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by

John A. DeRuntz Jr.
Staff Scientist

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Applied Mechanics Laboratory

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Discrete-Element Acoustic Analysis of Submerged Structures Using Doubly Asymptotic Approximations

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John A. DeRuntz Jr.
Staff Scientist

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Applied Mechanics Laboratory
Dept. 92-50, Bldg. 255
Lockheed Palo Alto Research Laboratory
3251 Hanover St., Palo Alto, Ca. 94304

ABSTRACT

Doubly Asymptotic Approximations have been found to offer significant advantages for the treatment of steady-state fluid-structure interaction in vibration, acoustic-radiation, and acoustic-scattering problems for complex submerged structures. This paper describes the theoretical foundations, development, and verification of two boundary-element/finite-element processors that implement this approach. The first processor is SWEEPS, which determines the structural response and surface pressure on a vibrating submerged body. The second is TARGET, which embodies a discretized form of the Helmholtz integral equation to obtain fluid pressures away from the body. To test these processors, two problems involving a spherical shell in an infinite fluid have been solved. The first problem is one of modal internal forcing; while the second is concerned with forcing by incident plane waves. The computational results exhibit excellent agreement with closed form solutions.

Additional keywords: USA (underwater shock analysis); computer program; underwater acoustics; computer program verification; date.

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Doubly Asymptotic Approximations have been found to offer significant advantages for the treatment of steady-state fluid-structure interaction in vibration, acoustic-radiation, and acoustic-scattering problems for complex submerged structures. This paper describes the theoretical foundations, development, and verification of two boundary-element/finite-element processors that implement this approach. The first processor is SWEEPS, which determines the structural response of and surface pressure on a vibrating submerged body. The second is TARGET, which embodies a discretized form of the Helmholtz integral equation to obtain fluid pressures away from the body. To test these processors, two problems involving a spherical shell in an infinite fluid have been solved. The first problem is one of modal internal forcing; while the second is concerned with forcing by incident plane waves. The computational results exhibit excellent agreement with closed form solutions.
Section 1

Introduction

This paper describes the theoretical foundations, development, and verification of two
discrete-element processors, SWEEPS and TARGET, that can treat the steady state
acoustic radiation and scattering of a resilient submerged structure. Both are built
around a database associated with the Underwater Shock Analysis (USA) code [1] as
they share a common approach to the fluid-structure interaction: the use of a boundary-
element method based on the Doubly Asymptotic Approximation (DAA) [2].

DAA methods have long proved their usefulness in underwater shock problems and it is
only recently that they have been applied to acoustics. The use of several forms of the
DAA to study the forced vibration of a submerged spherical shell [3] has shown that
the first order DAA is generally unsatisfactory for acoustics applications, however two
second order forms appear to perform quite well in most cases. These are the so-called
curvature corrected form DAA_2^c and the modal form DAA_2^m.

Although theoretically exact solutions to the underwater acoustics problem do exist
[4-7], the DAA approach uses a fluid mass matrix and a diagonal fluid area matrix that
do not depend upon frequency. This fact leads to a more efficient use of computational
resources when performing variable frequency calculations. In contrast, the governing
fluid matrices must be reformed in full for every frequency in the exact formulations. In
addition, the exact solution can be prone to the well known critical frequency problem
[4,6] that the DAA approach does not encounter, although the exact method discussed
in [7] does avoid the problem. Hence, it appears that DAA methods can be used to
advantage in underwater acoustics in that an increase in efficiency can offset some loss
of accuracy. Indeed, if preliminary results can be obtained with the use of DAA_2^c or
DAA_2^m and some frequency ranges prove to be of particular interest, then an exact
approach could always be used in this limited range.

The primary aim of this paper is to document the problem formulation used in the new
processors and, to verify the integrity of the software by solving both a radiation and
a scattering problem for the spherical shell. For information on the operation of the
code itself, and sample input and output, a usage primer has been written [8].

In the next section the governing finite-element, boundary-element equations are pre-
sented for the steady state vibration of a submerged structure excited either by a set of
internal forces with the same specified frequency but otherwise arbitrary magnitudes
and phases, or, by an infinite train of sinusoidal incident waves emanating from a spheri-
cal source with a specified frequency and magnitude. A simple and direct elimination
solution is then given for the structural displacement field and the wet surface scattered
pressures. This solution process forms the basis of the SWEEPS processor.

