OPTIMALITY ROBUSTNESS OF TESTS IN TWO POPULATION PROBLEMS*

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1. **Introduction.**

Recently a lot of work has been done in the area of robustness of multivariate tests. Robustness typically carries three meanings: null, nonnull, and optimality. Roughly speaking, a test is defined to be null robust if its null distribution remains the same for a class of distributions including the distribution under which the test is derived. Similarly we define the nonnull robustness of a test by the invariance of its nonnull distribution in a class of distributions including the underlying distribution. Finally, a test is called optimality robust if an optimal property the test enjoys can be extended to a class of distributions including the distribution under which the optimality holds. In a series of articles published by the present authors, the various relationships connecting these robustness concepts have been clarified, tools for establishing robustness of many multivariate tests have been developed and extensively used (see e.g. Kariya (1977, 1980, 1981a, 1981b), Kariya and Sinha (1985), Sinha (1984), Sinha and Giri (1985), Sinha and Das (1985)). It turns out that in all the cases considered so far optimality robustness along with null robustness hold for many familiar tests for a broad class of nonnormal distributions. However, it should be noted that in these approaches nonnormality at the sacrifice of independence has been considered. Therefore, in dealing with independent populations, this approach is not quite natural.

In this paper keeping (between) independence of the two populations we consider some testing problems involving location and scale parameters under some conditions on the underlying distributions. To be specific, in Section 2 we have considered the problem of testing the equality of two location parameters without any scale parameter and shown that the test based on the difference between the two sample means is conditional UMPI, conditionally given two ancillary statistics. However, it is not unconditionally so unless either the populations
are normal or there is only one observation from each population. In Section 3 we consider the problem of testing the equality of two scale parameters with or without the presence of location parameters and prove that the standard normal-theory UMPI F-test is optimality robust.

In Sections 4 and 5 analogous results have been obtained in association with standard tests considered in life-testing models (exponential distribution).

It should be pointed out that in these tests null robustness does not hold mainly because of independence.

As a technical tool, Wijsman's (1967) representation theorem is used.

2. Tests of Equality of Two Location Parameters (Non-Normal Case).

Let \( X^* : (n_1 + 1) \times 1 \) and \( Y^* : (n_2 + 1) \times 1 \) be independent random vectors with pdfs (with respect to Lebesgue measure)

\[
(2.1) \quad f_{\theta_1} (x^*) = q_1 [(x^* - \theta_1 e_1)' (x^* - \theta_1 e_1)], \quad f_{\theta_2} (y^*) = q_2 [(y^* - \theta_2 e_2)' (y^* - \theta_2 e_2)]
\]

where \( x^* \in R^{n_1+1}, y^* \in R^{n_2+1}, e_1 = (1, \ldots, 1)' \in R^{n_1+1}, e_2 = (1, \ldots, 1)' \in R^{n_2+1} \), \( \theta_1, \theta_2 \in R \) and \( q_1 \) and \( q_2 \) are pdfs over \( R^{n_1+1} \) and \( R^{n_2+1} \) respectively. Making a suitable orthogonal transformation, without any loss of generality, we can write

\[ x^* = (x, x_1'), \quad y^* = (y, y_1') \]

where \( x_1 \in R^{n_1}, y_1 \in R^{n_2} \) and assume that the pdfs are

\[
(2.2) \quad f_{\theta_1} (x^*) = q_1 [(x - \theta_1)^2 + x_1'^2], \quad f_{\theta_2} (y^*) = q_2 [(y - \theta_2)^2 + y_1'^2].
\]

