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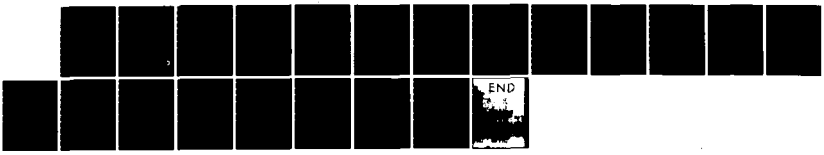

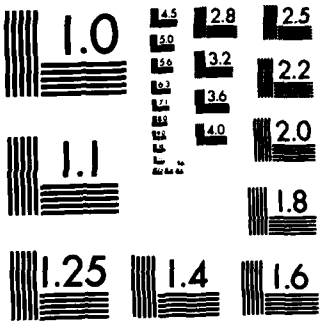

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ON THE SEMILATTICE OF WEAK ORDERS OF A SET*

M. F. JANOWITZ**

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§1. Introduction. Individual and collective preferences are often modelled using binary relations. Weak orders turn out to be especially useful for this purpose. A survey of this general area is provided by [3]. The references ([2], [9], [10], [12] and [13]) also are of interest. The idea is to let X denote a set of alternatives and then rank X by preference. Thus xPy means y is preferable to x. The resulting binary relation P is often a weak order on X, in that P is

(1) reflexive (xPx for all $x \in X$)

(2) transitive (xPy and $yPz \Rightarrow xPz$)

(3) total (for $x,y \in X$, xPy or yPx).

Such relations also arise naturally in digital image processing. In its most general form, a (monochromatic) digital picture is simply a rectangular array of numbers that have spatial as well as numerical significance. Multilevel threshholding of a picture involves choosing numbers $h_1, h_2, \ldots, h_k$ with $h_1 < h_2 < \ldots < h_k$ and labelling sites with 1 if their value does not exceed h_1, 2 if their value exceeds h_1 but not $h_2, \ldots$, etc. This process simply constructs a weak order on the picture.

Finally, weak orders arise in connection with the reconstruction of evolutionary trees. The underlying set here is a set of "evolutionary units", and if the goal is to construct a binary tree on X that in some sense estimates the true evolutionary history of the currently existing members of X, then one can view this as constructing a

nested sequence of weak orders on these members. For more details, the reader is referred to [5], [6], [7], [8] and [11].

Using the above examples for motivation, we now embark on a discussion of the order theoretic properties of the partially ordered set of weak orders on a set. In §3, this work will be related to earlier work in pure lattice theory, and in §4 the weak orders on a finite set will be characterized in an order theoretic and combinatorial setting.

§2. Weak Orders. Let X be a nonempty set, and $W(X)$ the set of weak orders on X, with $W(X)$ partially ordered by implication. Thus $P \leq Q$ in $W(X)$ if

$$xPy \text{ implies } xQy$$

for all $x,y \in X$. The atoms of $W(X)$ are then the linear orders of X, the largest element of $W(X)$ is the relation $X \times X$, and the coatoms are the weak orders that partition X into two classes. For J a proper subset of X, it will be convenient to let $C(J)$ be the coatom defined by $(x,y) \in C(J)$ if $\{x,y\} \subseteq J$, $\{x,y\} \subseteq X \setminus J$, or $x \in X \setminus J$ with $y \in J$.

A weak order P on X may also be thought of as an ordered partition of X. This is a pair $(\mathcal{P},<)$ where $\mathcal{P}$ is a partition of X, and "$<$" a linear ordering of the classes of $\mathcal{P}$. When $\mathcal{P}$ has only a finite number of distinct classes, it will often be convenient to

specify their ordering by simply listing them in ascending order. For example, one could write

$$C(J) = (X\backslash J)(J).$$

There is yet a third way of viewing weak orders. If $A, P \in W(X)$ with A an atom under P, then P may be viewed as a comgruence relation on the chain (X,A). The principal filter in $W(X)$ generated by A is then isomorphic to the lattice of congruence relations on (X,A).

We turn now to some elementary order theoretic properties of $W(X)$. We begin with the assumption that X is infinite and will later see what else can be said in the finite case. Accordingly, until further notice, it will be assumed that X denotes some fixed infinite set. Rather than stating the results formally as theorems, they will instead be listed as properties of $W(X)$.

