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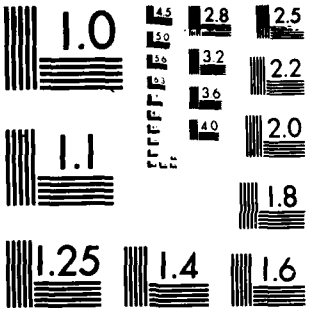

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LOCAL EPI-CONTINUITY AND LOCAL OPTIMIZATION

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ABSTRACT

One of the fundamental questions in nonlinear optimization is how optimization problems behave when the functions defining them change (e.g., by continuous deformation). Recently the study of epi-continuity has somewhat unified the results in this area. Here we show how to localize the concept of epi-continuity, and how to apply these localized ideas to ensure persistence and stability of local optimizing sets. We also show how these conditions follow from known properties of nonlinear programming problems.

AMS (MOS) Subject Classification: 90C30

Key Words: Nonlinear programming, stability, epi-continuity, epi-convergence, local optimizers.

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SIGNIFICANCE AND EXPLANATION

Many real-world problems in statistics, engineering, economics and other areas require the solution of optimization problems. Often these problems involve nonlinear functions, and when this is the case we can frequently find only local solutions (i.e., we know that there is no better solution near the solution point that we have found, but we are not sure what happens far away). Also, these problems are often subject to small changes in the problem data caused by, e.g., inaccurate information or problem evolution in time. Once we have found a local solution, it is natural to ask whether there will still be a local solution nearby if the problem data are slightly changed.

This paper develops an appropriate theoretical framework within which this question can be analyzed, and it provides relatively simple and verifiable conditions under which local solutions will persist in the sense just described.

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LOCAL EPI-CONTINUITY AND LOCAL OPTIMIZATION

Stephen M. Robinson

1. Introduction.

The question of stability in optimization deals with what happens to an optimization problem when the elements of the problem are in some way deformed. For example, if the original problem had optimal solutions, one might ask whether the perturbed problem has solutions (persistence) and, if so, whether they are in some sense close to those of the original problem if the deformations are in some sense small (stability). In general the answers to these questions are "no" and "no," so people have tried to find conditions to impose on the optimization problem so that the answers become "yes" and, frequently, so that the solutions are somehow well behaved as functions of the perturbation parameters. A comprehensive overview of much work in parametric optimization is given in the book by Bank *et al.* [2].

Recent development of the theory of epi-convergence has unified many of the approaches and results in stability analysis. A general survey of results in this area may be found in [1], and a general introduction in [8].

In this paper we show how to localize certain results found in the theory of epi-convergence, and how to use these localized results to develop useful criteria for persistence and stability of local minimizers. Moreover, we show that these criteria are implied by assumptions commonly used in optimization, such as constraint qualifications and second-order sufficient conditions. Thus, the results presented here can be applied whenever these assumptions hold.

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The basic technique used in this paper is a generalization of the method used in [6, Th. 3.1] to prove existence of solutions to smooth nonlinear programming problems whose data are perturbed. The results given here, however, apply to much more general problems in which no differentiability need be assumed.

The remainder of this paper is organized as follows: in Section 2 we develop the local epi-continuity results that we shall need here, and in Section 3 we apply these to study global solutions of perturbed minimization problems. Of course, similar results can be applied, mutatis mutandis, to maximization problems. Then in Section 4 we introduce the local optimization problem whose solutions we want to study, and we employ the results of Section 3 to develop criteria for persistence and stability of its local minimizers. Finally, in Section 5 we show how commonly used assumptions about nonlinear optimization problems imply the general criteria used in Section 4.

2. Local epi-continuity.

The ideas of epi-continuity and epi-convergence of functions have recently been used to unify and extend the study of variational problems; for further information and references consult [1] or [8]. Here we develop a local version of epi-continuity and establish some of its properties.

In order to deal with the presence of parameters in the functions we use, it will be convenient to employ functions of the form $f(p,x)$, where p is understood to be a parameter in a topological space P, and for each p the function $f(p,\cdot)$ is the object of interest for minimization or other purposes. We shall deal with the behavior of $f(p,\cdot)$ for p near some base value p_0. We can include the case of sequences in this framework by taking P to be the standard one-point compactification of $\{0,1,2,\ldots\}$ created by adjoining the point $p_0 = \infty$, with a base of neighborhoods of ∞ given by $V_n := \{n,n+1,\ldots\} \cup \{\infty\}$ for $n = 0,1,2,\ldots$.

In the situations we want to consider, the question of interest is to relate various properties of $f(p,\cdot)$ to the corresponding properties of $f(p_0,\cdot)$ as p converges to p_0. To work with such problems we need an appropriate concept of convergence for functions. Of course, ordinary pointwise convergence is available, but it is not particularly convenient for the study of minimization problems. A more useful idea is epi-convergence, which permits us to employ, in the definition of the limit of $f(p,\cdot)$ at x_0, information not only about $f(p,x_0)$ for p near p_0 but also about $f(p,x)$ for x near x_0. For our purposes it is convenient to introduce epi-convergence by means of two auxiliary functions associated with f; we define these next. For the moment we allow x to take values in an arbitrary topological space X; in Section 3 we shall specialize X to a portion of $\mathbb{R}^n$. We use $N(x)$ or $N(p)$ to denote the neighborhood system of x or p. All functions used here will be extended-real-valued, taking values in $\overline{\mathbb{R}} := [-\infty,+\infty]$ or sometimes in $(-\infty,+\infty]$. The effective domain of such a function (abbreviated "dom") is simply the set of points at which the function does not take the value $+\infty$. Also, for any function $\alpha : P \to \overline{\mathbb{R}}$, we set

$$\liminf_{p \to p_0} \alpha := \sup_{U \in \mathcal{N}(p_0)} \inf_{p \in U} \alpha(p)$$

and

$$\limsup_{p \to p_0} \alpha := \inf_{U \in \mathcal{N}(p_0)} \sup_{p \in U} \alpha(p) .$$

