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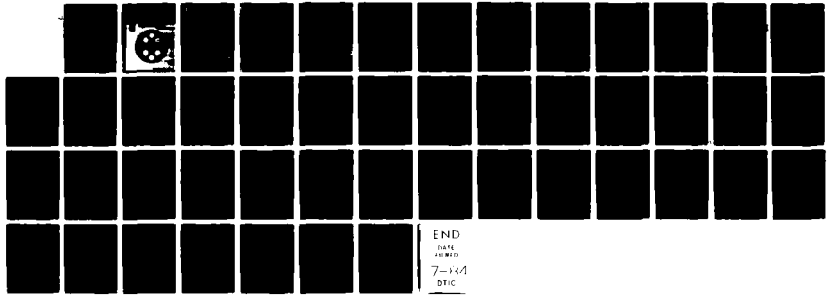

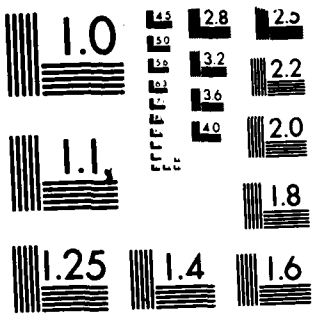

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ABELIAN INTEGRALS AND BIFURCATION THEORY

by

Jack Carr, Shui-Nee Chow and Jack K. Hale

April, 1984 LCDS Report #84-7

Lefschetz Center for Dynamical Systems

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ABELIAN INTEGRALS AND BIFURCATION THEORY

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Abstract

Conditions are given for uniqueness of limit cycles for autonomous equations in the plane. The results are applicable to codimension two bifurcations near equilibrium points for vector fields.

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§1. INTRODUCTION

Consider a system of autonomous differential equations in $\mathbb{R}^d$, $d \geq 2$, or in some infinite dimensional space. Suppose that there exists an equilibrium point which is non-hyperbolic and is doubly degenerate. In other words, it is a codimensional two singularity. More specifically, we assume that the linear variational equation in the center manifold has one of the following linear parts

$$A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega \\ 0 & 0 & -\omega & 0 \end{bmatrix}.$$

For more detail, see Arnold [1,2], Carr [8], Chow and Hale [9] and Guckenheimer and Holmes [11].

The classification of vector fields near a singularity of type A_1 for generic perturbations can be found in Arnold [1], Bogdanov [3,4,5], and, for symmetric perturbations, in Carr [8] and Takens [14]. Such classification is not simple. The main difficulty is to prove the uniqueness of the limit cycle. It is usually proved by using properties of elliptic integrals depending on a parameter (see Arnold [1],

Bogdanov [1,2,3], Carr [8], Chow and Hale [9], and Guckenheimer and Holmes [11]) and is related to the weakened Hilbert 16th problem (Arnold [1], p. 303). In [12,13], Il'jašenko showed that these properties of elliptic integrals could also be obtained by methods in algebraic geometry.

The classification of vector fields near a singularity of type A_2 or A_3 is far from being complete. Since the dimension of the system is ≥ 3, many new types of dynamical behavior can occur. The first step in any attempt at classification is to put the equations in normal form and then try to classify the truncated equations; namely, the normal form equations up through polynomials of degree $\leq k$ for some fixed integer k. Because of the nature of the linear parts A_2 or A_3, these equations are very symmetrical and the flow can be reduced to a polynomial equation in the plane. It is this polynomial equation which is the subject of this paper. For this equation, the existence and number of periodic orbits plays a fundamental role in the classification.

Most of the paper is concerned with the case A_2. In Section 2, for the case A_2 we show how one obtains the relevant equation in the plane and point out that the bifurcation diagram is easy to obtain everywhere except in the neighborhood of one point for which the unperturbed equation is Hamiltonian. The discussion of period orbits reduces to the consideration of the monotonicity properties of a certain function which is the ratio of Abelian type integrals. For certain values of the parameter, these are Abelian integrals and one can analyze this function by using methods from the theory of analytic

functions and algebraic geometry. In Section 3, we discuss a case of this type and show periodic orbits are unique. The method of proof follows earlier ideas used by Carr [8]. Cushman and Sanders [10] also have discussed this case and have shown very clearly the connections with the Picard-Fuchs theory of Abelian integrals. Bogdanov presented this result in [4] with a different proof. Arnold [1] states that the method of Il'yashenko [12,13] can be applied also to this case.

In Section 4, by using a more analytic proof, we show uniqueness in case A_2 for situations where the algebraic methods do not seem to be applicable.

In Section 5, we derive the bifurcation function for case A_3 and special values of the parameters. The general case is still open. Numerical evidence indicates uniqueness of the periodic orbit.

§2. NORMAL FORMS AND BIFURCATIONS

In this section, we suppose that we have a singularity of type A_2. In order to classify the behavior of vector fields near such a singularity, consider the equation in $\mathbb{R}^3$

$$\dot{u} = Au + F(u)$$

where $u \in \mathbb{R}^3$,

$$A = \begin{bmatrix} \alpha & 1 & 0 \\ -1 & \alpha & 0 \\ 0 & 0 & \beta \end{bmatrix}$$

α, $\beta \in \mathbb{R}$ are small parameters, $F = O(|u|^2)$, as $|u| \to 0$, and is smooth. By introducing cylindrical coordinates,

$$u_1 = r \cos \theta$$
$$u_2 = r \sin \theta$$
$$u_3 = s$$

and applying the theory of normal forms (method of averaging) (see, for example, [9]), one may assume without loss of generality that our differential system near $u = 0$ has the form

$$\dot{\theta} = 1 + f_k(r^2,s,\alpha,\beta) + O(k + 1)/r$$
$$\dot{r} = rg_k(r^2,s,\alpha,\beta) + O(k + 1)$$
$$\dot{s} = h_k(r^2,s,\alpha,\beta) + O(k + 1)$$

where $f_k(0,0,0) = g_k(0,0,0) = h_k(0,0,0) = \partial h_k(0,0,0)/\partial s = 0$, the functions f_k, rg_k, h_k are polynomials in r, z, α, β, of degree $\leq k$ and $O(k + 1)$ denotes terms which have order $k + 1$ uniformly in θ.

One important observation about these equations is that the terms up through order k are independent of θ and that the equations truncated up through order k always have the manifold $r = 0$ invariant under the flow.

An attempt to analyze the above equations consists of the following steps. First, ignore the terms of order $k + 1$ and analyze the truncated equations completely. Since the truncated equations are independent of θ, one can replace the time variable by θ and obtain an equation in the plane. Because the truncated equations correspond to a planar system, periodic orbits play a major role in the discussion of the flow for α,β parameters varying in a neighborhood of zero.

For $k = 3$, we will give below a complete discussion of the periodic orbits.

The next step in the analysis is to discuss the effect of perturbation terms $O(k + 1)$. These terms depend upon θ and also may not possess the symmetry properties in r that the truncated equations possess. This creates significant difficulties. A complete solution to the complete equations is not available at this time. We do not discuss this problem, but mention only that significant progress has been made by Broer [6], Broer and Vetger [7].

If we truncate the equations, replace t by θ and keep only terms up through the third degree, we obtain the system

$$r' = r(\alpha + \bar{a}s + \bar{d}r^2 + \bar{e}s^2)$$
$$s' = \beta s + \bar{b}s^2 + \bar{c}r^2 + \bar{f}r^2 s + \bar{g}s^3$$

where "$'$" $= d/d\theta$ and $\bar{a}, \bar{b}, \bar{c}, \bar{d}, \bar{e}, \bar{f}, \bar{g}$ are fixed constants. The terms neglected from the truncated equations can be shown to be insignificant (see [9]).

To obtain a simpler form for these equations we rescale the variables by letting

$$r = |\bar{b}|^{\frac{1}{2}} x$$
$$s = |\bar{c}|^{\frac{1}{2}} y$$
$$\theta = -\frac{1}{\bar{b}|\bar{c}|^{\frac{1}{2}}} t$$
$$\alpha = -\bar{b}|\bar{c}|^{\frac{1}{2}} \tilde{\alpha}$$
$$\beta = -\bar{b}|\bar{c}|^{\frac{1}{2}} \tilde{\beta}$$

to obtain the following:

$$\dot{x} = \tilde{\alpha}x + Bxy + \tilde{d}x^3 + \tilde{e}xy^2$$
$$\dot{y} = \tilde{\beta}y - y^2 - (\operatorname{sgn}\bar{b}\bar{c})x^2 + \tilde{f}x^2y + \tilde{g}y^3$$

where $B = -\bar{a}/\bar{b}$ and $\tilde{d}$, $\tilde{e}$, $\tilde{f}$, $\tilde{g}$ are constants.

