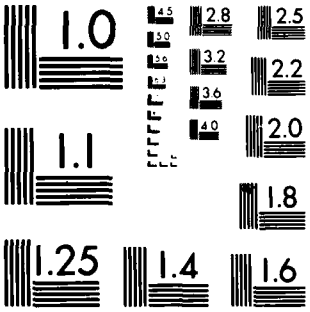
AD-A144 140 AN ANISOTROPIC LENS FOR LAUNCHING TEM WAVES ON A CONDUCTING CIRCULAR CONI..(U) NEW MEXICO UNIV ALBUQUERQUE DEPT OF MATHEMATICS AND STATISTIC.. 1/1

UNCLASSIFIED A P STONE ET AL. JUN 84 NOTE-285 F/G 12/1 NL

END
DATE
FILMED
9-84
DTIC



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS 1963 A

AFOSR-TR- 84-0600

Sensor and Simulation Notes

Note 285

June 1984

AD-A144 140

AN ANISOTROPIC LENS FOR LAUNCHING TEM WAVES ON A CONDUCTING CIRCULAR CONICAL SYSTEM

A. P. Stone *
Dept. of Mathematics and Statistics
University of New Mexico
Albuquerque, New Mexico 87131
and
C. E. Baum
Air Force Weapons Laboratory
Kirtland AFB
Albuquerque, New Mexico 87117.

ABSTRACT

A differential impedance and transit-time matching approach is used in the design of an anisotropic lens for launching TEM waves from a small source, through the lens, and onto a conducting circular conical system. This approach leads to a system of ordinary differential equations which may be solved exactly to obtain the lens parameters. An approximate solution, which would be applicable to a design procedure, is also given.

DTIC FILE COPY

DTIC
ELECTE
AUG 8 1984
S D
D

* Work of this author was supported in part by AFOSR under grant number AFOSR-83-0040.

Approved for public release;
distribution unlimited.

84 08 07 111

CONTENTS

Section		Page
I	Introduction: Remarks on EM Lens Design	3
II	Review of Results For a Lens Between Coaxial Circular Cylinders	5
III	Statement of the Problem: TEM Waves on Circular Conical Systems	10
IV	Brewster-angle Considerations	17
V	Determination of Lens Profile	24
	A. General Equations for θ' and $\epsilon_r(\theta')$	24
	B. Constraints on Z_c and ϵ_r	27
	C. Numerical Results	29
	Figures 5.1 ...	33
	Tables 5.1 ...	38
	D. Length of Boundary Curve	71
VI	Approximation of Exact Case by Uniform Isotropic ϵ_r	73
	Tables 6.1 ...	76
VII	Summary and Conclusions	82
	References	83

Accession For
NTIS GRA&I ☒
DTIC TAB ☐
Unannounced ☐
Justification
By
Distribution/
Availability Codes
Dist | Avail and/or Special
A/1

DTIC COPY INSPECTED 2

I. INTRODUCTION: Remarks on EM Lens Design

A technique commonly used in the solution of electromagnetic boundary value problems is that of expressing Maxwell's equations in orthogonal curvilinear coordinates and then solving for quantities which combine the physical components of fields with the scale factors of the coordinate transformation. The constitutive parameters of the medium are also combined with the scale factors with the hope that the coordinate transformation simplifies the problem. An extension of this technique was made by Baum in his research on EM lens design [1]. In his work he found many cases of isotropic inhomogeneous media for which certain types of electromagnetic wave propagation could be simply expressed. His differential geometric scaling method represented a design procedure for a certain kind of electromagnetic lens. Particular conductor geometries and media inhomogeneities were utilized to transition waves between conical and/or cylindrical transmission lines, and the transition was accomplished with neither reflection nor distortion of the wave. These lenses can then transmit arbitrary pulse waveforms without distortion. We note that the properties of such lenses, when combined with perfect conductors, are independent of frequency if it is assumed that the permittivity and permeability of the medium are real and frequency independent and that its conductivity is zero. This result is in contrast to lenses based on a geometrical optics approximation, such as the Luneberg lens which relies on the frequency being sufficiently high. Of course in practical applications the range (maximum and minimum values) of the permittivity and permeability, the spatial extent of the lens, and range of frequencies of interest would be important considerations.

Further research on this differential geometric approach to electromagnetic lens design was reported on by Stone in [2]. The main result obtained, for the case of transitioning TEM waves between cylindrical and conical transmission lines and isotropic inhomogeneous media, was that a class of solutions to the design problem exists. This class of solutions can be obtained from rotational coordinate systems arising from complex analytic transformations in the plane.

An important alternative way of viewing this differential geometric method in lens design is through a differential impedance matching and transit time matching approach. If a lens is to be inserted between two transmission lines it is clear that not only do impedances have to be matched but that the travel time for waves following paths of different lengths have to be equal if we want no reflection and no distortion of the propagated wave front. This impedance matching approach, which is described in section 2 of this paper, will be used in a specific lens design problem. This problem involves the design of a certain anisotropic lens suitable for launching TEM waves on a conducting circular conical system. Following some Brewster angle considerations in section 3, the statement of this problem is given in section 4, and an analytic solution is described in section 5. An approximate solution which would be applicable to a design procedure appears in section 6, and conclusions and summary appear in section 7.

This section is concluded with some comments regarding applications of EM lenses. Such transient lenses have application to EMP simulators [4] and energy transport in pulse power equipment. Thus included in these applications are highly directional high frequency antennae in which the spatial geometry and medium inhomogeneity are used to launch approximate TEM waves over a cross section with dimensions much larger than a wave length. In particular, in studying TEM waves on circular conical systems one major application of EM lenses would be their use in EMP simulators. For example, a possible way to simulate an electromagnetic pulse on a system in flight is to radiate a large amplitude pulse from an antenna to the systems. Hence one approach to such a simulator is to discharge a fast, high-voltage, capacitive generator into a large antenna which then radiates a narrow, large-amplitude pulse in the direction of the system under test. This type of simulator, which is of the dipole type, is useful in cases where

AIR FORCE OFFICE OF SCIENTIFIC RESEARCH
NOTICE OF
Chief, Technical Information Division

the simulator is far from the system under test compared to the size of the dipole structure [3],[4]. Another realized type of dipole simulator is a resistively loaded circular cone mounted on a ground plane. Such vertically polarized electric dipoles have been constructed at various locations [4].

II. REVIEW OF RESULTS FOR A LENS BETWEEN COAXIAL CIRCULAR CYLINDERS

An excellent illustration of the use of the impedance matching approach to a lens design problem appears in [5]. *In this reference the* authors consider two cylindrical coaxial waveguides of different size as illustrated in figure 2.1. The waveguide section of the left, which is region I, has inner and outer cylindrical radii denoted by $\rho = A$ and $\rho = B \equiv \lambda_t A$ respectively, while the waveguide section on the right, which is region III has its respective inner and outer radii given by $\rho = A'$ and $\rho = B' \equiv \lambda_t' B'$ with $A' > A$. Both waveguides are filled with the same simple uniform medium of constant ϵ and μ, and $\sigma \equiv 0$. Physical considerations will require that the transverse ratios λ_t, λ'_t be equal. The problem considered in [5] was that of finding a perfect matching section II between regions I and III such that a TEM wave incident from the left side could propagate into a TEM wave in region III without reflection and without distortion. In the lens region II a variable $\epsilon(\theta)$ and anisotropic conductivity σ was permitted, but the lens was to have the same fixed μ as in the cylindrical regions. One way of obtaining a perfect matching for regions I, II, and III is to insert coaxial conducting layers in all the regions with spacing d and thickness Δ of each sheath satisfying $\Delta << d$ (for negligible reduction of impedance by conductor thickness) and also satisfying $d << \lambda$ (for propagation of only TEM mode), where λ is the wavelength of the TEM wave. This condition is needed if there is to be no distortion. Moreover, for negligible loss to be introduced by the conductors one would also need

$$\sqrt{\frac{\omega\epsilon}{\sigma}}\, L << d \quad \text{if} \quad \frac{1}{\sqrt{\omega\mu\sigma}} << \Delta \tag{2.1}$$

and

$$\frac{L}{\sigma\Delta}\sqrt{\frac{\epsilon}{\mu}} << d \quad \text{if} \quad \Delta << \frac{1}{\sqrt{\omega\mu\sigma}} \tag{2.2}$$

where L is the longitudinal dimension in region II. Finally, for a TEM wave to propagate from I, through II, and into III without distortion a plane wave front in I should appear as a plane wave front in III: that is, the traveling time of waves along paths of different radii should be equal. Thus, (see figure 2.2) we require the travel times along $MM'\ M''$ and its infinitesimally changed version $OO'\ O''$ to be equal, and so we obtain

$$\begin{aligned} &\sqrt{\mu\epsilon(\theta)}\ [r_2(\theta)-r_1(\theta)] + \sqrt{\mu\epsilon}\ \Delta_1 \\ &= [r_2(\theta+d\theta) - r_1(\theta+d\theta)]\sqrt{\mu\epsilon(\theta+d\theta)} + \sqrt{\mu\epsilon}\ \Delta_2 \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} \Delta_1 &= r_1(\theta)\cos(\theta) - r_1(\theta+d\theta)\cos(\theta+d\theta) \\ \Delta_2 &= r_2(\theta)\cos(\theta) - r_2(\theta+d\theta)\cos(\theta+d\theta)\ . \end{aligned} \tag{2.4}$$

Some elementary calculus coupled with the fact that the $r_1(\theta)$ which describes the boundary between region I and II, and the $r_2(\theta)$ which describes the boundary between regions II and III,

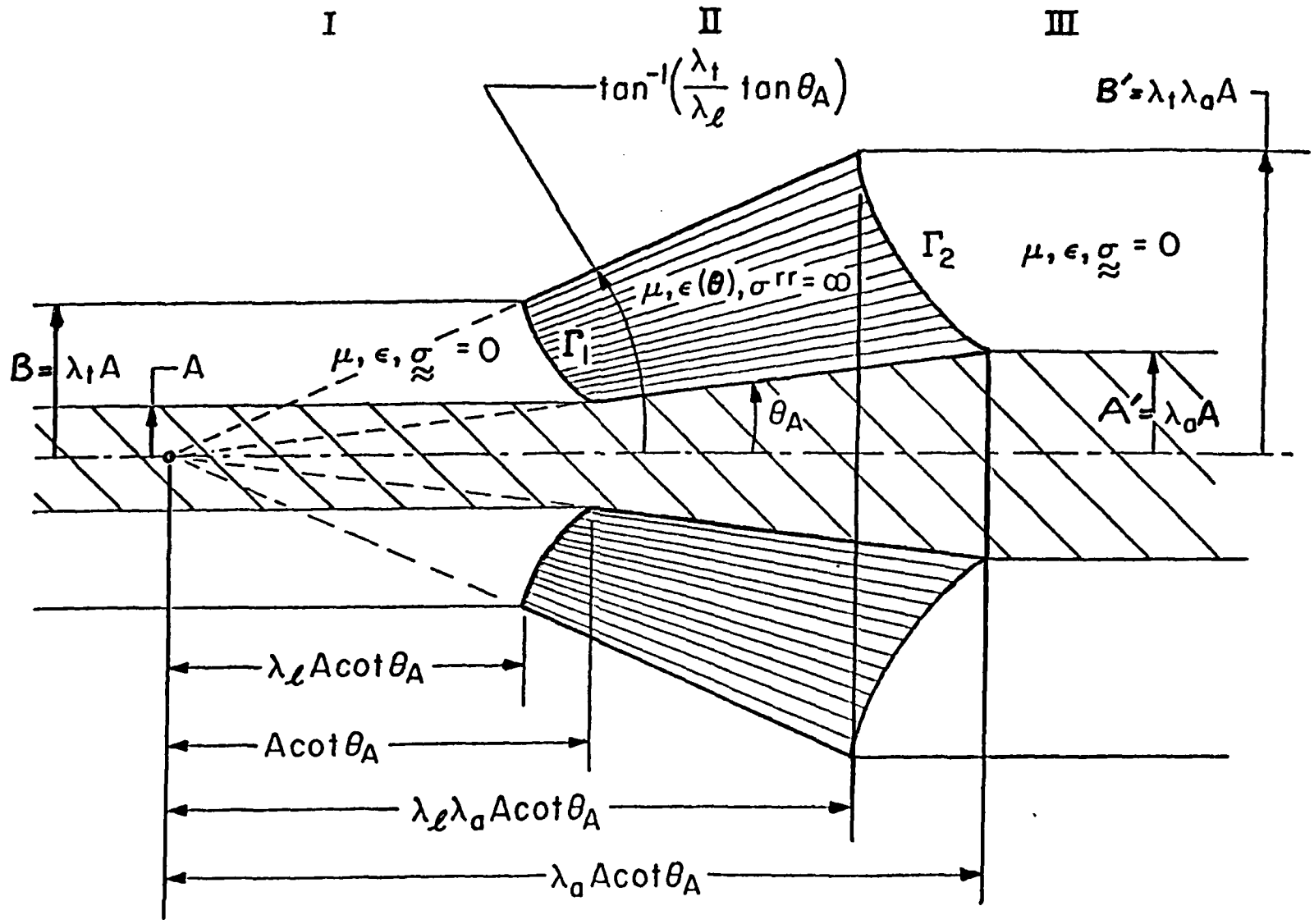


Figure 2.1: Coaxial Waveguide Geometry (SSN #169)

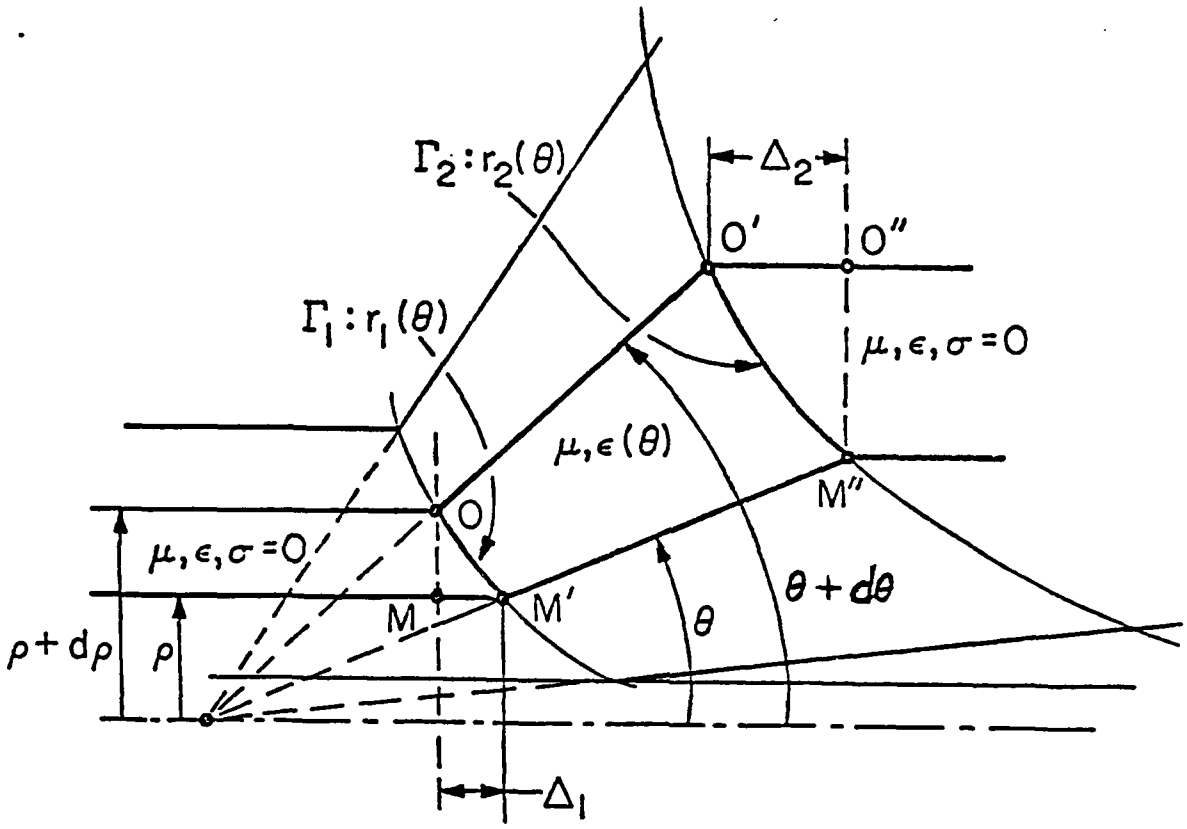


Figure 2.2: Travel Time Geometry (SSN #169)

are related by a constant factor, leads to a differential equation given by

$$r'(\theta)[cos(\theta) - \sqrt{\epsilon(\theta)/\epsilon}] - r(\theta)[sin(\theta) + \frac{(\epsilon(\theta)/\epsilon)'}{2\sqrt{\epsilon(\theta)/\epsilon}}] = 0 . \tag{2.5}$$

Similarly if we require impedances to be matched on the appropriate boundaries, then we are led to another differential equation through the infinitesimal matching on the boundaries between regions I and II and between regions II and III. Thus since the impedance for the TEM wave of a cylindrical coaxial waveguide is

$$Z_{cyl.} = \frac{V}{I} = \sqrt{\frac{\mu}{\epsilon}} \frac{ln(\rho_{outer}/\rho_{inner})}{2\pi} \tag{2.6}$$

and that for a conical coaxial waveguide is

$$Z_{con.} = \frac{V}{I} = \frac{\sqrt{\frac{\mu}{\epsilon}}}{2\pi} ln|\frac{\tan(\theta_{outer}/2)}{\tan(\theta_{inner}/2}| \tag{2.7}$$

one obtains for the infinitesimal matching on the boundary between regions I and II

$$\sqrt{\frac{\mu}{\epsilon}} ln(\frac{\rho+d\rho}{\rho}) = \sqrt{\frac{\mu}{\epsilon(\theta)}} ln\left|\frac{\tan(\frac{\theta+d\theta}{2})}{\tan(\frac{\theta}{2})}\right| . \tag{2.8}$$

Thus equation (2.8) leads to a differential equation

$$\frac{r'(\theta)}{r(\theta)} + \frac{1}{sin(\theta)}[cos(\theta) - \frac{1}{\sqrt{\epsilon(\theta)/\epsilon}}] = 0 \tag{2.9}$$

which is obtained by expanding the preceding equation and using $\rho = r\, sin(\theta)$. Both $r_1(\theta)$ which describes the boundary between regions I and II and $r_2(\theta)$ which describes the boundary between regions II and III will satisfy (2.9) since the ratio $r_2(\theta)/r_1(\theta)$ is a positive constant greater than 1 . We thus have a system of first order ordinary differential equations given by (2.5) and (2.9). We note that the choice in region II of $\epsilon(\theta)$ as a function of θ only is made because the TEM propagation is independent of ϕ and any dependence on r would give rise to a reflection. Solutions to the preceding system are obtained in [5] and have the form

$$r(\theta)sin(\theta) = c_2 \exp\left[\frac{-r(\theta)cos(\theta)}{c_1}\right] \tag{2.10}$$

and

$$\epsilon(\theta) = \epsilon\left[\frac{c_1+r(\theta)\cos(\theta)}{r(\theta)}\right]^2 \tag{2.11}$$

where c_1 and c_2 are determined by the geometry. Detailed plots of the boundary surfaces and $\epsilon(\theta)$ are included in [5]. In all of the above analysis only the geometry and medium were

considered in the impedance matching approach. That this matching obtained indeed matches a reflectionless and distortionless TEM wave from region I to III can be verified by a fields approach. One may consider, for example, TEM solutions to Maxwell's equations in regions I and II with appropriate boundary conditions and find an $\epsilon(\theta)$ which is consistent with that given by (2.11). A similar result is obtained by considering regions II and III, and hence the solutions to the system (2.5) and (2.9) do indeed give yield to perfect matchings. We note that this particular matching problem was also solved in [5] by the more general procedure of differential geometric scaling.

III. STATEMENT OF THE PROBLEM: TEM Waves on Circular Conical Systems

We consider a circular conical system as in figure 3.1. A source is located at a point P at a distance l below a ground plane AA', and a wave front launched at P is to propagate undistorted and unreflected through the lens region PRS into a conical region $QRSA$ with apex at 0. The basic problem is to determine both the shape of the lens and its constitutive parameters as well as the location of the source. The lens region is to be filled with a dielectric whose relative permittivity will be a function of a single variable θ'. This material can be transformer oil, whose relative permittivity is approximately 2.3, in which are immersed thin metal strips to achieve the design criteria. Thus this new problem is similar to that discussed in section 2 with the main difference being that we now have two regions instead of three.

The mathematical problem can be formulated as follows. We refer to figure 3.1 in which θ_b is the angle measured from the vertical axis to a point B in a conical region at distance r_b from the origin 0. The distance to B from point P is r_b' and the angle between the vertical axis and r_b' is θ_b'. Since the impedance for a TEM wave on a conical coaxial waveguide is

$$Z_c = \frac{V}{I} = \frac{Z_0}{2\pi} \ln\left|\frac{\tan(\frac{\theta_{outer}}{2})}{\tan(\frac{\theta_{inner}}{2})}\right| , \tag{3.1}$$

with $Z_0 = \sqrt{\frac{\mu}{\epsilon}}$, the infinitesimal matching on the boundary curve Γ becomes

$$\sqrt{\frac{\mu}{\epsilon}} \ln\left[\frac{\tan(\frac{\theta+d\theta}{2})}{\tan(\frac{\theta}{2})}\right] = \sqrt{\frac{\mu}{\epsilon(\theta')}} \ln\left[\frac{\tan(\frac{\theta'+d\theta'}{2})}{\tan(\frac{\theta'}{2})}\right] \tag{3.2}$$

where θ and θ' are taken on the boundary curve. To emphasize this point θ and θ' should be replaced by θ_b and θ_b', but we shall not do this in order to economize on notation. If both sides of the above equation are expanded, we obtain the differential equation, applicable on the boundary,

$$\frac{d\theta}{d\theta'} = \frac{1}{\sqrt{\epsilon_r(\theta')}} \frac{\sin(\theta)}{\sin(\theta')} , \tag{3.3}$$

where $\epsilon_r(\theta')$ denotes the relative permittivity (relative to ϵ).

Since the transit time to a sphere of radius r is to be conserved, we also have the expression $(r' \sqrt{\epsilon_r(\theta')} - r)$ independent of θ' (and also θ). Hence a second differential equation

$$\frac{d}{d\theta'}[r' \sqrt{\epsilon_r(\theta')} - r] = 0 \tag{3.4}$$

is obtained. On the boundary we can relate coordinates (see figure 3.2) as follows. Let

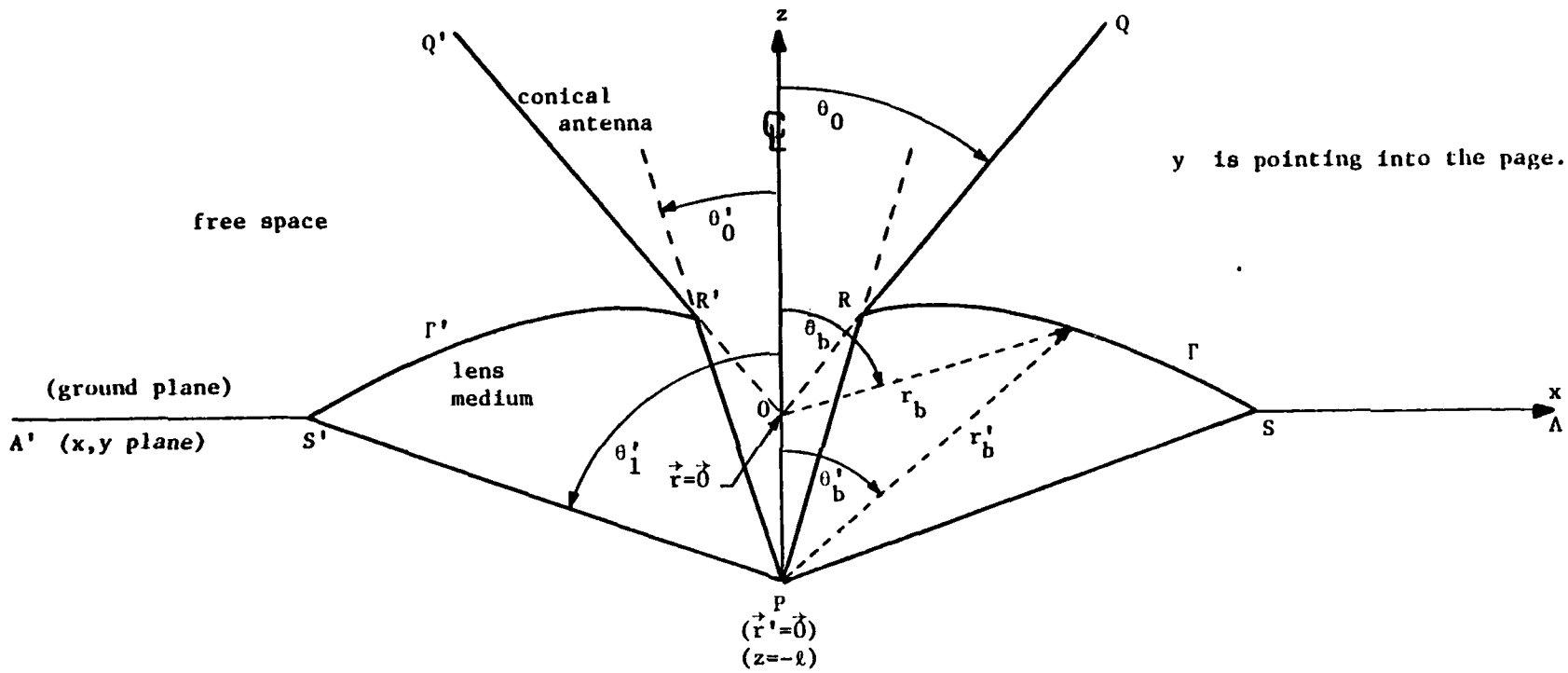


Figure 3.1. Geometry of Circular Conical Lens Feeding Circular Conical Antenna with Ground Plane.