DEFINITION 2.1: Let P and X be topological spaces and let $f : P \times X \to \bar{\mathbb{R}}$. For $p_0 \in P$ and $x_0 \in X$, the <u>epi-limit inferior of</u> f <u>at</u> x_0 <u>as</u> $p \to p_0$ is

$$(E_* f)(x_0) := \sup_{V \in \mathcal{N}(x_0)} \liminf_{p \to p_0} \inf_{x \in V} f(p,x)$$

and the <u>epi-limit superior of</u> f <u>at</u> x_0 <u>as</u> $p \to p_0$ is

$$(E^* f)(x_0) := \sup_{V \in \mathcal{N}(x_0)} \limsup_{p \to p_0} \inf_{x \in V} f(p,x) .$$

For more discussion of these functions, see [8].

Evidently $E_* f \leq E^* f$, but these two functions may not agree at any given x. The set (possibly empty) of x where they do agree is called the <u>domain of epi-continuity</u> of $f(p,\cdot)$, written $\mathcal{D}_E(f)$, and on this set we define the <u>epi-limit</u> of f as $p \to p_0$ to be their common value:

$$Ef(x) = E_* f(x) = E^* f(x), \quad \text{for} \quad x \in \mathcal{D}_E(f) .$$

The names given to $E_* f$ and $E^* f$ suggest using them to define a kind of semicontinuity. We do this in the next definition.

DEFINITION 2.2: Let f, p_0 and x_0 be as in Definition 2.1. We say f is <u>epi-lower semicontinuous</u> (e-lsc) at x_0 as $p \to p_0$ if $f(p_0,x_0) \leq E_* f(x_0)$, and <u>epi-upper semicontinuous</u> (e-usc) there if $f(p_0,x_0) \geq E^* f(x_0)$, whereas f is <u>epi-continuous</u> at x_0 if it is both e-lsc and e-usc there.

If f is epi-continuous at a point x_0 then clearly $x_0 \in \mathcal{D}_E(f)$, but the converse is false in general. However, a simple condition on $f(p_0,\cdot)$ removes this difficulty. We state this after the following proposition.

PROPOSITION 2.3: <u>Let</u> f, p_0, <u>and</u> x_0 <u>be as in Definition 2.1. Then</u>

$$\limsup_{(p,x)\to(p_0,x_0)} f(p,x) \geq E^*f(x_0) \geq \liminf_{x\to x_0} f(p_0,x) \geq \liminf_{(p,x)\to(p_0,x_0)} f(p,x) = E_*f(x_0). \quad (2.1)$$

PROOF: Choose two members V_1 and V_2 of $N(x_0)$. As $x_0 \in V_1 \cap V_2$, we have for any $p \in P$

$$\sup_{x\in V_1} g(p,x) \geq \inf_{x\in V_2} g(p,x) ,$$

and therefore

$$\inf_{U\in N(p_0)} \sup_{p\in U} \sup_{x\in V_1} g(p,x) \geq \inf_{U\in N(p_0)} \sup_{p\in U} \inf_{x\in U_2} g(p,x) .$$

Taking the infimum over $V_1 \in N(x_0)$ on the left and the supremum over $V_2 \in N(x_0)$ on the right, we obtain the first inequality. For the second, note that for each $U \in N(p_0)$ and each $V \in N(x_0)$ we have

$$\sup_{p\in U} \inf_{x\in V} f(p,x) \geq \inf_{x\in V} f(p_0,x) .$$

Hence

$$\inf_{U\in N(p_0)} \sup_{p\in U} \inf_{x\in V} f(p,x) \geq \inf_{x\in V} f(p_0,x) ,$$

and upon taking the supremum in V we get

$$E^*f(x_0) \geq \sup_{V\in N(x_0)} \inf_{x\in V} f(p_0,x) = \liminf_{x\to x_0} f(p_0,x) ,$$

which proves the second inequality. The third is trivial, and the equality on the right follows from

$$E_*f(x_0) = \sup_{V\in N(x_0)} \liminf_{p\to p_0} \inf_{x\in V} f(p,x)$$

$$= \sup_{V\in N(x_0)} \sup_{U\in N(p_0)} \inf_{p\in U} \inf_{x\in V} f(p,x) = \liminf_{(p,x)\to(p_0,x_0)} f(p,x) . \quad \blacksquare$$

COROLLARY 2.4: Let f, p_0, and x_0 be as in Definition 2.1, and suppose $f(p_0,\cdot)$ is lsc at x_0. Then the set of points at which f is epi-continuous is precisely $\mathcal{D}_E(f)$.

