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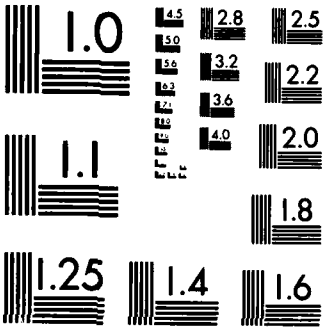

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Monterey, California

ESTIMATING THE PROBABILITY OF A DIFFUSING TARGET ENCOUNTERING A STATIONARY SENSOR

by

James N. Eagle

July 1985

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In this report two expressions are given for the probability of a diffusing target avoiding detection to time t by a sensor fixed at the center of a square region A. The target is constrained to always remain within A. The first expression results from an approximation to the exact solution of the diffusion equation, and the second from experimentation with a Monte Carlo simulation of the diffusion process.

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TAB
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Estimating the Probability of a Diffusing Target Encountering a Stationary Sensor

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In this report two expressions are given for the probability of a diffusing target avoiding detection to time t by a sensor fixed at the center of a square region R. The target is constrained to always remain within R. The first expression results from an approximation to the exact solution of the diffusion equation, and the second from experimentation with a Monte Carlo simulation of the diffusion process.

The sensor considered is a definite range law or "cookie cutter" detector. For such a sensor, there is a specified range, R, beyond which detection is impossible and inside which detection is certain.

1. Background

This work was begun with the intention of developing for the Naval War College, Newport, RI a simple expression for the probability of a diffusing target avoiding detection by a sensor conducting a moving, systematic search of the area R. It was soon realized that the special case of a stationary sensor had to be first addressed, and that this simpler problem was indeed nontrivial.

The importance of the stationary sensor problem goes beyond being a limiting case of the moving sensor problem. It may be used to provide estimates of the rate at which randomly moving targets will encounter

stationary objects with extended fields of influence, such as fixed acoustic sensors, sonobuoys, or possibly mines.

This stationary sensor problem might appear deceptively simple at first glance. After all, it could be argued, this problem is equivalent to that of a searcher with a cookie cutter sensor of range R conducting a random search for a stationary target. And Koopman[1946] and [1980] argued that the probability of nondetection to time t is exp(-2Rvt/A), where v is the speed of the random search. The problem is, of course, what to use for v when the searcher's path is a diffusion. One of the results of this work is an expression for the equivalent speed of such a "diffusion search" of a square area.

The initial, experimental results for the problem here addressed were obtained by Sislioglu[1984]. Sislioglu conducted Monte Carlo simulations with different target diffusion constants D, area sizes A, and sensor detection ranges R. He observed that when the initial distribution of the target was uniform over A, the probability of nondetection to time t, PND(t), was given approximately by

$$(1 - \pi R^2/A)\exp(-24.7\, RDt/A^{1.5}). \qquad (1)$$

In this report, some analytical support for Sislioglu's results is offered. Also a slight modification of (1) is suggested which agrees more closely with theory and expermental results when the area of the detection disk approaches the area of the search region A.

2. The Diffusion Equation

The probability density p(t) of a particle undergoing diffusion in any coordinate system must satisfy the diffusion equation

$$(D/2)\,\nabla^2 p = \partial p/\partial t, \qquad (2)$$

where D is the diffusion constant, and ∇^2 is the Laplacian operator. In Cartesian and polar coordinates, respectively, (2) becomes

$$(D/2)\,(\partial^2 p/\partial x^2 + \partial^2 p/\partial y^2) = \partial p/\partial t, \text{ and} \tag{3}$$

$$(D/2)\,(\partial^2 p/\partial r^2 + (1/r^2)(\partial^2 p/\partial \theta^2) + (1/r)(\partial p/\partial r)) = \partial p/\partial t. \tag{4}$$