For the case where $B \neq 0$, $\bar{b}\bar{c} < 0$, the bifurcation diagram in the $(\tilde{\alpha},\tilde{\beta})$ plane is easy to obtain and qualitatively does not depend upon the cubic terms. The bifurcations consist of elementary bifurcations of equilibrium points. For the complete equation, this means only Hopf bifurcations and the coalescing of periodic orbits. For a complete discussion, see [9].

The most interesting case is when $\bar{b}\bar{c} > 0$. It is convenient to introduce a change of variables. Choose γ close to $\tilde{\beta}/2$ so that $\tilde{\beta} - 2\gamma + 3d_4\gamma^2 = 0$ and the transformation $y \mapsto \gamma + y$ together with a rescaling of x,y leads to the more symmetric form

$$\dot{x} = \tilde{\alpha}x + Bxy + d_1x^3 + d_2xy^2$$
$$\dot{y} = \frac{\tilde{\beta}^2}{4} - y^2 - x^2 + d_3x^2y + d_4y^3$$

where we keep the same labeling for the constants, even though they are not exactly equal.

If we perform the scalings

$$x \to \varepsilon x, \quad y \to \varepsilon y, \quad \tilde{\beta} \to \varepsilon, \quad \alpha \to \varepsilon\lambda, \quad t \to \varepsilon^{-1}t$$

the new equations become

$$\begin{aligned} \dot{x} &= \lambda x + Bxy + \varepsilon[d_1x^3 + d_2xy^2] \\ \dot{y} &= \frac{1}{4} - y^2 - x^2 + \varepsilon[d_3x^2y + d_4y^3] \end{aligned} \tag{2.1}$$

and are in normal form.

Equation (2.1) must be discussed for all $x \geq 0$, $y \in \mathbb{R}$, $\lambda \in \mathbb{R}$, $B \in \mathbb{R}$, and ε in a neighborhood of $\varepsilon = 0$.

In the following, we assume $B > 0$.

Let us first discuss equation (2.1) for $\varepsilon = 0$; that is, the equation

$$\dot{x} = \lambda x + Bxy$$

$$\dot{y} = \frac{1}{4} - y^2 - x^2$$

There are always two equilibrium solutions $(0, \pm 1/2)$. The equilibrium point $(0,1/2)$ is a hyperbolic stable node if $\lambda < -B/2$ and a saddle point if $\lambda > -B/2$. The point $(0,-1/2)$ is a saddle point if $\lambda < B/2$ and a hyperbolic unstable node if $\lambda > B/2$. If $\lambda^2 < B^2/4$, there are other equilibrium points, $(\pm(1/4 - \lambda^2/B^2)^{1/2}, -\lambda/B)$. At the values $\lambda = \pm B/2$, there is a bifurcation of an equilibrium point into three equilibria, each of which is hyperbolic. This bifurcation still remains when $\varepsilon \neq 0$ is small. The equilibrium point $(\pm(1/4 - \lambda^2/B^2)^{1/2}, -\lambda/B)$ changes its stability properties at $\lambda = 0$, being a stable focus for $\lambda > 0$ and an unstable focus for $\lambda < 0$. The point $\lambda = 0$ thus becomes another place where bifurcations can occur.

It remains to analyze the behavior of the solutions of (2.1) for $\lambda = 0$, $\varepsilon = 0$. For $\lambda = 0$, $\varepsilon = 0$; that is, the equation

$$\begin{aligned} \dot{x} &= Bxy \\ \dot{y} &= \frac{1}{4} - y^2 - x^2 , \end{aligned} \tag{2.2}$$

there is a first integral

$$H = \frac{B}{2}\left[\frac{x^{q+1}}{4} - x^{q+1}y^2 - \frac{x^{q+3}}{1+B}\right] \tag{2.3}$$

where $1+q = 2/B$.

The fact that (2.1) for $\lambda = \varepsilon = 0$ has a first integral implies that cubic terms are necessary to resolve the complete bifurcation diagram near $\lambda = \varepsilon = 0$. To obtain this bifurcation diagram, one must discuss the periodic orbits of (2.1) for (λ,ε) small. The first integral of (2.2) is very useful in such an analysis. Let us briefly indicate how this is done.

It is convenient to reparametrize the orbits by replacing t by $x^q t$ to obtain the system

$$\begin{aligned} \dot{x} &= x^q\,[\lambda x + Bxy + \varepsilon(d_1x^3 + d_2xy^2)] \\ \dot{y} &= x^q[\tfrac{1}{4} - x^2 - y^2 + \varepsilon(d_3x^2y + d_4y^3)] \end{aligned} \tag{2.4}$$

The derivative of H along the solutions of (2.4) satisfies

$$\frac{dH}{dt} = \dot{y}x^{q+1}\lambda + \varepsilon\dot{y}(d_1x^3 + d_2xy^2)x^q - \varepsilon\dot{x}(d_3x^2y + d_4y^3)x^q \tag{2.5}$$

It is not difficult to see (Carr [8]) that a necessary and sufficient condition for an orbit $\Gamma = \Gamma(\lambda,\varepsilon)$ of (2.1) to be periodic in that

$$\int_\Gamma \dot{H}\,dt = 0\ . \tag{2.6}$$

Using (2.5) and (2.6), one obtains a bifurcation function for periodic orbits which involves Abelian integrals. (See Sections 3,4 and Appendices A,B). It is then shown that the number of periodic orbits

is then reduced to the discussion of the monotonicity properties of a scalar valued function which is the ratio of two Abeli · integrals. Under the assumption that $B = 2$, $d_1 = 1$, $d_2 = d_3 = d_4 = 0$ and by assuming the monotonicity of the above function, the complete bifurcation has been obtained (see, for example, Chow and Hale [9]).

In the following section, we will give a proof of this monotonicity hypothesis. Thus, this completes the bifurcation diagram for equation (2.1) with $B = 2$, $d_1 = 1$, $d_2 = d_3 = d_4 = 0$. In Section 4, the case $B \geq 1/2$ will be considered. The problem of monotonicity is still open for $0 < B < 1/2$. In Appendices A, B, we show that the results obtained are still valid for arbitrary but fixed values of the d_j not lying on a hyperplane in (d_1, d_2, d_3, d_4) space. Note that this is not surprising because, on a Riemann surface of finite genus, there are only a fixed number, depending on the genus, of linearly independent holomorphic differentials.

§3. UNIQUENESS THEOREM FOR $B = 2$

Consider the equation

$$\begin{aligned} \dot{x} &= \lambda x + 2xy + \varepsilon x^3 \\ \dot{y} &= \frac{1}{4} - y^2 - x^2 \end{aligned} \tag{3.1}$$

which is equation (2.1) with $B = 2$, $d_1 = 1$, $d_2 = d_3 = d_4 = 0$. Let

$$H = \frac{x}{4} - xy^2 - \frac{x^3}{3} .$$

When $\lambda = \varepsilon = 0$, (3.1) is a Hamiltonian system with the Hamiltonian H. Thus, solutions of (3.1) with $\lambda = \varepsilon = 0$ are parametrized by $H = c$ or

$$xy^2 = \frac{x}{4} - \frac{x^3}{3} - c \tag{3.2}$$

where $c \in \mathbb{R}$. We note that $c = 0$ corresponds to the heteroclinic orbit, $c = 1/12$ corresponds to the fixed point $(1/2,0)$ while $0 < c < 1/12$ corresponds to a periodic orbit.

As remarked in the previous section, a necessary and sufficient condition for an orbit $\Gamma = \Gamma(\lambda,\varepsilon)$ of (3.1) to be periodic is that

$$\int_\Gamma \dot{H}\, dt = 0$$

or

$$0 = \int_0^T \dot{y}(\lambda x + \varepsilon x^3)dt$$

$$= - \int_0^T y(\lambda\dot{x} + 3\varepsilon x^2\dot{x})dt$$

where $T > 0$ is the minimal period of the periodic orbit $\Gamma(\lambda,\varepsilon)$. By using the above equation and letting $\lambda = 3\varepsilon\mu$, one obtains a bifurcation function $G(\mu,\varepsilon,c)$ for periodic orbits which for $\varepsilon = 0$ is given by

$$G(\mu,0,c) = \mu J_0(c) + J_2(c) \ , \quad 0 < c < \frac{1}{12} \ ,$$

where

(3.2) $$J_0(c) = \int_{a_1}^{a_2} y dx$$

(3.3) $$J_2(c) = \int_{a_1}^{a_2} x^2 y dx$$

$$y = +\left(\frac{1}{4} - \frac{x^2}{3} - \frac{c}{x}\right)^{1/2}$$

and $0 < a_1 < 1/2$ and $1/2 < a_2 < \sqrt{3}/2$ are the two real positive roots of the equation

$$\frac{1}{4} - \frac{x^2}{3} - \frac{c}{x} = 0$$

for $0 < c < 1/12$. Let

$$P(c) = \frac{J_2(c)}{J_0(c)} \qquad 0 < c < 1/12 \tag{3.4}$$

It is shown in [9] that, if $P'(c) \neq 0$, then equation (2.1) has a unique asymptotically stable limit cycle for appropriate values of $(\lambda,\varepsilon) \neq (0,0)$. Our main result in this section is the following.