PROOF: If f is epi-continuous at some point, we already know that point is in $\mathcal{D}_E(f)$. Now suppose x_0 belongs to $\mathcal{D}_E(f)$, so that $E_*f(x_0) = E^*f(x_0)$. Then by (2.1) this common value is $\liminf_{x\to x_0} f(p_0,x)$. But by hypothesis this is $f(p_0,x_0)$, and thus f is epi-continuous at x_0. ∎

COROLLARY 2.5: Let f, p_0, and x_0 be as in Definition 2.1. Then f is epi-lower semicontinuous at x_0 if and only if f is lower semicontinuous at (p_0,x_0).

PROOF: $E_*f(x_0) = \liminf_{(p,x)\to(p_0,x_0)} f(p,x)$. ∎

COROLLARY 2.6: Let f and g be functions from $P \times X$ to $(-\infty,+\infty]$, and let $(p_0,x_0) \in P \times X$. If f and g are epi-lower semicontinuous at x_0, then so is $f + g$.

PROOF: $f + g$ is well defined since neither function takes $-\infty$. Now use Corollary 2.5 and the fact that addition preserves lower semicontinuity. ∎

These results show that epi-lower semicontinuity behaves very simply under addition, being nothing more than ordinary lower semicontinuity in disguise. This is not true of epi-upper semicontinuity, though, since the first inequality in (2.1) may be strict. However, the implication holds in one direction.

COROLLARY 2.7: Let f, p_0, and x_0 be as in Definition 2.1. If f is upper semicontinuous at (p_0,x_0), then f is epi-upper semicontinuous at x_0.

PROOF: Use Proposition 2.3. ∎

This result differs from Corollary 2.5 in that its converse is not true (take $P = X = \mathbf{R}$ with $g(p,x) = 1$ if x is irrational and 0 if x is rational). One consequence of this discrepancy is that the sum of two epi-usc functions need not be epi-usc. To see this, take $g(p,x)$ to be as in the last sentence, with

$$h(p,x) = \begin{cases} 0 & \text{if } p = 0 \text{ and } x = 0 , \\ 1 & \text{if } p \neq 0 \text{ and } x = 0 , \\ 1 & \text{if } x \text{ is rational but not zero} , \\ 0 & \text{if } x \text{ is irrational} . \end{cases}$$

If we choose any p and any $V \in N(0)$, we have $\inf_{x \in V} h(p,x) = 0$; hence $E^* h(0) = 0 = h(0,0)$, so h is epi-usc at 0. However, $(g + h)(p,x)$ is 0 at $(0,0)$ and 1 everywhere else, so if $p \neq 0$ then for any $V \in N(0)$ one has $\inf_{x \in V} g(p,x) = 1$. Thus $E^*(g + h)(x_0) = 1$, so $g + h$ is not epi-usc at 0. Therefore the analogue of Corollary 2.6 does not hold. However, by strengthening the hypothesis somewhat one can obtain a condition for epi-usc of $g + h$, as we now show.

PROPOSITION 2.8: <u>Suppose that</u> g <u>and</u> h <u>are functions from</u> $P \times X$ <u>to</u> $(-\infty,+\infty]$ <u>and that</u> $(p_0,x_0) \in P \times X$. <u>Let</u> h <u>be epi-usc at</u> x_0. <u>If</u> g <u>is usc at</u> (p_0,x_0) <u>relative to</u> dom h, <u>then</u> $g + h$ <u>is epi-usc at</u> x_0.

PROOF: If $(g + h)(p_0,x_0) = +\infty$ then the result is true, so we can assume $(p_0,x_0) \in \text{dom } g \cap \text{dom } h$. Choose $\varepsilon > 0$; it suffices to prove $E^*(g + h)(x_0) \leq (g + h)(p_0,x_0) + \varepsilon$.

Since g is usc at (p_0,x_0) relative to dom h, there exist $U_0 \in N(p_0)$ and $V_0 \in N(x_0)$ such that if $(p,x) \in (U_0 \times V_0) \cap \text{dom } h$ then $g(p,x) < g(p_0,x_0) + \frac{1}{2}\varepsilon < +\infty$. Also, we know that $h(p_0,x_0) \geq E^* h(x_0)$, so for each $V \in N(x_0)$ there is some $U \in N(p_0)$ such that for each $p \in U$ there exists $x \in V$ with

$$h(p,x) < h(p_0,x_0) + \frac{1}{2}\varepsilon < +\infty . \tag{2.2}$$

Now choose $U \in N(p_0)$ and $V \in N(x_0)$ with $U \subset U_0$ and $V \subset V_0$. If $p \in U$, there is some $x \in V$ satisfying (2.2), and the pair (p,x) then belongs to dom h. Therefore $(g + h)(p,x) < (g + h)(p_0,x_0) + \varepsilon$, so

$$\sup_{p \in U} \inf_{x \in V} (g + h)(p,x) \leq (g + h)(p_0,x_0) + \varepsilon . \tag{2.3}$$

Since taking sup-inf over $V \subset V_0$ and $U \subset U_0$ is equivalent to taking sup-inf over

$N(p_0) \times N(x_0)$, we have from (2.3)

$$E^*(g + h)(x_0) \leq (g + h)(p_0,x_0) + \varepsilon . \qquad \blacksquare$$

Proposition 2.8 leads to an epi-continuity result for a certain type of function that we shall use in the following sections. This function is constructed using a multifunction (multivalued function), to model the situation one faces in dealing with constrained optimization. If T is such a multifunction from P to X, we say it is lower semicontinuous (lsc) at a pair $(p_0,x_0) \in P \times X$ if $x_0 \in T(p_0)$ and, for each $V \in N(x_0)$ there is some $U \in N(p_0)$ such that for each $p \in U$, $V \cap T(p) \neq \emptyset$. If T is lsc at (p_0,x_0) for each $x_0 \in T(p_0)$, we say it is lsc at p_0. For more information on this latter type of lower semicontinuity, see [8]; however, the idea of lower semicontinuity at (p_0,x_0) will be more useful to us in what follows. This type of semicontinuity seems to have been introduced by Dolecki [3].