To find a unique function $p(x,y,t)$ or $p(r,\theta,t)$ satisfying these partial differential equations, spatial and temporal boundary conditions must be specified. Defining A to be a square with sides of length L in the first quadrant, the boundary conditions in Cartesian coordinates are

$$\partial p/\partial x|_{x=0} = \partial p/\partial x|_{x=L} = 0, \tag{5}$$

$$\partial p/\partial y|_{y=0} = \partial p/\partial y|_{y=L} = 0, \tag{6}$$

$$p(x,y,t) = 0,\ ((x - L/2)^2+(y - L/2)^2)^{1/2} \leqslant R, \text{ and} \tag{7}$$

$$p(x,y,0) = 1/A,\ ((x - L/2)^2+(y - L/2)^2)^{1/2} > R. \tag{8}$$

Equations (5) and (6) ensure that none of the target's probability mass "escapes" from A. That is, the boundaries of A are reflecting. Equation (7) requires that $p(x,y,t)$ be 0 on the detection disk. And (8) ensures that the initial distribution of the target over the search region is uniform and integrates to $(A-\pi R^2)/A$ (the probability that the target is not detected at time 0).

For any particular instance of the problem, finding a $p(x,y,t)$ satisfying (3) and the boundary conditions is not difficult using numerical methods. Such procedures are routinely used in heat transfer problems to solve the diffusion equation (called the conduction equation or the Fourrier equation in the physics and engineering literature) to determine the temperature distribution across imperfectly conducting solids. In fact, Pitts and Sissom[1977] give the example of a heated pipe in a square block of insulating material as one where the isotherms can be accurately estimated by hand plotting.

Although the problem is not hard to solve numerically, the square boundary of A combined with a circular sensor tend to make the analytical solution difficult to obtain. And without an analytical solution, it may be impossible to establish Sislioglu's observations in general. Making a change to the geometry, however, allows an analytical solution. Specifically, if the search region A is assumed to be a disk of area A instead of a square, then an exact solution in polar coordinates is possible.

It is noted that the ray solution method described by Mangel[1981] could presumably be used to solve the diffusion equation, at least approximately, for a square search area. Such a solution was not attempted since an exact solution was available for the circular search area case.

3. The Solution

The disk-within-a-disk geometry has a radial symmetry, thus reducing the problem to one dimension, r. The new problem is to find a function $p(r,t)$ satisfying

$$(D/2)\,(\partial^2 p/\partial r^2 + (1/r)(\partial p/\partial r)) = \partial p/\partial t,$$

subject to

$$\partial p/\partial r\big|_{r=R_A} = 0,$$

$$p(r,t) = 0,\ r \leqslant R,$$

$$p(r,0) = 1/A,\ r > R,$$

where $R_A = (A/\pi)^{1/2}$ is the radius of the transformed (i.e., circular) search area A. This problem has been solved in the physics literature.

Muskat[1937] and Muskat[1934] (a paper investigating the production rate of oil wells) give the solution as

$$p(r,t) = -(\pi/A) \sum_{n=1}^{\infty} k_n \; U(\alpha_n r) \exp(-D\, \alpha_n^2\, t/2) \tag{9}$$

where

$$k_n = J_0(\alpha_n R)\, J_1(\alpha_n R_A) \,/\, (J_0^2(\alpha_n R) - J_1^2(\alpha_n R_A)), \tag{10}$$

$$U(\alpha_n r) = Y_1(\alpha_n R_A)\, J_0(\alpha_n r) - J_1(\alpha_n R_A)\, Y_0(\alpha_n r), \tag{11}$$

and α_n is the nth positive root of $U(\alpha R)$. (That is, the nth smallest positive value of α satisfying $U(\alpha R) = 0$.) Also, J_0, J_1, Y_0, and Y_1 are respectively Bessel functions of the first kind order 0, first kind order 1, second kind order 0, and second kind order 1.