The problem of testing \( H_0: \theta_1 = \theta_2 \) vs \( H_1: \theta_1 > \theta_2 \) remains invariant under the group \( G \) of transformations, \( G = G_1 \times G_2 \), where \( G_1 \) is the group of translations:

\[ x \rightarrow x + c, \quad y \rightarrow y + c, \quad -\infty < c < \infty, \]

and \( G_2 \) is the group of orthogonal transformations:

\[ O(n_1) \times O(n_2) \] acting on \( x_1 \) and \( y_1 \) as:

\[ x_1 \rightarrow \Gamma_1 x_1, \quad y_1 \rightarrow \Gamma_2 y_1, \quad \Gamma_i \in O(n_i), \quad i = 1, 2. \]
Clearly \( T = (x-y, x', y') \) is a maximal invariant. Using Wijsman's (1967) representation theorem, the ratio of the pdfs of \( T \) under nonnull to null is obtained as

\[
R = \frac{dP^T_{H_1}}{dP^T_{H_0}} = \frac{\int_{-\infty}^{\infty} q_1[(x+c-\delta)^2 + x'^2]q_2[(y+c-\delta)^2 + y'^2]d\delta}{\int_{-\infty}^{\infty} q_1[(x+c-\delta)^2 + x'^2]q_2[(y+c-\delta)^2 + y'^2]d\delta}
\]

\[
R = \frac{\int_{-\infty}^{\infty} q_1[(c-\delta)^2 + x'^2]q_2[(c-\delta)^2 + y'^2]d\delta}{\int_{-\infty}^{\infty} q_1[(c-\delta)^2 + x'^2]q_2[(c-\delta)^2 + y'^2]d\delta}
\]

To derive an optimum invariant test based on \( R \), note that, conditionally given the two ancillary statistics \( x'x' \) and \( y'y' \), the ratio of the conditional pdfs of \( (x-y) \) under nonnull to null is precisely \( R \). However, from (2.3), \( R \) can be expressed as

\[
R = E\{q_1[(c-\delta)^2 + x'^2]/q_1[c^2 + x'^2]\}
\]

where \( E \) stands for expectation with respect to \( c \) having the pdf

\[
c \sim q_1[c^2 + x'^2]q_2[(c-z)^2 + y'^2]/\int_{-\infty}^{\infty} q_1[c^2 + x'^2]q_2[(c-z)^2 + y'^2]d\delta.
\]

We now make the following assumption.

\[
\text{Dist}^\text{Special}
\]
Assumption 2.1. \( q_1, q_2 \in Q = \{ q : \frac{q((c-a)^2+t)}{c+t} + c, \text{ for every } a > 0, t > 0 \} \).

It then follows that the family of distributions of \( c \) in (2.5), generated by the parameter \( z \), keeping \( x_i'x_i \) and \( y_i'y_i \) fixed, has MLR property in \( c \) by the assumption \( q_2 \in Q \) and hence \( R \) is nondecreasing in \( z \) by the assumption \( q_1 \in Q \) (Lehmann (1959), p. 74). We have, therefore, proved the following.

Theorem 2.1. For testing \( H_0: \theta_1 = \theta_2 \) vs \( H_1: \theta_1 > \theta_2 \) in the model (2.2), the test with a critical region \( z > k \) is UMPI in the class of conditional level-\( \alpha \) tests, conditionally given the two ancillary statistics \( x_i'x_i \) and \( y_i'y_i \), whenever \( q_1, q_2 \in Q \).

Remark 2.1. It is clear from (2.3) that when both \( q_1 \) and \( q_2 \) are normal, \( x_i'x_i \) and \( y_i'y_i \) can be ignored and the above test is UMPI unconditionally. For nonnormal \( q \)'s, sufficiency-invariance reduces \( (x, y, x_i'x_i, y_i'y_i) \) to \( (x-y, x_i'x_i, y_i'y_i) \) and the above conditional argument is necessary unless \( n_1 = n_2 = 0 \). This means the test is UMPI when there is only one observation from each population and \( q_1, q_2 \in Q \).

