3
\end{cases}
\]

(6.8)

We denote (6.8) by \( S = \psi(\mathcal{W}) \), where \( S = (S_1, S_2, S_3) \). For sufficiently large \( T \), we can take a set

\[
M_T = \{ \mathcal{W} : |w_i| \leq c_i T^\alpha, \ 0 < \alpha < 1/6, \ c_i > 0, \ i = 1, 2, 3 \}
\]

such that \( \psi \) is a \( C^\infty \)-mapping on \( M_T \).
By (6.6),
\[
\sup_{B \in \mathcal{B}} |P_B^T(\mathcal{W} \in \psi^{-1}(B \times \mathbb{R}^2)) - Q^{(3)}(\psi^{-1}(B \times \mathbb{R}^2))| = \frac{4}{3} \sup_{B \in \mathcal{B}} Q^{(3)}((\psi^{-1}(B \times \mathbb{R}^2))a^2) + o(T^{-1}).
\]  \hspace{1cm} (6.9)

We can see that
\[
Q^{(3)}(\psi^{-1}(B \times \mathbb{R}^2)) = \int_{\psi^{-1}(B \times \mathbb{R}^2)} q_T(\mathcal{W})d\mathcal{W}
\]
\[
= \int_{M_T \cap \psi^{-1}(B \times \mathbb{R}^2)} q_T(\mathcal{W})d\mathcal{W} + o(T^{-1})
\]
\[
= \int_{(B \times \mathbb{R}^2) \cap N_T} q_T(\psi^{-1}(\mathcal{S}))|J|d\mathcal{S} + o(T^{-1}),
\]  \hspace{1cm} (6.10)

where $N_T = \psi(M_T)$ and $|J|$ is the Jacobian. Since we can solve so that
\[
W_1 = I_T S_1 - \frac{S_1 S_2}{\sqrt{T}} + \frac{(J + K)S_1^2}{2\sqrt{T}} + \frac{JS_1 S_2}{IT} - \frac{S_1 S_3}{2T}
\]
\[
+ \frac{(-J^2 - JK)S_1^3}{2IT} + \frac{L + 3M + 6N + H S_1^3}{6T} + o(T^{-1}),
\]  \hspace{1cm} (6.11)

uniformly on $M_T$, it is not difficult to show that
\[
q_T(\psi^{-1}(\mathcal{S}))|J|
\]
\[
= N(I_T S_1)N(S_2, S_3; \Omega_2) \times \left[ I_T + \frac{p_1(\mathcal{S})}{\sqrt{T}} + \frac{p_2(\mathcal{S})}{T} + o(T^{-1}) \right],
\]  \hspace{1cm} (6.12)

uniformly on $N_T$, where $p_1(\mathcal{S})$ and $p_2(\mathcal{S})$ are polynomials of $\mathcal{S}$. Thus
we have
\[ Q^3_{\mathbb{W}}(T^{-1}(B \times \mathbb{R}^2)) = \int_{\{B \times \mathbb{R}^2\} \cap N_T} N(I_T S) N(S_2, S_3; \Omega_2) \]
\[ \times \{ I_T + \frac{P_1(S)}{\sqrt{T}} + \frac{P_2(S)}{T} + o(T^{-1}) \} dS + o(T^{-1}) \]
\[ = \int_B N(I_T S) \left[ \int_{\mathbb{R}^2} \left\{ J_T + \frac{P_1(S)}{\sqrt{T}} + \frac{P_2(S)}{T} \right\} \right. \]
\[ \times N(S_2, S_3; \Omega_2) dS_2 dS_3 dS_1 + o(T^{-1}) . \quad (6.13) \]

Calculating the square bracket in (6.13), and noting that
\[ Q^3_{\mathbb{W}}(T^{-1}((\mathbb{E} \times \mathbb{R}))^{B \epsilon}, \mathbb{R}^2) \]
\[ \leq Q^3_{\mathbb{W}}(T^{-1}((\mathbb{E} \times B) \times \mathbb{R}^2)) , \text{ for some } b > 0 , \]
we have
\[ \sup_{B \in \mathcal{A}_o} \left| \int_B \phi(x) p_3^T(x) dx \right| \]
\[ = \frac{4}{3} \sup_{B \in \mathcal{A}_o} \int_B \phi(x) p_3^T(x) dx + o(T^{-1}) , \quad (6.14) \]

