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If $X$ is a sequence of semimartingales, converging to a semimartingale $X$, and such that \([X,X]\) converges to \([X,X]\), then all higher order variations \([X^n]\) and all the iterated integrals of $X$ converge jointly to the respective functionals of $X$. 
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WEAK CONVERGENCE OF THE VARIATIONS, ITERATED INTEGRALS,
AND DOLEANS-DADE EXPONENTIALS OF SEQUENCES OF SEMIMARTINGALES

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WEAK CONVERGENCE OF THE VARIATIONS, ITERATED INTEGRALS, 
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Abstract

If X is a sequence of semimartingales, converging to a semimartingale X, and such that [ X, X ] converges to [X,X], then all higher order variations (n) and all the iterated integrals of X converge jointly to the respective functionals of X.

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1. Introduction

A. Let \( X_t \) be a sequence of semimartingales, with \( t \in [0,1] \), such that

\[
X_t \xrightarrow{w(J_1)} X,
\]

where \( X \) is a semimartingale, and \( \xrightarrow{w(J_1)} \) denotes weak convergence on \( D[0,1] \) with respect to the \( J_1 \)-Skorohod topology.

We investigate the convergence of the variations, iterated integrals and Doléans-Dade exponentials of \( X \), which are defined as follows: for a semimartingale,

\[
V_k(Y)_t = \begin{cases} 
Y_t & \text{for } k = 1 \\
[Y,Y]_t = \langle Y, Y \rangle_t + \sum_{s \leq t} (\Delta Y)_s^2, & \text{for } k = 2 \\
\sum_{s \leq t} (\Delta Y)_s^k, & \text{for } k \geq 3 
\end{cases}
\]

\[
I_k(Y)_t = \begin{cases} 
Y_t & \text{for } k = 1 \\
\int_{t}^{s} I_{k-1}(Y) \, dY_s, & \text{for } k \geq 2 
\end{cases}
\]

\[
E(\lambda Y)_t = \exp\left[\lambda Y_t - \frac{\lambda^2}{2} \sum_{s \leq t} (\Delta Y)_s\right] \prod_{s \leq t} I(\lambda \Delta Y)_s,
\]

where \( \ell(x) = (1+x) e^{-x} \).

\( V_k(Y), I_k(Y) \) and \( E(\lambda Y) \) are called respectively the variations, the iterated integrals and the Doléans-Dade exponential of the semimartingale \( Y \).

It is known that \( V_k, I_k \) and \( E \) are well defined for any semimartingale \( Y \) (see Meyer, 1976). These quantities are important in the theory of multiple integration with respect to \( Y_t \).

B. When \( X_t = \sum_{i=1}^{[nt]} X_{i,n} \), with \( X_{i,n} \) a triangular array, then

\[
V_k(X)_t = \sum_{i=1}^{[nt]} X_{i,n}^k.
\]
\[ I_k(X)_t = \sum_{1 \leq i_1 < \ldots < i_k \leq [nt]} X_{i_1, \ldots, i_k, n}, \]
and
\[ E(\lambda X)_t = \prod_{i=1}^{[nt]} (1 + \lambda X_{i, n}) = \sum_{k=0}^{[nt]} \lambda^k I_k(X)_t. \]

The problem of the convergence of these "moments", "symmetric statistics", and generating function of the symmetric statistics have been studied in [1], [3-5], [7], and [9].

C. From formula 41.1 of Meyer (1976), it follows that in the semimartingale context, just like in the discrete deterministic case, \( I_k, k = 1, \ldots, m \) and \( V_k, k = 1, \ldots, m \) can be represented as polynomials of \( n \) variables in one another (the Newton polynomials which relate sums of powers to the sums of products). Thus, the issue of the joint convergence of \( I_k, k = 1, \ldots, m \), and that of the convergence of \( V_k, k = 1, \ldots, m \), are equivalent.

D. \( X \xrightarrow{w(J_1)} X \) does not imply in general \([X, X] + [X, X] \), as the following deterministic example from Jacod (1983) shows:

\[ X_t = \frac{n^2}{n} \sum_{k=1}^{n^2} (-1)^k \text{ converges uniformly to } 0, \text{ but } [X, X]_t = \sum_{k=1}^{n} \frac{1}{n^2} + t. \]

E. However, the following result holds:

**Theorem 1:** The following three statements are equivalent.

(1.5) \( \frac{n \,(X, [X, X]) \xrightarrow{n \to \infty} (X, [X, X]) \),

(1.6) \( (V_1(X), \ldots, V_m(X)) \xrightarrow{n \to \infty} V_1(X), \ldots, V_m(X), \forall m \geq 2, \)

(1.7) \( (I_1(X), \ldots, I_m(X)) \xrightarrow{n \to \infty} I_1(X), \ldots, I_m(X), \forall m \geq 2. \)

They also imply:

(1.8) \( E(\lambda X) \xrightarrow{w(J_1)} E(\lambda X), \forall \lambda. \)
Corollary: If
\[ n \rightarrow X_{w(J^n)} \rightarrow X \]
and the condition of Jacod (1983) holds:
\[ \limsup_{b \to \infty} P\{\text{Var}(B_{h}^{n}, n) > b\} = 0 \]
(where \( h \) is a truncation function and \( B_{h}^{n} \) is the predictable projection of the truncated semimartingale \( X \)), then (1.5), (1.6), (1.7) and (1.8) hold.

