A SPLINE-BASED APPROXIMATION METHOD FOR INVERSE PROBLEMS FOR A HYPERBOLIC SYSTEM INCLUDING UNKNOWN BOUNDARY PARAMETERS

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Summary

We discuss a method for the estimation of unknown parameters (variable as well as constant) occurring in a hyperbolic system, in the context of a seismic application. We present both theoretical results and some numerical examples.

Introduction

We have developed a numerical algorithm, and a corresponding convergence theory to solve a one-dimensional "seismic" inverse problem. The response in certain classes of seismic experiments can be modeled by the following hyperbolic partial differential equation with associated boundary and initial conditions:

\[ \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = \alpha(x) \frac{\partial u}{\partial x} \quad t > 0, \quad x \in [0,1] \]

\[ \frac{\partial u}{\partial x}(t,0) + k_1 u(t,0) = s(t,k) \]

\[ \frac{\partial u}{\partial x}(t,1) + k_2 \frac{\partial u}{\partial x}(t,1) = 0 \] (1)

\[ u(0,x) = g(x) \]

\[ u(0,x) = h(x) \]

Here \( x \) represents depth below the surface of the earth, \( u \) represents displacement, \( \alpha(x) \) is the mass density of the medium (in the most general case an unknown), and \( q_0(x) \) is an unknown elastic modulus. The boundary condition at the surface \( (x=0) \) is an elastic boundary condition involving the unknown (negative constant) \( k_1 \) and an unknown source term \( s(t,k) \). For our treatment of the problem it is not necessary to assume that \( s \) is an impulse. In the numerical examples presented below, it has been assumed that \( s(t,k) \) has a known form, and only the unknown \( k_1 \) (a constant vector in \( \mathbb{R}^k \)) is to be identified, but this also is not essential. We show for example, that \( q_0(x) \) (similar remarks are valid for \( s \)) can be identified as a function without a priori knowledge of its shape. The ideas in this case are similar to those in [2] where problems with coefficients which are unknown functions of both space and time are discussed for parabolic equations. At the bottom boundary \( (x=1) \), an absorbing boundary condition is imposed, involving an unknown (positive constant) \( k_2 \). The purpose of this condition is to limit the interval of computation without producing artificial reflections; it allows down-going waves to pass through the boundary undisturbed, while annihilating up-going waves.

We assume we have data observations, \( \tilde{y}_1 \), corresponding to \( u(t_1,0) \), a solution of (1) evaluated at the surface. The inverse, or identification problem consists of minimizing a least-squares function

\[ J(q) = \sum_{n=1}^{N} \left( \tilde{y}_1 - u(t_1,0;q) \right)^2 \overline{\text{over an appropriately chosen}} \]

constraint set \( Q \). Here \( (t,x) = u(t,x;k) \) is the solution of (1) corresponding to \( q(x) = (q_1(x), q_2(x), k_1, k_2, k) \). We follow the general approach developed in [3] and [1]; we first formulate the identification problem in an abstract setting, then define a sequence of approximate finite dimensional identification problems, the solution of which generate parameter estimates which converge to a solution of the original identification problem.

Convergence

Motivated by the fact that our differential equation can be written as a system using the variables \( (u, v) \), we define a Hilbert space \( X(q) \equiv V(q) \times L^2(q) \) where \( V(q) \) is \( H^1(0,1) \) with inner product \( \langle v,w \rangle = \int_0^1 q \delta_{v,w} \delta_0 \) and \( L^2(q) \) is \( H^0(0,1) \) with inner product \( \langle v,w \rangle \equiv \int_0^1 q \delta_{v,w} \delta_0 \). The X(q) inner product is then given by \( \langle x,y \rangle = \langle x_1,y_1 \rangle + \langle x_2,y_2 \rangle \) where \( x = (x_1,x_2)^T, y = (y_1,y_2)^T \). After a straightforward transformation to a system with homogeneous boundary conditions, system (1) can be rewritten in \( X(q) \) as

\[ A(q)z(t) = G(q) \]

\[ z(0) = z_0(q) \]

where \( z(t) \in X(q) \) is identified with \( u(t,.) \), the boundary conditions are incorporated into the domain of the operator \( A(q) \) by defining \( V_q = \{ v \in V(q) \cap H^2(0,1) \mid Dv(0) + k_1 v(0) = 0 \} \) and \( \overline{\text{dom} A(q)} = \{ y \in V_q \mid H^1(0,1) \} \)

and \( A(q) \) is the unbounded linear operator defined by

\[ A(q)z(t) = \{ u(t,.) \} \]