The following lemma relates lower semicontinuity of T to an epi-semicontinuity property of the graph of T, which is defined to be the set

$$\Gamma(T) := \{(p,x) \in P \times X | x \in T(p)\} .$$

For a set $S \subset P \times X$, we define the indicator function ψ_S of S to be

$$\psi_S(p,x) := \begin{cases} 0 & \text{if } (p,x) \in S \\ +\infty & \text{if } (p,x) \notin S . \end{cases}$$

LEMMA 2.9: <u>Let</u> T <u>be a multifunction from</u> P <u>to</u> X, <u>and let</u> $(p_0,x_0) \in$ graph T. <u>Then</u> $\psi_{\Gamma(T)}$ <u>is epi-usc at</u> x_0 <u>if and only if</u> T <u>is lsc at</u> (p_0,x_0).

PROOF (if): Suppose T is lsc at (p_0,x_0). Let $V \in N(x_0)$, and find $U_0 \in N(p_0)$ so that if $p \in U_0$ then $T(p) \cap V \neq \phi$. If $p \in U_0$, then $\inf_{x \in N} \psi_{\Gamma(T)}(p,x) = 0$, so

$$\sup_{p \in U_0} \inf_{x \in V} \psi_{\Gamma(T)}(p,x) = 0 ,$$

and thus

$$\inf_{U \in N(p_0)} \sup_{p \in U} \inf_{x \in V} \psi_{\Gamma(T)}(p,x) = 0 .$$

As V was arbitrary in $N(x_0)$, this implies

$$0 = E^*\psi_{\Gamma(T)}(x_0) = \psi_{\Gamma(T)}(p_0,x_0) ,$$

so $\psi^{\Gamma(T)}$ is epi-usc at x_0.

(only if): Suppose $\psi_{\Gamma(T)}$ is epi-usc at x_0. Let $V \in N(x_0)$. We have

$$0 = \psi_{\Gamma(T)}(p_0,x_0) \geq E^*\psi_{\Gamma(T)}(x_0) ,$$

so in particular

$$\limsup_{p \to p_0} \inf_{x \in V} \psi_{\Gamma(T)}(p,x) \leq 0 .$$

However, since $\psi_{\Gamma(T)}$ takes only 0 and $+\infty$ this implies that for some $U \in N(p_0)$ and each $p \in U$, $\inf_{x \in V} \psi_{\Gamma(T)}(p,x) = 0$. But this means that there is some $x \in V$ with $x \in T(p)$, so $T(p) \cap V$ is nonempty for each $p \in U$: that is, T is lower semicontinuous at (p_0,x_0). ∎

The next proposition gives a criterion for epi-upper semicontinuity of a function that models a constrained optimization problem. The function $g(p,\cdot)$ represents the objective function, while $T(p)$ represents the feasible set.

PROPOSITION 2.10: <u>Let</u> $g : P \times X \to (-\infty,+\infty]$ <u>and let</u> T <u>be a multifunction from</u> P <u>to</u> X. <u>Define</u>

$$f(p,x) := \begin{cases} g(p,x) & \text{if } x \in T(p) \\ +\infty & \text{if } x \notin T(p) . \end{cases}$$

<u>Assume that</u> $x_0 \in T(p_0)$, <u>that</u> T <u>is lsc at</u> (p_0,x_0), <u>and that</u> g <u>is usc at</u> (p_0,x_0) <u>relative to</u> graph T. <u>Then</u> f <u>is epi-usc at</u> x_0.

PROOF: We have $f = g + \psi_{\Gamma(T)}$. As $(p_0,x_0) \in$ graph T, Lemma 2.9 implies that $\psi_{\Gamma(T)}$ is epi-usc at x_0. However, dom $\psi_{\Gamma(T)}$ = graph T, and g is usc relative to this set. The hypotheses of Proposition 2.8 are therefore satisfied, and we conclude that f is epi-usc at x_0. ∎

In this section we have presented several general results about epi-semicontinuity. These can, in turn, be used to derive other results about continuity properties of infima of functions and of the sets of points at which those infima are achieved. Those results are dealt with in Section 3.

3. Infima and minimizers of locally epi-continuous functions.

In this section we relate local epi-continuity properties of extended-real-valued functions to the behavior of the infima of those functions and of the sets of points at which those infima are attained. Throughout the section we work with a function $f : P \times X \to \bar{\mathbb{R}}$, and we define two other quantities associated with f, its marginal function $\phi(p) := \inf_{x \in X} f(p,x)$ (an extended-real-valued function) and its set of minimizers

$$\Phi(p) := \{x \in X \mid f(x,p) \leq \phi(p)\}$$

(a multifunction from P to X). Note that if $\phi(p) = +\infty$ then $\Phi(p) = X$, but this is misleading since in actual modeling situations we do not want to consider points x for which $f(p,x) = +\infty$. We shall therefore often avoid this situation by requiring that $f(p,\cdot)$ be proper (never $-\infty$ and not identically $+\infty$).