<u>Theorem 3.1</u> Let $P(c)$ be defined as above. We have $P'(c) > 0$ for $0 < c < 1/12$.

To prove this, let $c_1 = \sqrt{a_1}$, $c_2 = \sqrt{a_2}$ and

$$R(w) = \left(\frac{w^2}{4} - c - \frac{w^6}{3}\right)^{1/2}, \qquad 0 \leq w \leq \sqrt{\frac{\sqrt{3}}{2}} \tag{3.5}$$

Define

$$I_n = \int_{c_1}^{c_2} w^{2n} R(w)\,dw. \tag{3.6}$$

By setting $w^2 = x$ and using (3.2), (3.3), we have

$$2I_2 = J_2,\ 2I_0 = J_0, \qquad 0 < c < \frac{1}{12}$$

Hence,

$$P(c) = \frac{I_2(c)}{I_0(c)} \qquad 0 < c < \frac{1}{12} \tag{3.7}$$

We will use (3.7) to show Theorem 3.1.

<u>Lemma 3.2</u>

$$I_n'(c) = -\frac{1}{2}\int_{c_1}^{c_2} \frac{w^{2n}}{R(w)}\,dw$$

Proof. From (3.5), $2RdR/dc = -1$.

Lemma 3.3

$$\lim_{c \to \frac{1}{12}} P(c) = \frac{1}{4}$$

Proof.

$$\lim_{c \to \frac{1}{12}} P(c) = \lim_{c \to \frac{1}{12}} \frac{I_2'(c)}{I_0'(c)}$$

$$= \left(\frac{1}{2}\right)^2$$

Lemma 3.4

$$\lim_{c \to 0} P(c) = \frac{3}{16}$$

Proof. Integration.

Lemma 3.5 Let $I(c)$ be the column vector (I_0, I_1, I_2). Then $I(c)$ satisfies the differential equation

$$I(c) = \Lambda(c)\, I'(c) \tag{3.8}$$

where

$$\Lambda(c) = \begin{bmatrix} \frac{3}{2}c & -\frac{1}{4} & 0 \\ 0 & c & -\frac{1}{6} \\ \frac{3}{32}c & -\frac{3}{64} & \frac{3}{4}c \end{bmatrix}$$

and

$$\Delta = \det \Lambda(c) = \frac{9}{8}\left(c^2 - \frac{1}{144}\right)c$$

<u>Proof</u>. Let us first prove that $I_0(c) = 3cI_0'(c)/2 - I_1'(c)/4$. From the relation

$$2R\frac{dR}{dw} = \frac{1}{2}w - 2w^5$$

it follows that, as differential forms,

$$\begin{aligned} Rdw &= \frac{R^2}{R}dw \\ &= \frac{dw}{R}\left[-\frac{1}{3}w\left(-R\frac{dR}{dw} + \frac{1}{4}w\right) + \frac{w^2}{4} - c\right] \\ &= \frac{1}{3}wdR + \frac{dw}{R}\left[\frac{1}{6}w^2 - c\right] \end{aligned}$$

Integration by parts yields

$$I_0(c) = -\frac{1}{3}I_0(c) - \frac{1}{3}I_1'(c) + 2cI_0'(c)$$

which is the desired relation. Let us now shown that

$$6I_1 = 6cI_1' - I_2'$$

By the definition of I_1 and by integration by parts, we have

$$\begin{aligned} I_1 &= \int_{c_1}^{c_2} w^2\left(\frac{w^2}{4} - c - \frac{w^6}{3}\right)\frac{dw}{R} \\ &= \int_{c_1}^{c_2}\left(\frac{w^8}{3} - \frac{w^4}{12}\right)\frac{dw}{R} \quad . \end{aligned}$$

Hence,

$$\frac{2}{3}\int_{c_1}^{c_2} w^8 \frac{dw}{R} = \int_{c_1}^{c_2}\left(\frac{w^4}{3} - cw^2\right)\frac{dw}{R} ,$$

so that

$$I_1 = \int_{c_1}^{c_2}\left(\frac{w^4}{12} - \frac{cw^2}{2}\right)\frac{dw}{R} = cI_1' - \frac{I_2'}{6}$$

Finally, we show that

$$64\, I_2 = 6c\, I_0' - 3\, I_1' + 48\, cI_2'$$

Using the definition and integration by parts, one obtains

$$I_2 = \int_{c_1}^{c_2}\left(\frac{w^{10}}{5} - \frac{w^6}{20}\right)\frac{dw}{R}$$

$$= \int_{c_1}^{c_2} \frac{w^4R^2}{R}\, dw .$$

Hence,

$$\frac{8}{15}\int_{c_1}^{c_2} w^{10}\frac{dw}{R} = \int\left(\frac{6}{20}w^6 - cw^4\right)\frac{dw}{R} ,$$

so that

$$I_2 = \int_{c_1}^{c_2}\left(\frac{w^6}{16} - \frac{3cw^4}{8}\right)\frac{dw}{R} .$$

A similar argument on I_0 yields

$$\int_{c_1}^{c_2} w^6\frac{dw}{R} = \int_{c_1}^{c_2}\frac{3}{4}\left(\frac{w^2}{2} - c\right)\frac{dw}{R} .$$

Combining the above two equations, we obtain

$$16I_2 = \int_{c_1}^{c_2} \left(\frac{3w^2}{8} - \frac{3c}{4} - 6cw^4 \right) \frac{dw}{R} \quad .$$

which is the desired formula for I_2. This proves the lemma.

Lemma 3.5 is known as the Picard-Fuchs equation for the integrals I_0, I_1, I_2 and corresponds to Eq. (26) in Cushman and Sanders [10].

<u>Lemma 3.6</u>

$$\begin{bmatrix} I_0'' \\ I_2'' \end{bmatrix} = \frac{1}{8\Delta} \begin{bmatrix} -3c^2 & \frac{1}{12} \\ -\frac{3}{4}c^2 & 3c^2 \end{bmatrix} \begin{bmatrix} I_0' \\ I_2' \end{bmatrix} \tag{3.9}$$

<u>Proof</u>. By (3.8), $I''(c) = \Lambda^{-1}(c)(E - \Lambda'(c))I'(c)$ where E denotes the identity matrix. We have

$$\Lambda^{-1}(c) = \frac{1}{\Delta} \begin{bmatrix} \frac{3}{4}c^2 - \frac{1}{128} & \frac{3}{16}c & \frac{1}{24} \\ -\frac{1}{64}c & \frac{9}{8}c^2 & \frac{1}{4}c \\ -\frac{3}{32}c^2 & -\frac{3}{32}c & \frac{3}{2}c^2 \end{bmatrix}$$

and

$$E - \Lambda'(c) = \begin{bmatrix} -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{3}{32} & 0 & \frac{1}{4} \end{bmatrix}$$

Equation (3.9) follows from the above matrix equation.

Lemma 3.7 Let

$$Q(c) = \frac{I_2'(c)}{I_0'(c)}$$

Then,

$$Q' = \frac{1}{8\Delta}\left[-\frac{3}{4}c^2 + 6c^2Q - \frac{1}{12}Q^2\right] \tag{3.10}$$

Proof. (3.10) follows from (3.9).

Equation (3.10) corresponds to the Ricatti equation of Cushman and Sanders [10].

Lemma 3.8 If $0 < c < \frac{1}{12}$, $Q \in \mathbb{R}$, then

$$\frac{1}{8\Delta}\left[-\frac{3}{4}c^2 + 6c^2Q - \frac{1}{12}Q^2\right] > 0 .$$

Proof. The maximum of the quadratic form

$$-\frac{3}{4}c^2 + 6c^2Q - \frac{1}{12}Q^2$$

occurs at $Q = 36c^2$ and its maximal value is

$$-\frac{9}{12}(1 - 144c^2)c^2.$$

Since $0 < c < \frac{1}{12}$ and $\Delta < 0$ (Lemma 3.5), we have the desired result.