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$A(q) = \begin{pmatrix} 0 & 1 \\ (1/q_1)D(q_2D) & 0 \end{pmatrix}$

With $X(q)$ and $A(q)$ so chosen, for each $q$, $A(q)$ is a dissipative operator in $X(q)$ and it can be shown that in fact $A(q)$ is the generator of a $C_0$-semigroup $T(t;q)$ on $X(q)$. Standard semigroup theory can then be used to show that equation (2) has a unique mild solution:

$$z(t;q) = T(t;q)z_0(q) + \int_0^t T(t-s;q)G(s;q)ds$$

and the identification problem can be restated as

(ID) Minimize $J(q) = \sum_{i=1}^{n} \| \gamma_i - z_1(t_1) \|_{x=0}^2$ over $q \in Q$

subject to $z(-;q)$ satisfying (3), where $z_1$ represents the first component of $z$.

Before formulating the approximate identification problems, we first define finite dimensional subspaces $X^N(q)$. Let $S^2(\Delta^N)$ be the subspace of $C^2$ cubic splines (as in [5], pp. 78-81) corresponding to the partition $\Delta^N = \{x_i\}_{i=0}^N$, $x_i = i/N$, and then define $X^N(q)$ to be that subspace of $S^2(\Delta^N) \times S^2(\Delta^N)$ which satisfies the boundary conditions corresponding to $q$, i.e., $X^N(q) \cap \text{dom}A(q)$. The space $X^N(q)$ can be expressed as the span of a set of $2N+3$ basis elements, which are straightforward modifications of the standard spline basis elements of $S^2(\Delta^N) \times S^2(\Delta^N)$ (see [5] or [4, p. 38] for details). As a result of these modifications, the new basis elements, and thus the subspaces, depend on the unknown parameter $q$. It is clear then, that as we iterate on $q$, these spaces will change.

One assumption we make about the constraint set $Q$ is that each component is uniformly bounded above and below, implying that as $q$ ranges over $Q$, the $X(q)$ norms will be uniformly equivalent, and hence the spaces $X(q)$ will be equal as sets. With this in mind, let $P^N(q): X(q) \rightarrow X^N(q)$ be the orthogonal projection of $X(q)$ onto $X^N(q)$ in the $X(q)$ norm (for a precise statement of this, one should introduce the canonical isomorphism which associates elements of $X(q)$ with those in the equivalent space $X(q)$, but to shorten this presentation, we will omit such notation); whenever $q$ and $\hat{q}$ are the same the projection will be written as $P^N(q)$. Define $A^N(q) = P^N(q)A(q)P^N(q)$ and define the approximate system in $X^N(q)$ as

$$z^N(t;q) = A^N(q)z^N(t) + P^N(q)G(t;q)$$

$$z^N(0) = P^N(q)z_0(q).$$

The operator $A^N(q)$ inherits the dissipativity of $A(q)$, and is also the generator of a $C_0$-semigroup, $T^N(t;q)$ on $X(q)$. Moreover, we can establish the existence and uniqueness of mild solutions to (4) and write them as

$$z^N(t;q) = T^N(t;q)z_0(q) + \int_0^t T^N(t-s;q)G(s;q)ds.$$
while elements of \( \text{dom} A(\hat{q}) \) satisfy the boundary conditions corresponding to \( q \), and in general there is no inclusion relation between these sets. This necessitates the use of an operator \( x \): \( \hat{x}(\hat{q}) - x(q^N) \) which maps elements of \( \text{dom} A(\hat{q}) \) into those of \( \text{dom} A(q^N) \) so that it will be possible to compare these elements.

Once the Trotter-Kato Theorem has been used to show the convergence of the semigroups, it can be shown that also the (mild) solutions, \( z^N(t; q^N) \) of (4) converge in \( X(q^N) \) to the (mild) solution \( z(t; \hat{q}) \) of (2) (again, a precise statement of this convergence would require the use of the canonical isomorphism) whenever \( q^N \to \hat{q} \) in an appropriate sense. With this result and the following theorem (from [4] or [5]) it can be shown that \( q^N \) is a solution to the inverse problem.