(remember (6.9)). Here
\[ p_3^T(x) = 1 + \frac{a_1 x}{\sqrt{T}} + \frac{\gamma_1(x^3 - 3x)}{6\sqrt{T}} \]
\[ + \frac{1}{2} \left( \frac{\gamma^2}{T} + \frac{a_1^2}{T} \right) (x^2 - 1) + \left( \frac{\gamma_1}{24T} + \frac{\gamma_1 \gamma_1}{6T} \right) (x^4 - 6x^2 + 3) \]
\[ + \frac{\gamma_1^2}{72T} (x^6 - 15x^4 + 45x^2 - 15) , \quad (6.15) \]
where

\[ a_1 = - \frac{J + K}{2I^{3/2}} + \frac{u}{I^{1/2}} \]

\[ \gamma_1 = - \frac{3J + 2K}{I^{3/2}} \]

\[ \rho_2 = \frac{2\eta}{I} + \frac{A}{I} + \frac{7J^2 + 14JK + 5K^2}{2I^3} - \frac{L + 4N + H}{I^2} - \frac{2\mu(2J + K)}{I^2} \]

\[ \delta_1 = \frac{12(2J + K)(J + K)}{I^3} - \frac{4L + 12N + 3H}{I^2}, \]

where

\[ \text{Var}(Z_1(\theta)) = I(\theta) + \frac{\Delta(\theta)}{T} + o(T^{-1}), \tag{6.16} \]

\[ I_T(\theta) = I(\theta) - \frac{n(\theta)}{T} + o(T^{-1}). \tag{6.17} \]

Remembering (2.6) we have proved (2) of Theorem 1. More explicit forms of (6.15) for the exact maximum likelihood estimators are given in Taniguchi (1985).
7. Third order asymptotic properties of minimum contrast estimators.

Taniguchi (1985) discussed third order asymptotic properties of maximum likelihood estimators in the class of third order asymptotically median unbiased (AMU) estimators, and showed a certain optimality of maximum likelihood estimators. Using the Edgeworth expansions of minimum contrast estimators we can discuss their third order asymptotic properties in this class.

If an estimator \( \tilde{\theta}_T \) satisfies the equations

\[
\lim_{T \to \infty} T^{(k-1)/2} \left| \mathbb{P}^T \left[ (\sqrt{T(\tilde{\theta}_T - \theta}) \leq 0 \right] - 1/2 \right| = 0, \tag{7.1}
\]

\[
\lim_{T \to \infty} T^{(k-1)/2} \left| \mathbb{P}^T \left[ (\sqrt{T(\tilde{\theta}_T - \theta}) > 0 \right] - 1/2 \right| = 0, \tag{7.2}
\]

then \( \tilde{\theta}_T \) is called kth-order asymptotically median unbiased (kth-order AMU for short). We denote the set of kth-order AMU estimators by \( A_k \). In general the minimum contrast estimator \( \hat{\theta}_T \) is not third order AMU. To be so a modification of \( \hat{\theta}_T \) is required. The following theorem gives the validity of Edgeworth expansion for modified estimators of \( \hat{\theta}_T \).

**THEOREM 2.** Suppose that \( m(\theta) \) is a continuously twice differentiable function. Define

\[
\hat{\theta}_m = \hat{\theta}_T + \frac{1}{T} m(\hat{\theta}_T).
\]

Then

\[
\sup_{F \in \mathcal{F}_C} \left| \mathbb{P}_F^T \left[ \sqrt{T} \left( \hat{\theta}_m - \theta \right) \right] \in \mathcal{B} \right| \leq \int_{\mathcal{B}} \mathcal{H}(y) q^T_m(y) dy = o(T^{-1}). \tag{7.3}
\]
uniformly for \( \theta \in C \), where

\[
q_{m3}(y) = 1 + \frac{1}{\sqrt{T}}[\alpha_1 + \sqrt{T}m(\theta)]y + \frac{\gamma_1}{6\sqrt{T}}(y^3 - 3y)
+ \frac{1}{2T}[\rho_2 + \alpha_1^2 + 2m'(\theta) + \text{Im}(\theta)^2 + 2\alpha_1\sqrt{T}m(\theta)](y^2 - 1)
+ \frac{1}{T}[\frac{1}{2t} + \frac{\alpha_1\gamma_1}{6} + \frac{\gamma_1\sqrt{T}m(\theta)}{6}](y^4 - 6y^2 + 3)
+ \frac{\gamma_1^2}{72T}(y^6 - 15y^4 + 45y^2 - 15).
\]

[PROOF]

Since \( m(\cdot) \) is continuously twice differentiable, we have

\[
\sqrt{T}T(\hat{\theta}_m - \theta) = \sqrt{T}T(\hat{\theta}_T - \theta) + \frac{\sqrt{T}}{T}m(\theta)
+ \sqrt{T}T(\hat{\theta}_T - \theta) \cdot m'(\theta) + T^{-3/2}(\sqrt{T}T(\hat{\theta}_T - \theta) \cdot m''(\theta^*)/\sqrt{T},
\]

where \( \theta \leq \theta^* \leq \hat{\theta}_T \).