(P1) Every principal filter of $W(X)$ is a complete, compactly generated distributive lattice.

Proof: This follows from the fact that if A is an atom of $W(X)$, then the principal filter generated by A in $W(X)$ is isomorphic to the lattice of congruence relations of the chain (X,A).

(P2) $W(X)$ is a join semilattice.

Proof: This follows trivially from (P1).

(P3) $W(X)$ is atomistic and dually atomistic.

Proof: Let $P \prec Q$ in $W(X)$. There must then exist elements $x, y \in X$ such that xQy but not xPy. Define a mapping $\pi : X \to X$ by letting $\pi(x) = y$, $\pi(y) = x$ and $\pi(z) = z$ for $z \neq x, y$. If A is any atom under P, we take A_1 to be the atom specified by sA_1t if $\pi(s)A\pi(t)$. Then $A_1 \leq Q$ but $A_1 \not\leq P$ and this shows $W(X)$ to be atomistic. Dual atomicity follows from the fact that the lattice of congruence relations of a chain is dually atomistic.

(P4) Let J, K be proper subsets of X. $C(J) \wedge C(K)$ exists in $W(X)$ if and only if $J \subset K$ or $K \subset J$.

Proof: If $J \subset K$, it is easy to show that

$$C(J) \wedge C(K) = (X \backslash K)(K \backslash J)(J).$$

If J, K are not comparable, one can choose $x \in J \backslash K$ and $y \in K \backslash J$. If A were an atom under both $C(J)$ and $C(K)$, this would force both

$$xAy \quad \text{and} \quad yAx,$$

a contradiction.

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(P5) Let $(C_i)_{i \in I}$ be a family of coatoms of $W(X)$. If $\bigwedge_i C_i$ fails to exist, then there exist indices $i, j \in J$ such that $C_i \wedge C_j$ fails to exist.

Proof: We shall establish the contrapositive of (P5). Suppose that $C_i \wedge C_j$ exists for all indices i,j. Let $P = \cap_i C_i$, where $\cap$ denotes set intersection. Then P is a reflexive transitive relation on X. We would be done if we could show P to be total. If xPy fails, then xC_iy must fail for some $i \in J$, so yC_ix. Consider any other C_k $(k \in J)$. If $C_i = C(J_i)$ and $C_k = C(J_k)$, we know that J_i, J_k are comparable. Now $(x,y) \notin C_i$ forces $y \in X \setminus J_i$ with $x \in J_i$. If $J_i \subset J_k$, then $x \in J_k$ forces yC_kx. If $J_k \subset J_i$, then $y \in X \setminus J_k$ again forces yC_kx. Consequently, yPx.

The above result should be compared with [7], Theorem 2.4, p. 146. It might also be mentioned that (P5) applies to arbitrary families of weak orders not just to coatoms. This is an immediate consequence of $W(X)$ being dually atomistic. Before proceeding we shall need some notation. As we have been doing, set union and intersection will be denoted by $\cup$ and $\cap$, with $\vee,\wedge$ reserved for the join and meet operations in $W(X)$. Following the terminology introduced by Monjardet ([13], p.54) we agree for a binary relation Q on X to let

$$Q^c = \{(x,y):(x,y) \notin Q\}$$

denote the complement of Q, and

$$Q^d = \{(x,y):(y,x) \notin Q\}$$

the dual of Q. Notice that when Q is a weak order, Q^d is the strong preference relation associated with Q in that

$$(x,y) \in Q^d \iff (x,y) \in Q \quad \text{and} \quad (y,x) \notin Q .$$

(P6) *For* P,Q *in* $W(X)$ *there is a smallest weak order* R *over* P *such that* $R \wedge Q$ *exists. This weak order* R *will be denoted* $R = P \uparrow Q$.

Proof: Our goal will be to prove that R is the transitive closure of $P \cup (P^c \cap Q^d)$. The key fact here is the observation that

(1) $P \wedge Q$ *exists in* $W(X)$ *if and only if* $P^d \subseteq Q$ *and* $Q^d \subseteq P$.