**Theorem:** Assume \( Q \) is compact in the \( \mathbb{C} = \mathbb{R}^2 \times \mathbb{R} \) topology. If \( q = q_0(\hat{q}), q = N(q)z, q = z(q^N(t; q^N)) \), \( z \in X(q) \) are continuous in this same \( \mathbb{Q} \)-topology, with the latter uniformly in \( t \in [0, T] \), then

(i) there exists for each \( N \) a solution \( \hat{q}^N \) of \( (I^N) \) and the sequence \( \{q^N\} \) possesses a convergent subsequence \( q^{N_k} \to \hat{q} \).

(ii) If we further assume that, for any sequence \( \{q^j\} \) in \( q^N \) with \( q^j \to \hat{q} \), we have \( z^N(t; q^j) \to z(t; \hat{q}) \) as \( j \to \infty \) uniformly in \( t \in [0, T] \), then \( \hat{q} \) is a solution of \( (I^M) \).

The proofs and details of all the results stated above can be found in [5] and are variations of the general framework developed in [3].

**Numerical Examples**

In the examples to be presented below, the "data" has been generated using an independent finite difference scheme, where known "true" values of the parameters were preassigned. We begin each example with an initial guess \( q^0 \) and a choice of \( N \) and solve \( (I^N) \) to obtain \( q^N \). We then use this \( q^N \) as the initial guess for the next value of \( N \). All examples were produced either on an IBM 370 or a CDC 6600.

**Example 1:** For this example we parameterized \( q_2 \) as \( q_2(x) = 3/2 + (1/\pi) \tan^{-1}[q_2(1-x^2)] \), where \( q_2(1) \) and \( q_2(2) \) are to be estimated. We used \( s(t; k) = 0 \), and initial conditions \( \phi(x) = e^{x^2} \) and \( \psi(x) = 3e^x \). Data points were chosen at \( x = 0 \) and fifteen equally spaced time values in \( [0, 1] \). We obtained the following results:

<table>
<thead>
<tr>
<th>( N )</th>
<th>( q_{21} )</th>
<th>( q_{22} )</th>
<th>( \Gamma_1 )</th>
<th>( \Gamma_2 )</th>
<th>( J^N(q^N) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>5.873</td>
<td>0.503</td>
<td>-0.995</td>
<td>3.005</td>
<td>0.15 \times 10^{-3}</td>
</tr>
<tr>
<td>8</td>
<td>5.929</td>
<td>0.497</td>
<td>-1.001</td>
<td>3.001</td>
<td>0.12 \times 10^{-4}</td>
</tr>
</tbody>
</table>

**Example 2:** We added random noise to the data in this example, at a level of about 3%. We searched for \( q_2 \) as a constant, we used \( s(t; k) = k_1(1-e^{-5t})k_2 \), and used zero initial conditions. Data points were chosen at \( x = 0 \) and fifteen equally spaced time values in \( [0, 2] \). We obtained these results:
Example 3: In this example, we searched for $q_2(x)$ in the space of cubic splines; the true $q_2$ used was $q_2(x) = 1.5 + \tanh[6(x-h)]$. We used $s(t;\hat{k}) = 0$, and $e(x) = e^x, w(x) = -3e^x$. We did not search for the boundary parameters in this example; the true values, $k_1^* = -1.0, k_2^* = 3.0$ were used. The data points were chosen at seven equally spaced spatial values in $[0,1]$ and three equally spaced time values in $(0,1)$. Our initial guess for $q_2(x)$ was the constant function $q_2^0(x) = 6.0$. With $N=4$ (for the state approximation) and $M=1$ (coefficient approximation) we obtained an estimate, $q_4$, for our functional coefficient such that $|q_4 - q_2^0| = 0.099$, and $J(q_4) = 0.48 \times 10^{-2}$. We have several spatial observations in this example rather than only the one at the surface; this is more representative of problems that arise in treating data from "bore-hole" type of seismic experiments, in which receivers are located at various points down a well.

Acknowledgments

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References


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