The results given here are analogues of Corollaries 3.35 and 3.44 of [8] that use local hypotheses instead of the global hypotheses required in [8]. We shall see in Section 5 that the local hypotheses, particularly in the case of epi-upper semicontinuity, can be easier to verify than global ones.

Our first result deals with the behavior of ϕ when an epi-upper semicontinuity requirement holds.

PROPOSITION 3.1: Suppose that f is epi-usc at some $x_0 \in X$. Then $f(p_0,x_0) \geq \limsup_{p \to p_0} \phi(p)$.

PROOF: The epi-semicontinuity assumption implies that

$$f(p_0,x_0) \geq E^* f(x_0) = \sup_{V \in N(x_0)} \limsup_{p \to p_0} \inf_{x \in V} f(p,x)$$

$$\geq \limsup_{p \to p_0} \phi(p) ,$$

where we made the special choice $V = X$. ∎

Note that if in Proposition 3.1 we had taken $x_0 \in \Phi(p_0)$, then we would have found that ϕ was upper semicontinuous at p_0, since then $\phi(p_0) = f(p_0,x_0)$.

We now develop a simple criterion for the multifunction Φ to be <u>closed at</u> p_0, which means that $\Phi(p_0) = \bigcap_{U \in N(p_0)} \mathrm{cl}\, \Phi(U)$, where $\Phi(U)$ means $\bigcup_{p \in U} \Phi(p)$. This property is sometimes called <u>upper semicontinuity</u>, and further details about it may be found in [8] or [4]. To establish this property we briefly consider an extended real valued function $g(p,x)$ on $P \times X$ and its associated level-set multifunction $\Lambda_\alpha^g(p) := \{x \in X | g(p,x) \leq \alpha\}$. We want to point out that Λ_α^g is closed at $p_0 \in P$ if for each $x \in X$, g is lsc at (p_0,x). To see this, note that we always have $\Lambda_\alpha^g(p_0) \subset \bigcap_{U \in N(p_0)} \mathrm{cl}\, \Lambda_\alpha^g(U)$. To prove the opposite inclusion, suppose $x_0 \in \bigcap_{U \in N(p_0)} \mathrm{cl}\, \Lambda_\alpha^g(U)$. Then for each $U \in N(p_0)$ and each $V \in N(x_0)$, there are $p \in U$ and $x \in V$ with $g(p,x) \leq \alpha$. But then $\alpha \geq \liminf_{(p,x)\to(p_0,x_0)} g(p,x) \geq g(p_0,x_0)$, so $x_0 \in \Lambda_\alpha^g(p_0)$ and thus Λ_α^g is closed at p_0. This observation leads to the following criterion for Φ to be closed.

PROPOSITION 3.2: <u>Let</u> $f : P \times X \to (-\infty,+\infty]$ <u>and assume that</u> $f(p_0,\cdot)$ <u>is proper. Suppose that for each</u> $x \in X$ f <u>is epi-lsc at</u> x, <u>and that for some</u> $x_0 \in \Phi(p_0)$ f <u>is epi-usc at</u> x_0. <u>Then</u> $\Phi(p_0)$ <u>is finite,</u> Φ <u>is closed at</u> p_0, <u>and</u> $f(p,\cdot)$ <u>is proper for each</u> p <u>in some neighborhood</u> U <u>of</u> p_0.

PROOF: The second part of the hypothesis, together with Proposition 3.1, implies that Φ is usc at p_0, and therefore the function $g(p,x) := f(p,x) - \Phi(p)$ is lsc at (p_0,x) for each $x \in X$ (via Corollary 2.5). Note that our assumption that $f(p_0,\cdot)$ was proper and that $x_0 \in \Phi(p_0)$ implies that $\Phi(p_0)$ is finite. Therefore, by upper semicontinuity of Φ we have $\Phi(p) < +\infty$ for all p in some $U \in N(p_0)$, and so for such p $f(p,\cdot)$ is not identically $+\infty$. As f never takes $-\infty$ we see that $f(p,\cdot)$ is proper for each $p \in U$. In particular, this means that $g(p,\cdot)$ is well defined. Now we just observe that Φ is the level-set multifunction Λ_0^g associated with g. ∎

Propositions 3.1 and 3.2 involved upper semicontinuity of Φ at p_0, but it will also be convenient to have a criterion for lower semicontinuity there. This is developed

in the next proposition, which imposes a somewhat different requirement on f: there is no need for epi-upper semicontinuity, but we need a locally uniform compactness condition on the level sets.

PROPOSITION 3.3: Let $f : P \times X \to \bar{\mathbb{R}}$. Suppose that there are a compact set $K \subset X$ and a neighborhood $U \in \mathcal{N}(p_0)$ such that $\Lambda^f_{\phi(p_0)}(U) \subset K$, and suppose further that f is epi-lsc at each $x \in K$. Then ϕ is lsc at p_0.