Lemma 3.9 If $P'(c_0) = 0$ for some $0 < c_0 < \frac{1}{12}$, then $P''(c_0) < 0$.

Proof. By the definitions of P and Q, we have

$$I_0'P + I_0P' = I_2'$$

$$I_0''P + 2I_0P' + I_0P'' = I_2''$$

$$I_0'Q = I_2'$$

$$I_0''Q + I_0'Q' = I_2''$$

Since $P'(c_0) = 0$, we have $P''(c_0) = I_0'(c_0)Q'(c_0)/I_0(c_0)$. By (3.6) and Lemma 3.2, $I_0(c) > 0$, $I_0'(c_0) < 0$. By Lemmas 3.7 and 3.8, $Q'(c_0) > 0$. This implies $P''(c_0) < 0$.

Lemma 3.10 If $P'(c_0) = 0$ for some $0 < c_0 < \frac{1}{12}$ then $\frac{3}{16} < P(c_0) < \frac{1}{4}$.

Proof. By the first and last equations in (3.8),

$$3I_0 - 16I_2 = 3cI_0' - 12I_2'$$

If $P'(c_0) = 0$, then, by the above equation,

$$\frac{3}{16} - P(c_0) = \frac{3}{16}\, c\, \frac{I_0'(c)}{I_0(c)}\,(1 - 4P(c_0))$$

because $I_2'(c_0) = P(c_0)I_0'(c_0)$. Since $I_0'(c_0) < 0$, $P(c_0) - \frac{3}{16}$ has the same sign as $\frac{1}{4} - P(c_0)$, i.e., we must have $\frac{3}{16} < P(c_0) < \frac{1}{4}$.

Proof of Theorem 3.1

It follows easily from Lemmas 3.3, 3.4, 3.9 and 3.10.

§4. UNIQUENESS THEOREM FOR $B \geq \frac{1}{2}$.

In this section, we consider the general case $B \geq \frac{1}{2}$, $d_1 = 1$, $d_2 = d_3 = d_4 = 0$. From (2.1) the normal form is

$$\begin{aligned} \dot{x} &= \lambda x + Bxy + \varepsilon x^3 \\ \dot{y} &= \frac{1}{4} - x^2 - y^2 \end{aligned} \tag{4.1}$$

We note that the notations used in this section will be similar to those in §3. This will not cause any confusion. In fact, if we let $B = 2$, then both notations will coincide. However, the proofs of monotonicity in §3 and this section are different. The main reason to include the proof in §3 in this paper is to show that, for $B = 2$, one may be able to use complex variable topological arguments as in [10,12,13] to prove Theorem 3.1, while it does not seem likely in the present case.

Let

$$1 + q = \frac{2}{B} \quad . \tag{4.2}$$

Changing time scales, we obtain the following normal form

$$\begin{aligned} \dot{x} &= x^q(\lambda x + Bxy + \varepsilon x^3) \\ \dot{y} &= x^q(\tfrac{1}{4} - x^2 - y^2) \end{aligned} \tag{4.3}$$

Let

$$H = \frac{B}{2}\left(\frac{x^{q+1}}{4} - x^{q+1}y^2 - \frac{x^{q+3}}{B+1} \right)$$

Using (4.3), we have along solutions of (4.3)

$$\dot{H} = \dot{y}(x^{q+1}\lambda + \varepsilon x^{q+3}) \tag{4.4}$$

Let $\varepsilon\mu = \lambda$. As in §3, we obtain, for periodic solutions of (4.3), a bifurcation function $G(\mu,\varepsilon,c,B)$ which for $\varepsilon = 0$ is given by

$$G(\mu,0,c,B) = (q+1)\mu J_0(c,B) + (q+3)J_2(c,B), \qquad 0 < c < c_m,$$

where

$$c_m = c_m(B) = \left(\frac{1}{2}\right)^{q+1}\left(\frac{1}{4} - \frac{1}{4(1+B)}\right) , \tag{4.5}$$

$$J_0(c,B) = \int_{a_1}^{a_2} x^q y dx ,$$

$$J_2(c,B) = \int_{a_1}^{a_2} x^{q+2} y dx ,$$

$$y = \left(\frac{1}{4} - \frac{x^2}{1+B} - \frac{c}{x^{q+1}}\right)^{\frac{1}{2}} ,$$

and $0 < a_1 < \frac{1}{2} < a_2$ are the zeros of y. We note that $c = 0$ corresponds to the heteroclinic orbit in equation (4.1) (with $\varepsilon = \lambda = 0$) while $c_m(B)$ corresponds to the fixed point $(\frac{1}{2}, 0)$ in equation (4.1) with $\varepsilon = \lambda = 0$. Let

$$P(c,B) = \frac{J_2(c,B)}{J_0(c,B)} , \qquad 0 < c < c_m(B)$$

As in §3, let

$$c_1^B = a_1, \; c_2^B = a_2 \quad \text{and}$$

$$R(w) = \left(\frac{w^2}{4} - \frac{w^{2B+2}}{B+1} - c\right)^{\frac{1}{2}}, \qquad 0 < c < c_m . \tag{4.6}$$

Note that $R(c_1) = R(c_2) = 0$ and $0 < c_1 < c_2$. Define

$$I_n(c,B) = \int_{c_1}^{c_2} w^{nB} R(w) dw, \qquad n = 0,1,2,\ldots \tag{4.7}$$

We have

$$P(c,B) = \frac{I_2(c,B)}{I_0(c,B)} . \tag{4.8}$$

<u>Theorem 4.1</u> If $B \geq \frac{1}{2}$ and $0 < c < c_m$, then

$$\frac{\partial P}{\partial c}(c,B) > 0 .$$

By using this theorem, it will be possible to obtain a complete bifurcation diagram for equation (4.1) by following the methods in Chow and Hale [9]. We leave the details to the reader.

The following are needed for the proof of Theorem 4.1.

<u>Lemma 4.2</u> For $n = 0,1,2,$

$$\frac{\partial I_n}{\partial c}(c,B) = -\frac{1}{2}\int_{c_1}^{c_2} \frac{w^{nB}}{R(w)}\,dw . \tag{4.9}$$

<u>Proof.</u> Since

$$R^2 = \frac{w^2}{4} - \frac{w^{2B+2}}{B+1} - c ,$$

we have

$$2RR' = -1 ,$$

where "$'$" = "$\partial/\partial c$" .

<u>Lemma 4.3</u> Let

$$\xi = \xi(B) = \lim_{c\to 0} P(c,B), \quad \eta = \eta(B) = \lim_{c\to c_m} P(c,B) .$$

Then

$$\xi = \frac{B+1}{2(3B+2)} < \frac{1}{4} = \eta .$$

<u>Proof.</u> By (4.2), (4.6), (4.7) and (4.8), we have $(x = w^B)$,

$$\xi = \frac{I_2(0,B)}{I_0(0,B)}$$

$$= \frac{\int_{a_1}^{a_2} x^{q+2}\left(1 - \frac{4x^2}{B+1}\right)^{\frac{1}{2}}dx}{\int_{a_1}^{a_2} x^{q}\left(1 - \frac{4x^2}{B+1}\right)^{\frac{1}{2}}dx}$$

where $a_1 = 0$ and $a_2 = \sqrt{B+1}/2$. Hence,

$$\xi = \frac{\frac{B+1}{4}\int_0^{\pi/2} (\sin\theta)^{q+2}\cos^2\theta\, d\theta}{\int_0^{\pi/2} (\sin\theta)^{q}\cos^2\theta\, d\theta}$$

$$= \frac{B+1}{2(3B+2)} \quad .$$

Next,

$$\eta = \lim_{c\to c_m} \frac{\frac{\partial I_2}{\partial c}(c,B)}{\frac{\partial I_0}{\partial c}(c,B)} = \frac{1}{4} \quad .$$

This completes the proof.

We are not able to find a differential equation like (3.8) for the integrals $I_0(c,B)$, $I_1(c,B)$ and $I_2(c,B)$. It seems this is possible when B is an integer. Such equations for $B = 1$ could be found in Carr [8] (Lemma 4 in Chapter 4).

Define

$$S(c,B) = \int_{c_1}^{c_2} w^2 R(w)dw \quad . \tag{4.10}$$

Note that $I_1(c,2) = S(c,2)$. Hence, the case $B = 2$ in §3 gives a check on some of the following formulae.