Proof of (1): If A is a lower bound for P,Q in $W(X)$, then $(x,y) \in P^d$ implies $(y,x) \in P^c$, so $(y,x) \in A^c$. It follows that $(x,y) \in A$, whence $(x,y) \in Q$. By symmetry, $Q^d \subseteq P$. Suppose conversely that $P^d \subseteq Q$ and $Q^d \subseteq P$. Let $R = P \cap Q$. We need only show that $R \in W(X)$, and to do this, we need only argue that it is total. If $(x,y) \in R^c$, then $(y,x) \in R^d = (P \cap Q)^d = P^d \cup Q^d \subseteq P \cup Q = R$, as desired. To establish the theorem we note first that necessarily,

(2) $$Q^d \subseteq P \cup (P^c \cap Q^d).$$

Proof of (2): This follows trivially from the fact that

$$Q^d = (Q^d \cap P) \cup (Q^d \cap P^c).$$

Note next that

(3) $$[P \cup (P^c \cap Q^d)]^d \quad Q.$$

Proof of (3): Using the fact that ([13], p. 54) $R \to R^d$ is an involution on the lattice of binary relations on X, this follows trivially from

$$[P \cup (P^c \cap Q^d)]^d = P^d \cap (P^{cd} \cup Q^{dd})$$

$$= (P^d \cap P^{cd}) \cup (P^d \cap Q)$$

$$= \phi \cup (P^d \cap Q) \subseteq Q.$$

It can now be shown that

(4) <u>For</u> $R \geq P$, $R \wedge Q$ <u>exists in</u> $W(X)$ <u>if and only if</u> R <u>contains the transitive closure of</u> $P \cup (P^c \cap Q^d)$.

<u>Proof of (4)</u>: If R contains $P \cup (P^c \cap Q^d)$ then by (2), $Q^d \subseteq R$. Also, if $(x,y) \in R^d$, then $(y,x) \notin R$, so $(y,x) \notin P \cup (P^c \cap Q^d)$ and by (3) $(x,y) \in Q$. Thus $R^d \subseteq Q$ and (1) may not be applied to see that $R \wedge Q$ exists.

Suppose conversely that $R \wedge Q$ exists. By (1), $Q^d \subseteq R$, whence

$$P \cup (P^c \cap Q^d) \subseteq R.$$

The assertion of (P6) now follows from the fact that R contains $P \cup (P^c \cap Q^d)$ if and only if it contains its transitive closure, and the transitive closure of $P \cup (P^c \cap Q^d)$ is a weak order on X.

<u>Remark</u>: If P, Q are voting preferences then one can think of $P \uparrow Q$ as being the preference obtained by Q casting doubt on P; i.e., whenever there is a conflict between P and Q, the issue is settled by leaving that particular preference unresolved. The preference $(P \uparrow Q) \wedge Q$ may be thought of as the modification of Q by means of P wherever possible, and using the preference of Q wherever there is a conflict. Similar interpretations occur when dealing with digital pictures or evolutionary trees.

(P7) *If the partition associated with* P *has only finitely many distinct classes* $C_1, C_2, \ldots, C_k$, *then the interval under* P *in* $W(X)$ *is isomorphic to* $W(C_1) \times W(C_2) \times \ldots \times W(C_k)$.

Proof of 7: The desired isomorphism is easily shown to be

$$Q \to (Q_1, Q_2, \ldots, Q_k)$$

where Q_i = the restriction of Q to C_i.

At this point we impose the restriction that X be finite, and with no loss in generality, take $X = \{1,2,\ldots,n\}$, where $n > 1$ is a positive integer. In view of this, we shall write $W(n)$ in place of $W(X)$. We then have:

(F1) *The interval above any atom of* $W(n)$ *is isomorphic to the lattice of all subsets of an* $n - 1$ *element set.*

Proof of (F1). This follows from the proof of (P1).

(F2) *Say that a coatom is of type* i *if its largest class has* i *members. This produces* $n - 1$ *distinct types of coatoms having the following properties:*

(a) *A coatom* C *is of type* i *iff* $C \wedge B$ *exists for exactly* i *type* 1 *coatoms* B.

(b) *Every atom is the meet of exactly* $n - 1$ *coatoms - one of each type.*

Proof of (F2). These assertions follow from (P4).