PROOF: If $\phi(p_0) = -\infty$ the result follows; thus we may assume $\phi(p_0) > -\infty$. Let $\beta < \phi(p_0)$; if $x \in K$ then $f(p_0,x) \geq \phi(p_0) > \beta$, so by lower semicontinuity there are $V(x) \in \mathcal{N}(x)$ and $U(x) \in \mathcal{N}(p_0)$ such that if $(p',x') \in V(x) \times U(x)$ then $f(p',x') > \beta$. As K is compact, there are $x_1,\ldots,x_s \in K$ such that $K \subset \bigcup_{i=1}^{s} V(x_i)$. Let $U_0 := U \cap [\bigcap_{i=1}^{s} U(x_i)]$. If $p \in U_0$ and $x \in X$ then either $x \notin K$ (in which case $f(p,x) > \phi(p_0) > \beta$) or else there is some i with $x \in V(x_i)$. As $p \in U(x_i)$ we have $f(p,x) > \beta$ in this case too. Thus $\phi(p) \geq \beta$ for each $p \in U_0$, so ϕ is lsc at p_0. ∎

Propositions 3.1 through 3.3, as well as the results of [8], deal with global infima and global minimizers. In the next section we show how to adapt them to the analysis of local minimization.

4. Stability in local minimization.

In this section we develop a set of criteria for persistence and stability of local minimizers, based on the results of Section 3. However, to apply those results to local instead of global minimization we need to define in suitable generality the idea of a local minimizer. In particular, to obtain the results that we want, we need somehow to make sure that we look at enough minimizers near where we are working. This idea is made precise in the following definition.

DEFINITION 4.1: Let f be an extended real valued function on X. A nonempty subset M of $\mathbb{R}^n$ is a complete local minimizing set (CLM set) for f with respect to an open set $G \supset M$, if the set of minimizers of f on $\mathrm{cl}\, G$ is M.

Note that in this definition the function f must take the same value at each point of M, and that value must be strictly less than the value assumed by f at any point of the boundary of G. If M happens to be a singleton, it is usually called a strict local minimizer of g. Of course, the set of global minimizers of f is always a complete local minimizing set (take $G = X$).

Our strategy in dealing with local minimization will be to consider a parametrized function $f : P \times X \to \mathbb{R}$, and to impose certain hypotheses upon f on a CLM set of $f(p_0,\cdot)$. From these we shall then draw conclusions about local minimizers of $f(p,\cdot)$ for p near p_0. It will be convenient to state the hypotheses in the language of epi-continuity; in Section 5 we shall show how to translate these hypotheses into other forms convenient for dealing with optimization problems found in practice.

For some of our results in this section we shall need a local compactness condition on X. Of course, an immediate example of a situation in which such a condition holds is the case of optimization in $\mathbb{R}^n$.

The next proposition essentially adapts the results of Propositions 3.1 and 3.2 to the case of local minimization.

PROPOSITION 4.2: Let f be a function from $P \times X$ to $(-\infty,+\infty]$ and let G be an open set in X. Define for $p \in P$, $\theta(p) := \inf_{x \in \mathrm{cl}\, G} f(p,x)$ and

$\Theta(p) := \{x \in \text{cl } G \mid f(p,x) = \theta(p)\}$. <u>Assume that</u> $\Theta(p_0)$ <u>is a CLM set with respect to</u> G (i.e., $\Theta(p_0) \subset G$), that $f(p_0,\cdot)$ is proper, and that f <u>is epi-usc at some</u> $x_0 \in \Theta(p_0)$ <u>and epi-lsc at each</u> $x \in \text{cl } G$.

<u>Then</u> $\theta(p_0)$ <u>is finite</u>, θ <u>is usc at</u> p_0, Θ <u>is closed at</u> p_0, <u>and there is some</u> $U' \in N(p_0)$ <u>such that for each</u> $p \in U'$, <u>the restriction of</u> $f(p,\cdot)$ <u>to</u> $\text{cl } G$ <u>is proper</u>.

PROOF: Let $g := f + \psi_{P\times \text{cl } G}$; this is well defined since neither function ever takes $-\infty$. Since $\Theta(p_0) \subset G$ we have $\theta(p_0) < +\infty$, so $g(p_0,\cdot)$ is proper. Note that g is epi-usc at x_0, since $f = g$ on $P \times \text{cl } G$ and (p_0,x_0) belongs to the interior of that set. Since x_0 is a global minimizer of $g(p_0,\cdot)$ and θ is the marginal function of g, Proposition 3.1 tells us that θ is usc at p_0. Hence there is some $U \in N(p_0)$ such that for each $p \in U$ $\theta(p) < +\infty$, and therefore $\Theta(p)$ is the set of global minimizers of $g(p,\cdot)$.

Next we show that g is epi-lsc at each $x \in X$. This is surely true if $x \notin \text{cl } G$, since then $g(p',x') = +\infty$ for any $p' \in P$ and any x' near x. If $x \in \text{cl } G$, then since $g \geq f$ we have by epi-lsc of f,

$$g(p_0,x) = f(p_0,x) \leq \liminf_{(p',x')\to(p_0,x)} f(p',x')$$

$$\leq \liminf_{(p',x')\to(p_0,x)} g(p',x') ,$$

so g is epi-lsc at x by Corollary 2.5.

Now applying Proposition 3.2 (on $U \times X$), we conclude that $\theta(p_0)$ is finite, Θ is closed at p_0, and $g(p,\cdot)$ is proper for each p in some neighborhood U' of p_0. However, if $g(p,\cdot)$ is proper then so is the restriction of $f(p,\cdot)$ to $\text{cl } G$, since the two functions agree whenever the second is defined and since $g(p,\cdot) = +\infty$ off $\text{cl } G$. ∎

Proposition 4.2 is relatively weak in that it does not, for example, assert the existence of minimizers for $p \neq p_0$. Indeed, such an assertion would be false under the

hypotheses assumed there. We shall therefore strengthen those hypotheses by assuming some compactness conditions, and with the new hypotheses we shall be able to prove a much stronger theorem.