<u>Lemma 4.4</u> The following equations are satisfied by I_0, S, I_2:

$$\Phi \begin{bmatrix} I_0 \\ I_2 \end{bmatrix} = \psi \begin{bmatrix} I_0' \\ I_2' \end{bmatrix} - \frac{B}{2(B+2)} S' \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{4.11}$$

where " $'$ " = "$\partial/\partial c$", $0 < c < c_m$

$$\Phi = \begin{bmatrix} 1 & 0 \\ -1 & \frac{2(3B+2)}{B+1} \end{bmatrix} \quad \text{and}$$

$$\Psi = c \begin{bmatrix} \frac{2(B+1)}{B+2} & 0 \\ -1 & 4 \end{bmatrix} .$$

<u>Proof</u>. Integrating by parts,

$$I_0 = \int_{c_1}^{c_2} \left(\frac{w^2}{4} - \frac{w^{2B+2}}{B+1} - c \right) \frac{dw}{R}$$

$$= \int_{c_1}^{c_2} R\, dw$$

$$= -\int_{c_1}^{c_2} w\, dR$$

$$= \int_{c_1}^{c_2} \left(w^{2B+2} - \frac{w^2}{4} \right) \frac{dw}{R} .$$

Hence,

$$\frac{B+2}{B+1}\int_{c_1}^{c_2} \frac{w^{2B+2}}{R(w)}\,dw = \int_{c_1}^{c_2}\left(\frac{w^2}{2} - c\right)\frac{dw}{R} .$$

The first equation in (4.11) follows from the above equation. The second equation in (4.11) is obtained similarly.

<u>Lemma 4.5</u> For $0 < c < c_m$

$$\begin{bmatrix} I_0'' \\ I_2'' \end{bmatrix} = \delta \begin{bmatrix} -4 & 0 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} I_0' \\ I_2' \end{bmatrix} + \frac{\delta}{2}\begin{bmatrix} 4 \\ 1 \end{bmatrix} S'' \tag{4.12}$$

where $\delta = B/8c(B+1)$.

<u>Proof</u>. This is obtained by differentiating (4.11).

<u>Lemma 4.6</u> For $0 < c < c_m$

$$Q' = Q[(1 - 4Q)\frac{I_2''}{I_2'} + 16\delta Q] \tag{4.13}$$

where

$$Q = \frac{I_2'}{I_0'} \quad \text{and} \quad \delta = \frac{B}{8c(B+1)}$$

<u>Proof</u>. This follows from (4.12)

<u>Lemma 4.7</u> If $P'(c_0,B) = 0$ for some $0 < c_0 < c_m$, then $P(c_0,B) = Q(c_0,B)$ and $P''(c_0,B)$ has the opposite sign as $Q'(c_0,B)$.

<u>Proof</u>. See the proof of Lemma 3.9.

Lemma 4.8 If $P'(c_0,B) = 0$ for some $0 < c_0 < c_m$, then

$$P(c_0,B) - \xi = \frac{4c\xi I_2'(c_0,B)}{I_2(c_0,B)} \left[P(c_0,B) - \frac{1}{4} \right] \tag{4.14}$$

where ξ is given by Lemma 4.3.

Proof. At $c = c_0$, $I_2'(c_0,B) = P(c_0,B)I_0'(c_0,B)$ since $I_2 = PI_0$, we obtain (4.14) from the second equation in (4.11) by substituting the above values of $P(c_0,B)$.

Lemma 4.9 For $0 < c < c_m$

$$\xi < P(c,B) < \frac{1}{4} \tag{4.15}$$

Proof. If not, then we have either $P(c_0,B) \geq \frac{1}{4}$ or $\leq \xi$ for some $0 < c_0 < c_m$. By Lemma 4.3, we may assume that $P'(c_0,B) = 0$. Since $I_2 > 0$ and $I_2' < 0$, this contradicts (4.14).

Lemma 4.10 If $P'(c_0,B) = 0$, $0 < c_0 < c_m$, then $P''(c_0,B)$ has the same sign as the following expression evaluated at (c_0,B)

$$4c_0 I_2'' + Q(\tfrac{1}{4} - Q)^{-1} \frac{2B}{B+1} I_2' \tag{4.16}$$

Proof. By Lemma 4.7, $P''(c_0,B)$ has the sign opposite to $Q'(c_0,B)$ and $P(c_0,B) = Q(c_0,B)$. By Lemma 4.9 and (4.13), $P''(c_0,B)$ has the sign opposite to the following expression evaluated at (c_0,B):

$$(1 - 4Q) \frac{I_2''}{I_2'} + 16\delta Q$$

Since $I_2' < 0$ and $Q(c_0,B) < \frac{1}{4}$, the result follows, from the definition of δ.

<u>Lemma 4.11</u> If $P'(c_0,B) = 0$, $0 < c_0 < c_m$, then $P''(c_0,B)$ has the same sign as the following expression evaluated at (c_0,B)

$$4c_0 I_2'' + 4I_2' \left(\frac{I_2 - c_0 I_2'}{I_2} \right) \tag{4.17}$$

<u>Proof</u>. Let $P'(c_0,B) = 0$. Using (4.14), we have, for $c = c_0$,

$$P(\tfrac{1}{4} - P)^{-1} = \left(\frac{I_2 - c_0 I_2'}{I_2} \right) \frac{\xi}{\frac{1}{4} - \xi}$$

Substituting the above equation into (4.16), we obtain the desired result.

The proof of Theorem 4.1 will be given in the same spirit as before (Theorem 3.1), i.e., we will show that if $P'(c_0,B) = 0$, then $P''(c_0,B) > 0$. Thus, we have to estimate very carefully the sign of the expression (4.17).

To begin, define

$$r(w) = \frac{w^2}{4} - \frac{w^{2B+2}}{B+1} - c ,$$

i.e., $r(w) = R^2(w)$. Let

$$\bar{w} = \left(\frac{1}{2}\right)^{\frac{1}{B}} \quad \text{and} \quad \bar{r} = r(\bar{w}) = \frac{\bar{w}^2}{2} - \frac{\bar{w}^{2B+2}}{B+1} - c .$$

Note that $r(w) - \bar{r}$ is independent of c and $\partial r(\bar{w})/\partial w = 0$. Define

$$J_2(c,B) = \int_{c_1}^{c_2} w^{2B}(r(w) - \bar{r}) \, [r(w)]^{\frac{1}{2}} \, dw \tag{4.18}$$

where c_1 and c_2 are as in (4.7)

<u>Lemma 4.12</u> Let

$$f(w) = (r(w) - \bar{r})\left(2\frac{\partial^2 r}{\partial w^2} - \frac{4B}{w}\frac{\partial r}{\partial w}\right) - \left(\frac{\partial r}{\partial w}\right)^2$$

Then

$$-4\bar{r}I_2'' = \int_{c_1}^{c_2} \frac{w^{2B} f(w)}{\left(\frac{\partial r}{\partial w}\right)^2 \sqrt{r(w)}}\, dw \tag{4.19}$$

<u>Proof</u>. Note that $\bar{w}$ is independent of c and $\partial\bar{r}/\partial c = -1$. By (4.18),

$$J_2' = -\frac{1}{2}\int_{c_1}^{c_2} \frac{w^{2B}(r(w) - \bar{r})}{[r(w)]^{\frac{1}{2}}}\, dw \tag{4.20}$$

$$= -\frac{1}{2} I_2 - \bar{r}I_2' .$$

Also,

$$J_2 = \frac{2}{3}\int_{c_1}^{c_2} \frac{w^{2B}(r(w) - \bar{r})}{\left(\frac{\partial r}{\partial w}\right)}\, d(r^{3/2}(w)) \tag{4.21}$$

Note that $\partial r/\partial w$ has a simple zero at $w = \bar{w}$ while the factor $r(w) - \bar{r}$ has a double zero at $w = \bar{w}$. Thus, the following calculations are valid. From (4.21),

$$J_2 = -\frac{2}{3}\int_{c_1}^{c_2} \frac{w^{2B} r^{3/2} g(w)}{\left(\frac{\partial r}{\partial w}\right)^2}\, dw ,$$

where

$$g(w) = \left(\frac{2B}{w}\frac{\partial r}{\partial w} - \frac{\partial^2 r}{\partial w^2}\right)(r - \bar{r}) + \left(\frac{\partial r}{\partial w}\right)^2 . \tag{4.22}$$

Note that $g(w)$ does not depend on c. By differentiating the above equation with respect to c,

$$J_2' = \int_{c_1}^{c_2} \frac{w^{2B}\sqrt{r}g(w)}{\left(\frac{\partial r}{\partial w}\right)^2}\, dw$$

$$= -\frac{1}{2} I_2 - \bar{r} I_2'$$

because of (4.20). Thus,

$$-\frac{1}{2} I_2' + I_2' - \bar{r} I_2''$$

$$= -\frac{1}{2}\int_{c_1}^{c_2} \frac{w^{2B} g(w)}{\sqrt{r(w)}\left(\frac{\partial r}{\partial w}\right)^2}\, dw$$

Finally, using (4.9), i.e.

$$2I_2' = -\int_{c_1}^{c_2} \frac{w^{2B}\left(\frac{\partial r}{\partial w}\right)^2}{[r(w)]^{\frac{1}{2}}\left(\frac{\partial r}{\partial w}\right)^2}\, dw ,$$

We obtain (4.19) and that $f(w) = (\partial r/\partial w)^2 - 2g(w)$.