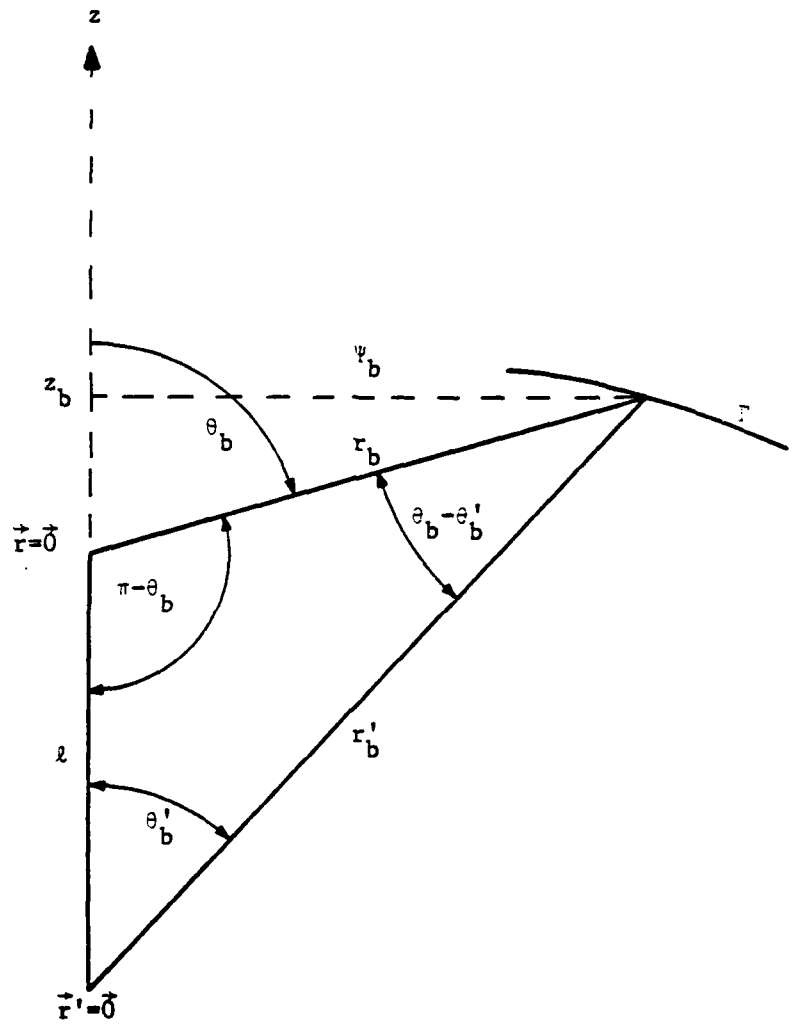


Figure 3.2. Relation of Coordinates on Lens Boundary Γ

$$\Psi = r\,sin(\theta) = r' \quad sin(\theta' \;)$$

$$z = r\,cos(\theta) = r' \quad cos(\theta' \;) - l\,. \tag{3.5}$$

The coordinates described by (3.5) are taken on the boundary curve Γ and so they should all appear at Ψ_b , z_b , etc., as they do in figure 3.2. Again, this is not done in the sequel in order to simplify the notation.

The law of sines then yields the equations

$$\frac{\sin(\theta-\theta' \;)}{l} = \frac{sin(\theta' \;)}{r} = \frac{\sin(\theta)}{r'} \tag{3.6}$$

from which we obtain

$$\frac{r}{l} = \frac{sin(\theta' \;)}{sin(\theta-\theta' \;)}$$

$$\frac{r'}{l} = \frac{\sin(\theta)}{sin(\theta-\theta' \;)}\,. \tag{3.7}$$

Note that the trigonometric equation

$$r^2 = (r' \;)^2 + l^2 - 2r' \; l\,cos(\theta' \;) \tag{3.8}$$

is just the law of cosines.

The ranges of θ and θ' will be $\theta_0 \leq \theta \leq \pi/2$ and $\theta_0' \leq \theta' \leq \theta_1'$, and we take $\epsilon_r = \epsilon_{r_0}$ at $\theta' = \theta_0'$ on the boundary. The value of ϵ_{r_0} should be a minimum of ϵ_r in the range of interest. Then for various choices of θ_0 our problem is to find θ_0' , θ_1' , $\frac{l}{r_0}$, θ in terms of θ' , and $\epsilon_r(\theta' \;)$. Thus in seeking solutions to the system of ordinary differential equations, obtained by infinitesimal impedance matching and transit time considerations,

$$\frac{d\theta}{d\theta'} = \frac{1}{\sqrt{\epsilon_r(\theta' \;)}}\,\frac{\sin(\theta)}{sin(\theta' \;)}$$

$$\frac{d}{d\theta'}[r' \;\sqrt{\epsilon_r(\theta' \;)} - r] = 0 \tag{3.9}$$

with geometric relations

$$\frac{sin(\theta-\theta')}{l} = \frac{sin(\theta')}{r} = \frac{\sin(\theta)}{r'}$$

$$r^2 = (r')^2 + l^2 - 2r'\ l\ cos(\theta') \tag{3.10}$$

certain initial data must be specified. Hence since the impedance Z_c of the conical line is given (see [3]) by the equation

$$Z_c = \frac{Z_0}{2\pi} ln(cot(\frac{\theta_0}{2})) \simeq 60\ ln(cot(\frac{\theta_0}{2})) \tag{3.11}$$

where $Z_0 = \sqrt{\frac{\mu}{\epsilon}} \simeq 377\Omega$ (in free space) the impedance is specified by choosing the angle θ_0. Thus if $\theta = \theta_0$, then $r = r_0$ and the value of r_0 is determined by the breakdown potential.

The initial angle θ_0' will be determined by the choice of ϵ_{r_0} if one imposes the condition

$$\frac{d}{d\theta'}\epsilon_r(\theta')|_{\theta'=\theta_0'} = 0 \tag{3.12}$$

While this condition is mathematically convenient, it has an engineering basis. As a practical matter one may wish to minimize the variation of $\epsilon_r(\theta')$ over the range of $\theta_0' \leq \theta' \leq \theta_1'$, thereby reducing the difficulty of the practical lens realization. To the extent that the ideal ϵ_r is uniform with respect to θ' one may consider approximating the ideal ϵ_r by a uniform ϵ_r in a practical realization; this situation is considered later in this paper. In particular for θ near θ_0 (and θ' near θ_0') the electric field is largest and electrical breakdown considerations are in general the most significant. The variations of ϵ_r with respect to θ' can be achieved with artificial - dielectric techniques, in particular by the presence of metallic conductors (ideally perfect conductors) which locally increase the average permittivity while being configured so as to not significantly reduce the local average permeability. Near $\theta' = \theta_0'$ an approximately uniform ϵ_r then as an approximation obviates the use of such artificial dielectric techniques (locally) in this region of maximum electric field in the free-space conical transmission line ($\theta_0 \leq \theta \leq \pi/2$). This approximately uniform ϵ_r near $\theta' = \theta_0'$ (or $\theta = \theta_0$, i.e., the inner boundary) also facilitates a physical interpretation in terms of the Brewster angle as is discussed in the next section.

As a result of this, a differential equation for $\epsilon_r(\theta')$ can be derived by substituting the trigonometric relations (3.10) into the second differential equation of the system (3.9) and carrying out the indicated differentiations. The result obtained is

$$\frac{d\epsilon_r}{d\theta'} = \frac{2}{sin(\theta-\theta')}[2\sqrt{\epsilon_r} - (1+\epsilon_r)cos(\theta-\theta')]\ , \tag{3.13}$$

and hence condition (3.12) yields the equation

$$\frac{2\sqrt{\epsilon_{r_0}}}{1+\epsilon_{r_0}} = cos(\theta_0 - \theta_0{}'\ ) \ , \tag{3.14}$$

which may be rewritten in the form

$$cos(\theta_0 - \theta_0{}'\ ) = sech[\frac{1}{2} ln(\epsilon_{r_0})] \tag{3.15}$$

Thus θ_0 , $\theta_0{}'$ and ϵ_{r_0} , are now known, and so the geometric relations (3.10) yield the values of $r_0{}'\ /r_0$ and l/r_0 . Hence, since the system (3.9) yields

$$r'\ \sqrt{\epsilon_r(\theta'\ )} - r = L \tag{3.16}$$

where L is a constant of integration, we obtain

$$L = r_0{}'\ \sqrt{\epsilon_{r_0}} - r_0 \ . \tag{3.17}$$

If we substitute (3.10) into (3.17) we find that

$$L = r_0[\frac{\sqrt{\epsilon_{r_0}} sin(\theta_0) - sin(\theta_0{}'\ )}{sin(\theta_0{}'\ )}]$$

and in terms of l we find that

$$\frac{L}{l} = [\sqrt{\epsilon_{r_0}} sin(\theta_0) - sin(\theta_0{}'\ )]/sin(\theta_0 - \theta_0{}'\ ) \ . \tag{3.18}$$

Since the quantity (L/l) appears in several places in our analysis, it is convenient to also express L/l in terms of the impedance Z_c given in (3.11). If (3.11) is rewritten as

$$tan(\frac{\theta_0}{2}) = exp\left(\frac{-2\pi Z_c}{Z_0}\right) \tag{3.19}$$

one finds, using standard trigonometric and hyperbolic relations, that

$$\begin{aligned} cos(\theta_0) &= tanh(\frac{2\pi Z_c}{Z_0}) \\ sin(\theta_0) &= sech\ (\frac{2\pi Z_c}{Z_0}) \ . \end{aligned} \tag{3.20}$$

One then finds from (3.14) coupled with some trigonometric relations that

$$cos(\theta_0{}' \ ) = \frac{1}{1+\epsilon_{r_0}}[(\epsilon_{r_0}-1)sech(\frac{2\pi Z_c}{Z_0}) + 2\sqrt{\epsilon_{r_0}}tanh\ (\frac{2\pi Z_c}{Z_0})]$$

$$sin(\theta_0{}' \ ) = \frac{1}{1+\epsilon_{r_0}}[(1-\epsilon_{r_0})\tanh(\frac{2\pi Z_c}{Z_0}) + 2\sqrt{\epsilon_{r_0}}\ sech\ (\frac{2\pi Z_c}{Z_0})] \tag{3.21}$$

Note that since $\theta_0{}' > 0$, the range of values for Z_c must necessarily be restricted. If the results expressed by (3.20) and (3.21) are substituted into (3.18), we find then that

$$\frac{L}{l} = \sqrt{\epsilon_{r_0}}sech(\frac{2\pi Z_c}{Z_0}) + tanh(\frac{2\pi Z_c}{Z_0}) \ . \tag{3.22}$$

Thus our lens problem has been formulated from infinitesimal impedance matching and travel time considerations with geometric conditions given by the differential equations (3.9) and trigonometric relations (3.10) as

$$\frac{d\theta}{d\theta'} = \frac{1}{\sqrt{\epsilon_r(\theta' \ )}}\ \frac{sin(\theta)}{sin(\theta' \ )} \tag{3.23}$$

$$\frac{d}{d\theta'}[r' \ \sqrt{\epsilon_r(\theta' \ )} - r] = 0 \tag{3.24}$$

$$\frac{sin(\theta-\theta' \ )}{l} = \frac{sin(\theta' \ )}{r} = \frac{sin(\theta)}{r'} \tag{3.25}$$

and

$$r^2 = (r' \ )^2 + l^2 - 2r' \ lcos(\theta' \ ) \ . \tag{3.26}$$

An analytic solution to the above system will be obtained in section 5.

IV. BREWSTER-ANGLE CONSIDERATIONS

One of the implications of (3.12) is that $\epsilon_r(\theta')$ is approximately uniform (or better asymptotically uniform) near $\theta' = \theta_0'$, the inner conical boundary ($\theta' = \theta_0'$ in the lens, $\theta = \theta_0$ outside it). With this insight let us consider what can be locally approximated as a plane wave propagating in a uniform dielectric medium. Referring back to the lens geometry in figure 3.1 *let us expand the picture near the meeting point of the lens/free-space boundary* Γ with the perfectly conducting circular conical boundaries of free space ($\theta = \theta_0$) and the lens ($\theta' = \theta_0'$). Figure 4.1 gives a picture showing a ray passing through Γ. Along this ray a TEM wave propagates with a TM polarization, i.e., the magnetic field is parallel to boundaries discussed above. In this polarization there exists the well-known Brewster angle ψ_B at which the wave passes through Γ with no distortion [6] given by

$$tan(\psi_B) = \epsilon_{r_0}^{-1/2} \tag{4.1}$$

where ψ_B is measured between Γ and the ray on the free-space side. On the lens side similarly ψ_B' is given by

$$tan(\psi_B') = \epsilon_{r_0}^{1/2} \tag{4.2}$$

which exhibits the fact that in going from one medium to the other, and then reversing ones path, the ratio of one dielectric constant to the other is inverted on path reversal, a rather symmetrical situation. Hence we have

$$\begin{aligned} tan(\psi_B)tan(\psi_B') &= 1 \\ \psi_B + \psi_B' &= \frac{\pi}{2} \end{aligned} \tag{4.3}$$

which is easily found by geometrical construction with a right triangle.

Referring to figure 4.1, consider the extension of the boundary $\theta' = \theta_0'$ which makes an angle of $\theta_0 - \theta_0'$ with respect to the boundary $\theta = \theta_0$. Furthermore, the extension of the incident ray direction makes an angle of $\psi_B' - \psi_B$ with respect to the transmitted ray direction. Since the two boundaries (on a plane of constant ϕ) are parallel to the local ray directions in the respective media, we conclude that the free-space boundary $\theta = \theta_0$ makes an angle of ψ_B with respect to Γ and that

$$\begin{aligned} \theta_0 - \theta_0' &= \psi_B' - \psi_B \\ &= arctan(\epsilon_{r_0}^{1/2}) - arctan(\epsilon_{r_0}^{-1/2}) \\ &= arctan(\frac{1}{2}[\epsilon_{r_0}^{1/2} - \epsilon_{r_0}^{-1/2}]) \\ &= function\ only\ of\ \epsilon_{r_0} \end{aligned} \tag{4.4}$$

using standard relations for trigonometric functions.

This immediately implies a practical limit on our lens geometry. Define

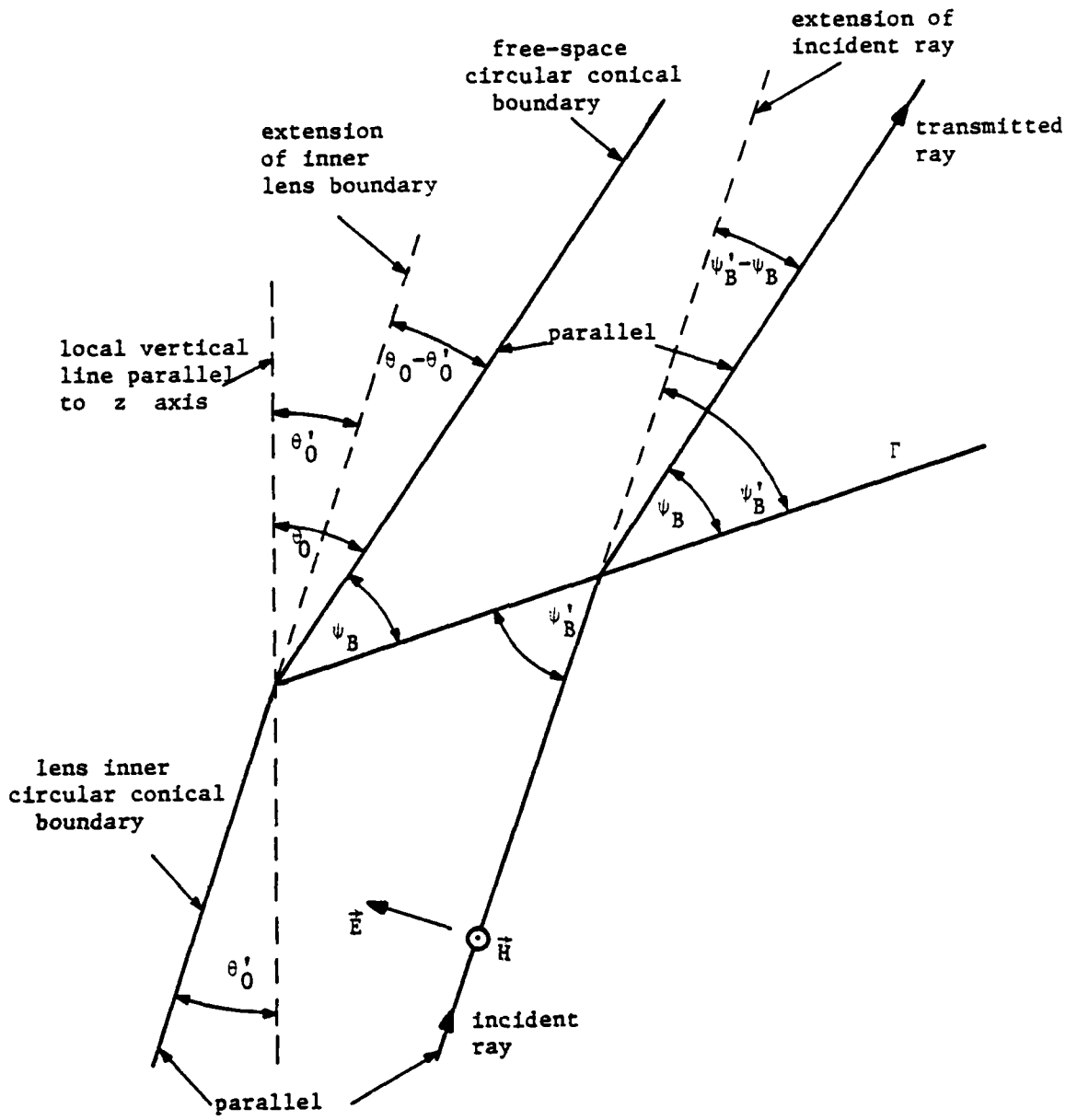


Figure 4.1. Brewster Angle Near Meeting of Three Boundaries

$$\theta_{0_{min}} \equiv arctan(\frac{1}{2}[\epsilon_{r_0}^{1/2} - \epsilon_{r_0}^{-1/2}]) \,. \tag{4.5}$$

Now, θ_0' is geometrically constrained as

$$\theta_0' > 0 \tag{4.6}$$

since for $\theta_0' \rightarrow 0+$ then $l \rightarrow \infty$ and negative θ_0' violates the Brewster angle condition near the boundary junction in Figure 4.1. Thus $\theta_{0_{min}}$ actually represents the minimum allowable value of θ_0, and fixing the value of θ_1 as $\pi/2$ the maximum value of the characteristic impedance of our conical system is then

$$Z_{c_{max}} = \frac{Z_0}{2\pi} \, ln(cot(\frac{\theta_{0_{min}}}{2})) \tag{4.7}$$

We thus have $Z_{c_{max}}$ expressed in geometric terms (i.e., in terms of $\theta_{0_{min}}$). It is also possible to express $Z_{c_{max}}$ in physical terms (i.e., in terms of ϵ_{r_0}) as follows. Since the identities

$$cot\left(\frac{x}{2}\right) = \frac{1+cos(x)}{sin(x)}$$

$$arctan(y) = arcsin\left(\frac{y}{\sqrt{1+y^2}}\right) = arccos\left(\frac{1}{\sqrt{1+y^2}}\right) \tag{4.8}$$

hold for appropriate x and y, we find that

$$Z_{c_{max}} = \frac{Z_0}{2\pi} \, ln\left(\frac{1+cos(\theta_{0_{min}})}{sin(\theta_{0_{min}})}\right) \tag{4.9}$$

where

$$\begin{aligned} \theta_{0_{min}} &= arctan \, \frac{1}{2}(\epsilon_{r_0}^{1/2} - \epsilon_{r_0}^{-1/2}) \\ &= arcsin\left(\frac{\epsilon_{r_0}^{1/2} - \epsilon_{r_0}^{-1/2}}{\epsilon_{r_0}^{1/2} + \epsilon_{r_0}^{-1/2}}\right) \\ &= arccos\left(\frac{2}{\epsilon_{r_0}^{1/2} + \epsilon_{r_0}^{-1/2}}\right) \end{aligned} \tag{4.10}$$

and hence

$$Z_{c_{max}} = \frac{Z_0}{2\pi} \ln\left[\frac{\epsilon_{r_0}^{1/2}+1}{\epsilon_{r_0}^{1/2}-1}\right] \tag{4.11}$$

The results obtained in (4.10) and (4.11) are plotted in figures 4.2 and 4.3. These figures give plots of minimum cone angle, $\theta_{0_{min}}$, and maximum cone impedance, $Z_{c_{max}}$, as a function of initial relative dielectric constant. Furthermore, we have the restricted ranges for θ and θ' as

$$\begin{aligned} \theta_0 - \theta_0' &= \theta_{0_{min}} \\ \theta_{0_{min}} \leq \theta &\leq \frac{\pi}{2} \\ 0 \leq \theta' &\leq \frac{\pi}{2} - \theta_{0_{min}} \end{aligned} \tag{4.12}$$

which are important for later graphs and tables.

These results also establish l. From figure 3.2 with $\theta = \theta_0$, $\theta' = \theta_0'$, we have from the law of sines

$$\frac{\sin(\theta_0')}{r_0} = \frac{\sin(\theta_0-\theta_0')}{l} . \tag{4.13}$$

Since $\theta_0' = \theta_0 - \theta_{0_{min}}$, we thus have

$$\frac{\sin(\theta_0-\theta_{0_{min}})}{r_0} = \frac{\sin(\theta_{0_{min}})}{l}$$

and so we may solve for $\frac{r_0}{l}$ and obtain

$$\begin{aligned} \frac{r_0}{l} &= \frac{\sin(\theta_0-\theta_{0_{min}})}{\sin(\theta_{0_{min}})} \\ &= \frac{\sin(\theta_0)\cos(\theta_{0_{min}}) - \cos(\theta_0)\sin(\theta_{0_{min}})}{\sin(\theta_{0_{min}})} \\ &= \sin(\theta_0)\cot(\theta_{0_{min}}) - \cos(\theta_0) , \end{aligned}$$

and hence

$$\frac{r_0}{l} = \frac{2}{\epsilon_{r_0}^{1/2}-\epsilon_{r_0}^{-1/2}} \sin(\theta_0) - \cos(\theta_0) . \tag{4.14}$$

Since θ_0 may also be expressed in terms of Z_c from (3.20), we also obtain

$$\frac{r_0}{l} = \frac{2}{\epsilon_{r_0}^{1/2}-\epsilon_{r_0}^{-1/2}} \operatorname{sech}\left(\frac{2\pi Z_c}{Z_0}\right) - \tanh\left(\frac{2\pi Z_c}{Z_0}\right) . \tag{4.15}$$

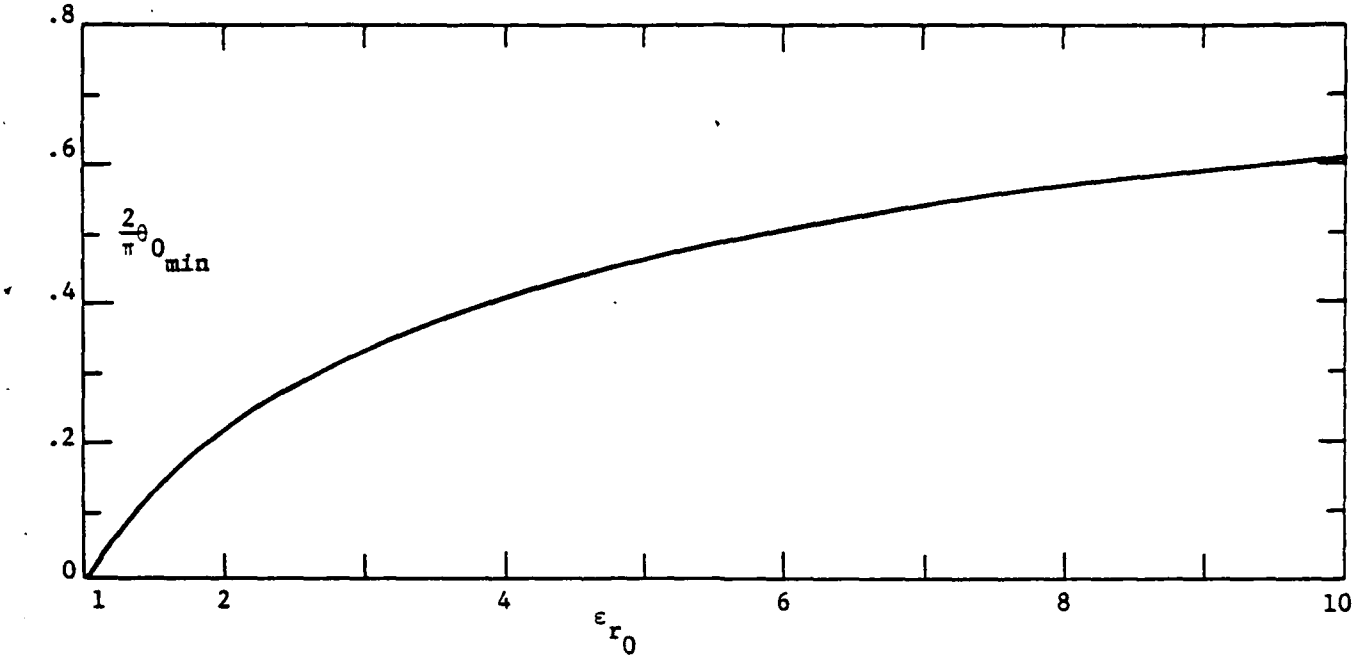


Figure 4.2. Minimum Cone Angle for Given Initial Dielectric Constant

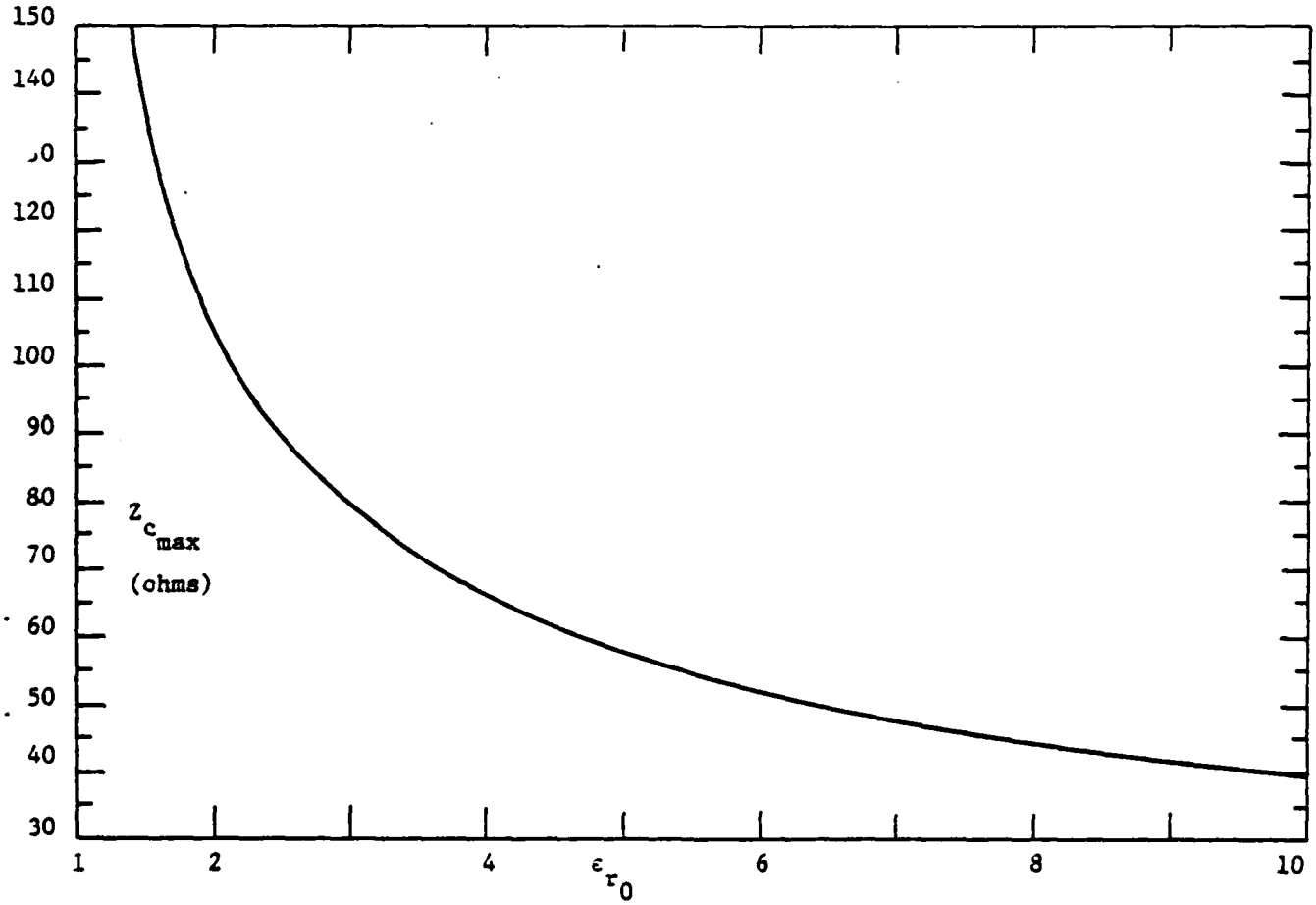


Figure 4.3. Maximum Cone Impedance for Given Initial Relative Dielectric Constant

Figure 4.4 thus gives plots of l/r_0 as a function of θ_0 (and also Z_c) with ϵ_{r_0} as a parameter. The corresponding data appears in table 5.2.