The solution (9)-(11) does not appear particularly easy to evaluate or interpret, being an infinite sum of Bessel functions. However for large t the solution simplifies considerably. As t becomes large, (9) becomes

$$(-\pi k_1 / A)\, U(\alpha_1 r) \exp(-D\, \alpha_1^2 t/2), \tag{12}$$

where α_1 is the smallest positive value of α such that $U(\alpha R) = 0$. The other exponential terms are ignored since they involve larger roots of $U(\alpha R)$. This means that as t becomes large, the decrease in p(r,t) for constant r becomes exponential at a rate of $D\alpha_1^2/2$.

When p(r,t) is specified, PND(t) is then given by

$$\int_0^{2\pi} \int_R^{R_A} p(r,t)\, r\, dr\, d\theta. \tag{13}$$

So for large t, PND(t) becomes

$$-(2\pi^2 k_1/R)\left\{\int_R^{R_R} U(\alpha_1 r)\, r\, dr\right\} \exp(-D\,\alpha_1^2 t/2) \tag{14}$$

$$= -(2\pi^2 k_1 R/(R\alpha_1))\{J_1(\alpha_1 R_R)\, Y_1(\alpha_1 R) - Y_1(\alpha_1 R_R)\, J_1(\alpha_1 R)\} \exp(-D\,\alpha_1^2 t/2), \tag{15}$$

$$= K \exp(-D\,\alpha_1^2 t/2),$$

where K is a constant which depends on the problem geometry.

Evaluation of the integral in (14) is straightforward given the change of variable $u=\alpha_1 r$ and the identities

$$\int x\, J_0(x)\, dx = x\, J_1(x) \quad \text{and} \quad \int x\, Y_0(x)\, dx = x\, Y_1(x).$$

Thus for large t, PND(t) decreases exponentially at the same rate as p(r,t). So Sisliogiu's observation of exponentially decreasing PND(t) (or, equivalently, a constant detection rate) appears asymptotically correct for large enough t.

Using Muskat's dimensionless notation, we can simplify the solution somewhat by defining

$$x_1 = \alpha_1 R \quad \text{and} \quad \rho = R_R/R.$$

Then for large t, PND(t) can be written as

$$K \exp(-D\, x_1^2\, t/(2R^2)), \tag{16}$$

where x_1 is the first positive x satisfying

$$Y_1(x\rho)\, J_0(x) - J_1(x\rho)\, Y_0(x) = 0; \tag{17}$$

K is given by

$$-(2\pi k_1/(x_1 \rho^2))\,\{J_1(x_1\rho)\, Y_1(x_1) - Y_1(x_1\rho)\, J_1(x_1)\}; \tag{18}$$

and k_1 is

$$J_0(x_1)\, J_1(x_1\rho) \,/\, (J_0^2(x_1) - J_1^2(x_1\rho)). \tag{19}$$

In the next section, experimentally determined PND(t) from a Monte Carlo simulation of the diffusion process will be compared with (16)-(19), Sislioglu's approximation (1), and a slight modification of (1).

4. Simulation Results and Conclusions

Figure 1. is a plot of PND(t) determined by Monte Carlo simulation of the diffusion process for a square area A of size 10,000 square nautical miles (nm^2), a diffusion constant D of 100 nm^2/hour, and detection radii R varying from 28.21 nm (ρ=2) to 3.76 nm (ρ=15). The time increment for these simulations was 1 minute, which resulted in the x and y displacements of the target in each increment being selected from independent normal distributions of mean 0 and variance 100/60. Figure 1. illustrates the degree to which the decrease in PND(t) is exponential. Plotted on a log scale, PND(t) appears very nearly linear. For small t, however, the decrease in PND(t) is faster than exponential. This is seen more clearly in Figure 2, which is an enlargement of the upper left-hand corner of Figure 1.