The following section briefly discusses the use of the Helmholtz integral equation in
order to find fluid pressures away from the wet surface of the structure. A discretized
form of this equation is implemented in the TARGET processor.
The next section summarizes the governing equation system and selected analytical DAA$_2$ solutions for the spherical shell. These are then called upon in the code verification section which follows. The discrete-element computational results are found to exhibit excellent agreement with the closed form solutions. Finally, some recommendations for future work are presented in the last section.
Section 2

Governing Equations For Wet Surface Solution

The interaction equations for a DAA time-harmonic vibration analysis of a submerged, linear-elastic structure may be written in matrix form as

\[
\begin{bmatrix}
E_{ss} & E_{sf} \\
E_{fs} & E_{ff}
\end{bmatrix}
\begin{bmatrix}
x \\
p_s
\end{bmatrix}
=
\begin{bmatrix}
g_s \\
p_f
\end{bmatrix},
\]

where

\[
\begin{align*}
E_{ss} &= -\omega^2 M_s + i\omega C_s + K_s, \\
E_{sf} &= G A_f, \\
E_{fs} &= \rho c \omega^2 (\omega M_f - i\Omega_f M_f) G^T, \\
E_{ff} &= -\omega^2 M_f + \rho c (i\omega A_f + \Omega_f A_f), \\
g_s &= f_s - G A_f p_f, \\
g_f &= \rho c \omega (\omega M_f - i\Omega_f M_f) u_f.
\end{align*}
\]

Here \( M_s, C_s, \) and \( K_s \) are the structural mass, damping, and stiffness matrices, respectively, \( G \) is the fluid-structure transformation matrix relating fluid node point forces normal to the wet surface of the structure to node point forces in the structural computational system, \( A_f \) is the wet surface fluid element area matrix, \( M_f \) is the wet surface fluid mass matrix, and \( \Omega_f \) is the wet surface fluid frequency matrix; \( x \) is the structural displacement vector, \( p_s \) is the wet surface scattered pressure vector, \( f \) is an applied structural force vector, \( p_f \) is an incident wave wet surface pressure vector, and \( u_f \) is an incident wave wet surface normal velocity vector. (If \( p_f = u_f = 0 \), the scattered pressure \( p_s \) reduces to the radiated pressure \( p_R \).) In addition, \( \rho \) and \( c \) are the fluid density and sound speed, respectively, \( i = \sqrt{-1} \), and \( \omega \) is the frequency of steady state vibration. A superscript \( T \) denotes matrix transposition.

The real, symmetric matrices \( M_s, C_s, \) and \( K_s \) can easily be generated by any finite-element structural analysis code, and, in the work reported upon here, STAGS (STress Analysis of General Shells) \cite{9}, has been used. The real, diagonal matrix \( A_f \) is trivially obtained, while the real, symmetric matrix \( M_f \) can be computed by the boundary-element method of \cite{10}. These two fluid arrays as well as the real, rectangular transformation matrix \( G \) are already produced by the FLUMAS processor of the USA code.

Finally, the real matrix \( \Omega_f \) may be obtained from either of two formulations, and is such that the matrix product \( \Omega_f A_f \) is symmetric. The development of \cite{2}, which is based upon the method of fluid boundary modes, gives

\[
\Omega_f^m = g p c A_f M_f^{-1},
\]

\( \text{(3)} \).
where \( g \) is a scalar parameter that can vary between zero and unity. \( g = 0 \) reduces (2) to the DAA\(_1\) equations, \( g = 1/2 \) appears to be best for the infinite cylindrical shell, and \( g = 1 \) is best for the spherical shell. On the other hand, the formulation of [11] does not contain any arbitrary parameters as in (3). It is based upon the method of matched asymptotic expansions, and, for the fitting procedure described in [3], yields

\[
\Omega_f^c = \rho c A_f M_f^{-1} - cK,
\]

where \( K \) is a diagonal matrix of wet surface mean local curvatures. It should be noted that both (3) and (4) do not involve any additional information that is not already provided by the FLUMAS processor, in particular, the mean local curvatures are used in the computation of \( M_f \) [10].