Note that in limiting cases, $\frac{l}{r_0} \to 0$ and $Z_{c_{max}} \to \infty$ as $\epsilon_{r_0} \to 1$ (for fixed $\theta_0 > \theta_{0_{min}}$) , while

$$Z_{c_{max}} = \frac{Z_0}{2\pi} \left[ln(1+\epsilon_{r_0}^{-1/2}) - ln(1-\epsilon_{r_0}^{-1/2})\right]$$

and hence

$$Z_{c_{max}} = \frac{Z_0}{\pi} \epsilon_{r_0}^{-1/2}[1 + O(\epsilon_{r_0}^{-1})] \ , \ as \ \ \epsilon_{r_0} \to \infty \ . \tag{4.16}$$

Note also that if ϵ_{r_0} is fixed and greater than 1 , then $l/r_0 \to \infty$ as $\theta_0 \to \theta_{0_{min}}$. These results indicate that in general the design of lenses will be facilitated by having ϵ_{r_0} near 1 and θ_0 large compared to $\theta_{0_{min}}$. These issues will be addressed in the next section.

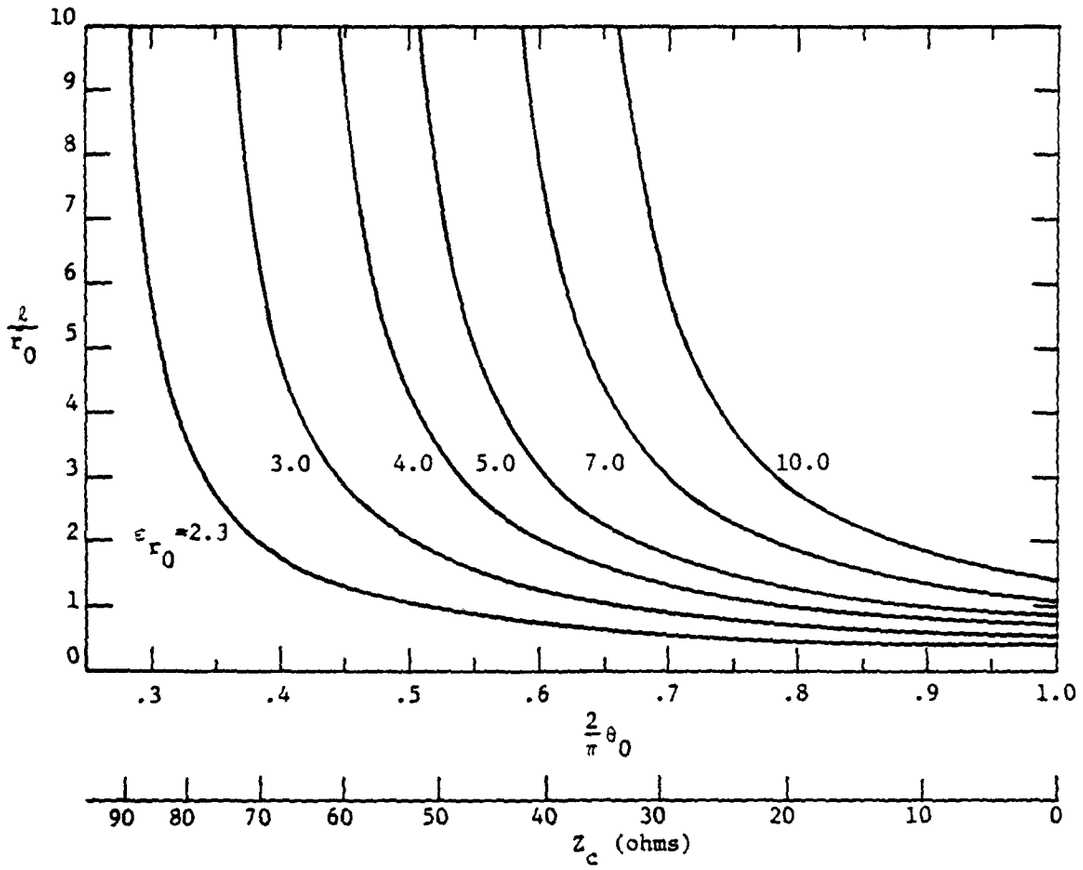


Figure 4.4. Apex of Conical Lens as a Function of Cone Angle with Initial Relative Dielectric Constant as a Parameter

V. DETERMINATION OF LENS PROFILE

A. *General equations for θ' and $\epsilon_r(\theta')$:*

In Section 3, our impedance matching approach led to a system of ordinary differential equations (3.23) and (3.24) and geometrical relations (3.25) and (3.26). Since the second equation (3.24) in this system can be immediately integrated to yield

$$r' \sqrt{\epsilon_r(\theta')} - r = L \tag{5.1}$$

where the constant L satisfies

$$L = r_0' \sqrt{\epsilon_{r_0}} - r_0 \tag{5.2}$$

we can solve (5.1) for $\sqrt{\epsilon_r(\theta')}$ and obtain

$$\sqrt{\epsilon_r'(\theta)} = (L+r)/r' \tag{5.3}$$

Thus, if (3.25) is used, we find

$$\sqrt{\epsilon_r'(\theta)} = \frac{L\,sin(\theta-\theta')+l\,sin(\theta')}{l\,sin(\theta)} \tag{5.4}$$

and if this result is substituted into differential equation (3.23), then we obtain

$$\frac{d\theta}{d\theta'} = \frac{l\,sin^2(\theta)}{L\,sin(\theta-\theta') + l\,sin(\theta')}\,\frac{1}{sin(\theta')} \tag{5.5}$$

The constants l and L are known if we specify θ_0 (or Z_c), r_0 , and ϵ_{r_0} as discussed in sections 3 and 4. Since θ_0 and θ_0' are given, the ordinary differential equation (5.5) will then have a unique solution for $\theta_0' \leq \theta' \leq \theta_1' < \frac{\pi}{2}$ where θ_1' is determined by the restriction that $\theta_1 = \frac{\pi}{2}$. Thus since θ' is known as a function of θ , the lens design problem has a solution since $\epsilon_r(\theta')$ is known explicitly through (5.4).

We now proceed with the construction of an exact solution to (5.5). If we make the substitution

$$e^{-T} = tan(\frac{\theta}{2}) \tag{5.6}$$

and use the resulting relations

$$sin(\theta) = \frac{2e^{-T}}{1 + e^{-2T}} = sech(T) \tag{5.7}$$

$$cos(\theta) = \frac{1 - e^{-2T}}{1 + e^{-2T}} = \frac{sinh(T)}{cosh(T)} \tag{5.8}$$

in (5.5), then we obtain a new differential equation namely,

$$\frac{dT}{dT'} = \frac{l\cosh(T')}{l\cosh(T) + L(\sinh(T') - \sinh(T))} . \tag{5.9}$$

Actually T represents the well known spherical TEM wave potential function with boundary values

$$T(\frac{\pi}{2}) = 0 \tag{5.10}$$

$$T(\theta_0) = ln(cot(\frac{\theta_0}{2})) . \tag{5.11}$$

Since we have from (3.22)

$$\frac{L}{l} = \sqrt{\epsilon_{r_0}}\, sech(\frac{2\pi Z_c}{Z_0}) + tanh(\frac{2\pi Z_c}{Z_0}) , \tag{5.12}$$

we can restrict Z_c so that $\frac{L}{l} \geq 1$ and hence rewrite (5.9) as

$$\frac{dT}{dT'} = \frac{l\cosh(T')}{L\sinh(T') - \sqrt{L^2-l^2}\sinh(T-a)} \tag{5.13}$$

where $a = arccosh(\frac{L}{\sqrt{L^2-l^2}})$. We then transform (5.12) into a linear, inhomogeneous differential equation by the substitutions

$$\begin{aligned} x &= \sinh(T-a) \\ y &= \sinh(T') . \end{aligned} \tag{5.14}$$

The resulting equation is then

$$\frac{dy}{dx} - \frac{L}{l}\frac{y}{\sqrt{1+x^2}} = -\frac{\sqrt{L^2-l^2}}{l}\frac{x}{\sqrt{1+x^2}} \tag{5.15}$$

which can also be written in the form

$$\frac{d}{dx}[y(x+\sqrt{1+x^2})^\alpha] = -\frac{\sqrt{L^2-l^2}}{l}\frac{x}{\sqrt{1+x^2}}[x+\sqrt{1+x^2}]^\alpha \tag{5.16}$$

where $\alpha = -L/l$. The integral of the right hand side of (5.16) reduces to an obvious standard form by a substitution $x = \sinh(u)$, with the result that the solution to (5.5) is

$$cot(\theta') = cot(\theta) + [cot(\theta_0') - cot(\theta_0)]\left[\frac{tan(\frac{\theta}{2})}{tan(\frac{\theta_0}{2})}\right]^{\alpha} \tag{5.17}$$

where the initial condition $\theta'(\theta_0) = \theta_0'$ has been incorporated. We may then reexpress (5.17) in a form in which ϵ_{r_0} and Z_c appear instead of θ_0 and θ_0'. This is done by substituting (3.19) through (3.21) into (5.17). Our solution then can be written as

$$cot(\theta') = cot(\theta) + \left\{\frac{(\epsilon_{r_0}-1)[(exp(\frac{2\pi Z_c}{Z_0}))tan(\frac{\theta}{2})]^{\alpha}}{(sech(\frac{2\pi Z_c}{Z_0}))[(1-\epsilon_{r_0})tanh(\frac{2\pi Z_c}{Z_0}) + 2\sqrt{\epsilon_{r_0}}sech(\frac{2\pi Z_c}{Z_0})]}\right\} \tag{5.18}$$

where $\alpha = -\frac{L}{l}$. Note that the constant α, which is expressed in terms of ϵ_{r_0}, θ_0, and θ_0' in (3.18), may be reexpressed in terms of ϵ_{r_0} and Z_c by using (3.20) and (3.21) with the result that

$$\alpha = -[\sqrt{\epsilon_{r_0}}sech(\frac{2\pi Z_c}{Z_0}) + tanh(\frac{2\pi Z_c}{Z_0})] . \tag{5.19}$$

Once θ' is known as a function of θ via (5.18), the relative permittivity $\epsilon_r(\theta')$ is also known through (5.4). Thus if the right hand side of (5.4) is expanded, one obtains

$$\sqrt{\epsilon_r(\theta')} = \frac{L}{l} sin(\theta')[cot(\theta') - cot(\theta)] + \frac{sin(\theta')}{sin(\theta)} \tag{5.20}$$

Thus (5.18) and (5.20) constitute an exact solution to our problem specified by (3.23) through (3.26).

The special case where $\theta = \theta_1 = \frac{\pi}{2}$ may be substituted into (5.18) and (5.20), with the result that

$$cot(\theta_1') = \frac{(\epsilon_{r_0}-1)\{\exp(\frac{2\pi Z_c}{Z_0})\alpha\}}{(sech(\frac{2\pi Z_c}{Z_0}))\{(1-\epsilon_{r_0})tanh(\frac{2\pi Z_c}{Z_0})+2\sqrt{\epsilon_{r_0}}sech(\frac{2\pi Z_c}{Z_0})\}} , \tag{5.21}$$

and

$$\sqrt{\epsilon_r(\theta_1')} = \{\sqrt{\epsilon_{r_0}}sech(\frac{2\pi Z_c}{Z_0}) + tanh(\frac{2\pi Z_c}{Z_0})\}cos(\theta_1') + sin(\theta_1') . \tag{5.22}$$

The preceding equations (5.21) and (5.22) will be useful in determining the lens profile in our subsequent numerical work.

B. *Constraints on Z_c and ϵ_r .*

In sections 3 and 4 it was noted that the condition (3.12), i.e.,

$$\frac{d\epsilon_r(\theta')}{d\theta'}\Big|_{\theta'=\theta_0'} = 0 \tag{5.23}$$

enabled us to specify θ_0' by specifying ϵ_{r_0} . Since $\epsilon_r(\theta')$, given by equation (5.4), is

$$\sqrt{\epsilon_r(\theta')} = \frac{L\,sin(\theta-\theta') + l\,sin(\theta')}{l\,sin(\theta)} \tag{5.24}$$

condition (5.23) implies $\epsilon_r(\theta')$ will be an increasing function of θ' for $\theta_0' \leq \theta' \leq \theta_1'$ provided we have

$$\frac{d\epsilon_r(\theta')}{d\theta'} \geq 0 \tag{5.25}$$

in the range $\theta_0' \leq \theta' \leq \theta_1'$. In view of differential equation (3.13), i.e.,

$$\frac{d\epsilon_r}{d\theta'} = \frac{2}{sin(\theta-\theta')}[2\sqrt{\epsilon_r}-(1+\epsilon_r)cos(\theta-\theta')] , \tag{5.26}$$

the condition (5.25) means that we must have

$$\frac{2\sqrt{\epsilon_r}}{1+\epsilon_r} \geq cos(\theta-\theta') \tag{5.27}$$

in the range of interest. The latter inequality however will in general not hold throughout this range, and consequently we impose a less restrictive condition, namely that

$$\epsilon_r(\theta') \geq \epsilon_{r_0} \tag{5.28}$$

for $\theta_0' \leq \theta' \leq \theta_1'$. This condition can be realized, for a given choice of ϵ_{r_0} , by restricting the range of the impedance Z_c so that

$$Z_{c_{min}} \leq Z_c \leq Z_{c_{max}} \tag{5.29}$$

where the upper and lower limits, Z_{c_1} and Z_{c_0} , are determined by the following considerations.

From the formulas for Z_c and $cos(\theta_0-\theta_0')$, given in (3.11) and (3.14), we note that θ_0 decreases as Z_c increases and that $(\theta_0-\theta_0')$ increases as ϵ_{r_0} increases. The geometry of our problem dictates that θ_0 , θ_0' , and $(\theta_0-\theta_0')$ be positive. Hence, for fixed ϵ_{r_0} , the quantity $(1-\epsilon_{r_0})\tanh(\frac{2\pi Z_c}{Z_0}) + 2\sqrt{\epsilon_{r_0}}sech(\frac{2\pi Z_c}{Z_0})$, which appears in the formulas for $sin(\theta_0')$ and

$cot(\theta'\ )$, (see (3.21) and (5.18)) should be non negative. Therefore we should require

$$sinh(\frac{2\pi Z_c}{Z_0}) \leq \frac{2\sqrt{\epsilon_{r_0}}}{(\epsilon_{r_0}-1)} . \tag{5.30}$$

If the inequality above is replaced by equality, then the resulting condition

$$sinh(\frac{2\pi Z_c}{Z_0}) = \frac{2\sqrt{\epsilon_{r_0}}}{\epsilon_{r_0}-1} \tag{5.31}$$

is exactly the condition that $cos(\theta_0'\ )$ be a maximum (i.e., $cos\ \theta_0' = 1$) for fixed ϵ_{r_0}. If we denote by $Z_{c_{max}}$ the value of Z_c which satisfies (5.31) for fixed ϵ_{r_0} , then we find that

$$tanh(\frac{2\pi Z_{c_{max}}}{Z_0}) = \frac{2\sqrt{\epsilon_{r_0}}}{1+\epsilon_{r_0}} = cos(\theta_0) \tag{5.32}$$

which is in agreement with the results based on Brewster-angle considerations in section 4. Hence, from (3.11), we obtain the result that

$$Z_{c_{max}} = \frac{Z_0}{2\pi}\ ln\left[\frac{\epsilon_{r_0}^{1/2}\ +\ 1}{\epsilon_{r_0}^{1/2}-1}\right] . \tag{5.33}$$

The value, $Z_{c_{max}}$, given by (5.33) is an upper bound for Z_c for a fixed choice of ϵ_{r_0}. Note that since

$$\frac{\epsilon_{r_0}^{1/2}+1}{\epsilon_{r_0}^{1/2}-1} = \frac{1+cos(\theta_{0_{min}})}{sin(\theta_{0_{min}})} = cot(\frac{\theta_{0_{min}}}{2})$$

we also obtain the formula

$$Z_{c_{max}} = \frac{Z_0}{2\pi}\ ln\left[cot(\frac{\theta_{0_{min}}}{2})\right] \tag{5.34}$$

as in (4.7).

To determine a lower bound $Z_{c_{min}}$ for Z_c , we refer to (5.21) and (5.22) which give $cot\ (\theta_1'\ )$ and $\sqrt{\epsilon_r(\theta_1'\ )}$ where θ_1' is the angle corresponding to $\theta_1 = \pi/2$. Since we want to satisfy the condition that $\epsilon_{r_1} \geq \epsilon_{r_0}$, we seek solutions Z_c to the transcendental equation

$$\sqrt{\epsilon_{r_1}} = \sqrt{\epsilon_{r_0}} = [\sqrt{\epsilon_{r_0}}sech(\frac{2\pi Z_c}{Z_0}) + tanh(\frac{2\pi Z_c}{Z_0})]cos\ (\theta_1'\ ) + sin\ (\theta_1'\ ) \tag{5.35}$$

where θ_1' is given by (5.21). Clearly $Z_c = Z_{c_{max}}$ will be one of these solutions, but there will be other solutions as well. If we set

$$a = (\epsilon_{r_0}-1)(cosh(\frac{2\pi Z_c}{Z_0}))\{\exp\left[\frac{2\pi Z_c}{Z_0}\right]\alpha\}$$

and

$$b = (1-\epsilon_{r_0})tanh(\frac{2\pi Z_c}{Z_0}) + 2\sqrt{\epsilon_{r_0}}sech(\frac{2\pi Z_c}{Z_0})$$

so that $cot(\theta') = a/b$ and rewrite (5.35) in the form

$$\sqrt{\epsilon_{r_0}}\sqrt{a^2+b^2} = [\sqrt{\epsilon_{r_0}}sech(\frac{2\pi Z_c}{Z_0}) + tanh(\frac{2\pi Z_c}{Z_0})]a + b \ , \tag{5.36}$$

then the first solution Z_c strictly less than $Z_{c_{max}}$ we will denote by $Z_{c_{min}}$. Thus we may obtain a range of Z_c, for fixed ϵ_{r_0}, for which the inequality

$$Z_{c_{min}} \leq Z_c \leq Z_{c_{max}} \tag{5.37}$$

is satisfied. Numerical values of $Z_{c_{min}}$ and $Z_{c_{max}}$, obtained by computer calculations, are given in table 5.1. We note that for this range of Z_c the constant L/l satisfies the condition $\frac{L}{l} \geq 1$ which is implicit in the result finally obtained in (5.18). Values of L/l for the range $Z_{c_{min}} \leq Z_c \leq Z_{c_{max}}$ with ϵ_{r_0} as a parameter appear in table 5.2.

C. *Numerical Results*

In this section we present numerical results based on the analytical results of section 5A. Equation (5.18) gives θ' as a function of θ with parameters ϵ_{r_0} and Z_c. Thus if we take $\epsilon_{r_0} = 2.3$ and take Z_c as a parameter with values ranging between 58Ω and 95Ω, then we obtain graphs of θ' versus θ as in figure 5.1. In this figure, as well as other figures, the scales for θ and θ' have been normalized by a factor of $\frac{2}{\pi}$. The corresponding numerical data appears in table 5.3. The range of impedances chosen corresponds to the inequality (5.37). For the same value of ϵ_{r_0}, namely 2.3, and the same range of impedances, graphs of ϵ_r versus θ' are given in figure 5.2 with the corresponding numerical data in table 5.4. These results are obtained from (5.20) which gives ϵ_r as a function of θ'.

If one takes $\theta_1 = \pi/2$, then the corresponding values of θ_1' and ϵ_r are given in (5.21) and (5.22). This value of ϵ_r is denoted by ϵ_{r_1}. Thus if we choose a particular ϵ_{r_0}, then θ_1' and ϵ_{r_1} are functions of the impedance Z_c, whose range is restricted by the choice of ϵ_{r_0}, and we may obtain plots of ϵ_{r_1} versus impedance with ϵ_{r_0} as parameter. These graphs appear in figure 5.3 with values of ϵ_{r_0} given by 2.3, 3.0, 4.0, 5.0, 7.0, and 10.0. The corresponding numerical data is in table 5.5.

Next, the determination of the curved boundary of the lens is made in the following analysis. In figure 3.2, we let z denote the height above the ground plane and Ψ the cylindrical radius. Thus we have

$$z = r\cos(\theta) = r'\ (\cos(\theta'\ ) - l$$

$$\Psi = r\sin(\theta) = r'\ \sin(\theta'\ ) \tag{5.38}$$

and we seek an expression for z in terms of Ψ and the constants ϵ_{r_0} and Z_c . To obtain this, note that the trigonometric relations (3.6) when substituted into $\Psi = r\sin(\theta)$ yield

$$\Psi = l\,\frac{\sin(\theta'\ )\sin(\theta)}{\sin(\theta-\theta'\ )} \tag{5.39}$$

and so

$$\frac{l}{\Psi} = \frac{\sin(\theta-\theta'\ )}{\sin(\theta)\sin(\theta'\ )} \tag{5.40}$$

Since we may rewrite (5.16) in the form

$$\frac{\sin(\theta-\theta'\ )}{\sin(\theta)\sin(\theta'\ )} = \frac{\sin(\theta_0-\theta_0'\ )}{\sin(\theta_0)\sin(\theta_0'\ )}\left[\frac{\tan(\frac{\theta_0}{2})}{\tan(\frac{\theta}{2})}\right]^{L/l} \tag{5.41}$$

we obtain the equation

$$\left[\tan(\frac{\theta}{2})\right] = (\frac{\Psi}{l})^{l/L}\left(\frac{\sin(\theta_0-\theta_0'\ )}{\sin(\theta_0)\sin(\theta_0'\ )}\right)^{l/L}\left[\tan(\frac{\theta_0}{2})\right] \tag{5.42}$$

by using (5.39) and (5.40) and solving for $\tan(\frac{\theta}{2})$. Since

$$\tan(\theta) = \frac{2\tan(\frac{\theta}{2})}{1-\tan^2(\frac{\theta}{2})} \tag{5.43}$$

is valid when $0 \leq \theta < \frac{\pi}{2}$, we find

$$\tan(\theta) = \frac{2x}{1-x^2} \tag{5.44}$$

where

$$x \equiv \left[\frac{\Psi}{l}\ \frac{\sin(\theta_0-\theta_0'\ )}{\sin(\theta_0)\sin(\theta_0'\ )}\right]^{l/L}\tan(\frac{\theta_0}{2})\ . \tag{5.45}$$

Since the trigonometric relations (3.6) are valid with $r_0 = 1$, the preceding expression for x

can be simplified with the result that

$$x \equiv \left(\frac{\Psi}{sin\ \theta_0}\right)^{l/L} \tan(\frac{\theta_0}{2}) \ . \tag{5.46}$$

Now (5.38) yields the result that

$$tan(\theta) = \frac{\Psi}{z} \tag{5.47}$$

and so (5.43) through (5.46) may be combined and we obtain

$$z = \Psi\left(\frac{1-x^2}{2x}\right) \tag{5.48}$$

where x is given by (5.45). Note that

$$\frac{1-x^2}{2x} = \frac{x^{-1}-x}{2} = \frac{e^{-lnx}-e^{lnx}}{2}$$

and since

$$\tan\left(\frac{\theta_0}{2}\right) = exp\left(\frac{-2\pi Z_c}{Z_0}\right)$$

$$sin\ \theta_0 = sech\left(\frac{2\pi Z_c}{Z_0}\right)$$

from (3.19) and (3.20), one obtains

$$z = \Psi\ sinh\left[\frac{l}{L}\ sech\left(\frac{2\pi Z_c}{Z_0}\right) - \frac{l}{L}\ ln\Psi + \left(\frac{2\pi Z_c}{Z_0}\right)\right] \tag{5.49}$$

where $\frac{L}{l} = \sqrt{\epsilon_{r_0}} sech\left(\frac{2\pi Z_c}{Z_0}\right) + tanh\left(\frac{2\pi Z_c}{Z_0}\right)$. We can now use (5.49) to obtain the graphs of figures 5.4 and 5.5 and the corresponding numerical results in tables 5.6 and 5.7. In figure 5.4 we take $\epsilon_{r_0} = 2.3$ and take Z_c as a parameter to obtain plots of the height z versus the cylindrical radius Ψ . We can also fix the value of Z_c at 60Ω and take ϵ_{r_0} as a parameter to obtain similar plots in figure 5.5. In each case $r_0 = 1$ and we obtain the lens profile as indicated in these figures.

One may also attempt to solve numerically the system of ordinary differential equations, given in (3.23) and (3.13), as

$$\frac{d\theta}{d\theta'} = \frac{1}{\sqrt{\epsilon_r(\theta')}} \frac{sin(\theta)}{sin(\theta')}$$

$$\frac{d\epsilon_r}{d\theta'} = \frac{2}{sin(\theta-\theta')} \left[2\sqrt{\epsilon_r} - (1+\epsilon_r)cos(\theta-\theta')\right] \tag{5.50}$$

with initial data given by θ_0 , θ_0' , and ϵ_{r_0} . This scheme was carried out by the use of a subroutine for the choice of $Z_c = 60\Omega$ and $\epsilon_{r_0} = 2.30$ and the results, which are consistent with our previous results, appear in table 5.8. This table shows the values of ϵ_r for $\theta_0 \leq \theta \leq \pi/2$ and $\theta_0' \leq \theta' \leq \theta_1'$ where the values of θ_0 and θ_0' are determined by the choices of ϵ_{r_0} and Z_c .