Sislioglu reported that PND(t) can be approximated by

$$(1 - \pi R^2/A)\ \exp(-\ 24.7\ RDt/A^{1.5}). \qquad (20)$$

The simulation results reported here indicate that a somewhat better approximation when ρ approaches 1 is

$$(1 - \pi R^2/A)\ \exp(-\ 24.7\ RDt/(A - \pi R^2)^{1.5}). \qquad (21)$$

That is, A in (20) is replaced with $A - \pi R^2$. Figure 3. compares PND(t) calculated by simulation with the estimates given by (21), (20), and the asymptotic estimate (16)-(19). The simulation data in Figure 3. are the

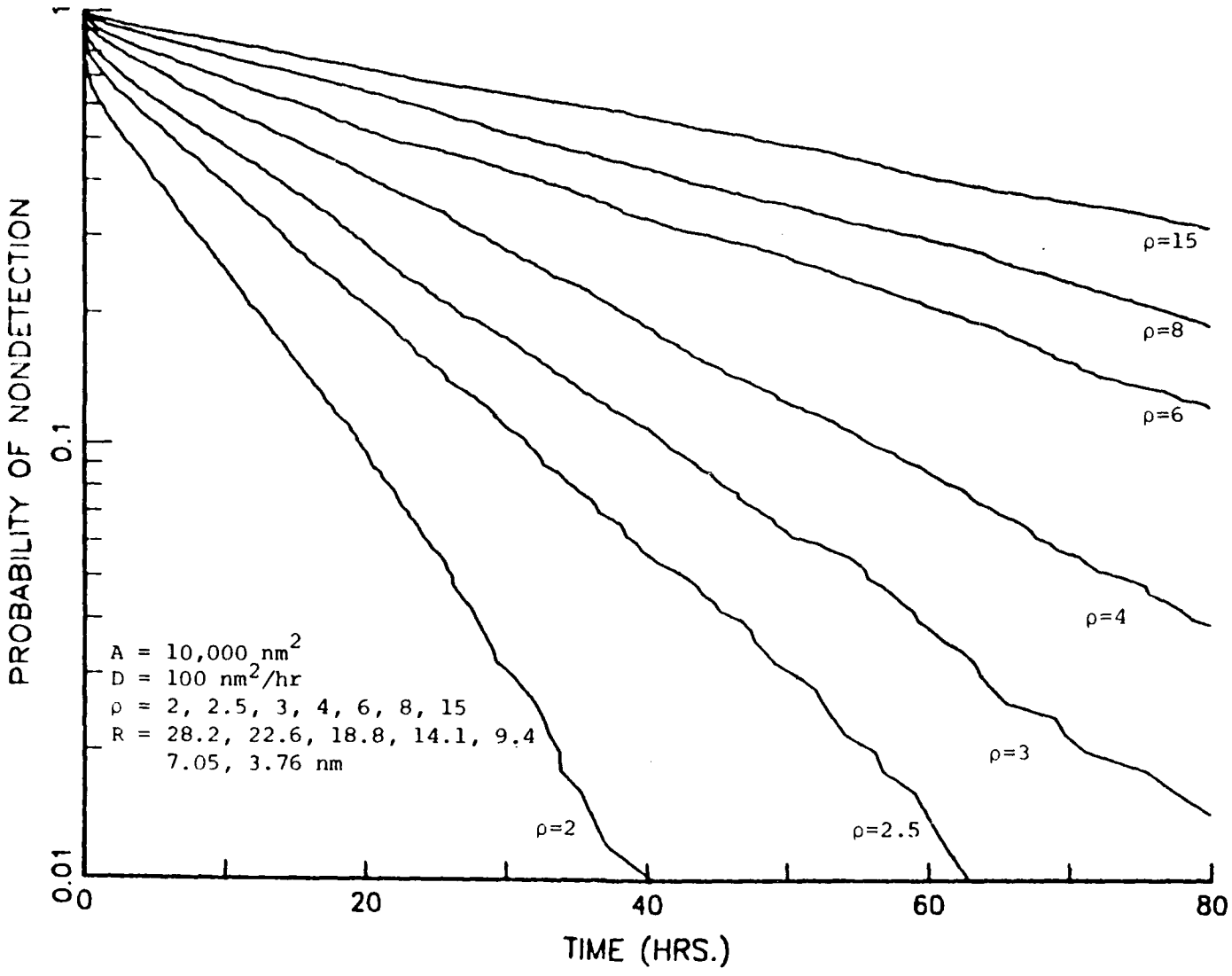

Figure 1. Probability of Nondetection vs. Time (PND(t))