For convenience, (3) and (4) can be combined into the one expression

\[
\Omega_f = g pc A_f M_f^{-1} - \delta cK,
\]

where, if \( \delta = 0 \) the DAA\(_2\) form is obtained; whereas, if \( \delta = g = 1 \) the DAA\(_2\) form is obtained. With this substitution, (2) then become

\[
\begin{align*}
E_{ss} &= -\omega^2 M_s + i\omega C_s - K_s, \\
E_{sf} &= G A_f, \\
E_{fs} &= \rho c i\omega^2 [(\omega + i\delta c K)M_f - igpc A_f]G^T, \\
E_{ff} &= -\omega^2 M_f + \rho c [(i\omega - \delta c K)A_f - gpc D_f], \\
g_s &= f_s - G A_f p_i, \\
g_f &= \rho c \omega [(\omega + i\delta c K)M_f - igpc A_f]u_i,
\end{align*}
\]

where \( D_f \) is the symmetric matrix given by

\[
D_f = A_f M_f^{-1} A.
\]

Perhaps the most important characteristic of (6) is that the matrices \( M_s, C_s, K_s, G, A_f, M_f \), and \( D_f \) are frequency-independent, so that they need only be computed once for a complete set of frequency-sweep calculations. This characteristic also renders (1) particularly amenable to incremental iterative methods of solution thus avoiding costly refactorization of the coefficient matrices at every frequency step. More will be said about this later.

To complete the governing equation system, the form of the right hand side forcing vectors in (2) must now be specified. The elements of the internal forcing vector \( f_s \) can be written as

\[
f_{si} = F_i e^{-i\theta_i},
\]
where $F_i$ and $\theta_i$ are the magnitude and phase angle respectively of the $i^{th}$ degree of freedom of $f_s$. Also, the elements of the external forcing vectors $p_i$ and $u_i$ can be given for a train of spherical incident waves as

$$p_{li} = p_0 \frac{S}{R_i} e^{-ik(R_i - S)},$$
$$u_{li} = \frac{p_{li}}{\rho c} (1 - i/kR_i) \gamma_i. \quad (9)$$

Here $S$ is the standoff, i.e., the distance between the origin of the spherical wave and the nearest point on the wet surface of the structure, $R_i$ is the distance from the origin of the spherical wave to the $i^{th}$ fluid node on the wet surface, and, $\gamma_i$ is the cosine of the angle between the vector corresponding to $R_i$ and the wet surface outward normal vector at the $i^{th}$ fluid node. $p_0$ is the amplitude of the incident pressure at the standoff distance, and $k$ is the wave number $\omega/c$.

Now that the governing equation system for the wet surface unknowns has been fully defined, equations (1) are rewritten by solving the first for $x$ and substituting into the second. In combination with the first of (1), these become

$$(E_{ff} - E_{fs}E_{sE}^{-1}E_{sf})p_S = g_f - E_{fs}E_{sE}^{-1}g_sc,$$
$$(E_{sE}x = g_s - E_{sf}p_S. \quad (10)$$

$p_S$ is found from the first of (10) while $x$ is then obtained from the second. This is the solution procedure currently implemented in the SWEEPS processor.
Section 3

Governing Equations For Far Field Pressure Solution

The pressure $p$ in the acoustic field is given by the Helmholtz wave equation

$$\nabla^2 p + k^2 p = 0,$$  \hspace{1cm} (11)

where $\nabla^2$ is the Laplacian operator. A convenient boundary integral solution to (11) can be given by particularizing Kirchhoff’s retarded potential formulation [12,13] to the steady state case to obtain the Helmholtz integral equation

$$p_{\text{sc}} = -\frac{1}{4\pi} \int_B \left[ \frac{\rho \omega^2}{r^2} \mathbf{x} \cdot \mathbf{r} + \frac{1}{r^2} \frac{\partial}{\partial n} (1 + ikr)(p_1 + p_S) \right] e^{-ikr} d\mathbf{B}, \hspace{1cm} (12)$$

where the explicit time dependence has been omitted, and, $p_{\text{sc}}$ is the scattered (or, radiated pressure if $p_1 = 0$) at any point $P$ outside of the wet surface boundary $B$, $r$ is the distance from $P$ to a point $Q$ on $B$, $n$ is the outward unit vector normal to $B$ at $Q$, and $\partial r, \partial n$ is the cosine of the angle between $r$ and $n$. It should be emphasized that (12) is an exact result and that the only approximation involved here is due to the fact that $x$ and $p_S$ are obtained on the wet surface by the DAA as described in the preceding section. A geometric approximation is also invoked when (12) is discretized by assuming that $x$, $p_1$, and $p_S$ are constant over each boundary element covering the surface $B$. The result becomes