(F3) There are $\binom{n}{i}$ coatoms of type i and each such coatom dominates exactly $i!(n-i)!$ atoms. In fact, the principal ideal generated by a type i coatom is order isomorphic to $W(i) \times W(n-i)$.

Proof of (F3). See property (P7).

(F4) For $n \geq 3$, the group of order automorphisms of $W(n)$ is isomorphic to $\underline{2} \times S_n$, where $\underline{2}$ is a group of order 2, and S_n is the group of permutations of $\{1,2,\ldots,n\}$.

Proof of (F4). For each $\pi \in S_n$, the correspondence $C(J) \to C(\pi(J))$ clearly extends to an order automorphism of $W(n)$, and these order automorphisms are distinct. The correspondence $(\mathcal{B},\leq) \to (\mathcal{B},\geq)$ is an order automorphism of order 2 that commutes with the order automorphisms induced by the elements of S_n. The subgroup of the group G of all order automorphisms of $W(n)$ generated by these mappings is thus clearly isomorphic to $\underline{2} \times S_n$. We would be done if we could just show that G has order $\leq 2 \times n!$. By F3, an order automorphism of $W(n)$ must map a type 1 coatom into either a type 1 or a type $n-1$ coatom. Suppose that $C(i)$ gets mapped to a type $n-1$ coatom by the order automorphism f. Choose $j,k \neq i$, and note that none of $fC(i) \wedge fC(j)$, $fC(i) \wedge fC(k)$, or $fC(j) \wedge fC(k)$ can exist. Since $fC(i)$ is type $n-1$, it follows that at least one of $fC(j)$ or $fC(k)$ must also be type $n-1$. It

then follows that the remaining one must be type $n - 1$. Here we have used the fact that a type $n - 1$ coatom has a meet with exactly 1 type 1 coatom. Thus if f sends one type 1 coatom to a type $n - 1$ coatom, then it must send <u>all</u> type 1 coatoms to type $n - 1$ coatoms. By P3, an order automorphism is completely determined by its effect on the coatoms of $W(n)$. There are only $2 \times n!$ ways that the type 1 coatoms can be mapped onto either themselves or the type $n - 1$ coatoms, and we are done. It should be noted that the result does indeed fail for $OP(2)$; for $OP(2)$ is a 3 element semilattice with 2 atoms and a unit element, so its group of automorphisms is simply $\underline{2}$.

3. <u>Relation to Residuated Mappings</u>. Before doing anything along these lines, some background material is needed. A mapping ϕ from a partially ordered set P to a partially ordered set Q is said to be <u>residuated</u> in case the preimage of a principal ideal of Q is necessarily a principal ideal of P; dually, ϕ is said to be residual if the preimage of a principal filter of Q is a principal filter of P. An alternate but illuminating definition for residuated mappings would state that $\phi: P \to Q$ is <u>residuated</u> if

(1) ϕ is <u>isotone</u> in that $a \leq b$ in P implies $\phi(a) \leq \phi(b)$ in Q,

and there is an isotone mapping $\phi^+: Q \to P$ such that

(2) $p \leq \phi^+\phi(p)$ and $q \geq \phi\phi^+(q)$ for all $p \in P$, $q \in Q$.

The mapping ϕ^+ is then residual and is completely determined by ϕ. To say that ϕ is <u>range-residuated</u> will be to say that the preimage of

every principal ideal of Q is either empty or a principal ideal of P. This is evidently equivalent to the assertion that ϕ is residuated if it is considered to be a mapping from P into the order filter of Q generated by its image. To say that $\phi:P \to Q$ is <u>range-closed</u> is to say that its image is <u>convex</u> in that $\phi(a) \leq q \leq \phi(b)$ implies the existence of an element p of P such that $q = \phi(p)$. A residuated mapping $\phi:P \to Q$ is said to be <u>dually range-closed</u> if the image in P of its associated residual mapping is convex. Finally, we agree to call ϕ <u>multiplicative</u> in case the existence of $a \wedge b$ in P implies that $\phi(a \wedge b)$ is the infimum in Q of $\{\phi(a),\phi(b)\}$; in other words, a multiplicative mapping preserves finite existing infima. For an introduction to the theory of residuated mappings, the reader might consult [4]. It will be convenient to let $M(P)$ denote the semigroup of all range-residuated multiplicative mappings on the partially ordered set P, and call P an M-<u>semilattice</u> in case for each $p \in P$ there correspond idempotents ϕ_p, ψ_p in $M(P)$ such that

(1) ϕ is range-closed with image the principal ideal generated by p,

(2) ψ is dually range-closed with the image of ψ^+ the principal filter generated by p.