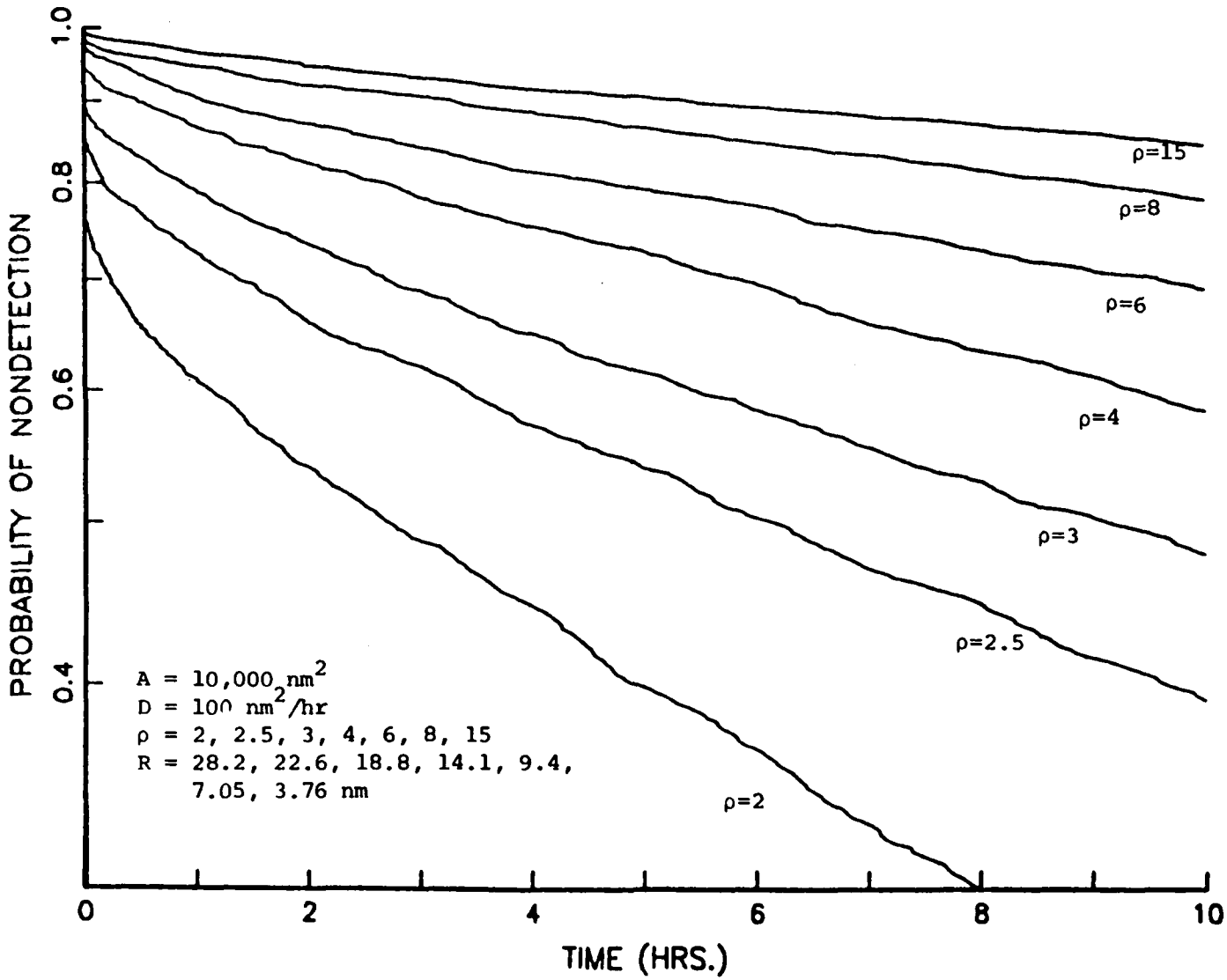

Figure 2. Probability of Nondetection vs. Time (PND(t))