$$p_{\text{sc}} = -\frac{1}{4\pi} \sum_{i=1}^N A_i \left[ \frac{\rho \omega^2 x_i}{R_i} + \frac{\gamma_i}{R_i} (1 + ikR_i)(p_{1i} + p_{Si}) \right] e^{-ik(R_i - S)}, \hspace{1cm} (13)$$

where $N$ is the total number of boundary elements on the wet surface. $A_i$ is the area of the $i^{th}$ element, while $S$, $R_i$, and $\gamma_i$ have already been defined in Section 2, except that here they pertain to the point at which the scattered pressure is to be calculated, rather than the incident wave source. (13) then forms the basis of the TARGET processor.
Section 4
Submerged Spherical Shell Solutions

The nondimensional equations of motion for the axisymmetric modal vibrations of a submerged, linearly elastic spherical shell are briefly summarized here from [3]. With the addition of the right hand side forcing terms due to an incident plane wave, they may be expressed in matrix form as

\[
\begin{bmatrix}
A_n - n(n+1)\omega^2 & B_n & 0 \\
B_n & C_n - \omega^2 & \mu \\
0 & \omega^2 Q_n(\omega) & R_n(\omega)
\end{bmatrix}
\begin{bmatrix}
v_n \\
w_n \\
p_n^S
\end{bmatrix} =
\begin{bmatrix}
\mu(p_n^F - p_n^I) \\
- i \omega Q_n(\omega) u_n^I
\end{bmatrix}.
\]

(14)

in which \(v_n\) and \(w_n\) are the meridional and radial components of modal shell displacement, respectively, \(p_n^S, p_n^F\) and \(p_n^I\) are the scattered, internally forced and incident components of modal surface pressure, respectively, and \(u_n^I\) is the fluid particle velocity due to the incident wave. \(\mu = \rho a/\rho_s h\), where \(\rho_s\) is the density of the shell material, and \(a\) and \(h\) are the shell's radius and thickness, respectively. In addition

\[
A_n = \gamma^2(1 + \epsilon)n(n + 1)\xi_n, \\
B_n = \gamma^2(1 + \nu + \epsilon\xi_n)n(n + 1), \\
C_n = \gamma^2[2(1 + \nu) + n(n + 1)\epsilon\xi_n].
\]

(15)

Here, \(\gamma = c_s/c, \epsilon = h^2/12a^2\), and \(\xi_n = n(n + 1) - (1 - \nu)\), where \(c_s\) and \(\nu\) are the plate velocity and Poisson's ratio for the shell material, respectively, and \(n\) is the modal index for meridional expansion in Legendre polynomials of the shell displacement, external pressure, and internal pressure fields.

The incident pressure and particle velocity can be written as \[14\]

\[
p_n^I = p_0(-1)^n(2n + 1)j_n(\omega), \\
u_n^I = p_0(-1)^{n-1}(2n - 1)j_n^\prime(\omega).
\]

(16)

\(p_0\) is the amplitude of the incident pressure and the \(j_n\) are the spherical Bessel functions of the first kind. The factor \((-1)^n\) occurs because the incident wave is assumed to impinge on the sphere at \(\theta = \pi\).

For the DAA_2 solutions required here, the polynomials \(Q_n(\omega)\) and \(R_n(\omega)\) take the following forms \[3\]

\[
\text{DAA}^m_2 : Q_n(\omega) = i\omega - g(n + 1), \quad R_n(\omega) = (i\omega)^2 + i\omega(n + 1) \cdot g(n + 1)^2.
\]

\[
\text{DAA}^c_2 : Q_n(\omega) = i\omega + n, \quad R_n(\omega) = (i\omega)^2 - i\omega(n + 1) \cdot n(n + 1).
\]

(17)
Finally, it should be noted that the nondimensionalization of (14) is based on the relations

\[ v = V/a, \quad w = W/a, \quad u = U/c, \quad p = P/\rho c^2, \quad \omega = \omega a/c, \quad (18) \]

where \( V, W, U, P \) and \( \omega \) are the appropriate dimensional variables.