Remark 2.2. If \( q \in Q \) is differentiable, \( Q \) is equivalent to

\[
(2.6) \quad \tilde{Q} = \{ q : \frac{cq'(c+t)}{c+t} + c, \quad t > 0 \}.
\]

An example of a \( q \in \tilde{Q} \) is provided by \( q(u) = e^{-u^r}, r > \frac{1}{2} \). It may be remarked that a \( t \)-density \( \not\in Q \).

Remark 2.3. Our attempt to derive a UMPI or even a LBI test when \( q_1 \) and \( q_2 \) involve an unknown common scale parameter is not successful unless \( q_1 \) and \( q_2 \) are normal.

It is possible that the familiar Fisher's \( t \)-test for this problem based on \( (x-y)/(x_i'x_i + y_i'y_i)^{1/2} \) is optimum only for normal densities.
Remark 2.4. The null pdf of $Z$ is given by

\begin{equation}
\frac{dP_{H_0}}{dz} = \int_{-\infty}^{\infty} q_1(c^2) q_2((c-z)^2) dc
\end{equation}

where $q_1(c^2) = \int_{\mathbb{R}^1} q_1(c^2 + x_i'x_i) dx_i$, $q_2((c-z)^2) = \int_{\mathbb{R}^2} q_2((c-z)^2 + y_1' y_1) dy_1$. Thus the null robustness of the test does not hold.

3. Test of Equality of Two Scale Parameters (Non-Normal Case).

Let $x_1$ be $n_1 \times 1$ random vector with pdf (with respect to Lebesgue measure)

\begin{equation}
f_{i_1}(x_i) = \sigma_i^{-n_i} q_i((x_i - \mu_{i_1} e_{i_1})' (x_i - \mu_{i_1} e_{i_1}) / \sigma_i^2) \quad (i = 1, 2),
\end{equation}

where $\mu_i \in \mathbb{R}$, $e_i = (1, \ldots, 1)' \in \mathbb{R}^{n_i}$ and $x_i \in \mathbb{R}^{n_i}$ and $x_2 \in \mathbb{R}^{n_2}$ are independent. Here we consider the problem of testing $H: \sigma_1^2 = \sigma_2^2$ vs $H: \sigma_1^2 > \sigma_2^2$. Of course, if $x_1$'s are both normal, then the $F$-test is UMPI. We shall show it is also UMPI in the above situation under a mild condition on $q_1, q_2$. This implies an optimality robustness of the $F$-test. Without essential loss of generality, assume $\mu_1 = \mu_2 = 0$. Then the problem remains invariant under the transformation $x_i \rightarrow cx_i$ $(i = 1, 2)$, where $c \in G = (0, \infty)$. Then the ratio $R$ of the pdfs of a maximal invariant under nonnull to null is given by $R = N/D$, where

\begin{equation}
N \equiv N(\sigma_1, \sigma_2) = \left[ \prod_{i=1}^{n_1} q_i \left( c^2 x_i'x_i / \sigma_i^2 \right) \right] c^{n-1} dc
\end{equation}

$n = n_1 + n_2$ and $D = N(\sigma_1, \sigma_1)$. In (3.2), transforming $c$ into $c(x_1'x_2)^{1/2}/\sigma_2$, $N$ becomes

\begin{equation}
N_y = (x_1'x_2)^{-n/2} \frac{n_1}{2} \int_0^{\infty} q_1(\gamma t^2 c^2) q_2(c^2) c^{n-1} dc
\end{equation}

where $\gamma = \sigma_2^2 / \sigma_1^2$ and $t = x_1'x_2 / x_1'x_1$. Note $D = N_1$. Further let $Q_n$ be the class of
functions from \([0, \infty)\) into \([0, \infty)\) such that for any \(q \in \mathcal{Q}_n\), \(\int R^n q(x^t x) dx = 1\) and \(q(\beta u)/q(u)\) is increasing in \(u\) for any \(0 < \beta < 1\). Then we obtain

**Theorem 3.1.** The test based on \(t = x_1^t x_1 / x_2^t x_2 > k\) is UMPI for testing \(\sigma_1^2 = \sigma_2^2\) vs \(\sigma_1^2 > \sigma_2^2\) under the pdf's in (3.1) for any \(q_1 \in \mathcal{Q}_n^1\), \(q_2 \in \mathcal{Q}_n^2\). But the null distribution of \(t\) depends on \(q_1^1\) and \(q_2^2\).