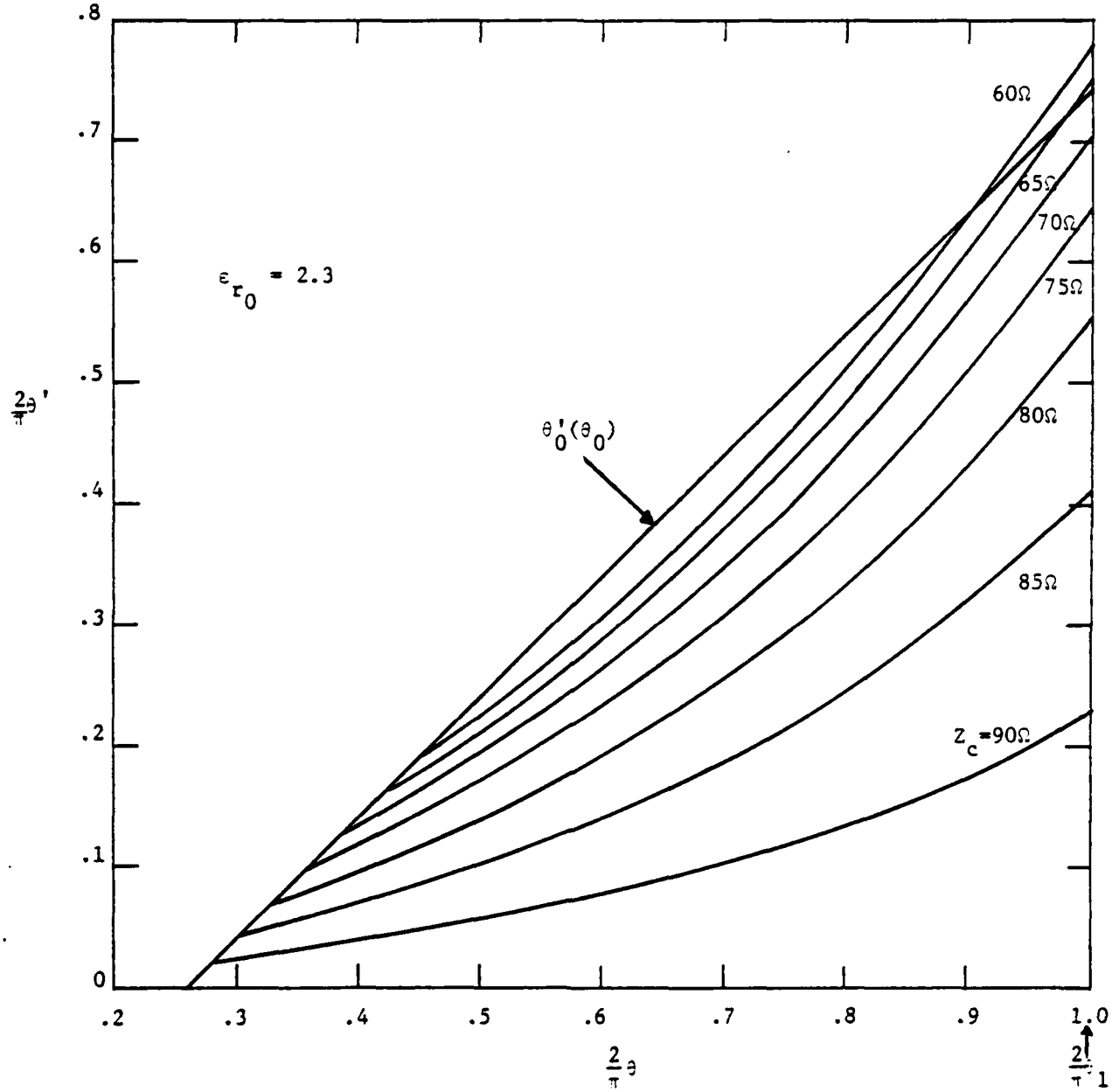


Figure 5.1. Relation Between Lens Angle θ' and Conical Antenna Angle θ for ε_{r_0} = 2.3 with Cone Impedance as a Parameter

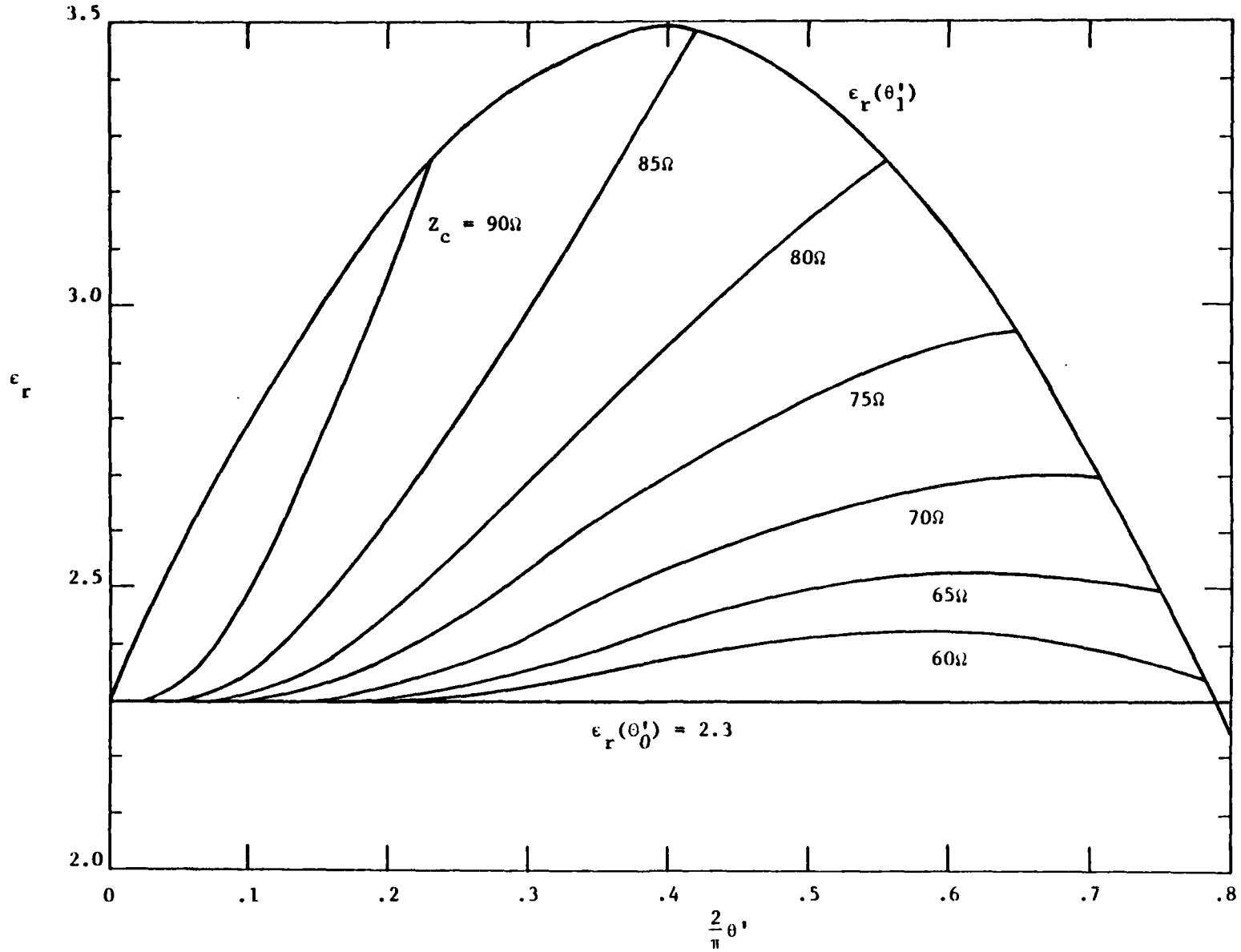


Figure 5.2. Relative Dielectric Constant as a Function of Lens Angle with Cone Impedance as a Parameter

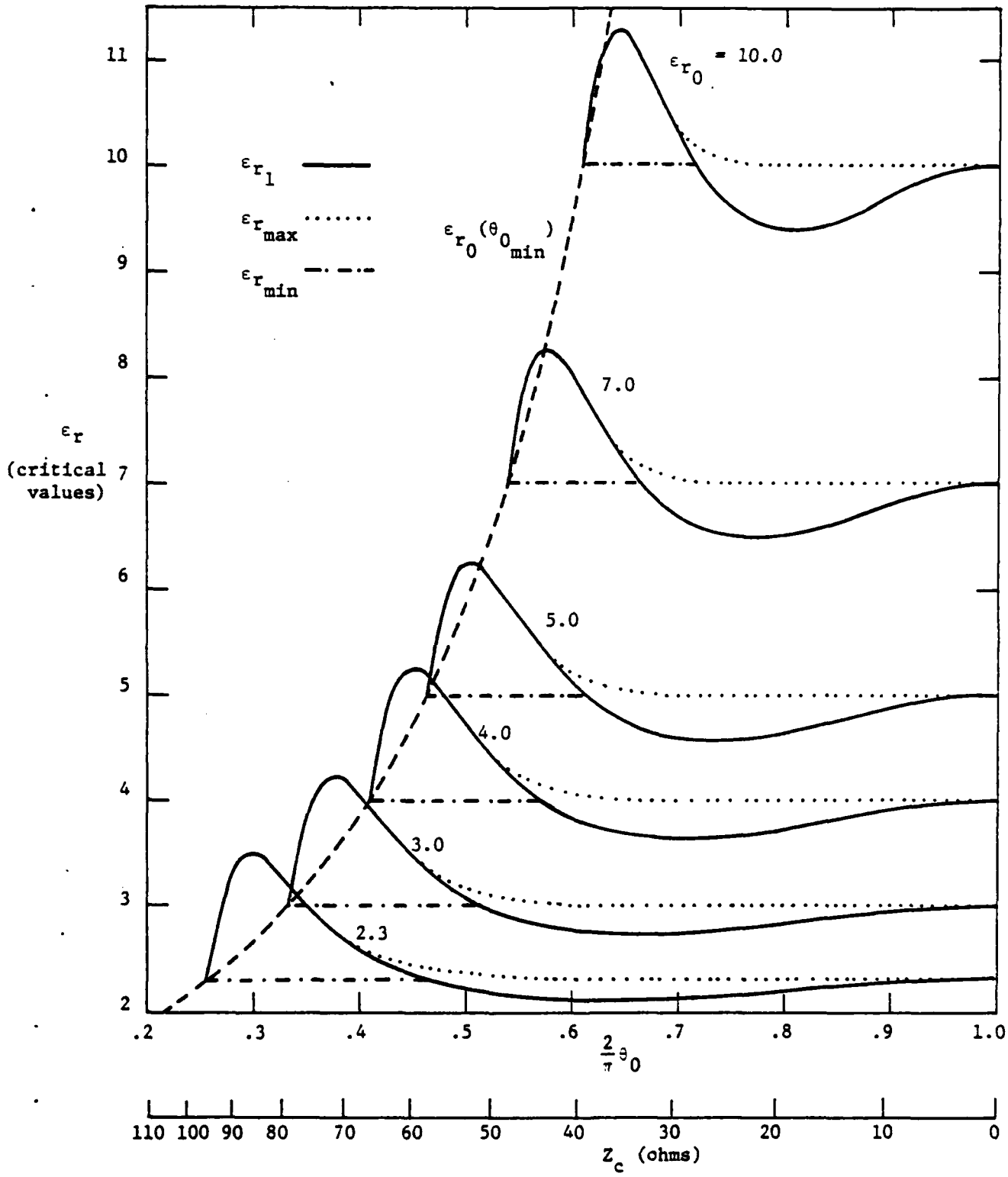


Figure 5.3. Maxima, Minima, and Boundary Values of the Relative Dielectric Constant as a Function of the Conical Antenna Angle θ_0 with Initial Dielectric Constant as a Parameter

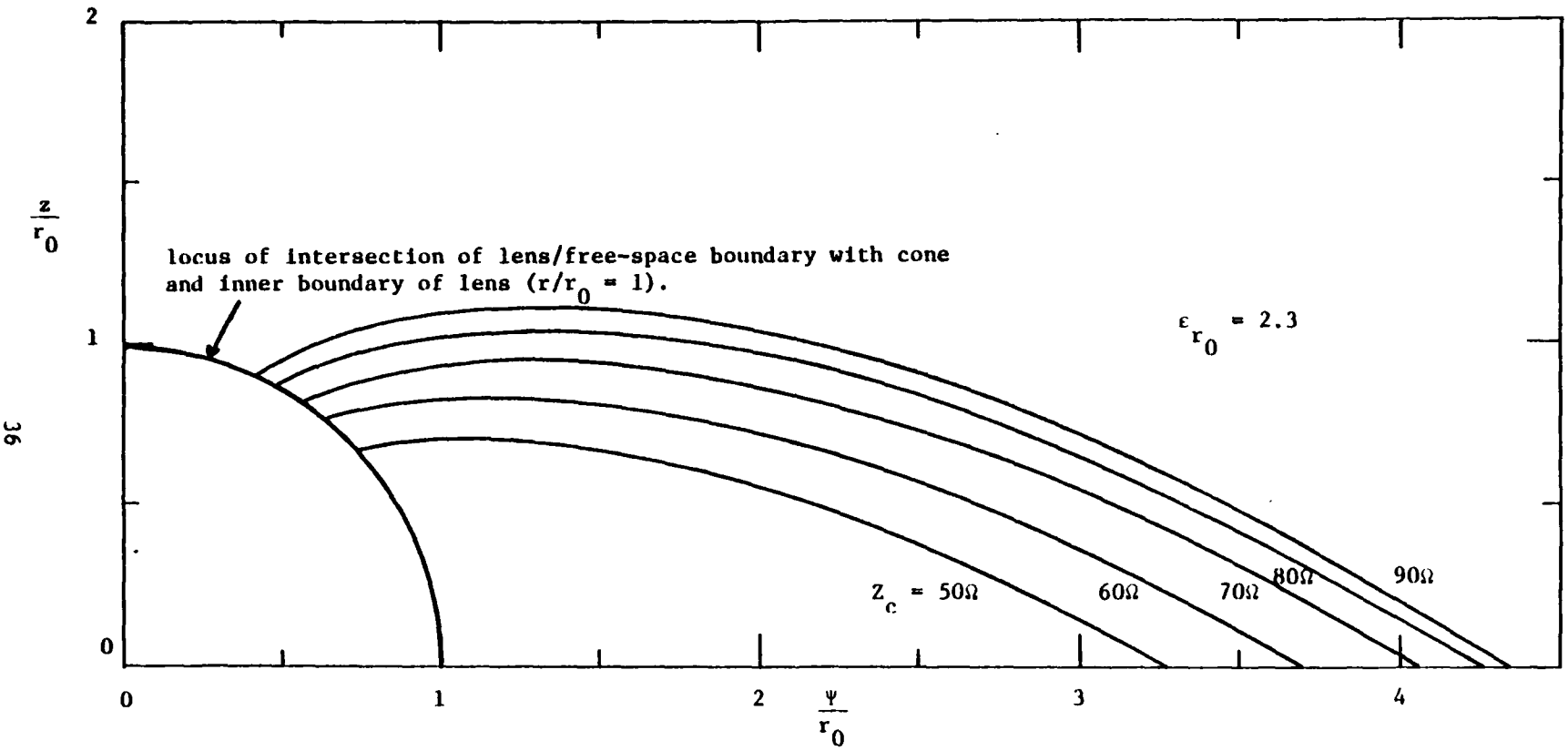


Figure 5.4. Shape of Lens/Free-Space Boundary for $\varepsilon_{r_0} = 2.3$ with Cone Impedance as a Parameter

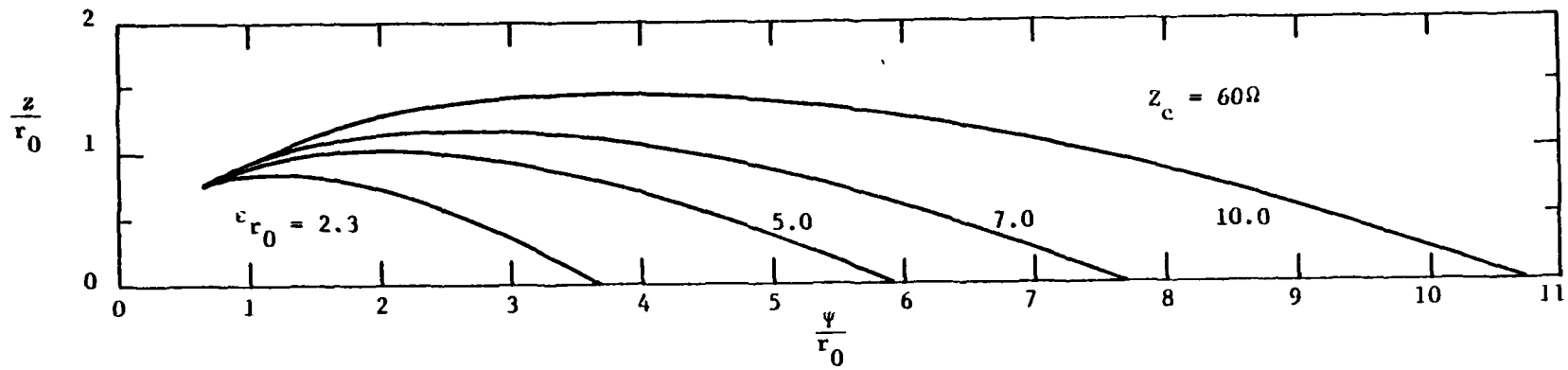


Figure 5.5. Shape of Lens/Free-Space Boundary for $Z_c = 60\Omega$ with Initial Relative Dielectric Constant as a Parameter

ε_{r_0}	$Z_{c_{min}}$	$Z_{c_{max}}$
2.30	58.110	95.006
3.00	50.735	79.018
4.00	43.840	65.917
5.00	39.163	57.745
7.00	33.050	47.722
10.00	27.624	39.294

Table 5.1: $Z_{c_{min}}$ and $Z_{c_{max}}$ versus ε_{r_0}

$\varepsilon_{r_0} = 2.30$

Z_c	L/ℓ	ℓ/r_0	L/r_0
58.11000	1.754518	1.249413	2.192117
59.00000	1.749824	1.287407	2.252737
60.00000	1.744417	1.332549	2.324522
61.00000	1.738876	1.380505	2.400528
62.00000	1.733209	1.431529	2.481139
63.00000	1.727423	1.485908	2.566793
64.00000	1.721527	1.543965	2.657978
65.00000	1.715529	1.606065	2.755250
66.00000	1.709434	1.672625	2.859242
67.00000	1.703251	1.744121	2.970676
68.00000	1.696986	1.821101	3.090383
69.00000	1.690646	1.904194	3.219319
70.00000	1.684237	1.994134	3.358595
71.00000	1.677766	2.091773	3.509505
72.00000	1.671238	2.198115	3.673573
73.00000	1.664660	2.314342	3.852593
74.00000	1.658037	2.441866	4.048703
75.00000	1.651374	2.582377	4.264470
76.00000	1.644677	2.737925	4.503002
77.00000	1.637951	2.911012	4.768096
78.00000	1.631201	3.104729	5.064438
79.00000	1.624431	3.322943	5.397894
80.00000	1.617647	3.570549	5.775887
81.00000	1.610851	3.853834	6.207953
82.00000	1.604049	4.181032	6.706580
83.00000	1.597244	4.563110	7.288402
84.00000	1.590441	5.015022	7.976096
85.00000	1.583642	5.557707	8.801418
86.00000	1.576852	6.221390	9.810210
87.00000	1.570073	7.051428	11.07126
88.00000	1.563309	8.119020	12.69254
89.00000	1.556563	9.542828	14.85402
90.00000	1.549838	11.53633	17.87945
91.00000	1.543137	14.52616	22.41584
92.00000	1.536461	19.50660	29.97113
93.00000	1.529814	29.45454	45.05996
94.00000	1.523198	59.18390	90.14879
95.00000	1.516615	9980.497	15136.57
95.00600	1.516575	3355443.	5088782.

Table 5.2: L/ℓ , ℓ/r_0 , L/r_0 versus cone impedance Z_c with ε_{r_0} as a parameter

Table 5.2

(continuation)

$\varepsilon_{r_0} = 3.00$

z_c	L/λ	λ/r_0	L/r_0
50.73500	1.944482	1.763757	3.429592
51.00000	1.942976	1.783050	3.464423
52.00000	1.937157	1.859413	3.601976
53.00000	1.931133	1.941889	3.750046
54.00000	1.924914	2.031208	3.909899
55.00000	1.918509	2.128221	4.083013
56.00000	1.911930	2.233930	4.271118
57.00000	1.905186	2.349516	4.476264
58.00000	1.898286	2.476387	4.700890
59.00000	1.891240	2.616230	4.947919
60.00000	1.884057	2.771090	5.220892
61.00000	1.876747	2.943463	5.524134
62.00000	1.869317	3.136432	5.862986
63.00000	1.861777	3.353855	6.244130
64.00000	1.854135	3.600612	6.676020
65.00000	1.846399	3.882974	7.169518
66.00000	1.838576	4.209145	7.738832
67.00000	1.830675	4.590054	8.402898
68.00000	1.822703	5.040604	9.187523
69.00000	1.814667	5.581646	10.12883
70.00000	1.806574	6.243280	11.27894
71.00000	1.798430	7.070620	12.71601
72.00000	1.790242	8.134506	14.56274
73.00000	1.782017	9.552853	17.02335
74.00000	1.773760	11.53758	20.46491
75.00000	1.765478	14.51162	25.61994
76.00000	1.757175	19.45852	34.19202
77.00000	1.748857	29.31204	51.26256
78.00000	1.740529	58.53878	101.8885
79.00000	1.732196	3433.030	5946.682

Table 5.2

(continuation)

$\varepsilon_{r_0} = 4.00$

Z_c	L/ℓ	ℓ/r_0	L/r_0
43.84000	2.187163	2.386710	5.220124
44.00000	2.186189	2.406013	5.260000
45.00000	2.179928	2.533545	5.522945
46.00000	2.173374	2.674242	5.812129
47.00000	2.166539	2.830190	6.131717
48.00000	2.159437	3.003936	6.486809
49.00000	2.152079	3.198635	6.883716
50.00000	2.144478	3.418231	7.330321
51.00000	2.136646	3.667728	7.836638
52.00000	2.128596	3.953563	8.415538
53.00000	2.120339	4.284173	9.083899
54.00000	2.111887	4.670824	9.864253
55.00000	2.103252	5.128909	10.78739
56.00000	2.094444	5.680033	11.89651
57.00000	2.085474	6.355490	13.25421
58.00000	2.076354	7.202386	14.95470
59.00000	2.067093	8.295081	17.14671
60.00000	2.057703	9.758166	20.07941
61.00000	2.048192	11.81765	24.20482
62.00000	2.038571	14.93032	30.43652
63.00000	2.028849	20.17943	40.94101
64.00000	2.019035	30.90882	62.40600
65.00000	2.009138	65.05243	130.6993

$\varepsilon_{r_0} = 5.00$

39.16300	2.405302	2.920503	7.024690
40.00000	2.399874	3.069721	7.366943
41.00000	2.393045	3.267981	7.820426
42.00000	2.385855	3.491914	8.331202
43.00000	2.378319	3.746737	8.910937
44.00000	2.370450	4.039188	9.574695
45.00000	2.362265	4.378114	10.34226
46.00000	2.353776	4.775383	11.24018
47.00000	2.344998	5.247278	12.30485
48.00000	2.335945	5.816756	13.58762
49.00000	2.326631	6.517292	15.16333
50.00000	2.317069	7.399636	17.14547
51.00000	2.307273	8.544646	19.71483
52.00000	2.297257	10.08943	23.17803
53.00000	2.287033	12.28674	28.10019
54.00000	2.276614	15.65921	35.64998
55.00000	2.266013	21.49093	48.69872
56.00000	2.255241	34.00879	76.69802
57.00000	2.244311	80.12195	179.8186
57.74500	2.236073	145888.8	326218.0

Table 5.2

(continuation)

$\varepsilon_{r_0} = 7.00$

Z_c	L/ℓ	ℓ/r_0	L/r_0
33.05000	2.790681	3.816293	10.65006
34.00000	2.784137	4.095514	11.40247
35.00000	2.776752	4.435141	12.31528
36.00000	2.768875	4.833435	13.38318
37.00000	2.760523	5.306806	14.64956
38.00000	2.751715	5.878414	16.17572
39.00000	2.742466	6.582036	18.05101
40.00000	2.732794	7.468962	20.41113
41.00000	2.722717	8.620969	23.47246
42.00000	2.712252	10.17705	27.60271
43.00000	2.701416	12.39387	33.48101
44.00000	2.690226	15.80405	42.51647
45.00000	2.678700	21.72275	58.18872
46.00000	2.666853	34.52038	92.06079
47.00000	2.654702	82.78300	219.7642

$\varepsilon_{r_0} = 10.00$

27.62400	3.284787	4.904465	16.11013
28.00000	3.282146	5.073945	16.65343
29.00000	3.274649	5.585564	18.29076
30.00000	3.266485	6.208265	20.27920
31.00000	3.257672	6.982218	22.74578
32.00000	3.248231	7.969625	25.88719
33.00000	3.238183	9.272207	30.02510
34.00000	3.227546	11.06856	35.72429
35.00000	3.216342	13.70364	44.07559
36.00000	3.204591	17.94128	57.49447
37.00000	3.192314	25.87715	82.60799
38.00000	3.179533	46.08517	146.5293
39.00000	3.166267	203.7826	645.2301
39.29400	3.162278	3355443.	1.0610844E+07

$Z_c = 60\Omega$

$\frac{2}{\pi}\theta$	θ	$\frac{2}{\pi}\theta'$	θ'	$\frac{2}{\pi}\theta$	θ	$\frac{2}{\pi}\theta'$	θ'
0.440	0.691	0.185	0.291	0.730	1.147	0.437	0.686
0.450	0.707	0.192	0.301	0.740	1.162	0.447	0.703
0.460	0.723	0.199	0.312	0.750	1.178	0.458	0.720
0.470	0.738	0.206	0.324	0.760	1.194	0.469	0.737
0.480	0.754	0.213	0.335	0.770	1.210	0.481	0.755
0.490	0.770	0.221	0.346	0.780	1.225	0.492	0.773
0.500	0.785	0.228	0.358	0.790	1.241	0.504	0.791
0.510	0.801	0.236	0.370	0.800	1.257	0.516	0.810
0.520	0.817	0.243	0.382	0.810	1.272	0.527	0.829
0.530	0.833	0.251	0.394	0.820	1.288	0.540	0.848
0.540	0.848	0.259	0.407	0.830	1.304	0.552	0.867
0.550	0.864	0.267	0.420	0.840	1.319	0.564	0.886
0.560	0.880	0.275	0.433	0.850	1.335	0.577	0.906
0.570	0.895	0.284	0.446	0.860	1.351	0.589	0.926
0.580	0.911	0.292	0.459	0.870	1.367	0.602	0.946
0.590	0.927	0.301	0.473	0.880	1.382	0.615	0.966
0.600	0.942	0.310	0.486	0.890	1.398	0.628	0.987
0.610	0.958	0.319	0.500	0.900	1.414	0.641	1.008
0.620	0.974	0.328	0.514	0.910	1.429	0.655	1.029
0.630	0.990	0.337	0.529	0.920	1.445	0.668	1.050
0.640	1.005	0.346	0.544	0.930	1.461	0.682	1.071
0.650	1.021	0.355	0.558	0.940	1.477	0.696	1.093
0.660	1.037	0.365	0.573	0.950	1.492	0.710	1.115
0.670	1.052	0.375	0.589	0.960	1.508	0.723	1.136
0.680	1.068	0.385	0.604	0.970	1.524	0.738	1.159
0.690	1.084	0.395	0.620	0.980	1.539	0.752	1.181
0.700	1.100	0.405	0.636	0.990	1.555	0.766	1.203
0.710	1.115	0.415	0.652	1.000	1.571	0.780	1.226
0.720	1.131	0.426	0.669				

Table 5.3: Lens angle θ' versus conical antenna angle θ for $\varepsilon_{r_0} = 2.30$, with cone impedance as a parameter

Table 5.3

(continuation)