The wet surface solution to the \( 3 \times 3 \) matrix system in (14) is easily obtained numerically as a function of the mode number \( n \). The pressure in the fluid field can then be found by using this solution in the infinite series expansion

\[ p_{\omega a}(r, \theta) = i \omega^2 \sum_{n=1}^{\infty} P_n(\cos \theta) \left( p_n^r + p_n^\theta \right) j_n'(\omega) - \omega w_n j_n(\omega) (j_n(\omega r) - i y_n(\omega r)), \quad (19) \]

where \( r \) and \( \theta \) are the radial and meridional polar spherical coordinates, respectively, and the nondimensional radius \( r \) is given in terms of the dimensional radius \( R \) as \( r = R/a \). \( P_n(\cos \theta) \) are the axisymmetric Legendre polynomials and \( y_n \) is the spherical Bessel function of the second kind. The minus sign multiplying \( y_n \) differs from [14] because the time dependence used here is \( e^{i\omega t} \) rather than \( e^{-i\omega t} \) that is used there. (19) follows directly from (4.14) and (8.10) of [14], although it is not explicitly derived there, since the exact solution for the sphere can also be expressed more simply in terms of the modal surface displacements or pressures alone. This is in contrast to that for an approximate wet surface solution like the DAA2.
Section 5

Code Verification Using The Spherical Shell Problem

Two problems involving analytical solutions for the submerged spherical shell are described in this section for the purpose of providing a basis of verification of the SWEEPS and TARGET processors. The first involves comparisons of the internally forced modal results of 3 with those produced by SWEEPS; while the second is a test of both SWEEPS and TARGET for an incident plane wave excitation that was motivated by 15.

The computational model chosen for the comparisons to be made here is a quarter of a sphere whose surface mesh consists of quadrilaterals only, all of which are very nearly square. It contains 96 elements and 113 node points and is shown in Figure 1 in an exploded view. It was constructed using a pre-processor developed especially for this purpose [16] and forms the basis of the structural element grid for STAGS, and also for the fluid element grid for the USA processor FLUMAS.

Internally Forced Problem

In this case the analytical DAA calculations of 3 are used to provide the basis for comparison with SWEEPS predictions of surface radial velocities and surface pressures. The modal vibration results presented in 3 include plots of these quantities as functions of frequency and clearly show the resonance and anti-resonance zones that are of primary interest in this problem. The particular computations reported there are carried out for the modal indices \( n = 0, 1, 2 \) and 3 and for two non-zero values of structural damping. In contrast, SWEEPS does not currently include structural damping and, of course, is constructed around a discrete-element model that can include many modes. To make the comparisons then, the governing equation system presented in 3 and summarized in the preceding section is solved for a selected number of frequencies without structural damping. Also, the surface distribution of the SWEEPS internal forcing function is specified to be that of the appropriate axisymmetric Legendre polynomial mode. The results presented in Figures 2 through 9 correspond to the parameter values used in 3

\[
\alpha = 100, \quad \rho_s/\rho = 7.67, \quad \nu = 0.3, \quad c_s = 3.53.
\]

and, as can be seen, the comparisons are excellent except for one point at \( \omega = 3 \) for \( n = 0 \). No immediate explanation for this minor discrepancy can be offered at this time.

Incident Plane Wave Excitation

The work reported upon in 15 is a feasibility study that examines the accuracy of several surface interaction approximations as applied to the underwater acoustic echo signal problem. That work makes use of the known exact and approximate modal
Figure 10  Backscattered Pressure At $r = 20$ For Submerged Spherical Shell
Figure 9  Surface Pressure Response For $n = 3$ Excitation Of Submerged Spherical Shell
Figure 8  Radial Velocity Response For $n = 3$ Excitation Of Submerged Spherical Shell

- Closed Form Solution
- SWEEPS Solution
Figure 7  Surface Pressure Response For n = 2 Excitation Of Submerged Spherical Shell
Figure 6  Radial Velocity Response For n = 2 Excitation Of Submerged Spherical Shell
Figure 5  Surface Pressure Response For n = 1 Excitation Of Submerged Spherical Shell
Figure 4  Radial Velocity Response For $n = 1$ Excitation Of Submerged Spherical Shell
Figure 3  Surface Pressure Response For $n = 0$ Excitation Of Submerged Spherical Shell
Figure 2  Radial Velocity Response For n = 0 Excitation Of Submerged Spherical Shell

- Closed Form Solution
- SWEEPS Solution


References


Section 6
Discussion and Conclusions

Based upon the excellent agreement between the discrete-element and closed form DAA\textsubscript{2} solutions obtained here, it may be concluded that the SWEEPS and TARGET processors are capable of reproducing the variety of phenomena that are inherent in underwater acoustics problems. In addition, since it has already been demonstrated elsewhere that the DAA\textsubscript{2} approach provides very good approximations to exact solutions, it is apparent that such discrete-element formulations will be valuable assets in the acoustic studies of submerged structures.