<u>Lemma 4.13</u> We have

$$-\frac{2\bar{r} I_2'}{I_2} \geq 1 \tag{4.23}$$

<u>Proof</u>. Recall that $\bar{r} = r(\bar{w})$ and that $\partial r(\bar{w})/\partial w = 0$. Thus $\bar{r}$ is the maximum of $r(w)$. From (4.20),

$$-\frac{1}{2} I_2 - \bar{r} I_2' \geq 0$$

Since $I_2 > 0$, the result follows.

<u>Lemma 4.14</u> If $B \geq \frac{1}{2}$ and $P'(c_0,B) = 0$, then $P''(c_0,B) > 0$.

<u>Proof</u>. Using (4.23) in Lemma 4.11, it is sufficient to show that the following expression

$$4c_0 \bar{r} I_2'' + (4\bar{r} + 2c_0) I_2'$$

is negative when it is evaluated at (c_0,B). By Lemma 4.12, it is sufficient to prove that

$$h(w) = c_0 f(w) + (2\bar{r} + c_0)\ (\partial r/\partial w)^2 \geq 0$$

for all $c_1 \leq w \leq c_2$. Since $2B \geq 1$ and $c_1 \geq 0$,

$$2\frac{\partial^2 r}{\partial w^2} - \frac{4B}{w}\ \frac{\partial r}{\partial w}$$

$$= 1 - 2B - 4w^{2B} \leq 0, \qquad c_1 \leq w \leq c_2$$

By (4.22),

$$h(w) = c_0(r(w) - \bar{r})\ (1 - 2B - 4w^{2B}) + 2\bar{r}(\partial r/\partial w)^2 .$$

Since $r(w) - \bar{r} \leq 0$ and $\bar{r} \geq 0$, we have that $h(w) \geq 0$ for $c_1 \leq w \leq c_2$. The result follows.

<u>Proof of Theorem 4.1</u> It follows from Lemma 4.9 and Lemma 4.14.

§5. BIFURCATION FUNCTION

Consider a singularity of type A_3 in the Introduction. This leads to the following equation in $\mathbb{R}^4$

$$\dot{v} = Av + G(v) \tag{5.1}$$

where $v \in \mathbb{R}^4$

$$A = \begin{bmatrix} \alpha & 1 & 0 & 0 \\ -1 & \alpha & 0 & 0 \\ 0 & 0 & \beta & \omega \\ 0 & 0 & -\omega & \beta \end{bmatrix}$$

$\alpha, \beta \in \mathbb{R}$ are small parameters, $\omega \in \mathbb{R}$, G is smooth and is of higher order at $v = 0$. If

$$v_1 = r_1 \cos \theta_1, \qquad v_2 = -r_1 \sin \theta_1 ,$$

$$v_3 = r_2 \cos \theta_2, \qquad v_4 = -r_2 \sin \theta_2 ,$$

then (5.1) is equivalent to the following equations

$$\begin{aligned} \dot{r}_1 &= \alpha r_1 + R_1(\theta_1,\theta_2,r_1,r_2) \\ \dot{r}_2 &= \beta r_2 + R_2(\theta_1,\theta_2,r_1,r_2) \\ \dot{\theta}_1 &= 1 + \Theta_1(\theta_1,\theta_2,r_1,r_2) \\ \dot{\theta}_2 &= \omega + \Theta_2(\theta_1,\theta_2,r_1,r_2) \end{aligned} \tag{5.2}$$

where each function is 2π-periodic in θ_1 and θ_2. Assuming certain nonresonance conditions on ω and symmetric conditions on $G(v)$, one can apply the method of averaging to (5.2) to obtain the following

equations

$$\dot{r}_1 = r_1(\alpha - \tilde{a}r_1^2 - \tilde{b}r_2^2) + \ldots$$

$$\dot{r}_2 = r_2(\beta + \tilde{c}r_1^2 + \tilde{d}r_2^2) + \ldots$$

$$\dot{\theta}_1 = 1 + \ldots$$

$$\dot{\theta}_2 = \omega + \ldots$$

where higher order terms are just indicated by the dots and $\tilde{a},\tilde{b},\tilde{c},\tilde{d}$ are constants.

As in §2, this leads to the consideration of the following equation in the plane

$$\begin{aligned} \dot{x} &= \frac{\tilde{b}+1}{\tilde{c}+1}\, x - \tilde{b}xy^2 - x^3 - \lambda x + \varepsilon x^5 \\ \dot{y} &= -y + \tilde{c}x^2y + y^3 \end{aligned} \tag{5.3}$$

for $x \geq 0$, $y \geq 0$, where λ,ε are small parameters $\tilde{b}$, $\tilde{c}$ are constants. Details may be found in Chow and Hale [9].

For the analysis of (5.3) the main question concerns the existence and number of limit cycles of (5.3) for $(\lambda,\varepsilon) \neq (0,0)$ near the origin. It will be shown that this is again related to the monotonicity of a scalar valued function. To do this, we must derive the bifurcation equation.

By scaling time $t \to 2px^{2p-1}y$ and writing

$$\frac{2(\tilde{c}+1)}{\tilde{b}\tilde{c}-1}\begin{bmatrix} -\lambda \\ \varepsilon \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \varepsilon_1 \end{bmatrix}$$

so that (5.3) yields

$$\dot{x} = 2px^{2p-1}y(\frac{\tau}{p}x - \tilde{b}xy^2 - x^3)$$

$$(5.4) \qquad + x^{2p-1}y(\lambda_1 x + \varepsilon_1 x^5) ,$$

$$\dot{y} = 2px^{2p-1}y(-y + \tilde{c}x^2y + y^3) ,$$

where

$$p = \frac{\tilde{c}+1}{\tilde{b}\tilde{c}-1} , \qquad \tau = \frac{1+\tilde{b}}{\tilde{b}\tilde{c}-1} .$$

For $\varepsilon_1 = \lambda_1 = 0$, the equation (5.4) has a first integral. This can be used to obtain the bifurcation function for periodic orbits. We have not been able to analyze completely the bifurcation function for this case. There seem to be regions in the $(\tilde{b},\tilde{c})$-parameter space where there is only one periodic orbit and other regions where there are more than one periodic orbit. Therefore, to illustrate non-uniqueness, we assume in the remainder of this section that $\tau = 1$, $p > 0$. These relations are equivalent to $\tilde{b} = 2/(\tilde{c} - 1)$, $\tilde{c} > 1$. Notice that this implies that $\tilde{b}\tilde{c} > 1$.

If one defines

$$H = x^{2p}y^2[1 - y^2 - px^2]$$

then (5.4) can be simplified as follows:

$$(5.5) \qquad \begin{aligned} \dot{x} &= \frac{\partial H}{\partial y} + x^{2p-1}y(\lambda_1 x + \varepsilon_1 x^5) \\ \dot{y} &= -\frac{\partial H}{\partial x} \end{aligned}$$

For $\lambda_1 = \varepsilon_1 = 0$, we have the following Hamiltonian system:

$$\dot{x} = \frac{\partial H}{\partial y}$$

(5.6)

$$\dot{y} = -\frac{\partial H}{\partial y}$$

Integral curves of (5.6) are given by the level curves $H = c$, i.e.,

$$H = x^{2p}y^2 - x^{2p}y^4 - px^{2p+2}y^2 = c ,$$

where $c \in \mathbb{R}$. Solving for y^2, we get

(5.7) $$y_{\pm}^2 = \frac{1}{2} - \frac{px^2}{2} \pm \sqrt{k(x)}$$

where

(5.8) $$k(x) = \frac{1}{4}(px^2 - 1)^2 - cx^{-2p} .$$

Let

(5.9) $$w = x^2, \qquad j(w) = k(x)$$

We now relate this to the periodic orbits of (5.6). Let $0 < w_1 < w_2$ be the zeros of $j(w)$. As w varies from w_1 to w_2, y_+^2 (given by (5.7) with the plus sign) goes through half of the periodic orbit $H = c$ which is above the line defined by:

(5.10) $$\ell: \quad 2y^2 = 1 - px^2$$

in the x^2y^2-plane (or wy^2-plane). This is illustrated in Figure 5.1.