$Z_c = 65\Omega$

$\frac{2}{\pi}\theta$	θ	$\frac{2}{\pi}\theta'$	θ'
0.420	0.660	0.161	0.252
0.430	0.675	0.167	0.262
0.440	0.691	0.173	0.272
0.450	0.707	0.180	0.282
0.460	0.723	0.186	0.292
0.470	0.738	0.193	0.303
0.480	0.754	0.200	0.313
0.490	0.770	0.206	0.324
0.500	0.785	0.213	0.335
0.510	0.801	0.221	0.346
0.520	0.817	0.228	0.358
0.530	0.833	0.235	0.369
0.540	0.848	0.243	0.381
0.550	0.864	0.250	0.393
0.560	0.880	0.258	0.405
0.570	0.895	0.266	0.418
0.580	0.911	0.274	0.430
0.590	0.927	0.282	0.443
0.600	0.942	0.290	0.456
0.610	0.958	0.299	0.470
0.620	0.974	0.307	0.483
0.630	0.990	0.316	0.497
0.640	1.005	0.325	0.511
0.650	1.021	0.334	0.525
0.660	1.037	0.343	0.539
0.670	1.052	0.353	0.554
0.680	1.068	0.362	0.569
0.690	1.084	0.372	0.584
0.700	1.100	0.381	0.599
0.710	1.115	0.391	0.615
0.720	1.131	0.401	0.631
0.730	1.147	0.412	0.647
0.740	1.162	0.422	0.663
0.750	1.178	0.433	0.680
0.760	1.194	0.443	0.697
0.770	1.210	0.454	0.714
0.780	1.225	0.465	0.731
0.790	1.241	0.477	0.749
0.800	1.257	0.488	0.767
0.810	1.272	0.500	0.785
0.820	1.288	0.511	0.803
0.830	1.304	0.523	0.822
0.840	1.319	0.536	0.841
0.850	1.335	0.548	0.860
0.860	1.351	0.560	0.880
0.870	1.367	0.573	0.900
0.880	1.382	0.586	0.920
0.890	1.398	0.599	0.940
0.900	1.414	0.612	0.961
0.910	1.429	0.625	0.982
0.920	1.445	0.638	1.003
0.930	1.461	0.652	1.024
0.940	1.477	0.666	1.046
0.950	1.492	0.680	1.068
0.960	1.508	0.694	1.090
0.970	1.524	0.708	1.112
0.980	1.539	0.722	1.134
0.990	1.555	0.737	1.157
1.000	1.571	0.751	1.180

Table 5.3

(continuation)

$Z_c = 70\Omega$

$\frac{2}{\pi}\theta$	θ	$\frac{2}{\pi}\theta'$	θ'
0.400	0.628	0.135	0.212
0.410	0.644	0.140	0.221
0.420	0.660	0.146	0.229
0.430	0.675	0.152	0.238
0.440	0.691	0.157	0.247
0.450	0.707	0.163	0.257
0.460	0.723	0.169	0.266
0.470	0.738	0.175	0.276
0.480	0.754	0.182	0.285
0.490	0.770	0.188	0.295
0.500	0.785	0.194	0.305
0.510	0.801	0.201	0.316
0.520	0.817	0.208	0.326
0.530	0.833	0.214	0.337
0.540	0.848	0.221	0.348
0.550	0.864	0.228	0.359
0.560	0.880	0.235	0.370
0.570	0.895	0.243	0.381
0.580	0.911	0.250	0.393
0.590	0.927	0.258	0.405
0.600	0.942	0.265	0.417
0.610	0.958	0.273	0.429
0.620	0.974	0.281	0.441
0.630	0.990	0.289	0.454
0.640	1.005	0.297	0.467
0.650	1.021	0.306	0.480
0.660	1.037	0.314	0.494
0.670	1.052	0.323	0.507
0.680	1.068	0.332	0.521
0.690	1.084	0.341	0.535
0.700	1.100	0.350	0.550
0.710	1.115	0.359	0.564
0.720	1.131	0.369	0.579
0.730	1.147	0.378	0.594
0.740	1.162	0.388	0.610
0.750	1.178	0.398	0.625
0.760	1.194	0.408	0.641
0.770	1.210	0.419	0.658
0.780	1.225	0.429	0.674
0.790	1.241	0.440	0.691
0.800	1.257	0.451	0.708
0.810	1.272	0.462	0.725
0.820	1.288	0.473	0.743
0.830	1.304	0.485	0.761
0.840	1.319	0.496	0.779
0.850	1.335	0.508	0.798
0.860	1.351	0.520	0.817
0.870	1.367	0.532	0.836
0.880	1.382	0.545	0.856
0.890	1.398	0.557	0.876
0.900	1.414	0.570	0.896
0.910	1.429	0.583	0.916
0.920	1.445	0.596	0.937
0.930	1.461	0.610	0.958
0.940	1.477	0.623	0.979
0.950	1.492	0.637	1.001
0.960	1.508	0.651	1.023
0.970	1.524	0.665	1.045
0.980	1.539	0.680	1.068
0.990	1.555	0.694	1.090
1.000	1.571	0.709	1.113

Table 5.3

(continuation)

$Z_c = 75\Omega$

$\frac{2}{\pi}\theta$	θ	$\frac{2}{\pi}\theta'$	θ'	$\frac{2}{\pi}\theta$	θ	$\frac{2}{\pi}\theta'$	θ'
0.360	0.565	0.100	0.156	0.690	1.084	0.300	0.472
0.370	0.581	0.104	0.163	0.700	1.100	0.309	0.485
0.380	0.597	0.109	0.171	0.710	1.115	0.317	0.498
0.390	0.613	0.113	0.178	0.720	1.131	0.326	0.511
0.400	0.628	0.118	0.185	0.730	1.147	0.334	0.525
0.410	0.644	0.123	0.193	0.740	1.162	0.343	0.539
0.420	0.660	0.128	0.200	0.750	1.178	0.352	0.553
0.430	0.675	0.133	0.208	0.760	1.194	0.362	0.568
0.440	0.691	0.138	0.216	0.770	1.210	0.371	0.583
0.450	0.707	0.143	0.224	0.780	1.225	0.381	0.598
0.460	0.723	0.148	0.233	0.790	1.241	0.391	0.613
0.470	0.738	0.153	0.241	0.800	1.257	0.401	0.629
0.480	0.754	0.159	0.250	0.810	1.272	0.411	0.645
0.490	0.770	0.164	0.258	0.820	1.288	0.421	0.662
0.500	0.785	0.170	0.267	0.830	1.304	0.432	0.678
0.510	0.801	0.176	0.276	0.840	1.319	0.443	0.695
0.520	0.817	0.182	0.285	0.850	1.335	0.454	0.713
0.530	0.833	0.188	0.295	0.860	1.351	0.465	0.731
0.540	0.848	0.194	0.304	0.870	1.367	0.477	0.749
0.550	0.864	0.200	0.314	0.880	1.382	0.488	0.767
0.560	0.880	0.206	0.324	0.890	1.398	0.500	0.786
0.570	0.895	0.213	0.334	0.900	1.414	0.513	0.805
0.580	0.911	0.219	0.345	0.910	1.429	0.525	0.825
0.590	0.927	0.226	0.355	0.920	1.445	0.538	0.844
0.600	0.942	0.233	0.366	0.930	1.461	0.550	0.865
0.610	0.958	0.240	0.377	0.940	1.477	0.564	0.885
0.620	0.974	0.247	0.388	0.950	1.492	0.577	0.906
0.630	0.990	0.254	0.399	0.960	1.508	0.591	0.928
0.640	1.005	0.261	0.411	0.970	1.524	0.604	0.949
0.650	1.021	0.269	0.422	0.980	1.539	0.618	0.971
0.660	1.037	0.277	0.434	0.990	1.555	0.633	0.994
0.670	1.052	0.284	0.447	1.000	1.571	0.647	1.017
0.680	1.068	0.292	0.459				

Table 5.3

(continuation)

$Z_c = 80\Omega$

$\frac{2}{\pi}\theta$	θ	$\frac{2}{\pi}\theta'$	θ'	$\frac{2}{\pi}\theta$	θ	$\frac{2}{\pi}\theta'$	θ'
0.330	0.518	0.071	0.112	0.670	1.052	0.235	0.369
0.340	0.534	0.074	0.117	0.680	1.068	0.241	0.379
0.350	0.550	0.078	0.122	0.690	1.084	0.248	0.390
0.360	0.565	0.082	0.128	0.700	1.100	0.255	0.401
0.370	0.581	0.085	0.134	0.710	1.115	0.262	0.412
0.380	0.597	0.089	0.140	0.720	1.131	0.270	0.423
0.390	0.613	0.093	0.146	0.730	1.147	0.277	0.435
0.400	0.628	0.097	0.152	0.740	1.162	0.285	0.447
0.410	0.644	0.101	0.158	0.750	1.178	0.292	0.459
0.420	0.660	0.105	0.164	0.760	1.194	0.300	0.472
0.430	0.675	0.109	0.171	0.770	1.210	0.308	0.484
0.440	0.691	0.113	0.177	0.780	1.225	0.317	0.497
0.450	0.707	0.117	0.184	0.790	1.241	0.325	0.511
0.460	0.723	0.121	0.191	0.800	1.257	0.334	0.524
0.470	0.738	0.126	0.198	0.810	1.272	0.343	0.538
0.480	0.754	0.130	0.205	0.820	1.288	0.352	0.552
0.490	0.770	0.135	0.212	0.830	1.304	0.361	0.567
0.500	0.785	0.140	0.219	0.840	1.319	0.370	0.582
0.510	0.801	0.144	0.227	0.850	1.335	0.380	0.597
0.520	0.817	0.149	0.234	0.860	1.351	0.390	0.613
0.530	0.833	0.154	0.242	0.870	1.367	0.400	0.629
0.540	0.848	0.159	0.250	0.880	1.382	0.411	0.645
0.550	0.864	0.164	0.258	0.890	1.398	0.421	0.662
0.560	0.880	0.170	0.266	0.900	1.414	0.432	0.679
0.570	0.895	0.175	0.275	0.910	1.429	0.443	0.696
0.580	0.911	0.180	0.283	0.920	1.445	0.455	0.714
0.590	0.927	0.186	0.292	0.930	1.461	0.466	0.732
0.600	0.942	0.192	0.301	0.940	1.477	0.478	0.751
0.610	0.958	0.197	0.310	0.950	1.492	0.490	0.770
0.620	0.974	0.203	0.319	0.960	1.508	0.503	0.790
0.630	0.990	0.209	0.329	0.970	1.524	0.516	0.810
0.640	1.005	0.216	0.339	0.980	1.539	0.529	0.831
0.650	1.021	0.222	0.348	0.990	1.555	0.542	0.852
0.660	1.037	0.228	0.358	1.000	1.571	0.556	0.873

Table 5.3

(continuation)

$Z_c = 85\Omega$

$\frac{2}{\pi}\theta$	θ	$\frac{2}{\pi}\theta'$	θ'
0.300	0.471	0.044	0.070
0.310	0.487	0.047	0.074
0.320	0.503	0.049	0.077
0.330	0.518	0.052	0.081
0.340	0.534	0.054	0.085
0.350	0.550	0.057	0.089
0.360	0.565	0.059	0.093
0.370	0.581	0.062	0.097
0.380	0.597	0.065	0.102
0.390	0.613	0.068	0.106
0.400	0.628	0.070	0.111
0.410	0.644	0.073	0.115
0.420	0.660	0.076	0.120
0.430	0.675	0.079	0.124
0.440	0.691	0.082	0.129
0.450	0.707	0.085	0.134
0.460	0.723	0.089	0.139
0.470	0.738	0.092	0.144
0.480	0.754	0.095	0.149
0.490	0.770	0.098	0.155
0.500	0.785	0.102	0.160
0.510	0.801	0.105	0.165
0.520	0.817	0.109	0.171
0.530	0.833	0.113	0.177
0.540	0.848	0.116	0.183
0.550	0.864	0.120	0.189
0.560	0.880	0.124	0.195
0.570	0.895	0.128	0.201
0.580	0.911	0.132	0.207
0.590	0.927	0.136	0.213
0.600	0.942	0.140	0.220
0.610	0.958	0.144	0.227
0.620	0.974	0.149	0.234
0.630	0.990	0.153	0.241
0.640	1.005	0.158	0.248
0.650	1.021	0.162	0.255

$\frac{2}{\pi}\theta$	θ	$\frac{2}{\pi}\theta'$	θ'
0.660	1.037	0.167	0.262
0.670	1.052	0.172	0.270
0.680	1.068	0.177	0.278
0.690	1.084	0.182	0.286
0.700	1.100	0.187	0.294
0.710	1.115	0.192	0.302
0.720	1.131	0.198	0.311
0.730	1.147	0.203	0.319
0.740	1.162	0.209	0.328
0.750	1.178	0.215	0.337
0.760	1.194	0.221	0.347
0.770	1.210	0.227	0.356
0.780	1.225	0.233	0.366
0.790	1.241	0.239	0.376
0.800	1.257	0.246	0.386
0.810	1.272	0.253	0.397
0.820	1.288	0.259	0.407
0.830	1.304	0.266	0.419
0.840	1.319	0.274	0.430
0.850	1.335	0.281	0.442
0.860	1.351	0.289	0.453
0.870	1.367	0.296	0.466
0.880	1.382	0.305	0.478
0.890	1.398	0.313	0.491
0.900	1.414	0.321	0.505
0.910	1.429	0.330	0.518
0.920	1.445	0.339	0.532
0.930	1.461	0.348	0.547
0.940	1.477	0.358	0.562
0.950	1.492	0.367	0.577
0.960	1.508	0.377	0.593
0.970	1.524	0.388	0.609
0.980	1.539	0.398	0.626
0.990	1.555	0.409	0.643
1.000	1.571	0.421	0.661

Table 5.3

(continuation)

$Z_c = 90\Omega$

$\frac{2}{\pi}\theta$	θ	$\frac{2}{\pi}\theta'$	θ'	$\frac{2}{\pi}\theta$	θ	$\frac{2}{\pi}\theta'$	θ'
0.270	0.424	0.021	0.032	0.640	1.005	0.086	0.135
0.280	0.440	0.022	0.034	0.650	1.021	0.089	0.139
0.290	0.456	0.023	0.036	0.660	1.037	0.091	0.143
0.300	0.471	0.024	0.038	0.670	1.052	0.094	0.147
0.310	0.487	0.026	0.040	0.680	1.068	0.097	0.152
0.320	0.503	0.027	0.042	0.690	1.084	0.099	0.156
0.330	0.518	0.028	0.044	0.700	1.100	0.102	0.160
0.340	0.534	0.030	0.047	0.710	1.115	0.105	0.165
0.350	0.550	0.031	0.049	0.720	1.131	0.108	0.170
0.360	0.565	0.032	0.051	0.730	1.147	0.111	0.174
0.370	0.581	0.034	0.053	0.740	1.162	0.114	0.179
0.380	0.597	0.035	0.056	0.750	1.178	0.117	0.184
0.390	0.613	0.037	0.058	0.760	1.194	0.120	0.189
0.400	0.628	0.038	0.060	0.770	1.210	0.124	0.194
0.410	0.644	0.040	0.063	0.780	1.225	0.127	0.200
0.420	0.660	0.042	0.065	0.790	1.241	0.130	0.205
0.430	0.675	0.043	0.068	0.800	1.257	0.134	0.211
0.440	0.691	0.045	0.071	0.810	1.272	0.138	0.216
0.450	0.707	0.047	0.073	0.820	1.288	0.141	0.222
0.460	0.723	0.048	0.076	0.830	1.304	0.145	0.228
0.470	0.738	0.050	0.079	0.840	1.319	0.149	0.234
0.480	0.754	0.052	0.082	0.850	1.335	0.153	0.241
0.490	0.770	0.054	0.085	0.860	1.351	0.157	0.247
0.500	0.785	0.056	0.087	0.870	1.367	0.161	0.254
0.510	0.801	0.058	0.090	0.880	1.382	0.166	0.260
0.520	0.817	0.060	0.094	0.890	1.398	0.170	0.268
0.530	0.833	0.062	0.097	0.900	1.414	0.175	0.275
0.540	0.848	0.064	0.100	0.910	1.429	0.180	0.282
0.550	0.864	0.066	0.103	0.920	1.445	0.185	0.290
0.560	0.880	0.068	0.106	0.930	1.461	0.190	0.298
0.570	0.895	0.070	0.110	0.940	1.477	0.195	0.306
0.580	0.911	0.072	0.113	0.950	1.492	0.200	0.314
0.590	0.927	0.074	0.117	0.960	1.508	0.206	0.323
0.600	0.942	0.077	0.120	0.970	1.524	0.211	0.332
0.610	0.958	0.079	0.124	0.980	1.539	0.217	0.341
0.620	0.974	0.081	0.128	0.990	1.555	0.223	0.350
0.630	0.990	0.084	0.131	1.000	1.571	0.229	0.360

$Z_c = 60\Omega$

$\frac{2}{\pi}\theta'$	θ'	ε_r	$\frac{2}{\pi}\theta'$	θ'	ε_r
0.191	0.300	2.300	0.510	0.800	2.413
0.200	0.314	2.300	0.520	0.817	2.416
0.210	0.329	2.301	0.530	0.832	2.417
0.220	0.345	2.303	0.540	0.848	2.419
0.230	0.361	2.305	0.550	0.864	2.420
0.240	0.376	2.307	0.560	0.880	2.421
0.250	0.393	2.310	0.570	0.896	2.421
0.260	0.409	2.314	0.580	0.911	2.421
0.270	0.424	2.317	0.590	0.926	2.421
0.280	0.440	2.321	0.600	0.942	2.421
0.290	0.455	2.325	0.610	0.959	2.420
0.300	0.471	2.329	0.620	0.973	2.419
0.310	0.486	2.333	0.630	0.990	2.417
0.320	0.502	2.338	0.640	1.006	2.415
0.330	0.519	2.343	0.650	1.021	2.412
0.340	0.534	2.347	0.660	1.037	2.410
0.350	0.550	2.352	0.670	1.052	2.407
0.360	0.566	2.357	0.680	1.069	2.403
0.370	0.582	2.362	0.690	1.084	2.399
0.380	0.598	2.366	0.700	1.099	2.394
0.390	0.612	2.370	0.710	1.116	2.389
0.400	0.628	2.375	0.720	1.131	2.384
0.410	0.643	2.379	0.730	1.146	2.379
0.420	0.660	2.384	0.740	1.162	2.373
0.430	0.675	2.388	0.750	1.178	2.366
0.440	0.691	2.392	0.760	1.194	2.359
0.450	0.706	2.396	0.770	1.210	2.351
0.460	0.722	2.399	0.780	1.226	2.344
0.470	0.738	2.403	0.790	1.241	2.336
0.480	0.755	2.406	0.800	1.257	2.327
0.490	0.769	2.408	0.810	1.273	2.319
0.500	0.786	2.411	0.812	1.275	2.317

Table 5.4: Relative dielectric constant versus conical antenna angle θ', with cone impedance Z_c as parameter.

Table 5.4

(continuation)

$Z_c = 65\Omega$

$\frac{2}{\pi}\theta'$	θ'	ε_r
0.158	0.248	2.300
0.160	0.252	2.300
0.170	0.267	2.301
0.180	0.283	2.302
0.190	0.299	2.305
0.200	0.314	2.308
0.210	0.329	2.311
0.220	0.345	2.316
0.230	0.361	2.321
0.240	0.377	2.326
0.250	0.393	2.332
0.260	0.408	2.338
0.270	0.424	2.344
0.280	0.440	2.351
0.290	0.455	2.357
0.300	0.472	2.365
0.310	0.487	2.372
0.320	0.503	2.379
0.330	0.519	2.386
0.340	0.534	2.393
0.350	0.551	2.401
0.360	0.566	2.408
0.370	0.581	2.415
0.380	0.597	2.422
0.390	0.612	2.430
0.400	0.629	2.437
0.410	0.644	2.444
0.420	0.659	2.450
0.430	0.676	2.457
0.440	0.691	2.463
0.450	0.707	2.469
0.460	0.722	2.475
0.470	0.738	2.481
0.480	0.753	2.487
0.490	0.769	2.492
0.500	0.786	2.497
0.510	0.800	2.501
0.520	0.816	2.506
0.530	0.832	2.509
0.540	0.848	2.513
0.550	0.864	2.516
0.560	0.880	2.519
0.570	0.895	2.522
0.580	0.911	2.524
0.590	0.926	2.525
0.600	0.942	2.527
0.610	0.958	2.527
0.620	0.973	2.528
0.630	0.989	2.528
0.640	1.005	2.527
0.650	1.022	2.526
0.660	1.037	2.525
0.670	1.052	2.523
0.680	1.069	2.521
0.690	1.084	2.518
0.700	1.100	2.515
0.710	1.116	2.511
0.720	1.131	2.507
0.730	1.147	2.503
0.740	1.163	2.497
0.750	1.178	2.492
0.760	1.194	2.486
0.770	1.209	2.479
0.780	1.225	2.472
0.790	1.240	2.465
0.800	1.257	2.457
0.810	1.272	2.449
0.820	1.289	2.439
0.827	1.299	2.434

Table 5.4

(continuation)

$Z_c = 70\Omega$

$\frac{2}{\pi}\theta'$	θ'	ε_r	$\frac{2}{\pi}\theta'$	θ'	ε_r
0.127	0.199	2.300	0.440	0.692	2.573
0.130	0.204	2.300	0.450	0.707	2.583
0.140	0.221	2.301	0.460	0.723	2.592
0.150	0.236	2.304	0.470	0.739	2.601
0.160	0.251	2.307	0.480	0.755	2.610
0.170	0.267	2.312	0.490	0.770	2.619
0.180	0.283	2.317	0.500	0.785	2.626
0.190	0.298	2.323	0.510	0.801	2.634
0.200	0.315	2.330	0.520	0.816	2.641
0.210	0.330	2.337	0.530	0.833	2.649
0.220	0.346	2.346	0.540	0.848	2.655
0.230	0.361	2.354	0.550	0.864	2.661
0.240	0.377	2.363	0.560	0.880	2.667
0.250	0.392	2.372	0.570	0.896	2.672
0.260	0.409	2.382	0.580	0.912	2.677
0.270	0.424	2.392	0.590	0.927	2.681
0.280	0.440	2.403	0.600	0.942	2.685
0.290	0.456	2.413	0.610	0.959	2.688
0.300	0.471	2.423	0.620	0.973	2.691
0.310	0.487	2.434	0.630	0.990	2.693
0.320	0.503	2.446	0.640	1.005	2.695
0.330	0.519	2.456	0.650	1.021	2.697
0.340	0.534	2.467	0.660	1.037	2.697
0.350	0.550	2.478	0.670	1.053	2.698
0.360	0.565	2.489	0.680	1.068	2.697
0.370	0.582	2.501	0.690	1.083	2.696
0.380	0.597	2.512	0.700	1.100	2.695
0.390	0.613	2.522	0.710	1.115	2.693
0.400	0.629	2.533	0.720	1.131	2.690
0.410	0.644	2.543	0.730	1.146	2.687
0.420	0.660	2.554	0.738	1.160	2.684
0.430	0.675	2.563			

Table 5.4

(continuation)

$Z_c = 75\Omega$

$\frac{2}{\pi}\theta'$	θ'	ε_r	$\frac{2}{\pi}\theta'$	θ'	ε_r
0.097	0.153	2.300	0.420	0.660	2.714
0.100	0.157	2.300	0.430	0.676	2.730
0.110	0.173	2.302	0.440	0.691	2.744
0.120	0.188	2.305	0.450	0.707	2.759
0.130	0.205	2.310	0.460	0.722	2.773
0.140	0.220	2.316	0.470	0.739	2.788
0.150	0.236	2.324	0.480	0.753	2.800
0.160	0.252	2.332	0.490	0.770	2.814
0.170	0.267	2.341	0.500	0.786	2.827
0.180	0.283	2.352	0.510	0.801	2.839
0.190	0.299	2.363	0.520	0.816	2.850
0.200	0.314	2.375	0.530	0.833	2.862
0.210	0.329	2.387	0.540	0.848	2.873
0.220	0.345	2.400	0.550	0.864	2.884
0.230	0.362	2.415	0.560	0.880	2.894
0.240	0.377	2.428	0.570	0.895	2.903
0.250	0.392	2.443	0.580	0.911	2.912
0.260	0.408	2.458	0.590	0.927	2.921
0.270	0.424	2.473	0.600	0.943	2.928
0.280	0.440	2.489	0.610	0.959	2.936
0.290	0.456	2.505	0.620	0.974	2.942
0.300	0.472	2.521	0.630	0.990	2.948
0.310	0.486	2.536	0.640	1.006	2.953
0.320	0.503	2.553	0.650	1.021	2.958
0.330	0.518	2.569	0.660	1.037	2.962
0.340	0.534	2.585	0.670	1.052	2.965
0.350	0.550	2.602	0.680	1.068	2.967
0.360	0.566	2.619	0.690	1.084	2.969
0.370	0.580	2.634	0.700	1.100	2.971
0.380	0.597	2.651	0.710	1.115	2.971
0.390	0.613	2.667	0.720	1.131	2.971
0.400	0.628	2.682	0.730	1.147	2.970
0.410	0.644	2.698	0.732	1.149	2.970

Table 5.4

(continuation)

$Z_c = 80\Omega$

$\frac{2}{\pi}\theta'$	θ'	ε_r	$\frac{2}{\pi}\theta'$	θ'	ε_r
0.070	0.110	2.300	0.340	0.534	2.783
0.080	0.126	2.301	0.350	0.549	2.808
0.090	0.142	2.306	0.360	0.566	2.833
0.100	0.157	2.312	0.370	0.582	2.858
0.110	0.173	2.321	0.380	0.597	2.883
0.120	0.188	2.331	0.390	0.613	2.906
0.130	0.204	2.342	0.400	0.628	2.931
0.140	0.220	2.356	0.410	0.644	2.955
0.150	0.236	2.371	0.420	0.660	2.978
0.160	0.252	2.387	0.430	0.675	3.001
0.170	0.267	2.403	0.440	0.692	3.025
0.180	0.283	2.422	0.450	0.707	3.048
0.190	0.298	2.439	0.460	0.722	3.070
0.200	0.315	2.460	0.470	0.738	3.091
0.210	0.331	2.480	0.480	0.754	3.113
0.220	0.346	2.500	0.490	0.770	3.135
0.230	0.361	2.522	0.500	0.785	3.154
0.240	0.377	2.544	0.510	0.800	3.174
0.250	0.392	2.566	0.520	0.817	3.195
0.260	0.409	2.591	0.530	0.832	3.213
0.270	0.423	2.612	0.540	0.849	3.232
0.280	0.440	2.637	0.550	0.865	3.250
0.290	0.455	2.660	0.560	0.880	3.266
0.300	0.471	2.684	0.570	0.895	3.283
0.310	0.487	2.709	0.580	0.911	3.298
0.320	0.503	2.735	0.590	0.927	3.313
0.330	0.518	2.758	0.596	0.936	3.322

Table 5.4

(continuation)