Our goal will be to characterize M-semilattices. It will turn out to be convenient to first investigate the "arrow" operation on a join semilattice.

Theorem 1. Let P be a join semilattice with no smallest element. Suppose that every filter of P is a distributive lattice, and that for every a,b in P, $a \uparrow b$ is the smallest element above a that has a meet with b. Then:

(1) $a \leq a \uparrow b$.

(2) $a = a \uparrow b$ iff $a \wedge b$ exists.

(3) $a \leq b$ implies $a \uparrow c \leq b \uparrow c$.

(4) $b \geq c$ implies $a \uparrow b \geq a \uparrow c$.

(5) If $a \wedge b$ exists, then $(a \wedge b) \uparrow c = (a \uparrow c) \wedge (b \uparrow c)$.

(6) $(a \vee b) \uparrow c = (a \uparrow c) \vee (b \uparrow c)$.

(7) $(a \uparrow b) \uparrow c = (a \uparrow b) \vee (a \uparrow c)$.

(8) If $b \wedge c$ exists, then $a \uparrow (b \wedge c) = (a \uparrow b) \vee (a \uparrow c)$.

Proof: The first 4 items are trivial.

(5) If $x \geq a \wedge b$ and $x \wedge c$ exists, then $x \vee a \geq a \uparrow c$ and $x \vee b \geq b \uparrow c$. Since $x = (x \vee a) \wedge (x \vee b)$, this shows that $x \geq (a \uparrow c) \wedge (b \uparrow c)$. In particular, this is true for $x = (a \wedge b) \uparrow c$. The reverse inequality follows from (3).

(6) Is clear.

(7) If $x \geq (a \uparrow b) \uparrow c$, then $x \geq a \uparrow b$ and $x \wedge c$ exists, so from $x \geq a$, we have also that $x \geq a \uparrow c$. Hence $x \geq (a \uparrow b) \vee (a \uparrow c)$. If $x \geq (a \uparrow b) \vee (a \uparrow c)$, then $x \geq a \uparrow b$ and $x \wedge c$ exists, so $x \geq (a \uparrow b) \uparrow c$.

(8) If $x \geq a$ and $x \wedge b \wedge c$ exists, then $x \wedge b$ and $x \wedge c$ must exist, so $x \geq (a \uparrow b) \vee (a \uparrow c)$. If $x \geq (a \uparrow b) \vee (a \uparrow c)$, then $x \geq a$ and $x \wedge b$, $x \wedge c$ exist. Since also $b \wedge c$ exists, we have that $x \wedge b \wedge c$ exists and $x \geq a \uparrow (b \wedge c)$.

We turn now to the characterization of M-semilattices.

Theorem 2. *Necessary and sufficient conditions for a partially ordered set* P *to be an* M-*semilattice are:*

(i) *Every principal filter of* P *be an implicative lattice,*

(ii) P *be a join semilattice.*

(iii) *Given* $a,b \in P$, *there is an element* $a \uparrow b$ *such that* $a \leq a \uparrow b$ *and* $x \wedge b$ *exists for* $x \geq a$ *iff* $x \geq a \uparrow b$.

Proof: Necessity. Evidently P must have a largest element 1, by [4], Theorem 13.1, p. 119, P is a join semilattice. Now let $a,p \in P$ with $\phi \in M(P)$ a range-closed idempotent whose image is $(p]$, the principal ideal generated by p. If a,p have a lower bound x, then $x = \phi(x)$, and by [4], *Theorem* 13.1, *p.*119, $a \wedge p$ *exists and equals* $\phi^{+}(a)$. In fact

$$\phi(a) = \phi(a) \wedge \phi(p) = \phi(a \wedge p) = a \wedge p. \tag{1}$$

In particular, since $\phi^{+}\phi(a)$ and p have $\phi(a)$ as a lower bound, we have