**Proof.** Clearly \(t\) is a maximal invariant under \(G\). Under the null hypothesis, the pdf of \(t\) is directly shown to be

\[
d_{H_0}^t = a t^{(n-2)/2} \int_0^\infty q_1(tc^2)q_2(c^2)c^{n-1} dc
\]

where \(a\) is a constant depending on \(n_1\) and \(n_2\) only and \(p_{H_0}^t\) is the distribution of \(t\) under \(H_0: \sigma_1^2 = \sigma_2^2\). Hence from \(R = d_{H_0}^t / d_{H_1}^t = N_1/N_0\) with \(N_1\) in (3.3) and (3.4), the pdf of \(t\) under \((\sigma_1, \sigma_2)\) is given by

\[
h(t|\gamma)dt = Rd_{H_0}^t = a t^{(n-2)/2} \int_0^\infty q_1(tc^2)q_2(c^2)c^{n-1} dc] dt,
\]

so \(h(t|\gamma)/h(t|1) = R\). Further \(R\) in this sense is expressed as

\[
R = \gamma^{n_1/2} E[q_1(\gamma c^2)/q_1(c^2)]
\]

where the expectation is taken with respect to the pdf

\[
k_1^t(c^2) = q_1(c^2)q_2(c^2/t)c^{n-1} \int_0^\infty q_1(c^2)q_2(c^2/t)c^{n-1} dc.
\]

Here we show that \(k_1^t(c^2)\) with \(t > 0\) has a monotone likelihood ratio property in \(c^2\). In fact, since \(q_2 \in \mathcal{Q}_n^2\), \(k_1^t(c^2)/k_1^t(c^2) = q_2(c^2/t_2)/q_2(c^2/t_1)\) is increasing in \(c^2\) for \(t_2 > t_1\). Therefore, since \(q_1(\gamma c^2)/q_1(c^2)\) in the inside of the expectation of (3.6) is increasing in \(c^2\) by assumption, its expectation is increasing in \(t\) (see Lehmann (1959) p. 74). This implies \(R\) in (3.6) is an increasing function of \(t\). Consequently by the Neyman-Pearson Lemma, the test based on \(t > k\) is UMP.
Since \( t \) does not depend on \( q_i \)'s, it is UMPI for all \( q_1 \in Q_1, q_2 \in Q_2 \). But the null distribution of \( t \) in (3.4) cannot be free from \( q_i \)'s. This proves our result.

**Remark 3.1.** If \( u_1 = u_2 = 0 \) is not assumed, the increasingness of \( q(\beta u)/q(u) \) is simply replaced by that of \( \bar{q}(\beta u)/q(u) \), where \( \bar{q}(u) = \int_{-\infty}^{\infty} q(u+x^2)dx \).

**Remark 3.2.** A crucial distinction of our result from Corollary 4.3 in Kariya (1981) lies in assuming the independence of \( x_1 \) and \( x_2 \).

**Remark 3.3.** The null distribution of \( t \) is given by (3.5) with \( \gamma = 1 \). Hence for specific \( q_1 \) and \( q_2 \), significance points of \( t \) are obtained from it.

**Remark 3.4.** If \( q \) is differentiable, a necessary and sufficient condition for \( q(\beta u)/q(u) \) to be increasing in \( u \) for \( 0 < \beta < 1 \) is given by \( q'((u) < 0 \) and \( \frac{uq'(u)}{q(x)} + u \). This condition is satisfied for a multivariate \( t \)-distribution and also for more general normal mixtures.