$Z_c = 85\Omega$

$\frac{2}{\pi}\theta'$	θ'	ε_r	$\frac{2}{\pi}\theta'$	θ'	ε_r
0.045	0.071	2.300	0.240	0.378	2.767
0.050	0.079	2.301	0.250	0.392	2.801
0.060	0.095	2.305	0.260	0.409	2.840
0.070	0.110	2.314	0.270	0.424	2.878
0.080	0.126	2.326	0.280	0.440	2.915
0.090	0.142	2.341	0.290	0.456	2.955
0.100	0.157	2.357	0.300	0.471	2.994
0.110	0.174	2.378	0.310	0.487	3.035
0.120	0.189	2.399	0.320	0.502	3.072
0.130	0.205	2.422	0.330	0.518	3.112
0.140	0.220	2.446	0.340	0.533	3.151
0.150	0.236	2.474	0.350	0.550	3.193
0.160	0.251	2.500	0.360	0.565	3.232
0.170	0.267	2.530	0.370	0.581	3.275
0.180	0.283	2.561	0.380	0.597	3.315
0.190	0.299	2.594	0.390	0.613	3.355
0.200	0.315	2.626	0.400	0.628	3.393
0.210	0.329	2.658	0.410	0.644	3.434
0.220	0.346	2.695	0.420	0.660	3.475
0.230	0.361	2.728	0.424	0.667	3.490

$\varepsilon_{r_0} = 2.30$

$\frac{2}{\pi}\theta_0$	θ_0	Z_c	ε_{r_1}	$\frac{2}{\pi}\theta_0$	θ_0	Z_c	ε_{r_1}
0.150	0.236	128.04	2.358	0.580	0.911	42.81	2.116
0.160	0.251	124.13	2.457	0.590	0.927	41.63	2.111
0.170	0.267	120.45	2.568	0.600	0.942	40.46	2.108
0.180	0.283	116.98	2.688	0.610	0.958	39.30	2.106
0.190	0.298	113.69	2.815	0.620	0.974	38.15	2.105
0.200	0.314	110.56	2.941	0.630	0.990	37.02	2.104
0.210	0.330	107.59	3.049	0.640	1.005	35.90	2.105
0.220	0.346	104.74	3.113	0.650	1.021	34.79	2.106
0.230	0.361	102.02	3.094	0.660	1.037	33.69	2.108
0.240	0.377	99.40	2.945	0.670	1.052	32.60	2.111
0.250	0.393	96.89	2.638	0.680	1.068	31.52	2.114
0.260	0.408	94.48	2.426	0.690	1.084	30.45	2.118
0.270	0.424	92.14	2.922	0.700	1.100	29.38	2.123
0.280	0.440	89.89	3.263	0.710	1.115	28.33	2.128
0.290	0.456	87.72	3.437	0.720	1.131	27.28	2.133
0.300	0.471	85.61	3.480	0.730	1.147	26.25	2.139
0.310	0.487	83.56	3.438	0.740	1.162	25.22	2.145
0.320	0.503	81.58	3.349	0.750	1.178	24.19	2.152
0.330	0.518	79.65	3.239	0.760	1.194	23.18	2.159
0.340	0.534	77.77	3.124	0.770	1.210	22.16	2.166
0.350	0.550	75.94	3.012	0.780	1.225	21.16	2.174
0.360	0.565	74.16	2.908	0.790	1.241	20.16	2.181
0.370	0.581	72.43	2.813	0.800	1.257	19.17	2.189
0.380	0.597	70.73	2.728	0.810	1.272	18.18	2.197
0.390	0.613	69.07	2.651	0.820	1.288	17.20	2.205
0.400	0.628	67.45	2.583	0.830	1.304	16.22	2.212
0.410	0.644	65.86	2.522	0.840	1.319	15.24	2.220
0.420	0.660	64.31	2.468	0.850	1.335	14.27	2.228
0.430	0.675	62.79	2.420	0.860	1.351	13.30	2.236
0.440	0.691	61.30	2.377	0.870	1.367	12.34	2.243
0.450	0.707	59.83	2.340	0.880	1.382	11.38	2.250
0.460	0.723	58.39	2.306	0.890	1.398	10.42	2.257
0.470	0.738	56.98	2.276	0.900	1.414	9.46	2.264
0.480	0.754	55.59	2.250	0.910	1.429	8.51	2.270
0.490	0.770	54.23	2.227	0.920	1.445	7.56	2.276
0.500	0.785	52.88	2.206	0.930	1.461	6.61	2.281
0.510	0.801	51.56	2.188	0.940	1.477	5.66	2.286
0.520	0.817	50.26	2.173	0.950	1.492	4.72	2.290
0.530	0.833	48.97	2.159	0.960	1.508	3.77	2.293
0.540	0.848	47.71	2.147	0.970	1.524	2.83	2.296
0.550	0.864	46.46	2.137	0.980	1.539	1.89	2.298
0.560	0.880	45.23	2.129	0.990	1.555	0.94	2.300
0.570	0.895	44.01	2.122	1.000	1.571	0.00	2.300

Table 5.5: Relative dielectric constant as a function of conical antenna angle θ_0 (or cone impedance Z_c), with ε_{r_0} as parameter. The relative dielectric constant is ε_{r_1}, calculated at $\theta_1 = \pi/2$.

Table 5.5

(continuation)

$\varepsilon_{r_0} = 3.00$

$\frac{2}{\pi}\vartheta_0$	ϑ_0	Z_c	ε_{r_1}	$\frac{2}{\pi}\vartheta_0$	ϑ_0	Z_c	ε_{r_1}
0.330	0.518	79.65	3.153	0.670	1.052	32.60	2.728
0.340	0.534	77.77	3.367	0.680	1.068	31.52	2.729
0.350	0.550	75.94	3.800	0.690	1.084	30.45	2.732
0.360	0.565	74.16	4.070	0.700	1.100	29.38	2.736
0.370	0.581	72.43	4.192	0.710	1.115	28.33	2.741
0.380	0.597	70.73	4.204	0.720	1.131	27.28	2.747
0.390	0.613	69.07	4.144	0.730	1.147	26.25	2.754
0.400	0.628	67.45	4.044	0.740	1.162	25.22	2.762
0.410	0.644	65.86	3.926	0.750	1.178	24.19	2.770
0.420	0.660	64.31	3.803	0.760	1.194	23.18	2.780
0.430	0.675	62.79	3.684	0.770	1.210	22.16	2.790
0.440	0.691	61.30	3.571	0.780	1.225	21.16	2.801
0.450	0.707	59.83	3.467	0.790	1.241	20.16	2.812
0.460	0.723	58.39	3.373	0.800	1.257	19.17	2.823
0.470	0.738	56.98	3.288	0.810	1.272	18.18	2.835
0.480	0.754	55.59	3.212	0.820	1.288	17.20	2.847
0.490	0.770	54.23	3.144	0.830	1.304	16.22	2.859
0.500	0.785	52.88	3.084	0.840	1.319	15.24	2.871
0.510	0.801	51.56	3.030	0.850	1.335	14.27	2.883
0.520	0.817	50.26	2.983	0.860	1.351	13.30	2.895
0.530	0.833	48.97	2.942	0.870	1.367	12.34	2.907
0.540	0.848	47.71	2.905	0.880	1.382	11.38	2.919
0.550	0.864	46.46	2.873	0.890	1.398	10.42	2.930
0.560	0.880	45.23	2.845	0.900	1.414	9.46	2.940
0.570	0.895	44.01	2.821	0.910	1.429	8.51	2.951
0.580	0.911	42.81	2.801	0.920	1.445	7.56	2.960
0.590	0.927	41.63	2.783	0.930	1.461	6.61	2.969
0.600	0.942	40.46	2.769	0.940	1.477	5.66	2.976
0.610	0.958	39.30	2.756	0.950	1.492	4.72	2.983
0.620	0.974	38.15	2.747	0.960	1.508	3.77	2.989
0.630	0.990	37.02	2.739	0.970	1.524	2.83	2.994
0.640	1.005	35.90	2.734	0.980	1.539	1.89	2.997
0.650	1.021	34.79	2.730	0.990	1.555	0.94	2.999
0.660	1.037	33.69	2.728	1.000	1.571	0.00	3.000

Table 5.5

(continuation)

$\varepsilon_{r_0} = 4.00$

$\frac{2}{\pi}\theta_0$	θ_0	Z_c	ε_{r_1}	$\frac{2}{\pi}\theta_0$	θ_0	Z_c	ε_{r_1}
0.400	0.628	67.45	4.423	0.710	1.115	28.33	3.643
0.410	0.644	65.86	4.021	0.720	1.131	27.28	3.645
0.420	0.660	64.31	4.570	0.730	1.147	26.25	3.650
0.430	0.675	62.79	4.950	0.740	1.162	25.22	3.657
0.440	0.691	61.30	5.161	0.750	1.178	24.19	3.666
0.450	0.707	59.83	5.233	0.760	1.194	23.18	3.676
0.460	0.723	58.39	5.207	0.770	1.210	22.16	3.688
0.470	0.738	56.98	5.120	0.780	1.225	21.16	3.701
0.480	0.754	55.59	5.000	0.790	1.241	20.16	3.716
0.490	0.770	54.23	4.866	0.800	1.257	19.17	3.731
0.500	0.785	52.88	4.729	0.810	1.272	18.18	3.748
0.510	0.801	51.56	4.597	0.820	1.288	17.20	3.765
0.520	0.817	50.26	4.473	0.830	1.304	16.22	3.782
0.530	0.833	48.97	4.359	0.840	1.319	15.24	3.800
0.540	0.848	47.71	4.256	0.850	1.335	14.27	3.818
0.550	0.864	46.46	4.164	0.860	1.351	13.30	3.836
0.560	0.880	45.23	4.082	0.870	1.367	12.34	3.854
0.570	0.895	44.01	4.010	0.880	1.382	11.38	3.872
0.580	0.911	42.81	3.946	0.890	1.398	10.42	3.889
0.590	0.927	41.63	3.890	0.900	1.414	9.46	3.906
0.600	0.942	40.46	3.842	0.910	1.429	8.51	3.922
0.610	0.958	39.30	3.800	0.920	1.445	7.56	3.936
0.620	0.974	38.15	3.764	0.930	1.461	6.61	3.950
0.630	0.990	37.02	3.734	0.940	1.477	5.66	3.962
0.640	1.005	35.90	3.709	0.950	1.492	4.72	3.973
0.650	1.021	34.79	3.688	0.960	1.508	3.77	3.983
0.660	1.037	33.69	3.672	0.970	1.524	2.83	3.990
0.670	1.052	32.60	3.660	0.980	1.539	1.89	3.995
0.680	1.068	31.52	3.651	0.990	1.555	0.94	3.999
0.690	1.084	30.45	3.645	1.000	1.571	0.00	4.000
0.700	1.100	29.38	3.643				

Table 5.5

(continuation)

$\varepsilon_{r_0} = 5.00$

$\frac{2}{\pi}\theta_0$	θ_0	Z_c	ε_{r_1}	$\frac{2}{\pi}\theta_0$	θ_0	Z_c	ε_{r_1}
0.460	0.723	58.39	5.229	0.740	1.162	25.22	4.577
0.470	0.738	56.98	5.341	0.750	1.178	24.19	4.581
0.480	0.754	55.59	5.822	0.760	1.194	23.18	4.589
0.490	0.770	54.23	6.111	0.770	1.210	22.16	4.600
0.500	0.785	52.88	6.236	0.780	1.225	21.16	4.613
0.510	0.801	51.56	6.239	0.790	1.241	20.16	4.628
0.520	0.817	50.26	6.162	0.800	1.257	19.17	4.646
0.530	0.833	48.97	6.039	0.810	1.272	18.18	4.665
0.540	0.848	47.71	5.894	0.820	1.288	17.20	4.686
0.550	0.864	46.46	5.742	0.830	1.304	16.22	4.708
0.560	0.880	45.23	5.594	0.840	1.319	15.24	4.730
0.570	0.895	44.01	5.453	0.850	1.335	14.27	4.754
0.580	0.911	42.81	5.324	0.860	1.351	13.30	4.777
0.590	0.927	41.63	5.207	0.870	1.367	12.34	4.801
0.600	0.942	40.46	5.103	0.880	1.382	11.38	4.825
0.610	0.958	39.30	5.010	0.890	1.398	10.42	4.848
0.620	0.974	38.15	4.929	0.900	1.414	9.46	4.870
0.630	0.990	37.02	4.859	0.910	1.429	8.51	4.892
0.640	1.005	35.90	4.798	0.920	1.445	7.56	4.912
0.650	1.021	34.79	4.746	0.930	1.461	6.61	4.931
0.660	1.037	33.69	4.703	0.940	1.477	5.66	4.948
0.670	1.052	32.60	4.667	0.950	1.492	4.72	4.963
0.680	1.068	31.52	4.638	0.960	1.508	3.77	4.976
0.690	1.084	30.45	4.615	0.970	1.524	2.83	4.986
0.700	1.100	29.38	4.598	0.980	1.539	1.89	4.994
0.710	1.115	28.33	4.586	0.990	1.555	0.94	4.998
0.720	1.131	27.28	4.579	1.000	1.571	0.00	5.000
0.730	1.147	26.25	4.576				

Table 5.5

(continuation)

$\varepsilon_{r_0} = 7.00$

$\frac{2}{\pi}\theta_0$	θ_0	Z_c	ε_{r_1}	$\frac{2}{\pi}\theta_0$	θ_0	Z_c	ε_{r_1}
0.530	0.833	48.97	7.494	0.780	1.225	21.16	6.482
0.540	0.848	47.71	7.008	0.790	1.241	20.16	6.491
0.550	0.864	46.46	7.649	0.800	1.257	19.17	6.505
0.560	0.880	45.23	8.047	0.810	1.272	18.18	6.524
0.570	0.895	44.01	8.234	0.820	1.288	17.20	6.547
0.580	0.911	42.81	8.263	0.830	1.304	16.22	6.573
0.590	0.927	41.63	8.187	0.840	1.319	15.24	6.601
0.600	0.942	40.46	8.048	0.850	1.335	14.27	6.632
0.610	0.958	39.30	7.879	0.860	1.351	13.30	6.665
0.620	0.974	38.15	7.700	0.870	1.367	12.34	6.698
0.630	0.990	37.02	7.524	0.880	1.382	11.38	6.732
0.640	1.005	35.90	7.358	0.890	1.398	10.42	6.766
0.650	1.021	34.79	7.207	0.900	1.414	9.46	6.800
0.660	1.037	33.69	7.071	0.910	1.429	8.51	6.832
0.670	1.052	32.60	6.952	0.920	1.445	7.56	6.863
0.680	1.068	31.52	6.849	0.930	1.461	6.61	6.892
0.690	1.084	30.45	6.761	0.940	1.477	5.66	6.918
0.700	1.100	29.38	6.687	0.950	1.492	4.72	6.942
0.710	1.115	28.33	6.626	0.960	1.508	3.77	6.962
0.720	1.131	27.28	6.577	0.970	1.524	2.83	6.978
0.730	1.147	26.25	6.540	0.980	1.539	1.89	6.990
0.740	1.162	25.22	6.512	0.990	1.555	0.94	6.997
0.750	1.178	24.19	6.493	1.000	1.571	0.00	7.000
0.760	1.194	23.18	6.482	1.010	1.587	-0.94	6.997
0.770	1.210	22.16	6.479				

Table 5.5

(continuation)

$\varepsilon_{r_0} = 10.00$

$\frac{2}{\pi}\theta_0$	θ_0	Z_c	ε_{r_1}
0.610	0.958	39.30	10.002
0.620	0.974	38.15	10.717
0.630	0.990	37.02	11.122
0.640	1.005	35.90	11.274
0.650	1.021	34.79	11.251
0.660	1.037	33.69	11.118
0.670	1.052	32.60	10.928
0.680	1.068	31.52	10.714
0.690	1.084	30.45	10.499
0.700	1.100	29.38	10.296
0.710	1.115	28.33	10.112
0.720	1.131	27.28	9.950
0.730	1.147	26.25	9.810
0.740	1.162	25.22	9.693
0.750	1.178	24.19	9.597
0.760	1.194	23.18	9.522
0.770	1.210	22.16	9.465
0.780	1.225	21.16	9.424
0.790	1.241	20.16	9.399
0.800	1.257	19.17	9.387
0.810	1.272	18.18	9.387
0.820	1.288	17.20	9.397
0.830	1.304	16.22	9.417
0.840	1.319	15.24	9.444
0.850	1.335	14.27	9.477
0.860	1.351	13.30	9.516
0.870	1.367	12.34	9.558
0.880	1.382	11.38	9.603
0.890	1.398	10.42	9.650
0.900	1.414	9.46	9.698
0.910	1.429	8.51	9.745
0.920	1.445	7.56	9.791
0.930	1.461	6.61	9.834
0.940	1.477	5.66	9.874
0.950	1.492	4.72	9.910
0.960	1.508	3.77	9.941
0.970	1.524	2.83	9.966
0.980	1.539	1.89	9.984
0.990	1.555	0.94	9.996
1.000	1.571	0.00	10.000

$Z_c = 50\Omega$		$Z_c = 60\Omega$		$Z_c = 70\Omega$	
Ψ/r_0	z/r_0	Ψ/r_0	z/r_0	Ψ/r_0	z/r_0
0.5000	0.6233	0.5000	0.7092	0.5000	0.7935
0.6000	0.6541	0.6000	0.7467	0.6000	0.8358
0.7000	0.6767	0.7000	0.7757	0.7000	0.8691
0.8000	0.6925	0.8000	0.7976	0.8000	0.8952
0.9000	0.7024	0.9000	0.8135	0.9000	0.9150
1.0000	0.7071	1.0000	0.8240	1.0000	0.9294
1.1000	0.7072	1.1000	0.8299	1.1000	0.9390
1.2000	0.7030	1.2000	0.8314	1.2000	0.9441
1.3000	0.6950	1.3000	0.8291	1.3000	0.9453
1.4000	0.6834	1.4000	0.8231	1.4000	0.9428
1.5000	0.6685	1.5000	0.8137	1.5000	0.9369
1.6000	0.6504	1.6000	0.8012	1.6000	0.9278
1.7000	0.6293	1.7000	0.7857	1.7000	0.9157
1.8000	0.6054	1.8000	0.7674	1.8000	0.9007
1.9000	0.5788	1.9000	0.7464	1.9000	0.8830
2.0000	0.5496	2.0000	0.7228	2.0000	0.8628
2.1000	0.5180	2.1000	0.6968	2.1000	0.8400
2.2000	0.4840	2.2000	0.6684	2.2000	0.8149
2.3000	0.4477	2.3000	0.6378	2.3000	0.7875
2.4000	0.4091	2.4000	0.6049	2.4000	0.7579
2.5000	0.3685	2.5000	0.5700	2.5000	0.7262
2.6000	0.3258	2.6000	0.5330	2.6000	0.6924
2.7000	0.2810	2.7000	0.4940	2.7000	0.6566
2.8000	0.2343	2.8000	0.4530	2.8000	0.6188
2.9000	0.1857	2.9000	0.4102	2.9000	0.5792
3.0000	0.1353	3.0000	0.3655	3.0000	0.5377
3.1000	0.0830	3.1000	0.3191	3.1000	0.4944
3.2000	0.0289	3.2000	0.2709	3.2000	0.4493
		3.3000	0.2210	3.3000	0.4026
		3.4000	0.1694	3.4000	0.3541
		3.5000	0.1161	3.5000	0.3040
		3.6000	0.0613	3.6000	0.2523
		3.7000	0.0049	3.7000	0.1990
				3.8000	0.1441
				3.9000	0.0878
				4.0000	0.0299

Table 5.6: Shape of lens/free-space boundary for $\varepsilon_{r_0} = 2.30$, with cone impedance Z_c as parameter. The table gives z/r_0 versus Ψ/r_0 where Ψ is the cylindrical radius and z is the height above the ground plane.

Table 5.6

(continuation)

$Z_c = 80\Omega$		$Z_c = 90\Omega$	
Ψ/r_0	z/r_0	Ψ/r_0	z/r_0
0.5000	0.8736	0.5000	0.9471
0.6000	0.9186	0.6000	0.9928
0.7000	0.9544	0.7000	1.0292
0.8000	0.9827	0.8000	1.0579
0.9000	1.0046	0.9000	1.0801
1.0000	1.0209	1.0000	1.0966
1.1000	1.0322	1.1000	1.1081
1.2000	1.0391	1.2000	1.1150
1.3000	1.0420	1.3000	1.1179
1.4000	1.0411	1.4000	1.1169
1.5000	1.0367	1.5000	1.1124
1.6000	1.0291	1.6000	1.1046
1.7000	1.0184	1.7000	1.0938
1.8000	1.0048	1.8000	1.0799
1.9000	0.9885	1.9000	1.0633
2.0000	0.9696	2.0000	1.0440
2.1000	0.9481	2.1000	1.0222
2.2000	0.9242	2.2000	0.9979
2.3000	0.8981	2.3000	0.9712
2.4000	0.8696	2.4000	0.9423
2.5000	0.8391	2.5000	0.9111
2.6000	0.8064	2.6000	0.8778
2.7000	0.7717	2.7000	0.8425
2.8000	0.7350	2.8000	0.8051
2.9000	0.6963	2.9000	0.7657
3.0000	0.6559	3.0000	0.7244
3.1000	0.6135	3.1000	0.6813
3.2000	0.5694	3.2000	0.6363
3.3000	0.5236	3.3000	0.5895
3.4000	0.4760	3.4000	0.5409
3.5000	0.4267	3.5000	0.4906
3.6000	0.3758	3.6000	0.4387
3.7000	0.3233	3.7000	0.3850
3.8000	0.2692	3.8000	0.3297
3.9000	0.2135	3.9000	0.2729
4.0000	0.1563	4.0000	0.2144
4.1000	0.0976	4.1000	0.1543
		4.2000	0.0928
		4.3000	0.0297

$\varepsilon_{r_0} = 2.30$

Ψ/r_0	z/r_0	Ψ/r_0	z/r_0
0.6500	0.7622	2.2500	0.6534
0.7000	0.7757	2.3000	0.6378
0.7500	0.7875	2.3500	0.6216
0.8000	0.7976	2.4000	0.6049
0.8500	0.8062	2.4500	0.5877
0.9000	0.8135	2.5000	0.5700
0.9500	0.8194	2.5500	0.5517
1.0000	0.8240	2.6000	0.5330
1.0500	0.8275	2.6500	0.5137
1.1000	0.8299	2.7000	0.4940
1.1500	0.8312	2.7500	0.4737
1.2000	0.8314	2.8000	0.4530
1.2500	0.8307	2.8500	0.4318
1.3000	0.8291	2.9000	0.4102
1.3500	0.8265	2.9500	0.3881
1.4000	0.8231	3.0000	0.3655
1.4500	0.8188	3.0500	0.3425
1.5000	0.8137	3.1000	0.3191
1.5500	0.8079	3.1500	0.2952
1.6000	0.8012	3.2000	0.2709
1.6500	0.7938	3.2500	0.2461
1.7000	0.7857	3.3000	0.2210
1.7500	0.7769	3.3500	0.1954
1.8000	0.7674	3.4000	0.1694
1.8500	0.7572	3.4500	0.1430
1.9000	0.7464	3.5000	0.1161
1.9500	0.7349	3.5500	0.0889
2.0000	0.7228	3.6000	0.0613
2.0500	0.7101	3.6500	0.0333
2.1000	0.6968	3.7000	0.0049
2.1500	0.6829		
2.2000	0.6684		

Table 5.7: Shape of lens/free-space boundary for cone impedance $Z_c = 60\Omega$, with ε_{r_0} as a parameter. The table gives z/r_0 versus Ψ/r_0 where Ψ is the cylindrical radius and z is the height above the ground plane.