We use this to get the bifurcation equation for periodic solutions of (5.5) when $(\lambda_1,\varepsilon_1) \neq (0,0)$. By differentiating along solutions of (5.5), we have

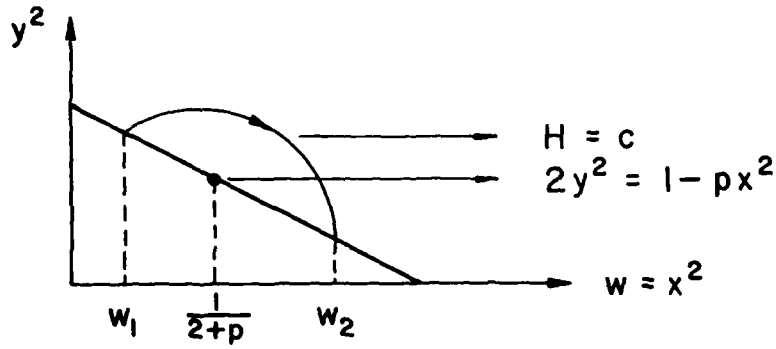


Fig. 5.1

$$\frac{dH}{dt} = x^{2p-1}y(\lambda_1 x + \varepsilon_1 x^5)\,\frac{\partial H}{\partial x}$$

$$= -\,(\lambda_1 x + \varepsilon_1 x^5)\; x^{2p-1}\,\frac{d}{dt}\left(\frac{y^2}{2}\right).$$

Integrating by parts from time t_1 to time t_2,

$$H(x(t_2),y(t_2)) - H(x(t_1),y(t_1))$$

$$= -\,(\lambda_1 x + \varepsilon_1 x^5)x^{2p-1}\;\frac{y^2}{2}\;\Bigg|_{t_1}^{t_2}$$

$$+ \int_{x(t_1)}^{x(t_2)} \frac{y^2(x)}{2}\left[2p\lambda_1 x^{2p-1} + \varepsilon_1(2p+4)x^{2p+3}\right]dx$$

$$= L(x,y)\;\Bigg|_{t_1}^{t_2} + \int_{x(t_1)}^{x(t_2)} M(x,y)dx, \text{ say.}$$

Let $t_1 = 0$. Assume $(x(0),y(0))$ is on the line ℓ defined in (5.10). Let $t_2^+ > 0$ $(t_2^- < 0)$ be the first forward (backward) time that the periodic orbit $(x(t),y(t))$ meets the line ℓ. Hence,

$$H(x(t_2^{\pm}),y(t_2^{\pm})) - H(x(0),y(0))$$

$$= L(x,y_{\pm})\Big|_0^{t_2^{\pm}} + \int_{x(0)}^{x(t_2^{\pm})} M(x,y_{\pm})dx$$

$$+ 0(|\lambda_1 + \varepsilon_1|^2),$$

where $y_{\pm}$ is given by (5.7). Thus, periodicity of $(x(t),y(t))$, i.e., $H(x(t_2^+),y(t_2^+)) = H(x(t_2^-),y(t_2^-))$, is equivalent to

$$\int_{x(0)}^{x(t_2^+)} M(x,y_+)dx = \int_{x(0)}^{x(t_2^-)} M(x,y_-)dx + 0(|\lambda_1 + \varepsilon_1|^2) . \tag{5.10}$$

By (5.7) and, by using the correct signs, (5.10) is equivalent to the following:

$$0 = \int_{x_1}^{x_2} \sqrt{k(x)} \left[2p\lambda_1 x^{2p-1} + \varepsilon_1(2p + 4)x^{2p+3}\right] dx + 0(|\lambda_1 + \varepsilon_1|^2)$$

where $x_1 = x(0)$ and $x_2 = x(t_2^+) = x(t_2^-)$. Let $\lambda_1 = \varepsilon_1\mu_1$. By dividing the above equation by ε_1, we obtain the bifurcation equation

$$0 = G(\mu_1,\varepsilon_1,c)$$

$$= \int_{x_1}^{x_2} \sqrt{k(x)} \left[2p\mu_1 x^{2p-1} + (2p + 4)x^{2p+3}\right] dx + 0(|\varepsilon|)$$

By (5.8) and (5.9),

$$G(\mu_1,0,c) \tag{5.11}$$

$$= \frac{1}{2}\int_{w_1}^{w_2}\left[\mu_1 p w^{\frac{p}{2}-1} + (p+2)w^{\frac{p}{2}+1}\right]R(w)dw$$

where

$$R(w) = [w^p(pw-1)^2 - 4c]^{\frac{1}{2}} . \tag{5.12}$$

Define

$$J_n(c) = \int_{w_1}^{w_2} w^{\frac{p}{2}-1+2n} R(w)dw, \quad n = 0,1 .$$

Consider the function

$$Y(c) = \frac{J_1(c)}{J_0(c)} \tag{5.13}$$

We summarize the above discussion into the following.

<u>Theorem 5.1</u> If $(\lambda_1,\varepsilon_1)$ is sufficiently small, then every periodic orbit of (5.5) must intersect the line ℓ defined by (5.10) with $0 < x^2 < 1/2+p$. Furthermore, there exists a smooth function $\mu_1^*(\varepsilon_1,c)$ such that a necessary and sufficient condition for $(x(t),y(t))$ to be a periodic solution of (5.5) with $(x(0),y(0)) \in \ell$, $0 < x^2(0) < 1/2+p$, $H(x(0),y(0)) = c$, is that $\mu_1 = \mu_1^*(\varepsilon_1,c)$, where $\lambda_1 = \varepsilon_1\mu_1$. Furthermore,

$$\mu_1^*(0,c) = -Y(c)\frac{p+2}{p} .$$

To discuss the behavior of the function $Y(c)$ as a function of c, observe that the range of values of c is $[0,c^*]$ where $c^* > 0$ and