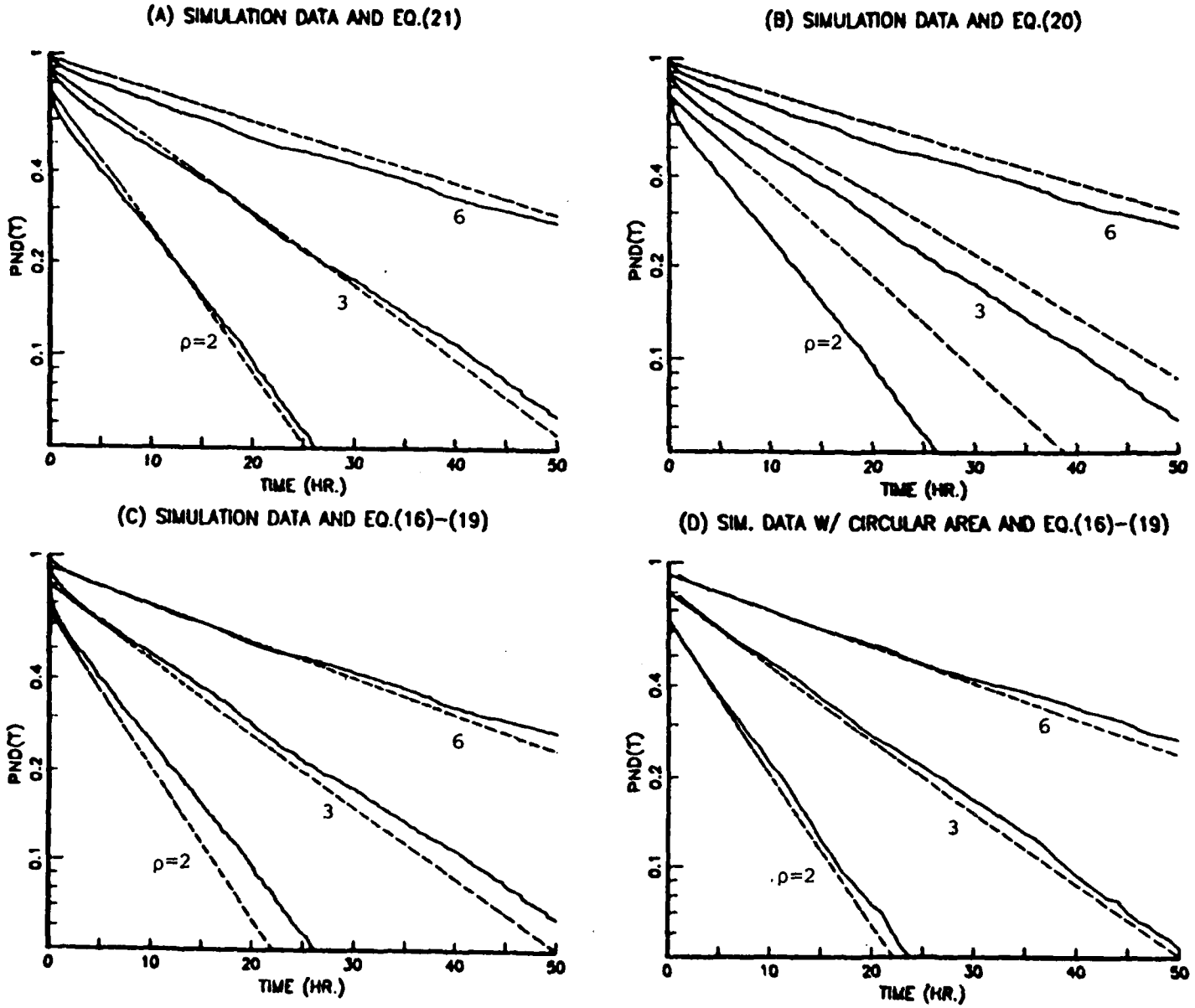

Figure 3. Comparison of Several PND(t) Approximations

same as in Figures 1. and 2. except that only results for ρ of 2, 3, and 6 are shown. In Figure 3. the simulated PND(t) is shown as a solid line, while the estimated PND(t) is shown as dashed.

Figure 3.(C) indicates that, especially for small ρ, PND(t) determined by simulation does not decrease as rapidly as predicted by (16)-(19). The explanation is believed to be that for ρ sufficiently close to 1, a circular search region does not provide a reasonable approximation of a square region of the same area. To test this explanation, the three simulations plotted in Figure 3. were repeated with circular search areas. The results, shown in Figure 3.(D)., indicate a closer agreement between the theoretical and observed data. But still the fit is not exact. This is somewhat disturbing, but might be explained by the discrete manner in which the diffusion path is simulated. The simulated diffusion path is approximated by a series of points. Detection must occur exactly at the points and can not occur between them. The distance between adjacent points is the random variable $(\Delta X^2+\Delta Y^2)^{1/2}$, where ΔX and ΔY are the independent, normally distributed x and y displacements. It is possible for the simulated path to jump across the edge of the detection disk without achieving detection, even though the line segment connecting the points lies partly on the disk. This will tend to reduce the simulated detection rate below that of the diffusion being approximated.

Figure 4. shows a plot of x_1 vs. ρ for values of ρ from 2 to 15. By using (16), these values of x_1 determine the theoretical asymptotic detection rate as