$$\phi(a) = \phi\phi^{+}\phi(a) = \phi^{+}\phi(a) \wedge p. \tag{2}$$

Now if $x \geq a$ and $x \wedge p$ exists, then by (1),

$$\phi(x) = x \wedge p \geq \phi(a).$$

Hence

$$x \geq \phi(a) \vee a. \tag{3}$$

If, on the contrary, (3) holds, then clearly $x \wedge p$ exists. Thus we may take

$$(4) \qquad a \uparrow p = \phi(a) \vee a.$$

The fact that every principal filter of P is an implicative lattice now follows from the observation that if a,p have a lower bound in P then $\phi(a) = a \wedge p$. Thus $a \to a \wedge p$ is residuated in any principal filter F of P, and this is precisely the assertion that F be an implicative lattice.

For the converse assume (i), (ii) and (iii). Let $p \in P$. The mapping $\psi(x) = x \vee p$ will serve as a dually range-closed idempotent member of $M(P)$ with $\psi^+:[p,1] \to P$ given by $\psi^+(y) = y$ for $y \geq p$. The mapping ψ is multiplicative because every principal filter of P is a distributive lattice. To define the desired range-closed idempotent ϕ in $M(P)$, take

$$(5) \qquad \phi(a) = (a \uparrow p) \wedge p.$$

For $b \in P$, if the preimage of $(b]$ under ϕ is not empty, then $\phi(x) \leq b,p$ implies the existence of $b \wedge p$. Working in the implicative lattice $[b \wedge p,1]$, let b^* be defined by the rule $t \wedge p \leq b$ iff $t \leq b^*$. Then for $a \in P$, $\phi(a) = (a \uparrow p) \wedge p \leq b$ implies $[(a \uparrow p) \vee b] \wedge p \leq b$, so $a \leq (a \uparrow p) \vee b \leq b^*$. If conversely $a \leq b^*$, then $a \vee (b \wedge p) \leq b^*$ shows $[a \vee (b \wedge p)] \wedge p \leq b$. Since $a \uparrow p \leq a \vee (b \wedge p) \leq b^*$, it follows that $\phi(a) = (a \uparrow p) \wedge p \leq b^* \wedge p \leq b$. Thus the preimage of $(b]$ under ϕ is $(b^*]$. This shows ϕ to be

range-residuated. Clearly ϕ is idempotent and its image is $(p]$. To see that ϕ is multiplicative we apply Theorem 1 (5) to see that if $a \wedge b$ exists in P, then

$$\phi(a \wedge b) = [(a \wedge b) \uparrow p] \wedge p = (a \uparrow p) \wedge (b \uparrow p) \wedge p$$

and this shows that $\phi(a \wedge b) = \phi(a) \wedge \phi(b)$.

The point to all this is contained in

COROLLARY 3. $W(X)$ is an M-semilattice.

§4. A Characterization of $W(n)$. For a partially ordered set P, let us agree to call $a,b \in P$ related in case they have a common lower bound, and call them unrelated otherwise. We then have

Theorem 6. Let L be a poset having a largest element 1, but no smallest element. Suppose L satisfies the following conditions:

(1) L is atomic.

(2) For every atom p of L, $[p,1]$ is isomorphic to the lattice of all subsets of an $n-1$ element set.

(3) Among the coatoms of L there is a family S of n coatoms that is maximal with respect to being pairwise unrelated. Call these coatoms special, and suppose they are such that:

(3a) Corresponding to each proper subset J of S, there is a unique coatom $c(J)$ that is related to all $s \in J$ and unrelated to every $s \notin J$. Every coatom is of this form.

(3b) $\wedge_i c(J_i)$ <u>exists iff the family</u> $\{J_i\}$ <u>are pairwise comparable as sets.</u>

<u>Then</u> L <u>is isomorphic to</u> $W(n)$.