By (1) of Theorem 1 we have

\[
P_\theta^T[\{\sqrt{T}T(\hat{\theta}_T - \theta) \cdot m'(\theta^*)/\sqrt{T} > T^{2\alpha} \} = o(T^{-1}),
\]

for \( 0 < \alpha < 1/4 \). Putting \( \rho_T = T^{2\alpha-1/2} \) in Lemma 4, we have only to derive the Edgeworth expansion for \( a_TU_T + s_T \), where

\[
a_T = \{1 + m'(\theta)/T\}, \quad U_T = \sqrt{T}T(\hat{\theta}_T - \theta) \quad \text{and} \quad s_T = \frac{\sqrt{T}}{T}m(\theta). \quad \text{By Theorem 1},
\]

we have

\[
\sup_{B \in \mathcal{B}} |P_\theta^T[U_T \in B] - \int_B \varphi(x)p_T^T(x) dx| = o(T^{-1}).
\]

Lemma 4 implies that
Also we have
\[
\sup_{B \in \mathcal{B}_0} |\mathbb{P}_T(\sqrt{T(\hat{\theta}_m - \theta)} \in B) - \mathbb{P}_T[\mathbb{a}_T^* U_T + s_T \in B]| = o(T^{-1}). \tag{7.7}
\]

Transforming \( y = a_T x + s_T \), it is not difficult to show
\[
\int_{a_T x + s_T \in B} \varphi(x)p_T^3(x)dx = \int_B \varphi(y)q_m^3(y)dy + o(T^{-1}). \tag{7.9}
\]

The relations (7.7), (7.8) and (7.9) imply our assertion. \( \Box \)

For \( m(\theta) = \frac{K(\theta)}{6I(\theta)^2} - \frac{\mu(\theta)}{I(\theta)} \), we denote \( \hat{\theta}_m^* = \hat{\theta}_m^* \). In this case we have

**COROLLARY 2.**

\[
\sup_{B \in \mathcal{B}_0} |\mathbb{P}_T(\sqrt{T(\hat{\theta}_m^* - \theta)} \in B) - \mathbb{P}_T[\mathbb{a}_T^* U_T + s_T \in B]| = o(T^{-1}). \tag{7.10}
\]
REMARK

Of course \( \hat{\theta}_T^* \) belongs to \( A_2 \), and we can see that the asymptotic distribution of \( \hat{\theta}_T^* \) (up to second order) coincides with that of the second order efficient estimator (see Taniguchi (1983) or (1985)).

REMARK

It is easy to check that \( \hat{\theta}_T^* \) is third order AMU. Also it is \( e_d = 2\pi + \Delta - 2\mu' \) that depends on the minimum contrast estimator.

Let \( \hat{\theta}_1^*(i = 1, 2) \) be the modified minimum contrast estimators with \( \mu_1, \Delta_1, \eta_1 \) and \( m_1(i = 1, 2) \) in place of \( \mu, \Delta, \eta \) and \( m \), respectively. Then we have

**COROLLARY 3.** For \( B = (-a, a) \), \( a > 0 \),

\[
\lim_{T \to \infty} T\left[ P^T_\theta[\sqrt{T}(\hat{\theta}_1^* - \theta) \in B] - P^T_\theta[\sqrt{T}(\hat{\theta}_2^* - \theta) \in B]\right] = \frac{1}{2} a \phi(a) \{2(\pi_2 - \eta_1) + \Delta_2 - \Delta_1 + 2(\eta_1 - \mu_1)\}, \quad (7.11)
\]

Thus if \( 2\pi_1 + \Delta_1 - 2\mu_1' \) is smaller than \( 2\pi_2 + \Delta_2 - 2\mu_2' \), then \( \hat{\theta}_1^* \) is better than \( \hat{\theta}_2^* \) in third order sense.

**EXAMPLE 3.** Let \( \{X_t\} \) be a Gaussian autoregressive process with the spectral density

\[
g_\theta(\lambda) = \frac{\sigma^2}{2\pi} \frac{1}{|1 - \theta e^{i\lambda}|^2},
\]

where \( |\theta| < 1 \).

Let \( \hat{\theta}_1^* \) be the modified maximum likelihood estimator of \( \theta \) defined in Example 1). Also let \( \hat{\theta}_2^* \) be the modified quasi-maximum
likelihood estimator of $\theta$ (defined in Example 2). Then we have

$$\mu_1 = 0, \quad \Delta_1 = \frac{3\theta^2 - 1}{(1 - \theta^2)^2}, \quad \eta_1 = -\Delta_1, \quad \eta_2 = 0.$$ 

For this case, the right-hand side of (7.11) is equal to 

$$\frac{4}{1} a(\phi(a)) \frac{\ell^2}{1 - \theta^2} \geq 0,$$

which coincides with the result of Fujikoshi and Ochi (1984). That is, $\hat{\theta}_1^*$ is better than $\hat{\theta}_2^*$.

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REFERENCES


**Title**: Validity of Edgeworth Expansions of Minimum Contrast Estimators for Gaussian ARMA Processes

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**Abstract**: Let \( \{X_t\} \) be a Gaussian ARMA process with spectral density \( f_g(\lambda) \), where \( \theta \) is an unknown parameter. To estimate \( \theta \) we propose a minimum contrast estimation method which includes the maximum likelihood method and the quasi-maximum likelihood method as special cases. Let \( \hat{\theta}_m \) be the minimum contrast esti-
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