4. **Test of Equality of Two Location Parameters (Non-Exponential Case).**

Let \( X: n_1 \times 1 \) and \( Y: n_2 \times 1 \) be independent random vectors with pdfs (with respect to Lebesgue measure)

\[
\begin{align*}
\theta_1 = q_1[\sum_{i=1}^{n_1} (x_i - \theta_1)], \\
\theta_2 = q_2[\sum_{i=1}^{n_2} (y_i - \theta_2)], \\
x_i \geq \theta_1, y_i \geq \theta_2, \forall i; x \in \mathbb{R}, y \in \mathbb{R}, \theta_1, \theta_2 \in \mathbb{R}.
\end{align*}
\]

The problem of testing \( H_0: \theta_1 = \theta_2 \) vs \( H_1: \theta_1 > \theta_2 \) remains invariant under the group \( G \) of translations acting as \( x_i \rightarrow x_i + c, y_i \rightarrow y_i + c, i = 1, \ldots, n, \theta_1 \rightarrow \theta_1 + c, \theta_2 \rightarrow \theta_2 + c, -\infty < c < \infty \). Let \( x(1) = \min x_1, y(1) = \min y_1, t_1 = \sum_{i=1}^{n_1} (x_i - x(1)), \)
\[ t_2 = \sum_{i=1}^{n_2} (y_i - y(1)). \]

Applying Wijsman's (1967) theorem, the ratio \( R \) of the pdfs of a maximal invariant \( T = (x(1), y(1), t_1, t_2) \) under nonnull to null is obtained as
To derive an optimum invariant test based on $R$, note that, conditionally given the two ancillary statistics $t_1$ and $t_2$, the ratio of the conditional pdfs of $z$ under nonnull to null is precisely $R$. Moreover, from (4.2), $R$ can be expressed as

(4.3) \[ R = E\{q_1[(t_1+c-\delta)]/q_1[(t_1+c)]:I_{[c>\delta]} \} \]

where $I$ is the indicator function and $E$ above stands for expectation with respect to $c$ having the pdf,

(4.4) $c \sim q_1[(t_1+c)]q_2+\frac{n_2}{n_1}(c-z)]/\int_{c>\max(0,z)} q_1[(t_1+c)]q_2+\frac{n_2}{n_1}(c-z)]dc, c>\max(0,z)$. 

\[
(4.2) \quad R = \frac{\int_{c>\max(0,z)} q_1[(t_1+c-\delta)]/q_1[(t_1+c)]q_2+\frac{n_2}{n_1}(c-z)]dc}{\int_{c>\max(0,z)} q_1[(t_1+c-\delta)]/q_1[(t_1+c)]q_2+\frac{n_2}{n_1}(c-z)]dc, c>\max(0,z)} \]
We now make the following assumption.

**Assumption 4.1.** \( q_1, q_2 \in \mathcal{Q} = \{ q : q(t+c-a)/q(t+c) + c \text{ for } c > a, \forall t \geq 0 \} \).

It then follows that the family of distributions of \( c \) in (4.4), generated by \( z \) for fixed \( t_1, t_2 \), has MLR property in \( c \) by the assumption \( q_2 \in \mathcal{Q} \) and hence \( R \) is nondecreasing in \( z \) by the assumption \( q_1 \in \mathcal{Q} \) [vide Lehmann (1959), p. 74]. This proves the following result.

**Theorem 4.1.** For testing \( H_0 : \theta_1 = \theta_2 \) vs \( H_1 : \theta_1 > \theta_2 \) in the model (4.1), the test with the critical region \( z > k \) is UMPI in the class of conditional level-\( \alpha \) tests, conditionally given the two ancillary statistics \( t_1 \) and \( t_2 \), whenever \( q_1, q_2 \in \mathcal{Q} \).