Proof: There are n special coatoms, so they may be labeled $c(1), c(2), \ldots, c(n)$. Define $\theta: L \to W(n)$ by $\theta(c(J)) = C(J)$, with $\theta(1)$ the unit element of $W(n)$. For any other element x of L, by (2), x has a unique representation as $x = \wedge_i c(J_i)$ where there are at most $n - 1$ coatoms $c(J_i)$. By (3b), the J_i's are pairwise comparable, so by property (P6), $\wedge_i C(J_i)$ exists in $W(n)$. Define $\theta(x) = \wedge_i C(J_i)$. By the uniqueness of the representation of x as the meet of a family of coatoms, θ is well defined. To see that it is onto, note that if $D = \wedge_i (C(J_i)$ exists in $W(n)$, the argument we just made can be reversed to conclude that $\wedge_i c(J_i)$ exists in L. By construction, $\theta(\wedge_i c(J_i)) = \wedge_i C(J_i))$. The proof is completed by observing that $x \geq y$ in L is equivalent to the assertion that the set of coatoms above x be contained in the set above y, and this is clearly equivalent to $\theta(x) \geq \theta(y)$.

Though the above characterization is easy to establish, it is in many ways unsatisfactory. For one thing, conditions (3a) and (3b) are extremely powerful; for another, they are combinatorial as well as being order theoretic. It would be interesting to weaken these conditions, though as we shall soon see, they cannot be entirely eliminated. It would also be of interest to have a characterization that is more order theoretic. In that $W(n)$ is a semiBoolean algebra a characterization along the lines of [1], Theorem 7.17, p. 244 would also be appropriate.

Before closing, it is illuminating to consider some examples. Each example is of a poset with height 3. Only the coatoms will be shown; a connection between a pair of coatoms will indicate that there is an atom beneath them. In that each atom will be under exactly two coatoms, this completely specifies the poset. Each example will have a set of 3 special coatoms. These will be indicated by open circles, with the remaining coatoms denoted by closed circles, Fig. 1 is the diagram for OP(3). Referring to the condition of Theorem 6, the example in Fig. 2 satisfies (1), (2), (3), (3a), but not (3b); the example in Fig. 3 satisfies all conditions except (3a).

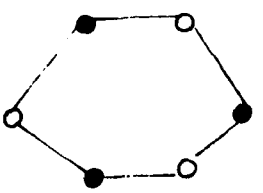

Fig. 1 OP(3)

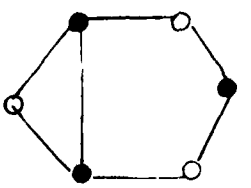

Fig. 2

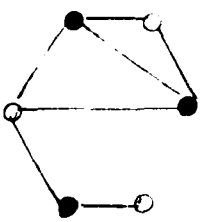

Fig. 3

REFERENCES

[1] ABBOTT, J. C., Sets lattices and Boolean algebras, Allyn and Bacon, Boston, 1967.

[2] ARROW, K. J., Social choice and individual values, Wiley, New York, 1963, second edition.

[3] BARTHÉLEMY, J. P., FLAMENT, D., and MONJARDET, B., Ordered sets and social sciences, in "Ordered Sets", ed. I. Rival, Reidel, Dordecht, 1981, pp. 721-758.

[4] BLYTH, T. S. and JANOWITZ, M. F., Residuation Theory, Pergamon, London, 1972.

[5] ESTABROOK, G. F., JOHNSON, C. S. Jr., and McMORRIS, F. R., An idealized concept of the true cladistic character, Math. Biosci., 23 (1975), pp. 263-272.

[6] _____, A mathematical foundation for the analysis of cladistic character compatibility, ibid., 29 (1976), pp. 181-187.

[7] _____, An algebraic analysis of cladistic characters, Discrete Math., 16 (1976), pp. 141-147.

[8] ESTABROOK, G. F., and McMORRIS, F. R., When are two qualitative taxonomic characters compatible? J. Math. Biology, 4 (1977), pp. 195-200.

[9] FISHBURN, P. C., The theory of social choice, Princeton, 1973.

[10] _____, Mathematics of decision theory, Mouton, The Hague, 1972.

[11] McMORRIS, F. R., On the compatibility of binary qualitative taxonomic characters, Bull. Math. Biology, 39 (1977), pp. 133-138.

[12] MIRKIN, B. G., Group choice, Winston, Washington, 1979.

[13] MONJARDET, B., Axiomatiques et proprietes des quasi-ordres, Math. Sci. Hum. 16 (1978), pp. 51-82.

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Order theoretic properties of the semilattice of weak orders on a set are investigated. In the finite case a characterization is provided. Applications are given to such diverse fields as voting preference, cladistics, and digital image processing.

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