We first point out that if X is locally compact and $\Theta(p_0)$ is a compact CLM set with respect to some open set G', then it is possible to find an open set G with $\Theta(p_0) \subset G \subset G'$, such that $\mathrm{cl}\, G$ is compact. Thus in the statement of our next theorem we can suppose with no loss of generality that $\mathrm{cl}\, G$ is compact. To see why this is so, let us associate with each $x \in \Theta(p_0)$ a compact *neighborhood* $V(x)$ with $\mathrm{int}\, V \subset G'$. As $\Theta(p_0)$ is compact, there is a finite set $x_1,\ldots,x_s$ with $G := \bigcup_{i=1}^{s} \mathrm{int}\, V(x_i) \supset \Theta(p_0)$. Then $\mathrm{cl}\, G$ is compact, and $\Theta(p_0)$ remains a CLM set with respect to G.

THEOREM 4.3: Assume the hypotheses of Proposition 4.2, and suppose in addition that X is locally compact, that $\mathrm{cl}\, G$ and $\Theta(p_0)$ are compact, and that f is lsc on $P \times \mathrm{cl}\, G$.

Then $\Theta(p_0)$ is finite, θ is continuous at p_0, and Θ is closed at p_0. Further, there is some $U'' \in N(p_0)$ such that for each $p \in U''$, $f(p,\cdot)$ restricted to $\mathrm{cl}\, G$ is proper and $\Theta(p)$ is a nonempty, compact CLM set for $f(p,\cdot)$ with respect to G.

PROOF: We know from Proposition 4.2 that $\theta(p_0)$ is finite, θ is usc at p_0, Θ is closed at p_0 and $f(p,\cdot)$ restricted to $\mathrm{cl}\, G$ is proper for each p in some $U' \in N(p_0)$. Thus we need to prove that θ is lsc at p_0 and that the last assertion about $\Theta(p)$ is true.

To prove lower semicontinuity of θ we observe that because of the way in which g was defined we have $\Lambda^g_{\theta(p_0)}(P) \subset \mathrm{cl}\, G$, and we saw in the proof of Proposition 4.2 that g was epi-lsc everywhere on X. Therefore Proposition 3.3 applies, and we conclude that θ is lsc at p_0, hence actually continuous there.

For the last assertion, note that the set $\Theta(p)$ is nonempty and compact for each $p \in P$ since $f(p,\cdot)$ is lsc on $\mathrm{cl}\, G$. To complete the proof we show that for some $U'' \in N(p_0)$ with $U'' \subset U'$ we have $\mathrm{cl}\, \Theta(U'') \subset G$; then surely $\Theta(p)$ is a CLM set with respect to G for each $p \in U''$.

The property claimed is certainly true if $G = X$. If $G \neq X$ then G has a boundary, and for each $U \in N(p_0)$ we let $T(U)$ be the intersection of $\mathrm{cl}\ \Theta(U)$ with the boundary of G. The sets $T(U)$ are closed subsets of the compact boundary of G. But since Θ is closed at p_0 we have $\Theta(p_0) = \bigcap_{U \in N(p_0)} \mathrm{cl}\ \Theta(U)$, which implies that the intersection of the $T(u)$ is the intersection of $\Theta(p_0)$ with the boundary of G, and this latter intersection is empty since $\Theta(p_0)$ is a CLM set with respect to G. It follows that there is a finite family $U_1, \ldots, U_t$ in $N(p_0)$ with the property that $\bigcap_{i=1}^{t} T(U_i) = \emptyset$. If we define U'' to be $U' \cap (\bigcap_{i=1}^{t} U_i)$, then $T(U'') = \emptyset$, so we have $\mathrm{cl}\ \Theta(U'') \subset G$. ∎

In order to apply Theorem 4.3 to a specific problem, we really need to know only four things:

1) $\Theta(p_0)$ is a compact CLM set with respect to G,

2) $f(p_0, \cdot)$ is proper,

3) f is epi-usc at some $x_0 \in \Theta(p_0)$, and

4) f is lsc on $P \times \mathrm{cl}\ G$.

Properties (2) and (4) can be checked using information about the functions appearing in the optimization problem being solved. Properties (1) and (3) are usually enforced by making regularity assumptions about the "base case" represented by p_0. In the next section we discuss how these properties can be obtained in a typical nonlinear programming problem.

5. How to verify the hypotheses of Theorem 4.3.

In this section we show how to apply Theorem 4.3 to a typical nonlinear programming problem. Of course, there are many other types of problems to which the results of that theorem can be applied, but this application will serve to illustrate how the hypotheses of Theorem 4.3 can be verified in practice.