**Remark 4.1.** It is clear from (4.2) that when both \( q_1 \) and \( q_2 \) are exponential, \( t_1 \) and \( t_2 \) can be ignored and the above test is UMPI unconditionally. For non-exponential \( q \)'s, sufficiency-invariance reduces \( (x_{(1)}, y_{(1)}, t_1, t_2) \) to \( (x_{(1)} - y_{(1)}), t_1, t_2) \) and the above conditional argument is necessary unless \( n_1 = n_2 = 1 \) in which case \( t_1 \) and \( t_2 \) become vacuous.

**Remark 4.2.** If \( q \in \mathcal{Q} \) is differentiable, \( \mathcal{Q} \) is equivalent to

\[
(4.5) \quad \tilde{\mathcal{Q}} = \{ q : cq'(t+c)/q(t+c) + c, \forall t \geq 0 \}.
\]

An example of a \( q \in \tilde{\mathcal{Q}} \) is provided by \( q(u) = e^{-u^r}, r > 0 \).

**Remark 4.3.** The null pdf of \( z \) is given by

\[
(4.6) \quad \frac{dP_z^z}{dz} \bigg|_{H_0} = \int_{c > \max(0, z)} q_1[c]q_2[c-z]dc
\]

where \( q_1(c) = \int_0^{t_1} q_1[b+t_1]t_1^2 \ dt_1, \ q_2(c-z) = \int_0^{\infty} q_2[c-z+t_2]t_2^2 \ dt_2 \). Thus the null robustness of the test does not hold.
5. Test of Equality of Two Scale Parameters (Non-Exponential Case).

Let \( X: n_1 \times 1 \) and \( Y: n_2 \times 1 \) be independent random vectors with pdfs (with respect to Lebesgue measure)

\[
(5.1) \quad f_{\mu_1, \sigma_1}(x) = \sigma_1 q_1 \left[ \frac{1}{\sigma_1} \sum_{i=1}^{n_1} \frac{(x_i - \mu_1)}{\sigma_1} \right], \quad f_{\mu_2, \sigma_2}(y) = \sigma_2 q_2 \left[ \frac{1}{\sigma_2} \sum_{i=1}^{n_2} \frac{(y_i - \mu_2)}{\sigma_2} \right]
\]

\[ x_1 > \mu_1, \; y_1 > \mu_2, \; \mu_i, \; -\infty < \mu_1, \mu_2 < \infty, \; \sigma_1, \sigma_2 > 0. \]

We consider the problem of testing \( H_0: \sigma_1 = \sigma_2 \) vs \( H_1: \sigma_1 > \sigma_2 \) when \( \mu_1 \) and \( \mu_2 \) are unknown. The problem remains invariant under the group \( G \) of transformations:

\[ x_1 = ax_1 + c_1, \quad y_1 = ay_1 + c_2, \quad \mu_1 = a\mu_1 + c_1, \quad \mu_2 = a\mu_2 + c_2, \quad \sigma_1 = a\sigma_1, \quad \sigma_2 = a\sigma_2, \quad -\infty < c_1, c_2 < \infty, \; a > 0. \]

Let \( x(1) = \min x_i, \; y(1) = \min y_i, \; t_1 = \frac{1}{n_1} \sum (x_i - x(1)), \; t_2 = \frac{1}{n_2} \sum (y_i - y(1)) \). Applying Wijsman's (1967) theorem, the ratio of the pdfs of a maximal invariant \( Z = t_1/t_2 \) under nonnull to null is obtained as

\[
(5.2) \quad R = \frac{dP_{H_1}}{dP_{H_0}} = \frac{\int_{-\infty}^{\infty} q_1 \left[ \int_{-\infty}^{\infty} q_2 \left[ \frac{1}{\sigma_1} \sum_{i=1}^{n_1} \frac{(x_i - x(1))}{\sigma_1} \right] \frac{1}{\sigma_2} \sum_{i=1}^{n_2} \frac{(y_i - y(1))}{\sigma_2} \right] \; da \; dc_1 \; dc_2}{\int_{-\infty}^{\infty} q_1 \left[ \int_{-\infty}^{\infty} q_2 \left[ \frac{1}{\sigma_1} \sum_{i=1}^{n_1} \frac{(x_i - x(1))}{\sigma_1} \right] \frac{1}{\sigma_2} \sum_{i=1}^{n_2} \frac{(y_i - y(1))}{\sigma_2} \right] \; da \; dc_1 \; dc_2}
\]