Table 5.7

(continuation)

$\varepsilon_{r_0} = 3.00$

Ψ/r_0	z/r_0	Ψ/r_0	z/r_0
0.6500	0.7623	2.5000	0.7181
0.7000	0.7791	2.5500	0.7046
0.7500	0.7942	2.6000	0.6907
0.8000	0.8077	2.6500	0.6762
0.8500	0.8198	2.7000	0.6614
0.9000	0.8305	2.7500	0.6461
0.9500	0.8399	2.8000	0.6304
1.0000	0.8481	2.8500	0.6142
1.0500	0.8551	2.9000	0.5976
1.1000	0.8611	2.9500	0.5806
1.1500	0.8660	3.0000	0.5632
1.2000	0.8699	3.0500	0.5454
1.2500	0.8730	3.1000	0.5272
1.3000	0.8751	3.1500	0.5086
1.3500	0.8763	3.2000	0.4896
1.4000	0.8767	3.2500	0.4702
1.4500	0.8763	3.3000	0.4504
1.5000	0.8751	3.3500	0.4303
1.5500	0.8732	3.4000	0.4097
1.6000	0.8705	3.4500	0.3888
1.6500	0.8672	3.5000	0.3676
1.7000	0.8632	3.5500	0.3460
1.7500	0.8584	3.6000	0.3240
1.8000	0.8531	3.6500	0.3017
1.8500	0.8471	3.7000	0.2790
1.9000	0.8405	3.7500	0.2560
1.9500	0.8333	3.8000	0.2326
2.0000	0.8255	3.8500	0.2090
2.0500	0.8172	3.9000	0.1849
2.1000	0.8083	3.9500	0.1606
2.1500	0.7988	4.0000	0.1359
2.2000	0.7888	4.0500	0.1108
2.2500	0.7783	4.1000	0.0855
2.3000	0.7672	4.1500	0.0598
2.3500	0.7557	4.2000	0.0339
2.4000	0.7436	4.2500	0.0076
2.4500	0.7311		

Table 5.7

(continuation)

$\varepsilon_{r_0} = 4.00$

Ψ/r_0	z/r_0	Ψ/r_0	z/r_0
0.6500	0.7624	2.8500	0.8091
0.7000	0.7827	2.9000	0.7979
0.7500	0.8014	2.9500	0.7862
0.8000	0.8185	3.0000	0.7742
0.8500	0.8342	3.0500	0.7618
0.9000	0.8486	3.1000	0.7491
0.9500	0.8617	3.1500	0.7360
1.0000	0.8737	3.2000	0.7226
1.0500	0.8846	3.2500	0.7088
1.1000	0.8944	3.3000	0.6946
1.1500	0.9033	3.3500	0.6802
1.2000	0.9112	3.4000	0.6653
1.2500	0.9182	3.4500	0.6502
1.3000	0.9244	3.5000	0.6347
1.3500	0.9297	3.5500	0.6190
1.4000	0.9342	3.6000	0.6028
1.4500	0.9380	3.6500	0.5864
1.5000	0.9410	3.7000	0.5697
1.5500	0.9434	3.7500	0.5527
1.6000	0.9450	3.8000	0.5353
1.6500	0.9460	3.8500	0.5177
1.7000	0.9463	3.9000	0.4997
1.7500	0.9461	3.9500	0.4815
1.8000	0.9452	4.0000	0.4630
1.8500	0.9437	4.0500	0.4441
1.9000	0.9416	4.1000	0.4250
1.9500	0.9390	4.1500	0.4056
2.0000	0.9358	4.2000	0.3860
2.0500	0.9321	4.2500	0.3660
2.1000	0.9279	4.3000	0.3458
2.1500	0.9232	4.3500	0.3253
2.2000	0.9180	4.4000	0.3046
2.2500	0.9123	4.4500	0.2835
2.3000	0.9061	4.5000	0.2622
2.3500	0.8994	4.5500	0.2407
2.4000	0.8923	4.6000	0.2189
2.4500	0.8848	4.6500	0.1968
2.5000	0.8768	4.7000	0.1745
2.5500	0.8684	4.7500	0.1519
2.6000	0.8595	4.8000	0.1290
2.6500	0.8502	4.8500	0.1059
2.7000	0.8405	4.9000	0.0826
2.7500	0.8305	4.9500	0.0590
2.8000	0.8200	5.0000	0.0352

Table 5.7

(continuation)

$\varepsilon_{r_0} = 5.00$

Ψ/r_0	z/r_0	Ψ/r_0	z/r_0
0.6500	0.7625	3.3000	0.8804
0.7000	0.7855	3.3500	0.8701
0.7500	0.8068	3.4000	0.8596
0.8000	0.8266	3.4500	0.8488
0.8500	0.8451	3.5000	0.8376
0.9000	0.8623	3.5500	0.8262
0.9500	0.8783	3.6000	0.8145
1.0000	0.8932	3.6500	0.8024
1.0500	0.9070	3.7000	0.7901
1.1000	0.9198	3.7500	0.7775
1.1500	0.9316	3.8000	0.7647
1.2000	0.9426	3.8500	0.7515
1.2500	0.9527	3.9000	0.7381
1.3000	0.9620	3.9500	0.7244
1.3500	0.9704	4.0000	0.7105
1.4000	0.9781	4.0500	0.6962
1.4500	0.9851	4.1000	0.6817
1.5000	0.9914	4.1500	0.6670
1.5500	0.9970	4.2000	0.6520
1.6000	1.0019	4.2500	0.6368
1.6500	1.0063	4.3000	0.6212
1.7000	1.0100	4.3500	0.6055
1.7500	1.0131	4.4000	0.5895
1.8000	1.0156	4.4500	0.5733
1.8500	1.0175	4.5000	0.5568
1.9000	1.0190	4.5500	0.5401
1.9500	1.0199	4.6000	0.5231
2.0000	1.0202	4.6500	0.5059
2.0500	1.0201	4.7000	0.4885
2.1000	1.0195	4.7500	0.4708
2.1500	1.0184	4.8000	0.4529
2.2000	1.0168	4.8500	0.4348
2.2500	1.0147	4.9000	0.4165
2.3000	1.0123	4.9500	0.3979
2.3500	1.0093	5.0000	0.3791
2.4000	1.0060	5.0500	0.3601
2.4500	1.0022	5.1000	0.3409
2.5000	0.9980	5.1500	0.3215
2.5500	0.9934	5.2000	0.3018
2.6000	0.9884	5.2500	0.2820
2.6500	0.9830	5.3000	0.2619
2.7000	0.9773	5.3500	0.2416
2.7500	0.9711	5.4000	0.2211
2.8000	0.9646	5.4500	0.2005
2.8500	0.9577	5.5000	0.1796
2.9000	0.9505	5.5500	0.1585
2.9500	0.9429	5.6000	0.1372
3.0000	0.9350	5.6500	0.1157
3.0500	0.9267	5.7000	0.0940
3.1000	0.9181	5.7500	0.0721
3.1500	0.9091	5.8000	0.0500
3.2000	0.8999	5.8500	0.0277
3.2500	0.8903	5.9000	0.0052

Table 5.7

(continuation)

$\varepsilon_{r_0} = 7.00$

Ψ/r_0	z/r_0	Ψ/r_0	z/r_0	Ψ/r_0	z/r_0
0.6500	0.7627	3.0500	1.1690	5.4000	0.7790
0.7000	0.7894	3.1000	1.1664	5.4500	0.7658
0.7500	0.8146	3.1500	1.1635	5.5000	0.7523
0.8000	0.8384	3.2000	1.1603	5.5500	0.7387
0.8500	0.8609	3.2500	1.1568	5.6000	0.7249
0.9000	0.8823	3.3000	1.1530	5.6500	0.7109
0.9500	0.9024	3.3500	1.1490	5.7000	0.6968
1.0000	0.9216	3.4000	1.1446	5.7500	0.6824
1.0500	0.9397	3.4500	1.1400	5.8000	0.6680
1.1000	0.9569	3.5000	1.1351	5.8500	0.6533
1.1500	0.9732	3.5500	1.1300	5.9000	0.6385
1.2000	0.9886	3.6000	1.1246	5.9500	0.6235
1.2500	1.0033	3.6500	1.1189	6.0000	0.6084
1.3000	1.0171	3.7000	1.1130	6.0500	0.5931
1.3500	1.0302	3.7500	1.1068	6.1000	0.5776
1.4000	1.0426	3.8000	1.1004	6.1500	0.5620
1.4500	1.0544	3.8500	1.0938	6.2000	0.5462
1.5000	1.0654	3.9000	1.0868	6.2500	0.5302
1.5500	1.0759	3.9500	1.0797	6.3000	0.5141
1.6000	1.0857	4.0000	1.0723	6.3500	0.4979
1.6500	1.0949	4.0500	1.0647	6.4000	0.4815
1.7000	1.1036	4.1000	1.0569	6.4500	0.4649
1.7500	1.1117	4.1500	1.0488	6.5000	0.4482
1.8000	1.1193	4.2000	1.0405	6.5500	0.4313
1.8500	1.1263	4.2500	1.0319	6.6000	0.4143
1.9000	1.1329	4.3000	1.0232	6.6500	0.3971
1.9500	1.1389	4.3500	1.0142	6.7000	0.3798
2.0000	1.1445	4.4000	1.0050	6.7500	0.3623
2.0500	1.1496	4.4500	0.9956	6.8000	0.3447
2.1000	1.1543	4.5000	0.9860	6.8500	0.3270
2.1500	1.1585	4.5500	0.9762	6.9000	0.3091
2.2000	1.1623	4.6000	0.9662	6.9500	0.2910
2.2500	1.1657	4.6500	0.9560	7.0000	0.2728
2.3000	1.1686	4.7000	0.9455	7.0500	0.2545
2.3500	1.1712	4.7500	0.9349	7.1000	0.2360
2.4000	1.1734	4.8000	0.9240	7.1500	0.2174
2.4500	1.1751	4.8500	0.9130	7.2000	0.1987
2.5000	1.1765	4.9000	0.9018	7.2500	0.1798
2.5500	1.1776	4.9500	0.8903	7.3000	0.1608
2.6000	1.1782	5.0000	0.8787	7.3500	0.1416
2.6500	1.1786	5.0500	0.8669	7.4000	0.1223
2.7000	1.1785	5.1000	0.8549	7.4500	0.1029
2.7500	1.1781	5.1500	0.8427	7.5000	0.0833
2.8000	1.1774	5.2000	0.8304	7.5500	0.0636
2.8500	1.1764	5.2500	0.8178	7.6000	0.0437
2.9000	1.1750	5.3000	0.8051	7.6500	0.0238
2.9500	1.1733	5.3500	0.7921	7.7000	0.0037
3.0000	1.1713				

Table 5.7

(continuation)

$\varepsilon_{r_0} = 10.00$

Ψ/r_0	z/r_0	Ψ/r_0	z/r_0	Ψ/r_0	z/r_0	Ψ/r_0	z/r_0
6500	0.7628	3.2000	1.4254	5.7500	1.2952	8.3000	0.7714
0.7000	0.7933	3.2500	1.4281	5.8000	1.2882	8.3500	0.7582
0.7500	0.8224	3.3000	1.4305	5.8500	1.2810	8.4000	0.7448
0.8000	0.8502	3.3500	1.4326	5.9000	1.2737	8.4500	0.7313
0.8500	0.8768	3.4000	1.4345	5.9500	1.2662	8.5000	0.7177
0.9000	0.9023	3.4500	1.4362	6.0000	1.2586	8.5500	0.7041
0.9500	0.9267	3.5000	1.4376	6.0500	1.2508	8.6000	0.6903
1.0000	0.9501	3.5500	1.4388	6.1000	1.2429	8.6500	0.6764
1.0500	0.9727	3.6000	1.4398	6.1500	1.2349	8.7000	0.6624
1.1000	0.9943	3.6500	1.4405	6.2000	1.2267	8.7500	0.6483
1.1500	1.0151	3.7000	1.4410	6.2500	1.2184	8.8000	0.6342
1.2000	1.0351	3.7500	1.4413	6.3000	1.2100	8.8500	0.6199
1.2500	1.0544	3.8000	1.4413	6.3500	1.2014	8.9000	0.6055
1.3000	1.0730	3.8500	1.4412	6.4000	1.1927	8.9500	0.5910
1.3500	1.0908	3.9000	1.4408	6.4500	1.1838	9.0000	0.5764
1.4000	1.1080	3.9500	1.4402	6.5000	1.1749	9.0500	0.5618
1.4500	1.1246	4.0000	1.4394	6.5500	1.1657	9.1000	0.5470
1.5000	1.1406	4.0500	1.4384	6.6000	1.1565	9.1500	0.5321
1.5500	1.1560	4.1000	1.4371	6.6500	1.1472	9.2000	0.5172
1.6000	1.1708	4.1500	1.4357	6.7000	1.1377	9.2500	0.5021
1.6500	1.1851	4.2000	1.4341	6.7500	1.1281	9.3000	0.4870
1.7000	1.1988	4.2500	1.4323	6.8000	1.1183	9.3500	0.4717
1.7500	1.2121	4.3000	1.4303	6.8500	1.1084	9.4000	0.4564
1.8000	1.2248	4.3500	1.4280	6.9000	1.0985	9.4500	0.4410
1.8500	1.2371	4.4000	1.4256	6.9500	1.0883	9.5000	0.4255
1.9000	1.2489	4.4500	1.4230	7.0000	1.0781	9.5500	0.4099
1.9500	1.2602	4.5000	1.4203	7.0500	1.0677	9.6000	0.3941
2.0000	1.2711	4.5500	1.4173	7.1000	1.0573	9.6500	0.3784
2.0500	1.2816	4.6000	1.4142	7.1500	1.0467	9.7000	0.3625
2.1000	1.2917	4.6500	1.4108	7.2000	1.0360	9.7500	0.3465
2.1500	1.3014	4.7000	1.4073	7.2500	1.0251	9.8000	0.3304
2.2000	1.3107	4.7500	1.4036	7.3000	1.0142	9.8500	0.3143
2.2500	1.3196	4.8000	1.3998	7.3500	1.0031	9.9000	0.2980
2.3000	1.3281	4.8500	1.3957	7.4000	0.9919	9.9500	0.2817
2.3500	1.3362	4.9000	1.3915	7.4500	0.9806	10.0000	0.2653
2.4000	1.3440	4.9500	1.3872	7.5000	0.9692	10.0500	0.2487
2.4500	1.3514	5.0000	1.3826	7.5500	0.9577	10.1000	0.2321
2.5000	1.3585	5.0500	1.3779	7.6000	0.9460	10.1500	0.2155
2.5500	1.3653	5.1000	1.3730	7.6500	0.9343	10.2000	0.1987
2.6000	1.3717	5.1500	1.3680	7.7000	0.9224	10.2500	0.1818
2.6500	1.3778	5.2000	1.3628	7.7500	0.9104	10.3000	0.1649
2.7000	1.3836	5.2500	1.3574	7.8000	0.8983	10.3500	0.1478
2.7500	1.3891	5.3000	1.3519	7.8500	0.8861	10.4000	0.1307
2.8000	1.3943	5.3500	1.3462	7.9000	0.8738	10.4500	0.1135
2.8500	1.3992	5.4000	1.3404	7.9500	0.8614	10.5000	0.0962
2.9000	1.4038	5.4500	1.3344	8.0000	0.8488	10.5500	0.0789
2.9500	1.4081	5.5000	1.3282	8.0500	0.8362	10.6000	0.0614
3.0000	1.4121	5.5500	1.3219	8.1000	0.8235	10.6500	0.0439
3.0500	1.4158	5.6000	1.3155	8.1500	0.8106	10.7000	0.0262
3.1000	1.4193	5.6500	1.3089	8.2000	0.7977	10.7500	0.0085
3.1500	1.4225	5.7000	1.3021	8.2500	0.7846		

θ'	θ	ε_r
0.300	0.705027	2.3000
0.310	0.719372	2.3001
0.320	0.733503	2.3006
0.330	0.747428	2.3012
0.340	0.761153	2.3021
0.350	0.774685	2.3033
0.360	0.788029	2.3046
0.370	0.801193	2.3061
0.380	0.814180	2.3078
0.390	0.826997	2.3096
0.400	0.839648	2.3116
0.410	0.852137	2.3138
0.420	0.864470	2.3160
0.430	0.876651	2.3184
0.440	0.888684	2.3209
0.450	0.900572	2.3234
0.460	0.912320	2.3261
0.470	0.923930	2.3288
0.480	0.935408	2.3316
0.490	0.946755	2.3344
0.500	0.957976	2.3373
0.510	0.969073	2.3402
0.520	0.980050	2.3431
0.530	0.990909	2.3461
0.540	1.001653	2.3491
0.550	1.012285	2.3521
0.560	1.022808	2.3551
0.570	1.033225	2.3581
0.580	1.043537	2.3611
0.590	1.053747	2.3640
0.600	1.063858	2.3670
0.610	1.073872	2.3699
0.620	1.083791	2.3728
0.630	1.093617	2.3756
0.640	1.103353	2.3784
0.650	1.113001	2.3812
0.660	1.122562	2.3839
0.670	1.132039	2.3865
0.680	1.141433	2.3891
0.690	1.150747	2.3916
0.700	1.159982	2.3940
0.710	1.169140	2.3964
0.720	1.178223	2.3987
0.730	1.187233	2.4009
0.740	1.196171	2.4029
0.750	1.205039	2.4050
0.760	1.213838	2.4069
0.770	1.222570	2.4087
0.780	1.231237	2.4104
0.790	1.239840	2.4120
0.800	1.248381	2.4134
0.810	1.256860	2.4148
0.820	1.265280	2.4160
0.830	1.273642	2.4172
0.840	1.281948	2.4182
0.850	1.290197	2.4191
0.860	1.298393	2.4198
0.870	1.306536	2.4204
0.880	1.314627	2.4209
0.890	1.322668	2.4213
0.900	1.330660	2.4215
0.910	1.338604	2.4216
0.920	1.346501	2.4215
0.930	1.354353	2.4213
0.940	1.362160	2.4209
0.950	1.369924	2.4204
0.960	1.377646	2.4198
0.970	1.385326	2.4189
0.980	1.392967	2.4180
0.990	1.400568	2.4169
1.000	1.408131	2.4156
1.010	1.415658	2.4142
1.020	1.423148	2.4126
1.030	1.430604	2.4108
1.040	1.438025	2.4089
1.050	1.445414	2.4069
1.060	1.452770	2.4046
1.070	1.460095	2.4023
1.080	1.467389	2.3997
1.090	1.474655	2.3970
1.100	1.481892	2.3941
1.110	1.489101	2.3911
1.120	1.496283	2.3879
1.130	1.503439	2.3845
1.140	1.510571	2.3809
1.150	1.517678	2.3772
1.160	1.524762	2.3734
1.170	1.531823	2.3693
1.180	1.538862	2.3651
1.190	1.545880	2.3608
1.200	1.552878	2.3563
1.210	1.559857	2.3516
1.220	1.566817	2.3467
1.230	1.573759	2.3417

Table 5.8. Numerical solution to the system (5.50) with $\varepsilon_{r_0} = 2.30$ and $Z_c = 60\Omega$

D. *Length of Boundary Curve*

We conclude this section with a derivation of an exact formula for the length of the curved boundary of the lens. The length is given by a rather complicated definite integral which can be evaluated numerically. We proceed with our derivation. From the geometry indicated in figure 3.2 we have:

$$\Psi = r\,cos(\frac{\pi}{2} - \theta) = r\,sin(\theta)$$

$$z = r\,sin(\frac{\pi}{2} - \theta) = r\,cos(\theta) \tag{5.51}$$

Hence, since $(ds)^2 = (d\Psi)^2 + (dz)^2$, we obtain

$$(ds)^2 = r^2(d\theta)^2 + (dr)^2 \tag{5.52}$$

and so if r can be expressed as a function of θ along the boundary curve we can determine its length. Since (5.18) can be reexpressed in the form

$$cot(\theta) - cot(\theta') = \frac{sin(\theta-\theta')}{sin(\theta)\cdot sin(\theta')} = cot(\theta_1')\cdot\tan^{\alpha}(\frac{\theta}{2}) \tag{5.53}$$

the law of sines (see(3.25)) leads us to

$$r = \frac{l\,cot^{\alpha}(\theta/2)}{cot(\theta_1')\cdot sin(\theta)} \tag{5.54}$$

where $\alpha = -L/l$ and θ_1' is the angle corresponding to $\theta_1 = \frac{\pi}{2}$. The preceding result can also be written as

$$r = \frac{l}{cot(\theta_1')}\left(\frac{1+cos(\theta)}{sin(\theta)}\right)^{\alpha}\frac{1}{sin(\theta)} . \tag{5.55}$$

Since

$$\frac{dr}{d\theta} = \frac{1}{cot(\theta_1')}\left(\frac{1+cos(\theta)}{sin(\theta)}\right)^{\alpha}\frac{[L-l\,cos(\theta)]}{\sin^2(\theta)} \tag{5.56}$$

we then find by a straightforward calculation that

$$(ds)^2 = \frac{[L^2+l^2-2lLcos(\theta)]}{cot^2(\theta_1')\cdot\sin^4(\theta)}\left(\frac{1+\cos(\theta)}{sin(\theta)}\right)^{2\alpha}(d\theta)^2 . \tag{5.57}$$

Thus the total length of the boundary Γ , denoted by $||\Gamma||$, is given by integration of (5.57) over the range $\theta_0 \leq \theta \leq \pi/2$. The result is

$$||\Gamma|| = \frac{1}{cot(\theta_1')} \int_{\theta_0}^{\pi/2} \frac{\sqrt{L^2+l^2-2lLcos(\theta)}}{\sin^2(\theta)} \left(\frac{1+cos(\theta)}{sin(\theta)} \right)^{\alpha} d\theta . \tag{5.58}$$

The curvature, κ , of Γ may also be calculated from the differential geometric formula

$$\kappa = \frac{d^2z}{d\Psi^2} / [1 + \left(\frac{dz}{d\Psi} \right)^2]^{3/2} \tag{5.59}$$

which, in view of (5.51), becomes

$$\kappa = \frac{r^2 + 2\left(\frac{dr}{d\theta} \right)^2 - r\left(\frac{d^2r}{d\theta^2} \right)}{[r^2 + \left(\frac{dr}{d\theta} \right)^2]^{3/2}} . \tag{5.60}$$

Thus, by a routine calculation, one obtains

$$\kappa = \frac{L\ cot(\theta_1')\, sin^2(\theta)[L - lcos(\theta)]}{[L^2+l^2Lcos(\theta)]^{3/2}} \left(\frac{sin(\theta)}{1+cos(\theta)} \right)^{\alpha} . \tag{5.61}$$

VI. APPROXIMATION OF EXACT CASE BY UNIFORM ISOTROPIC ϵ_r

Now that we have found the exact solution to our lens problem involving an $\epsilon_r(\theta')$, let us recall that our constraint of zero electric field in the r' direction requires an anisotropic medium which is in effect perfectly conducting (or has an infinite dielectric constant) in the r' direction. Practically, this can be realized by a set of (thin) conducting sheets in the form of circular cones which are surfaces of constant θ'; these sheets should be closely spaced in terms of wavelength, particularly in regions where the required $\frac{d\epsilon_r(\theta')}{d\theta'}$ is not small. By extending ridges of thin metal from these surfaces in the $\pm\theta'$ direction (surfaces of constant r') the electric field can be loaded to increase the effective ϵ_r by what are known as artificial-dielectric techniques. No conducting sheets should be placed on surfaces of constant ϕ or otherwise to interfere with H_ϕ because such would decrease the effective permeability which we have assumed to be a constant.

In this section we address the design problem of obtaining an average value of ϵ_r which approximates in some sense the exact case. For a given set of initial conditions (i.e., given ϵ_{r_0} and Z_c) we will have an inequality

$$\epsilon_{r_{min}} \leq \epsilon_{r_{avg}} \leq \epsilon_{r_{max}} \tag{6.1}$$

where typically $\epsilon_{r_{min}} = \epsilon_{r_0}$, and $\epsilon_{r_{max}}$ is determined by the initial data. Let us choose the value of $\epsilon_{r_{avg}}$ to be determined by matching the impedances in the lens and free-space conical systems. Thus, if ϵ_{r_0} and Z_c are specified, we then know θ_0 and θ_0' from (3.19) and (3.21). Hence if $\theta_1 = \frac{\pi}{2}$, then the corresponding values of ϵ_{r_1} and θ_1' are known. Thus, we seek a value $\epsilon_{r_{avg}}$ which satisfies the relations

$$Z_c = \frac{Z_0}{2\pi} \ln\left[\cot\left(\frac{\theta_0}{2}\right)\right] = \frac{1}{\sqrt{\epsilon_{r_{avg}}}} \frac{Z_0}{2\pi} \ln\left[\frac{\cot\left(\frac{\theta_0'}{2}\right)}{\cot\left(\frac{\theta_1'}{2}\right)}\right]. \tag{6.2}$$

It is important to realize that ϵ_{r_1} and $\epsilon_{r_{max}}$ need not in general be equal unless $\epsilon_r(\theta')$ is a non-decreasing function of θ' for the range of interest. For impedances $Z_c \geq 70\Omega$ and for $\epsilon_{r_0} = 2.3$ we do actually have $\epsilon_{r_1} = \epsilon_{r_{max}}$ (see table 6.1 and figure 5.2). We note that the condition (see (3.13)) that $\epsilon_r(\theta')$ be increasing is the condition that

$$\frac{2\sqrt{\epsilon_r}}{1+\epsilon_r} \geq \cos(\theta - \theta'). \tag{6.3}$$

This condition will in general not be realized over the entire range $\theta_0' \leq \theta' \leq \theta_1'$. For example, when $\epsilon_{r_0} = \epsilon_{r_{min}} = 2.3$ and $Z_c = 60\Omega$, calculations using the data from table 5.8 shows that condition (6.3) holds only for $(.300) \leq \theta' \leq (.910)$, while the range of interest for θ' is $(.300) \leq \theta' \leq 1.226$. For these values of ϵ_{r_0} and Z_c we have $\epsilon_{r_{max}} = 2.42$, while $\epsilon_{r_1} = 2.34$. The value of $\epsilon_{r_{avg}}$, from (6.2) is 2.36. This example and an examination of table 6.1 suggest that, since the variation of ϵ_r for some ranges of θ_0 and

ϵ_{r_0} is small, we may dispense with the guiding artificial dielectric structures in the lens and use some approximation in our design procedure. Specifically, in our example above, let us replace our value of 2.30 for ϵ_{r_0} with a value $\epsilon_{r_0} = 2.24$ and take $Z_c = 60\Omega$. These conditions would then yield a new solution to the system described by (3.23) through (3.26), and then a new value of $\epsilon_{r_{avg}}$ could be computed by (6.2). In this instance we find $\epsilon_{r_{avg}} = 2.30$. This example, while only an illustration, could form a basis for an iterative scheme yielding an approximate design procedure. Table 6.2 shows a tabulation of ϵ_r versus θ' and also z/r_o versus Ψ/r_o for the initial values $\epsilon_{r_0} = 2.24$ and $z_c = 60\Omega$. For these initial values we also have $l/r_o = 1.24565$. Table 6.3 gives a numerical solution to the system (5.50) by showing a tabulation of ϵ_r and θ versus θ' for the same initial values.

The value of $\epsilon_{r_{avg}}$ will of course give rise to a transit time error which is given in distance units by the formula

$$\frac{\Delta}{r_0} = \frac{r'(\theta')}{r_0}[\sqrt{\epsilon_{r_{avg}}} - \sqrt{\epsilon_r(\theta')} \tag{6.4}$$

where Δ represents an error as θ' varies. We find the transit time error in distance units is approximately

$$\Delta = r_o\sqrt{\epsilon_{r_{avg}}}\left[1 - \sqrt{\frac{\epsilon_r(\theta')}{\epsilon_{r_{avg}}}}\right]$$

$$\cong \frac{r_o}{2}\frac{1}{\sqrt{\epsilon_{r_{avg}}}}\left[\epsilon_{r_{avg}} - \epsilon_r(\theta')\right] \tag{6.5}$$

Thus, for example, when $\epsilon_{r_0} = 2.3$ and $Z_c = 60\Omega$, we have $2.30 \leq \epsilon_r(\theta') \leq 2.42$ and the maximum transit time error in distance units is approximately

$$\frac{r_o}{2}\frac{1}{\sqrt{2.36}}[2.36 - 2.30] = 0.0196\, r_o$$

which is approximately 2% of r_o (which can be quite small in some practical cases). Numerical results are given in table 6.1, where $\epsilon_{r_{avg}}$ is specified, along with approximate transit time errors, for various choices of ϵ_{r_0} and Z_c . The values of $\epsilon_{r_{avg}}$ were obtained from numerical solutions to the system (5.50).

Finally, as a separate issue, we note that it is also possible to obtain an upper bound on the values of ϵ_{r_1} as a function of ϵ_{r_0} and Z_c by the following analysis. From (5.22), we have

$$\sqrt{\epsilon_{r_1}} = [\sqrt{\epsilon_{r_0}}\, sech(\frac{2\pi Z_c}{Z_0}) + (\frac{2\pi Z_c}{Z_0})]cos(\theta_1') + sin(\theta_1') \tag{6.6}$$

and so, if $a_1 \equiv \sqrt{\epsilon_{r_0}}sech(\frac{2\pi Z_c}{Z_0}) + tanh(\frac{2\pi Z_c}{Z_0})$, then

$$\sqrt{\epsilon_{r_1}} = \sqrt{1+a_1^2}\left[sin(\theta_1' + \gamma_1)\right] \tag{6.7}$$

where $\gamma_1 = arctan(a_1)$. Hence by a easy calculation which uses the obvious inequality $sin(\theta_1' + \gamma_1) \leq 1$ and the equations

$$\begin{aligned} 2\sqrt{\epsilon_{r_0}} &= (\epsilon_{r_0}+1)cos(\theta_0-\theta_0') \\ (\epsilon_{r_0}-1) &= (\epsilon_{r_0}+1)sin(\theta_0-\theta_0') \end{aligned} \tag{6.8}$$

we obtain

$$\sqrt{\epsilon_{r_1}} \leq \sqrt{3+\epsilon_{r_0}\sin(\theta_0)cos(\theta_0')}\,. \tag{6.9}$$

Thus, we find that

$$\epsilon_{r_1} \leq 3 + \epsilon_{r_0}(sech(\frac{2\pi Z_c}{Z_0})) \tag{6.10}$$

gives a rough upper bound for ϵ_{r_1} for a given ϵ_{r_0} and Z_c .