The problem we shall consider is

$$\begin{aligned} &\text{minimize}_x\ e(p,x) \\ &\text{subject to}\quad g(p,x) \leq 0\ , \\ &\qquad\qquad h(p,x) = 0\ , \\ &\qquad\qquad\quad x \in C\ , \end{aligned} \tag{5.1}$$

where C is a closed convex set in $\mathbb{R}^n$, Ω is an open set in $\mathbb{R}^n$, P is a topological space and e, g, and h are continuous functions from $P \times \Omega$ into $\mathbb{R}$, $\mathbb{R}^r$, and $\mathbb{R}^q$ respectively. We let $(p_0,x_0) \in P \times \Omega$ and we assume that for each $p \in P$, $g(p,\cdot)$ and $h(p,\cdot)$ are Fréchet differentiable on Ω. First derivatives of $g(p,\cdot)$ and $h(p,\cdot)$ with respect to x will be denoted by $g_x(p,\cdot)$ and $h_x(p,\cdot)$. We assume that the functions g_x and h_x are continuous at (p_0,x_0).

Suppose now that (5.1) has a local minimizer at x_0 when $p = p_0$. We are interested in conditions, preferably verifiable, under which we can apply Theorem 4.3 to (5.1) to gain information about local minimization in (5.1) when p varies near p_0. In order to develop such conditions, we first define an essential objective function $f : P \times \mathbb{R}^n \to (-\infty,+\infty]$ by

$$f(p,x) := \begin{cases} e(p,x) & \text{if}\quad x \in C \cap \Omega,\ g(p,x) \leq 0 \quad\text{and}\quad h(p,x) = 0 \\ +\infty & \text{otherwise}\ . \end{cases} \tag{5.2}$$

We note that f is evidently lsc on $P \times \Omega$, and $f(p_0,\cdot)$ is proper. Next we make two key assumptions about (5.1). These are explained in detail in the next two paragraphs.

The first assumption we shall make is that the constraints of (5.1) are <u>regular</u> at x_0 for $p = p_0$. This means that

$$0 \in \operatorname{int} \begin{bmatrix} g(p_0,x_0) \\ h(p_0,x_0) \end{bmatrix} + \begin{bmatrix} g_x(p_0,x_0) \\ h_x(p_0,x_0) \end{bmatrix} (C - x_0) + \begin{bmatrix} \mathbb{R}^r_+ \\ \{0\}^q \end{bmatrix} ; \tag{5.3}$$

if C happens to be $\mathbb{R}^n$ then (5.2) is just the well known Mangasarian-Fromovitz constraint qualification. Various properties of regularity are treated in [5] and [7]; in particular, it is shown in [5, Th. 1] that if we define a multifunction $T : P \to \mathbb{R}^n$ by

$$T(p) := \{x \in C \cap \Omega \mid g(p,x) \leqq 0,\ h(p,x) = 0\} ,$$

then the hypothesis of regularity (5.3) implies that T is lower semicontinuous at (p_0,x_0). This, together with Proposition 2.11 above and the fact that $e(p,x)$ is continuous, permits us to conclude that the essential objective function f defined in (5.2) is epi-usc at x_0.

Our next assumption is that the level set of $f(p_0,\cdot)$ corresponding to $f(p_0,x_0)$ is compact and is contained in Ω. Specifically, we assume that there is an open bounded set U with $\operatorname{cl} U \subset \Omega$, such that if $x \in (C \cap \Omega)\backslash U$ with $g(x,p_0) \leqq 0$ and $h(x,p_0) = 0$, then $e(x,p_0) > e(x_0,p_0)$: that is, any feasible point of (5.1) (with $p = p_0$) that does not belong to U gives a worse value of the objective function than does x_0. We also assume that x_0 is indeed a minimizer of $f(p_0,\cdot)$ with respect to $\operatorname{cl} U$: that is, that U is a CLM set for $f(p_0,\cdot)$.

One way of ensuring that this assumption holds is by assuming the existence of second derivatives and invoking the generalized second-order sufficient condition introduced in [6]. In that case we know from [6, Th. 2.2] that we can take U to be a suitably small neighborhood of x_0, and that there will then exist some positive μ with the property that for each x in U that is feasible for (5.1) (with $p = p_0$), one has

$$e(p_0,x) \geqq e(p_0,x_0) + \mu\|x - x_0\|^2 . \tag{5.4}$$

However, there are also many less stringent hypotheses under which the assumptions of the last paragraph hold. For example, if the function $f(p_0,\cdot)$ is convex then we need only assume that it has a compact level set corresponding to $f(p_0,x_0)$, since the local minimizer x_0 will then necessarily be global and hence a CLM set U will exist.

With the above assumptions, the function f will satisfy the hypotheses of Theorem 4.3. We can therefore conclude immediately from that theorem that for each p in some neighborhood of p_0 the nonlinear optimization problem (5.1) has a local minimizer in U: in fact, the set of such local minimizers will be a nonempty, compact CLM set with respect to U. (Of course, for a convex problem this will be the global minimizing set.) Further, that set will be closed at p_0 as a multifunction of p, and the locally optimal objective value will be continuous at p_0. Thus, in this case the verification of the hypotheses of Theorem 4.3 involves two familiar tools of nonlinear programming: a suitable constraint qualification and a compactness condition on the level sets of the essential objective function. These conditions, moreover, have to be satisfied only by the unperturbed problem with $p = p_0$, and conclusions about the perturbed problem for all small perturbations will immediately follow.

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One of the fundamental questions in nonlinear optimization is how optimization problems behave when the functions defining them change (e.g., by continuous deformation). Recently the study of epi-continuity has somewhat unified the results in this area. Here we show how to localize the concept of epi-continuity, and how to apply these localized ideas to ensure persistence and stability of local optimizing sets. We also show how these conditions follow from known properties of nonlinear programming problems.

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