\[ \int_{0}^{\infty} \frac{\tilde{q}_1(\delta)}{\sigma_1} \frac{\tilde{q}_2(\delta)}{\sigma_2} \; da, \quad \tilde{q}(x) = \int_{0}^{\infty} q(x + u) \\ \int_{0}^{\infty} \frac{\tilde{q}_1(\delta)}{\sigma_1} \frac{\tilde{q}_2(\delta)}{\sigma_2} \; da, \quad \delta = \frac{\sigma_2}{\sigma_1}, \quad z = \frac{t_1}{t_2} \]

\[ \int_{0}^{\infty} \frac{\tilde{q}_1(\delta z)}{\sigma_1} \frac{\tilde{q}_2(\delta)}{\sigma_2} \; da, \quad \delta = \frac{\sigma_2}{\sigma_1}, \quad z = \frac{t_1}{t_2} \]
where E above stands for expectation with respect to a having the pdf

\begin{equation}
\int_{0}^{\infty} \tilde{q}_1[a\delta] \tilde{q}_2[az^{-1}] da = \frac{-(n_2-1)}{\delta} \int_{0}^{\infty} \tilde{q}_1[a] \tilde{q}_2[az^{-1}] da
\end{equation}

\begin{equation}
= \delta^{-(n_2-1)} \text{E}[\tilde{q}_1[a\delta]/\tilde{q}_1[a]]
\end{equation}

We now make the following assumption.

**Assumption 5.1.** \( \tilde{q}_1, \tilde{q}_2 \in \tilde{Q} = \{ \tilde{q}: \tilde{q}[(ad)/\tilde{q}[a]+ a^2] \text{ for } 0 < d < 1 \}. \)

It then follows (vide Lehmann (1959), p. 74) that the family (5.3) has MLR property in \( a \) and hence \( R \) is nondecreasing in \( z \). Thus we have proved the following.

**Theorem 5.1.** For testing \( H_0: \sigma_1 = \sigma_2 \) vs. \( \sigma_1 > \sigma_2 \) in the model (5.1), the test with the critical region \( z > k \) is UMPI whenever \( \tilde{q}_1, \tilde{q}_2 \in \tilde{Q} \).

**Remark 5.1.** If either \( \nu_1 \) or \( \nu_2 \) or both are known, \( t_1 \) and \( t_2 \) are suitably redefined and the assumption 5.1 is modified accordingly.

**Remark 5.2.** The null pdf of \( z \) is given by \( z^{-(n_2+3)/2} \int_{0}^{\infty} \tilde{q}_1[a] \tilde{q}_2[az^{-1}] da \).

Hence for specific \( q_1 \) and \( q_2 \) significance points of \( z \) are obtained from it.

**Remark 5.3.** If \( \tilde{q} \) admits a derivative, \( \tilde{Q} \) is equivalent to \( \tilde{Q} = \{ q: aq'(a)/q(a) + a > 0 \} \).

An example of a \( \tilde{q} \in \tilde{Q} \) is provided by \( \tilde{q}(u) = e^{-u^r}, r > 0 \).
REFERENCES


The problems of testing the equality of two location parameters (without the presence of scale parameters) and the equality of two scale parameters (with or without the presence of location parameters) based on two independent samples are considered. It is shown that, under some mild conditions on the underlying distributions, the standard optimum invariant tests derived under normality or exponentiality continue to be optimum for a wide class of distributions. However, the tests are not null robust. As a technical tool,
(Abstract) continued...

Wijsman's (Fifth Berkeley Symposium, Vol. 1, 1967) representation theorem is used.