$\varepsilon_{r_0} = 2.30$

Z_c	θ_0	θ_0'	θ_1'	ε_{r_1}	$\varepsilon_{r_{max}}$	$\varepsilon_{r_{avg}}$	Percent of r_0 transit time error
58.1100	0.7257	0.3208	1.2393	2.3000	2.3473	2.36	1.63
59.0000	0.7159	0.3110	1.2332	2.3199	2.3545	2.41	1.63
60.0000	0.7050	0.3001	1.2258	2.3439	2.3630	2.42	1.96
61.0000	0.6943	0.2894	1.2179	2.3696	2.3717	2.44	2.27
62.0000	0.6837	0.2788	1.2094	2.3971	2.3808	2.46	2.59
63.0000	0.6732	0.2683	1.2002	2.4265	2.3902	2.48	2.91
64.0000	0.6629	0.2580	1.1904	2.4579	2.3998	2.50	3.23
65.0000	0.6527	0.2478	1.1798	2.4913	2.4097	2.53	3.54
66.0000	0.6427	0.2378	1.1683	2.5270	2.4199	2.56	3.86
67.0000	0.6327	0.2278	1.1561	2.5649	2.4302	2.59	4.17
68.0000	0.6230	0.2180	1.1429	2.6052	2.4408	2.62	4.48
69.0000	0.6133	0.2084	1.1286	2.6479	2.4515	2.66	4.79
70.0000	0.6038	0.1989	1.1133	2.6931	2.4623	2.70	5.10
71.0000	0.5944	0.1895	1.0968	2.7408	2.4731	2.74	5.41
72.0000	0.5851	0.1802	1.0790	2.7911	2.4840	2.79	5.72
73.0000	0.5760	0.1710	1.0598	2.8438	2.4947	2.84	6.02
74.0000	0.5669	0.1620	1.0390	2.8990	2.5053	2.90	6.63
75.0000	0.5581	0.1531	1.0166	2.9563	2.5155	2.96	6.93
76.0000	0.5493	0.1444	0.9923	3.0156	2.5254	3.02	7.23
77.0000	0.5407	0.1357	0.9660	3.0763	2.5347	3.08	7.23
78.0000	0.5321	0.1272	0.9376	3.1378	2.5433	3.14	7.83
79.0000	0.5237	0.1188	0.9067	3.1993	2.5510	3.20	7.83
80.0000	0.5155	0.1106	0.8733	3.2596	2.5576	3.26	8.13
81.0000	0.5073	0.1024	0.8371	3.3173	2.5628	3.32	8.13
82.0000	0.4993	0.0944	0.7980	3.3703	2.5662	3.37	8.42
83.0000	0.4914	0.0864	0.7556	3.4162	2.5677	3.42	8.42
84.0000	0.4835	0.0786	0.7098	3.4522	2.5668	3.45	8.13
85.0000	0.4759	0.0709	0.6605	3.4748	2.5632	3.47	8.13
86.0000	0.4683	0.0634	0.6076	3.4802	2.5564	3.48	8.13
87.0000	0.4608	0.0559	0.5510	3.4642	2.5461	3.46	7.83
88.0000	0.4535	0.0485	0.4907	3.4228	2.5318	3.42	7.23
89.0000	0.4462	0.0413	0.4271	3.3524	2.5131	3.35	6.63
90.0000	0.4391	0.0342	0.3603	3.2504	2.4898	3.25	6.02
91.0000	0.4320	0.0271	0.2909	3.1156	2.4616	3.12	5.10
92.0000	0.4251	0.0202	0.2194	2.9491	2.4284	2.95	4.17
93.0000	0.4183	0.0134	0.1466	2.7540	2.3903	2.75	2.91
94.0000	0.4116	0.0067	0.0734	2.5357	2.3475	2.54	1.63
95.0000	0.4050	0.0000	0.0004	2.3014	2.3003	2.30	0.00
95.0060	0.4049	0.0000	0.0000	2.3000	2.3015	2.30	0.00

Table 6.1: Relative dielectric constant versus cone impedance (corresponding to $\theta_1 = \pi/2$), with ε_{r_0} as a parameter. In the range $\theta_0' \le \theta' \le \theta_1'$ (with fixed Z_c) the values of $\varepsilon_{r_{max}}$ and $\varepsilon_{r_{avg}}$ are determined respectively by equation (6.2) and an examination of computer data such as is contained in table 5.8.

Table 6.1

(continuation)

$\varepsilon_{r_0} = 3.00$

Z_c	θ_0	θ_0'	θ_1'	ε_{r_1}	$\varepsilon_{r_{max}}$	$\varepsilon_{r_{avg}}$	Percent of r_0 transit time error
50.7350	0.8110	0.2874	1.1315	3.0000	3.0587	3.12	1.43
51.0000	0.8078	0.2842	1.1289	3.0096	3.0620	3.12	1.71
52.0000	0.7959	0.2723	1.1187	3.0476	3.0747	3.15	2.00
53.0000	0.7840	0.2604	1.1076	3.0888	3.0879	3.17	2.56
54.0000	0.7723	0.2487	1.0954	3.1335	3.1015	3.20	2.84
55.0000	0.7608	0.2372	1.0820	3.1816	3.1156	3.23	3.40
56.0000	0.7493	0.2257	1.0675	3.2334	3.1302	3.27	3.67
57.0000	0.7380	0.2144	1.0516	3.2890	3.1450	3.32	4.23
58.0000	0.7269	0.2033	1.0342	3.3484	3.1601	3.36	4.50
59.0000	0.7159	0.1923	1.0153	3.4116	3.1753	3.42	5.05
60.0000	0.7050	0.1814	0.9945	3.4786	3.1905	3.48	5.33
61.0000	0.6943	0.1707	0.9719	3.5493	3.2056	3.55	5.86
62.0000	0.6837	0.1601	0.9471	3.6233	3.2204	3.62	6.13
63.0000	0.6732	0.1496	0.9199	3.7001	3.2346	3.70	6.67
64.0000	0.6629	0.1393	0.8902	3.7788	3.2480	3.78	6.93
65.0000	0.6527	0.1291	0.8577	3.8583	3.2603	3.86	7.20
66.0000	0.6427	0.1191	0.8220	3.9368	3.2710	3.94	7.47
67.0000	0.6327	0.1091	0.7829	4.0122	3.2798	4.01	7.73
68.0000	0.6230	0.0994	0.7402	4.0813	3.2862	4.08	7.99
69.0000	0.6133	0.0897	0.6934	4.1403	3.2894	4.14	7.99
70.0000	0.6038	0.0802	0.6424	4.1843	3.2889	4.18	7.99
71.0000	0.5944	0.0708	0.5870	4.2076	3.2840	4.21	7.73
72.0000	0.5851	0.0615	0.5270	4.2037	3.2738	4.20	7.47
73.0000	0.5760	0.0524	0.4624	4.1656	3.2576	4.17	7.20
74.0000	0.5669	0.0434	0.3933	4.0868	3.2347	4.09	6.67
75.0000	0.5581	0.0345	0.3202	3.9621	3.2042	3.96	5.86
76.0000	0.5493	0.0257	0.2435	3.7887	3.1658	3.79	4.77
77.0000	0.5407	0.0171	0.1642	3.5675	3.1191	3.57	3.40
78.0000	0.5321	0.0085	0.0831	3.3034	3.0641	3.30	1.71
79.0000	0.5237	0.0001	0.0014	3.0054	3.0012	3.00	0.00
79.0175	0.5236	0.0000	0.0000	3.0000	3.0003	3.00	0.00

Table 6.1

(continuation)

$\varepsilon_{r_0} = 4.00$

Z_c	θ_0	θ_0'	θ_1'	ε_{r_1}	$\varepsilon_{r_{max}}$	$\varepsilon_{r_{avg}}$	Percent of r_0 transit time error
43.8400	0.8976	0.2541	1.0176	4.0000	4.0692		1.73
43.0000	0.9086	0.2651	1.0278	3.9553	4.0546		1.24
44.0000	0.8955	0.2520	1.0155	4.0089	4.0721	4.14	1.73
45.0000	0.8826	0.2391	1.0018	4.0679	4.0904	4.18	2.23
46.0000	0.8698	0.2263	0.9866	4.1326	4.1096	4.22	2.71
47.0000	0.8571	0.2136	0.9696	4.2030	4.1294	4.26	3.20
48.0000	0.8446	0.2011	0.9507	4.2793	4.1497	4.32	3.68
49.0000	0.8322	0.1887	0.9298	4.3615	4.1704	4.38	4.16
50.0000	0.8199	0.1764	0.9066	4.4494	4.1912	4.46	4.64
51.0000	0.8078	0.1643	0.8808	4.5425	4.2119	4.54	5.12
52.0000	0.7959	0.1524	0.8522	4.6400	4.2319	4.64	5.59
53.0000	0.7840	0.1405	0.8204	4.7408	4.2510	4.74	6.06
54.0000	0.7723	0.1288	0.7852	4.8427	4.2686	4.84	6.53
55.0000	0.7608	0.1173	0.7461	4.9431	4.2839	4.94	6.77
56.0000	0.7493	0.1058	0.7027	5.0380	4.2963	5.04	7.23
57.0000	0.7380	0.0945	0.6547	5.1219	4.3046	5.12	7.23
58.0000	0.7269	0.0834	0.6017	5.1881	4.3079	5.19	7.47
59.0000	0.7159	0.0724	0.5434	5.2276	4.3049	5.23	7.23
60.0000	0.7050	0.0615	0.4794	5.2303	4.2940	5.23	7.00
61.0000	0.6943	0.0508	0.4098	5.1849	4.2740	5.18	6.53
62.0000	0.6837	0.0402	0.3347	5.0802	4.2433	5.08	5.83
63.0000	0.6732	0.0297	0.2544	4.9073	4.2006	4.91	4.88
64.0000	0.6629	0.0194	0.1699	4.6615	4.1449	4.66	3.68
65.0000	0.6527	0.0092	0.0821	4.3447	4.0756	4.34	1.98
65.9167	0.6435	0.0000	0.0000	4.0000	4.0000	4.00	0.00

$\varepsilon_{r_0} = 5.00$

39.1630	0.9600	0.2303	0.9326	5.0001	5.0761		1.77
39.0000	0.9623	0.2325	0.9351	4.9881	5.0724	5.15	1.55
40.0000	0.9486	0.2189	0.9192	5.0649	5.0956	5.19	2.21
41.0000	0.9352	0.2054	0.9012	5.1498	5.1198	5.24	2.65
42.0000	0.9218	0.1921	0.8810	5.2427	5.1448	5.30	3.30
43.0000	0.9086	0.1789	0.8582	5.3435	5.1704	5.37	3.74
44.0000	0.8955	0.1658	0.8326	5.4518	5.1961	5.46	4.39
45.0000	0.8826	0.1529	0.8038	5.5665	5.2215	5.57	4.81
46.0000	0.8698	0.1401	0.7715	5.6862	5.2460	5.69	5.46
47.0000	0.8571	0.1274	0.7352	5.8081	5.2686	5.81	5.88
48.0000	0.8446	0.1149	0.6945	5.9285	5.2886	5.93	6.30
49.0000	0.8322	0.1025	0.6490	6.0418	5.3047	6.04	6.52
50.0000	0.8199	0.0902	0.5981	6.1401	5.3154	6.14	6.94
51.0000	0.8078	0.0781	0.5413	6.2134	5.3191	6.21	6.94
52.0000	0.7959	0.0661	0.4784	6.2486	5.3137	6.25	6.73
53.0000	0.7840	0.0543	0.4089	6.2305	5.2971	6.23	6.52
54.0000	0.7723	0.0426	0.3329	6.1425	5.2670	6.14	5.88
55.0000	0.7608	0.0310	0.2507	5.9695	5.2211	5.97	4.81
56.0000	0.7493	0.0196	0.1629	5.7005	5.1574	5.69	3.52
57.0000	0.7380	0.0083	0.0707	5.3331	5.0746	5.33	1.55
57.7454	0.7297	0.0000	0.0000	5.0000	5.0028	5.00	0.00

Table 6.1

(continuation)

$\varepsilon_{r_0} = 7.00$

z_c	θ_0	θ_0'	θ_1'	ε_{r_1}	$\varepsilon_{r_{max}}$	$\varepsilon_{r_{avg}}$	Percent of r_0 transit time error
33.0500	1.0459	0.1978	0.8120	6.9999	7.0844		1.50
33.0000	1.0466	0.1985	0.8130	6.9945	7.0828		1.50
34.0000	1.0322	0.1842	0.7930	7.1084	7.1151	7.23	2.25
35.0000	1.0180	0.1699	0.7699	7.2348	7.1486	7.30	2.80
36.0000	1.0039	0.1558	0.7433	7.3729	7.1828	7.40	3.36
37.0000	0.9899	0.1418	0.7127	7.5211	7.2168	7.53	4.09
38.0000	0.9760	0.1279	0.6778	7.6761	7.2496	7.68	4.64
39.0000	0.9623	0.1142	0.6378	7.8330	7.2796	7.83	5.19
40.0000	0.9486	0.1006	0.5922	7.9841	7.3049	7.98	5.55
41.0000	0.9352	0.0871	0.5404	8.1182	7.3234	8.12	5.91
42.0000	0.9218	0.0738	0.4816	8.2194	7.3319	8.22	6.09
43.0000	0.9086	0.0606	0.4154	8.2670	7.3272	8.27	6.09
44.0000	0.8955	0.0475	0.3413	8.2356	7.3051	8.24	5.73
45.0000	0.8826	0.0345	0.2592	8.0969	7.2614	8.10	4.82
46.0000	0.8698	0.0217	0.1694	7.8247	7.1916	7.82	3.54
47.0000	0.8571	0.0091	0.0729	7.4012	7.0920	7.40	1.69
47.7219	0.8481	0.0000	0.0000	7.0000	7.0001		0.00

$\varepsilon_{r_0} = 10.00$

27.6240	1.1258	0.1676	0.6957	10.0000	10.0912		1.42
27.0000	1.1353	0.1770	0.7087	9.9092	10.0653	10.14	1.10
28.0000	1.1202	0.1620	0.6872	10.0585	10.1073	10.21	1.73
29.0000	1.1053	0.1470	0.6617	10.2271	10.1513	10.31	2.35
30.0000	1.0904	0.1322	0.6315	10.4127	10.1961	10.44	3.13
31.0000	1.0757	0.1175	0.5960	10.6102	10.2400	10.61	3.75
32.0000	1.0611	0.1028	0.5545	10.8114	10.2803	10.81	4.37
33.0000	1.0466	0.0884	0.5061	11.0028	10.3138	11.00	4.83
34.0000	1.0322	0.0740	0.4499	11.1638	10.3361	11.16	5.29
35.0000	1.0180	0.0597	0.3852	11.2654	10.3418	11.27	5.29
36.0000	1.0039	0.0456	0.3111	11.2686	10.3241	11.27	4.98
37.0000	0.9899	0.0316	0.2273	11.1259	10.2751	11.13	4.37
38.0000	0.9760	0.0178	0.1338	10.7881	10.1865	10.79	2.98
39.0000	0.9623	0.0040	0.0315	10.2157	10.0502	10.22	0.79
39.2940	0.9582	0.0000	0.0000	10.0000	9.9968		0.00

$\varepsilon_{r_0} = 2.24$

$Z_c = 60\Omega$

$\frac{2}{\pi}\theta'$	θ'	ε_r	$\frac{2}{\pi}\theta'$	θ'	ε_r	Ψ/r_0	z/r_0
0.199	0.312	2.240	0.550	0.864	2.345	0.5000	0.7102
0.200	0.314	2.240	0.560	0.880	2.345	0.6000	0.7470
0.210	0.330	2.240	0.570	0.895	2.346	0.7000	0.7753
0.220	0.345	2.241	0.580	0.911	2.346	0.8000	0.7966
0.230	0.361	2.243	0.590	0.926	2.345	0.9000	0.8118
0.240	0.377	2.245	0.600	0.942	2.345	1.0000	0.8216
0.250	0.392	2.247	0.610	0.958	2.344	1.1000	0.8268
0.260	0.408	2.250	0.620	0.973	2.343	1.2000	0.8276
0.270	0.424	2.253	0.630	0.990	2.341	1.3000	0.8245
0.280	0.440	2.256	0.640	1.006	2.339	1.4000	0.8178
0.290	0.455	2.259	0.650	1.021	2.337	1.5000	0.8076
0.300	0.471	2.263	0.660	1.036	2.334	1.6000	0.7943
0.310	0.487	2.267	0.670	1.052	2.331	1.7000	0.7780
0.320	0.503	2.271	0.680	1.068	2.327	1.8000	0.7589
0.330	0.518	2.275	0.690	1.084	2.323	1.9000	0.7370
0.340	0.534	2.279	0.700	1.099	2.319	2.0000	0.7126
0.350	0.550	2.283	0.710	1.116	2.315	2.1000	0.6857
0.360	0.565	2.288	0.720	1.131	2.310	2.2000	0.6564
0.370	0.581	2.292	0.730	1.146	2.305	2.3000	0.6249
0.380	0.597	2.296	0.740	1.163	2.298	2.4000	0.5911
0.390	0.612	2.300	0.750	1.178	2.293	2.5000	0.5552
0.400	0.628	2.304	0.760	1.194	2.286	2.6000	0.5172
0.410	0.643	2.308	0.770	1.209	2.279	2.7000	0.4772
0.420	0.660	2.312	0.780	1.225	2.272	2.8000	0.4353
0.430	0.676	2.316	0.790	1.240	2.265	2.9000	0.3915
0.440	0.691	2.319	0.800	1.256	2.257	3.0000	0.3458
0.450	0.707	2.323	0.810	1.272	2.248	3.1000	0.2982
0.460	0.722	2.326	0.820	1.288	2.240	3.2000	0.2490
0.470	0.739	2.329	0.830	1.303	2.230	3.3000	0.1980
0.480	0.754	2.332	0.840	1.319	2.221	3.4000	0.1453
0.490	0.769	2.334	0.850	1.335	2.211	3.5000	0.0909
0.500	0.785	2.337	0.860	1.351	2.201	3.6000	0.0349
0.510	0.810	2.339	0.870	1.367	2.190		
0.520	0.816	2.341	0.880	1.382	2.180		
0.530	0.832	2.342	0.882	1.386	2.177		
0.540	0.849	2.344					

Table 6.2: Relative dielectric constant ε_r versus conical antenna angle $\frac{2}{\pi}\theta'$ with $\varepsilon_{r_0} = 2.24$ and come impedance $Z_c = 60\Omega$, and z/r_0 versus Ψ/r_0 with $\varepsilon_{r_0} = 2.24$ and $Z_c = 60\Omega$.

θ'	θ	ε_r
0.312	0.705027	2.2400
0.322	0.719020	2.2401
0.332	0.732816	2.2405
0.342	0.746423	2.2411
0.352	0.759845	2.2419
0.362	0.773089	2.2429
0.372	0.786160	2.2441
0.382	0.799063	2.2455
0.392	0.811803	2.2470
0.402	0.824385	2.2487
0.412	0.836813	2.2505
0.422	0.849090	2.2524
0.432	0.861222	2.2545
0.442	0.873212	2.2566
0.452	0.885063	2.2588
0.462	0.896779	2.2611
0.472	0.908364	2.2635
0.482	0.919821	2.2659
0.492	0.931153	2.2684
0.502	0.942363	2.2710
0.512	0.953454	2.2736
0.522	0.964429	2.2762
0.532	0.975291	2.2788
0.542	0.986042	2.2815
0.552	0.996685	2.2841
0.562	1.007222	2.2868
0.572	1.017657	2.2895
0.582	1.027990	2.2922
0.592	1.038226	2.2948
0.602	1.048365	2.2975
0.612	1.058410	2.3001
0.622	1.068364	2.3027
0.632	1.078228	2.3052
0.642	1.088004	2.3077
0.652	1.097694	2.3102
0.662	1.107301	2.3126
0.672	1.116826	2.3150
0.682	1.126271	2.3173
0.692	1.135637	2.3195
0.702	1.144927	2.3217
0.712	1.154142	2.3238
0.722	1.163284	2.3258
0.732	1.172355	2.3278
0.742	1.181356	2.3297
0.752	1.190289	2.3314
0.762	1.199154	2.3331
0.772	1.207955	2.3347
0.782	1.216692	2.3362
0.792	1.225366	2.3376
0.802	1.233980	2.3389
0.812	1.242534	2.3401
0.822	1.251031	2.3412
0.832	1.259470	2.3422
0.842	1.267854	2.3430
0.852	1.276184	2.3438
0.862	1.284461	2.3444
0.872	1.292686	2.3449
0.882	1.300860	2.3453
0.892	1.308986	2.3455
0.902	1.317064	2.3457
0.912	1.325094	2.3456
0.922	1.333079	2.3455
0.932	1.341019	2.3452
0.942	1.348916	2.3448
0.952	1.356770	2.3443
0.962	1.364583	2.3436
0.972	1.372356	2.3427
0.982	1.380089	2.3418
0.992	1.387784	2.3406
1.002	1.395442	2.3394
1.012	1.403063	2.3380
1.022	1.410649	2.3364
1.032	1.418201	2.3347
1.042	1.425719	2.3329
1.052	1.433205	2.3308
1.062	1.440660	2.3287
1.072	1.448083	2.3264
1.082	1.455477	2.3239
1.092	1.462842	2.3213
1.102	1.407179	2.3185
1.112	1.477489	2.3156
1.122	1.484773	2.3125
1.132	1.492031	2.3093
1.142	1.499264	2.3059
1.152	1.506474	2.3024
1.162	1.513660	2.2987
1.172	1.520825	2.2949
1.182	1.527968	2.2909
1.192	1.535090	2.2867
1.202	1.542192	2.2824
1.212	1.549275	2.2779
1.222	1.556340	2.2733
1.232	1.563387	2.2686
1.242	1.570418	2.2636
1.252	1.577432	2.2586

Table 6.3. Numerical solution to the system (5.50) with $\varepsilon_{r_0} = 2.24$ and $Z_c = 60\Omega$

VII. SUMMARY AND CONCLUSIONS

In this paper a differential impedance and transit-time matching approach was used to obtain a design of an anisotropic lens for transmitting TEM waves launched from a small source onto a conducting circular conical system. The mathematical problem was formulated by a pair of differential equations and associated geometric conditions which appear in section 3 ((3.23) through (3.26)). An exact solution appears in section 5 (5.18) through (5.20)).

It should be noted that for engineering convenience one would like the relative permittivity $\epsilon_r(\theta')$ of the lens to be as uniform (i.e., insensitive to θ') as possible. This is first achieved by making the derivative of ϵ_r with respect to θ' zero at the inner lens boundary. Then by appropriate selection of Z_c (and hence θ_0) for a given ϵ_{r_0} (at the inner boundary) one can minimize the required ideal variation of ϵ_r over the entire lens, in some cases to quite small values. In such cases one may approximate the ideal $\epsilon_r(\theta')$ by a constant equal to $\epsilon_{r_{avg}}$. If one arranges the value of $\epsilon_{r_{avg}}$ to be that of the dielectric that one has at hand (e.g., transformer oil) by appropriate choice of ϵ_{r_0} then one has a rather simply constructible lens. These remarks are illustrated by the example of section 6.

We conclude by commenting that the present paper has considered a lens for a particularly simple geometry. Even so the solution was not so simple. One would like to consider such lenses for more complex geometries of practical interest for launching TEM waves on various types of transmission structures.

ACKNOWLEDGEMENT

The assistance of Albert Gabaldon, AFWL, in the preparation of the figures and numerical calculations is gratefully acknowledged.

REFERENCES

1. C.E. Baum, "A scaling technique for the design of idealized electromagnetic lens", Sensor and Simulation Note 64, August 1968, Kirtland AFB, AF Weapons Laboratory, Albuquerque, NM 87117.

2. A.P. Stone, "A differential geometric approach to electromagnetic lens design", Sensor and Simulation Note 282, April 1983, Kirtland AFB, AF Weapons Laboratory, Albuquerque, NM 87117. (also to appear under same title in Electromagnetics).

3. C.E. Baum, "A circular conical antenna simulator", Sensor and Simulation Note 36, March 1967, Kirtland AFB, AF Weapons Laboratory, Albuquerque, NM 87117.

4. C.E. Baum, "EMP simulators for various types of nuclear EMP environments: An interim categorization", Sensor and Simulation Note 240 (also in IEEE Transactions on Antennas and Propagation, Vol. AP-26, No. 1, Jan. 1978, pp. 35-53, and in IEEE Transactions on EM Compatibility, Feb. 1978).

5. T.C. Mo, C.H. Papas, and C.E. Baum, "Differential geometry scaling method for electromagnetic field and its applications to coaxial waveguide functions", Sensor and Simulation Note 169, March 1973 (also as a paper by the same authors under the title of "General scaling method for electromagnetic field with application to a matching problem", J. Math. Physics, Vol. 14, pp. 479-483, April 1973.

6. C.E. Baum, "The Brewster angle wave matcher", Sensor and Simulation Note 37, March 1967, Kirtland AFB, AF Weapons Laboratory, Albuquerque, NM 87117.

SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

1a REPORT SECURITY CLASSIFICATION UNCLASSIFIED	1b RESTRICTIVE MARKINGS
2a SECURITY CLASSIFICATION AUTHORITY	3 DISTRIBUTION/AVAILABILITY OF REPORT Approved for public release; distribution unlimited.
2b DECLASSIFICATION/DOWNGRADING SCHEDULE	
4 PERFORMING ORGANIZATION REPORT NUMBER(S)	5 MONITORING ORGANIZATION REPORT NUMBER(S) AFOSR-TR- 84-0600
6a NAME OF PERFORMING ORGANIZATION University of New Mexico 6b OFFICE SYMBOL (If applicable)	7a. NAME OF MONITORING ORGANIZATION Air Force Office of Scientific Research
6c. ADDRESS (City, State, and ZIP Code) Dept of Mathematics & Statistics Albuquerque NM 87131	7b. ADDRESS (City, State, and ZIP Code) Directorate of Mathematical & Information Sciences, AFOSR, Bolling AFB DC 20332
8a. NAME OF FUNDING/SPONSORING ORGANIZATION AFOSR 8b OFFICE SYMBOL (If applicable) NM	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER AFOSR-83-0040
8c. ADDRESS (City, State, and ZIP Code) Bolling AFB DC 20332	10. SOURCE OF FUNDING NUMBERS PROGRAM ELEMENT NO. 61102F; PROJECT NO. 2304; TASK NO. D9; WORK UNIT ACCESSION NO.

11. TITLE (Include Security Classification)
AN ANISOTROPIC LENS FOR LAUNCHING TEM WAVES ON A CONDUCTING CIRCULAR SYSTEM.

12. PERSONAL AUTHOR(S)
Alexander P. Stone and C.E. Baum*

13a. TYPE OF REPORT	13b. TIME COVERED	14. DATE OF REPORT (Year, Month, Day)	15. PAGE COUNT
Reprint	FROM _____ TO _____	June 1984	83

16. SUPPLEMENTARY NOTATION
Sensor and Simulation Notes, Note 285, June 1984.
*Air Force Weapons Lab, Kirtland AFB, Albuquerque NM 87117.

17. COSATI CODES			18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number)
FIELD	GROUP	SUB-GROUP	

19. ABSTRACT (Continue on reverse if necessary and identify by block number)

A differential impedance and transit-time matching approach is used in the design of an anisotropic lens for launching TEM waves from a small source, through the lens, and onto a conducting circular conical system. This approach leads to a system of ordinary differential equations which may be solved exactly to obtain the lens parameters. An approximate solution, which would be applicable to a design procedure, is also given.

20 DISTRIBUTION/AVAILABILITY OF ABSTRACT ☒ UNCLASSIFIED/UNLIMITED ☐ SAME AS RPT. ☐ DTIC USERS	21. ABSTRACT SECURITY CLASSIFICATION UNCLASSIFIED
22a NAME OF RESPONSIBLE INDIVIDUAL Dr. Robert N. Buchal	22b TELEPHONE (Include Area Code) (202) 767-4939 / 22c OFFICE SYMBOL NM

DD FORM 1473, 84 MAR — 83 APR edition may be used until exhausted. All other editions are obsolete.

SECURITY CLASSIFICATION OF THIS PAGE
UNCLASSIFIED