$$D\, x_1^2/(2R^2). \tag{22}$$

Table 1. lists the asympotic detection rates determined by (22), (21), (20),

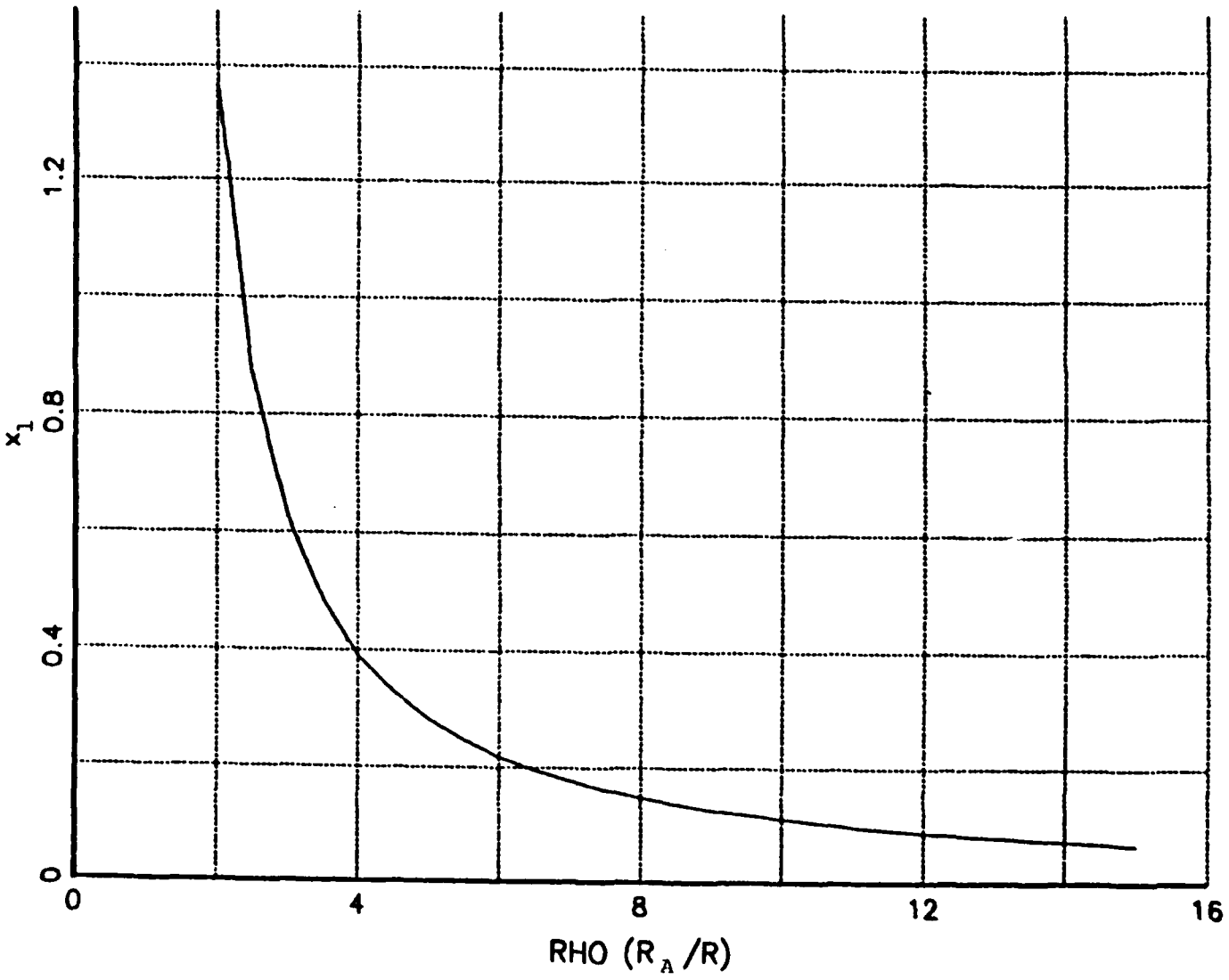

Figure 4. A Plot of x_1 vs. ρ

and an overall (i.e., from time 0) best fit rate calculated by least-squares fitting of the simulation data.

ρ	$x_1(\rho)$	$Dx_1^2/(2R^2)$ (Eq. 22)	$24.7\ RD/(A-\pi R^2)^{1.5}$ (Eq. 21)	$24.7\ RD/A^{1.5}$ (Eq. 20)	Simulation Best Fit
2	1.361	.116	.107	.0697	.102
2.5	.866	.0736	.0724	.0557	.0665
3	.626	.0554	.0554	.0465	.0507
4	.389	.0381	.0384	.0384	.0388
6	.218	.0269	.0242	.0232	.0250
8	.147	.0217	.0178	.0174	.0202
10	.111	.0194	.0141	.0139	.01704
15	.0661	.0154	.00935	.00929	.0137

Table 1. Detection Rates for Various ρ.

The simulation data suggest the following two conclusions:

a. Equations (16)-(19) give a reasonable estimate PND(t), but the fit deteriorates as ρ decreases to 1.

b. For small ρ, (21) gives a better fit to the data than does (20). For large ρ, both (20) and (21) underestimate the observed asymptotic detection rate.

We conclude with a few comments on random search. As mentioned earlier, Koopman's random search model predicts a detection rate of 2Rv/A for searcher with speed v and detection range R conducting a random search of an area of size A. It seems reasonable that a diffusion path should be "random" enough for this model to be appropriate. In fact, we have seen that the detection rate, while not

constant, approaches a constant value for large t. Setting $2Rv/A$ equal to the exponential term in (21) and solving for speed v gives

$$12.35\ AD/(A - \pi R^2)^{1.5}$$

as an approximate equivalent speed for a searcher conducting a "diffusion search" of a square area.

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