Proof: cf. Jacod (1983), Theorem 5.1.1, (1.9) and (1.10) imply (1.5).

2. Proofs

Introduce the following notation: For any real number \( x \),
\[ x^\geq_a := x^*1[|x| \geq a] \]
\[ x^<_a := x^*1[|x| < a] \]

We establish now the following:

Lemma 1: a) Suppose \( X \) are semimartingales such that
\[ \lim_{b \to \infty} \lim_{n \to \infty} P\{X, X \geq b\} = 0, \]
and let \( f(x) \) be any real function such that \( f(x) = o(x^2) \), as \( x \to 0 \). Then, for all \( \varepsilon \),
\[ \lim_{a \to 0} \lim_{n \to \infty} P\{\sum_{s=1}^{n} |f(X^a_{s})| \geq \varepsilon \} = 0. \]

b) If the assumptions of a) hold, \( X_{w(J^n)} \rightarrow X \) and \( f \) is a continuous, vector valued function, then:
\[ \sum_{s \leq t} f(\Delta X^\geq_{s})_{w(J^n)} \rightarrow \sum_{s \leq t} f(\Delta X_{s}). \]
Proof: a) Note first that \( \sum_{s \leq t} |f(\Delta X_s)| < \infty \), since \( \sum_{s \leq t} \Delta X_s < \infty \). Let now 
\( g(a) = \sup_{x \leq a} |f(x)|/x^2 \). Then,

\[
P\left( \sum_{s \leq 1} |f(\Delta X_s)| > \varepsilon \right) \leq P\left( \sum_{s \leq 1} (\Delta X_s)^2 g(a) > \varepsilon \right)
\]

\( (n)(n) \quad P\{ \sum_{s \leq 1} (\Delta X_s)^2 g(a) > \varepsilon \} \leq P\{ \sum_{s \leq 1} |f(\Delta X_s)| > \varepsilon /g(a) \} \).

Since \( g(a) \to 0 \), (2.2) follows from (2.1).

b) Let \( U(X) = \{u > 0 : P\{|\Delta X_0| \neq u, \text{for all } t\} = 0\} \). \( U(X) \) is dense in \( \mathbb{R}_+ \). For any \( a \in U(X) \), and \( f \) continuous, the functional

\[
S_f^a(Z) = \sum_{s \leq t} f(\Delta Z_s) > a
\]

is \( J_1 \) continuous a.s. (dist \((X))\). Thus, \( X \xrightarrow{w(J_1)} X \) implies for \( a \in U(X) \)

\[
S_f^a(X) \xrightarrow{w(J_1)} S_f^a(X).
\]

Also,

\[
S_f^a(X) \xrightarrow{a.s. J_1} S_f(X)_t := \sum_{s \leq t} f(\Delta X_s).
\]

The result follows now by (2.2) and Theorem 4.2 of Billingsley (1968).

Proof of Theorem 1:

By Lemma 1b, we have (1.5) \( \Rightarrow \) (1.6), and in fact the same type of argument yields (1.5) \( \Rightarrow \) (1.8), as follows: Assume for convenience \( \lambda = 1 \) and 

\( 1 \in U(X) \), let 

\[
f(x) = [\ell n(l+x) - x + \frac{x^2}{2}]_{|x| \leq 1},
\]

and let \( T : D[0,1] \to D[0,1] \) be defined by:

\[
T(Z)_t := \prod_{s \leq t} \ell (\Delta Z_s^1) = \prod_{s \leq t} (1 + \Delta Z_s^1 + \frac{1}{2}(\Delta Z_s^1)^2).
\]
Since the Doléans-Dade exponential
\[
E(X)_t = \exp\{X_t - \frac{1}{2}[X,X]_t + \sum_{s \leq t} f(\Delta X^s_{\leq t}) \cdot T(X)\}_t,
\]
it remains only to note that the functional:
\[
X^a : D^{(2)}[0,1] \rightarrow D^{(4)}[0,1]
\]
\[
X(Z_1, Z_2) = (Z_1, Z_2, S^a_t(Z_1), T_{Z_1})
\]
is continuous a.s., if both spaces are endowed with the respective \(J_1\) topologies. Letting then \(a \rightarrow 0\), as in the proof of Lemma 1, one gets:
\[
\begin{align*}
(n) & \rightarrow (n) \rightarrow (n) \rightarrow (n) \rightarrow (n)
\end{align*}
\]
\[
(\text{w}(J_1)) \rightarrow (X_t, [X,X]_t, \sum_{s \leq t} f(\Delta X^s_{\leq t}), \sum_{s \leq t} f(\Delta X^s_{> t})),
\]
since \(\ell n(1+x) - x + \frac{x^2}{2} = o(x^2)\), and since (1.5) implies (2.1). Finally, applying the continuous functional
\[
\rho : D^{(4)}[0,1] \rightarrow D[0,1],
\]
\[
\rho(Z_1, Z_2, Z_3, Z_4) = \exp[Z_1 - \frac{1}{2}Z_2 + Z_3] \cdot Z_4,
\]
we get that
\[
E(\lambda X) \xrightarrow{w(J_1)} E(\lambda X).
\]
Since (1.6) is equivalent to (1.7) (by the use of the polynomial mapping), and (1.6) trivially implies (1.5), Theorem 1 is proved. \(\Box\)
References