The results of this paper do raise an interesting question in that the discrete-element computations seem to retain accuracy to higher frequencies than would be expected. The reason for this serendipitous behavior is not yet understood.

With regard to software details, SWEEPS and TARGET currently have some minor restrictions on their usage. Common to both is the requirement of an infinite fluid, i.e., the presence of a free surface is not treated. In addition, structural damping has not yet been implemented in SWEEPS. The inclusion of these two capabilities requires some additional but straightforward code enhancement.

At this time, the computational algorithm for SWEEPS is a direct elimination solution of the structural and fluid equations, hence the CPU time is a linear function of the number of frequencies desired. It is also roughly proportional to $N^3$, where $N$ is the number of fluid degrees of freedom. Sample execution times for a single frequency for 57, 113, and 270 fluid DOF problems are 1, 5, and 60 CPU minutes respectively on the VAX 11/780. In order to reduce such expense for multi-frequency calculations, an incremental iterative scheme is planned that can eliminate costly complex matrix factorizations at every frequency. It should also be emphasized that an increase in efficiency in the SWEEPS processor, through a change in the algorithm, is possible only because of the frequency independent matrices that are inherent in the DAA approach. Such an algorithm would have no effect upon current exact solution methods.

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solutions for the spherical shell irradiated with a plane incident wave similar to those given in the preceding section (see [3] for the exact polynomials \( Q_n(\tilde{\omega}) \) and \( R_n(\tilde{\omega}) \)). In particular, the backscattered pressure is computed at \( r = 20, \theta = \pi \) and is plotted as a function of frequency in terms of the *echo function* defined as

\[
f_{\infty} = 2r \frac{p_{\text{sc}}(r, \theta)}{p_0} e^{-ikr}
\]

It concludes that the DAA\(^{2m}\) with \( g = 1 \) is a very good approximation to the exact solution over the entire frequency range studied there: \( 0 < \tilde{\omega} < 16 \).

For the purposes of the current work, comparisons will be drawn between discrete-element and closed form DAA\(^{2m}\) solutions using (19) of the preceding section rather than make use of any results already presented in [15]. The reason for this is that the scattered pressure in the fluid was obtained only approximately there, using numerical integration of the Helmholtz integral, and some differences were clearly evident between those and the current discrete-element results, particularly in the range \( 2 < \tilde{\omega} < 6 \). Using (19) then, the \( P_n(cos\theta) \) reduce to \(( -1)^n\) for the case \( \theta = \pi \) in order to compute the backscattered pressure. In addition, the convergence criterion is chosen so that the last modal increment to \(|f_{\infty}|\) is less than \( 10^{-8}\) times the current value of \(|f_{\infty}|\). The results presented in Figure 10 correspond to the parameter values

\[
a/h = 39.5, \quad \rho_s/\rho = 2.7, \quad \nu = 0.355, \quad c_s/c = 3.7928,
\]

that were used in [15]. As can be seen, the agreement is excellent over most of the frequency range and it is only when the surface discretization is very coarse in relation to the wavelength of the incident pressure that the SWEEPS calculations begin to diverge wildly for \( \tilde{\omega} > 14 \). Indeed, it is surprising the results are so good even out to \( \tilde{\omega} = 12 \), since at this frequency there are only 3 mesh points per wavelength.

It is easily demonstrated that these results are not particularly dependent upon well defined low order modes that have a sufficiently large number of mesh points on a modal wavelength. The convergence of \(|f_{\infty}|\) is shown in Figure 11 as a function of mode number, and it is seen that \( n = 12 \) is still a significant contributor. For the discrete-element solution, this mode has slightly less than 3 points per wavelength around the sphere.

Incidently, since the results of Figure 10 agree so well for two so completely different numerical methods, it would appear that there is a minor numerical flaw in some of the results of [15].
Figure 11  Modal Convergence For $\omega = 12$ For Submerged Spherical Shell