(5.14) $$w_1(0) = 0, \quad w_2(0) = \frac{1}{p}$$

(5.15) $$w_1(c^*) = w_2(c^*) = \frac{1}{p+2}$$

Using (5.14), one can easily integrate $J_1(0), J_0(0)$ to obtain

$$Y(0) = \frac{p+1}{p(p+2)(p+3)}$$

Using (5.15), one observes that

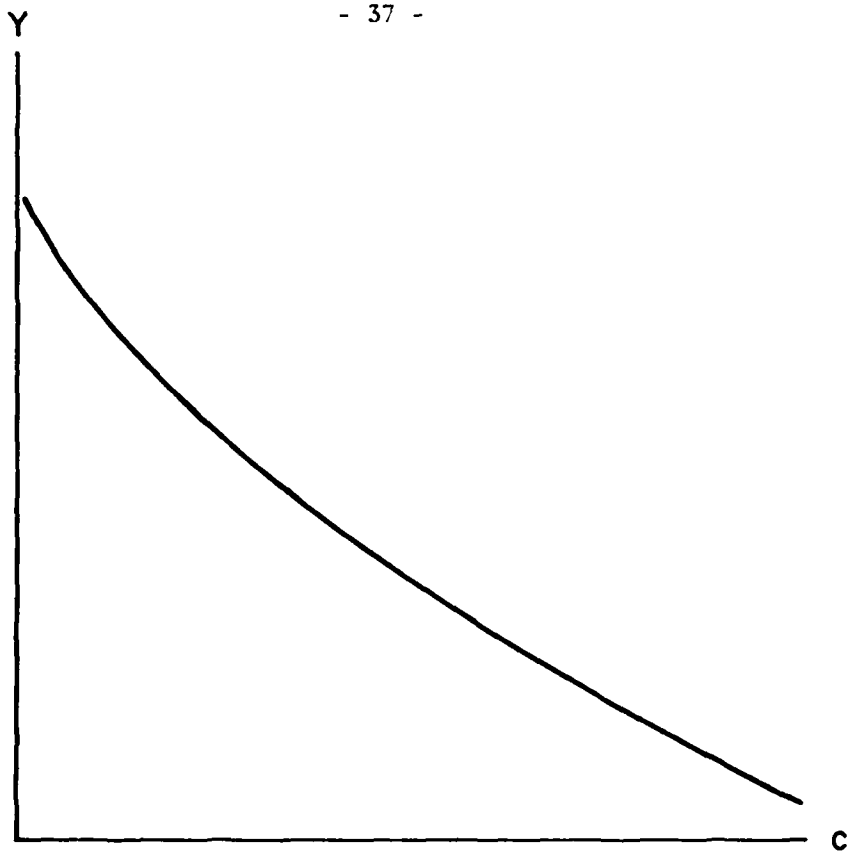
$$Y(c^*) = \frac{1}{(p+2)^2}$$

and

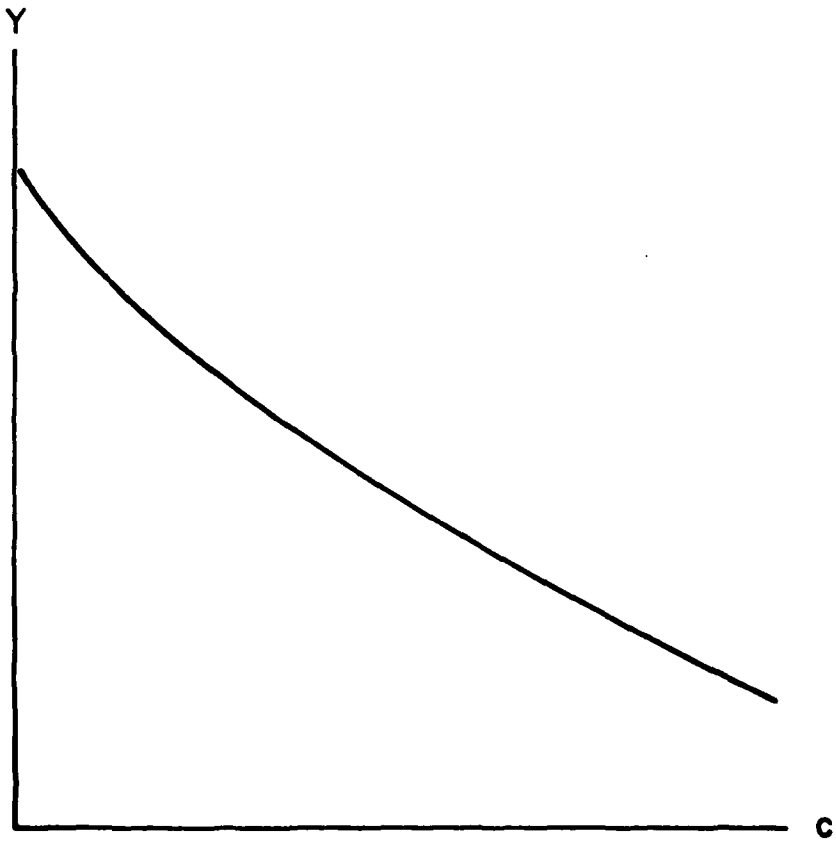
(5.16) $$Y(0) - Y(c^*) = \frac{2}{p(p+2)^2(p+3)} > 0$$

Numerical evidence indicates that $Y(c)$ is monotone and thus periodic orbits are unique. (See Fig. 5.2.)

It would be interesting to obtain the bifurcation function for $\tau \neq 1$ and also to determine numerically or analytically if there are regions in parameter space where there is nonuniqueness of periodic orbits.



p = 2



p = 1

Fig. 5.2

APPENDIX A

If $B = 2$, then the normal form is

$$\text{(A1)} \qquad \begin{aligned} \dot{x} &= \lambda x + 2xy + \varepsilon(d_1 x^3 + d_2 xy^2) \\ \dot{y} &= \frac{1}{4} - y^2 - x^2 + \varepsilon(d_3 x^2 y + d_4 y^3) \end{aligned}$$

where d_1, d_2, d_3 and d_4 are constants. In §2 and §3, we assumed that $d_1 = 1$, $d_2 = d_3 = d_4 = 0$. As in [8] or [9], we will obtain a bifurcation function $G(\mu,\varepsilon,c)$ which for $\varepsilon = 0$ is given by

$$\text{(A2)} \qquad \begin{aligned} G(\mu,0,c) &= \mu \int_{a_1}^{a_2} y\,dx + K_2 \int_{a_1}^{a_2} x^2 y\,dx + K_0 \int_{a_1}^{a_2} y\,dx \\ &= \mu J_0 + K_2 J_2 + K_0 J_0 \end{aligned}$$

where $\lambda = \varepsilon\mu$, J_0, J_2 are defined by (3.2) and (3.3) and

$$\text{(A3)} \qquad \begin{aligned} K_2 &= 3(d_1 - d_4) - (d_2 - d_3) \\ K_0 &= \frac{d_2}{4} + \frac{3}{4} d_4 \end{aligned}$$

To show (A2), we note that an orbit Γ of (A1) is periodic with minimal period $T > 0$ if and only if

$$0 = \int_\Gamma \dot{H}\,dt,$$

where

$$H = \frac{x}{4} - xy^2 - \frac{x^3}{3},$$

and (x,y) is the periodic solution Γ of (A1). By using $\lambda = \mu\varepsilon$, one obtains the following

(A4)
$$2G(\mu,0,c)$$
$$= \int_0^T \dot{y}(\mu x + d_1 x^3 + d_2 xy^2)dt$$
$$- \int_0^T \dot{x}(d_3 x^2 y + d_4 y^3)dt$$

where $(x(t),y(t))$ is the solution of

(A5)
$$\dot{x} = 2xy$$
$$\dot{y} = \frac{1}{4} - x^2 - y^2$$

with energy $H(x(t),y(t)) = c$. Since all the terms in (A4) are given by J_0 and J_2 (see (3.2) and (3.3)) except the following expressions

$$\frac{1}{2}\int_0^T y^3\dot{x}dt = \int_{a_1}^{a_2} y^3 dx$$

and

$$\frac{1}{2}\int_0^T xy^2\dot{y}dt = -\frac{1}{3}\int_{a_1}^{a_2} y^3 dx ,$$

we will show that

$$\int_{a_1}^{a_3} y^3 dx$$

can be expressed in terms of J_0 and J_2.

By (A5),

$$2xy\,\frac{dy}{dx} = \frac{1}{4} - x^2 - y^2.$$

Integrating by parts,

$$\int_{a_1}^{a_2} y^3 dx$$

$$= \int_{a_1}^{a_2} - 3xy^2 \frac{dy}{dx} dx$$

$$= - \frac{3}{2} \int_{a_1}^{a_2} y(\frac{1}{4} - x^2 - y^2) dx$$

$$= \frac{3}{4} J_0 - 3J_2 .$$

This gives (A2) and (A3).

If $K_2 \neq 0$, then by Theorem 3.1 we have a unique asymptotically stable limit cycle of (A1) for appropriate values of $(\lambda,\varepsilon) \neq (0,0)$.

APPENDIX B

If $B \neq 2$, the normal form is given by

$$\text{(C1)} \qquad \begin{aligned} \dot{x} &= \lambda x + Bxy + \varepsilon(d_1x^3 + d_2xy^2) \\ \dot{y} &= \frac{1}{4} - x^2 - y^2 + \varepsilon(d_3x^2y + d_4y^3) \end{aligned}$$

where d_1, d_2, d_3 and d_4 are constants. Changing time scales, we write (C1) as

$$\text{(C2)} \qquad \begin{aligned} \dot{x} &= x^q[\lambda x + Bxy + \varepsilon(d_1x^3 + d_2xy^2)] \\ \dot{y} &= x^q[\tfrac{1}{4} - x^2 - y^2 + \varepsilon(d_3x^2y + d_4y^3)] \ . \end{aligned}$$

Let

$$H = \frac{B}{2}\left[\frac{x^{q+1}}{4} - x^{q+1}y^2 - \frac{x^{q+3}}{1+B}\right] ,$$

where $1+q = 2/B$. Using (C2), it follows that

$$\begin{aligned} \frac{dH}{dt} = \dot{y}x^{q+1}\lambda &+ \varepsilon\dot{y}(d_1x^3 + d_2xy^2)x^q \\ &- \varepsilon\dot{x}(d_3x^2y + d_4y^3)x^q \ . \end{aligned}$$

As in Appendix A, the bifurcation function $G(\mu,\varepsilon,c,B)$, $\varepsilon\mu = \lambda$, which for $\varepsilon = 0$ is given by

$$\begin{aligned} &G(\mu,0,c,B) \\ &= (q+1)\mu J_0 + K_2J_2 + K_0J_0 \ , \end{aligned}$$

where

$$K_2 = (q+3)d_1 - (q+1)d_2 + d_3 - 3d_4 \ ,$$

$$K_0 = (d_2(q+1) + 3d_4)/4 \ ,$$

and J_0 and J_2 are defined in §4. This shows that there is no loss of generality by working with equation (4.1).

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