AN INVESTIGATION OF OPTIMAL AIMPOINTS FOR MULTIPLE NUCLEAR WEAPONS AGAINST INSTALLATIONS IN A TARGET COMPLEX

THESIS

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Captain, USAF

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FOR MULTIPLE NUCLEAR WEAPONS AGAINST
INSTALLATIONS IN A TARGET COMPLEX

THESIS

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Title: AN INVESTIGATION OF OPTIMAL AIMPOINTS FOR MULTIPLE NUCLEAR WEAPONS AGAINST INSTALLATIONS IN A TARGET COMPLEX

Thesis Advisor: Ivy D. Cook, Lt Col, USAF
Strategic nuclear targeting studies generally include more target installations than there are weapons. Hence, a weapon is not assigned to an installation, but rather, to a Desired Ground Zero (DGZ). The objective of this study was to investigate optimal DGZs for multiple nuclear weapons against installations in a target complex. To accomplish this, it was necessary to develop the target Complex Expected Damage Function (CEDF) maximization algorithm. The algorithm locates optimal DGZs by maximizing the CEDF; the CEDF is a nonlinear function of 2m variables, the \((x_i, y_i)\) DGZ coordinates for each of the m weapons.

The algorithm uses two CEDF models and two optimization techniques. These models use DIA Physical Vulnerability System probability of damage models. The CEP-Included model includes each weapon's CEP; the simpler CEP-Excluded model assumes each weapon's CEP equals 0. An analytical expression for the gradient of the CEP-Excluded model was calculated; the algorithm maximizes this CEDF using a conjugate gradient with restarts search technique. The algorithm maximizes the CEP-Included CEDF using a direct search technique, Powell's method of conjugate directions.

This investigation characterized three factors that affect the optimal DGZ locations for multiple nuclear weapons in a target complex. The first factor was gradient symmetry; this symmetry resulted from either a geographically symmetric target complex or collocated weapons. The second factor was weapon CEP. Maximization of the two CEDF models produced slightly different optimal DGZs; this difference depended on a weapon's CEP and the CEDF model. The third factor was the initial DGZ location prior to CEDF maximization. The algorithm located different CEDF local maximums depending on the initial DGZ condition. However, the investigation revealed that the most successful initial DGZ condition is to use the coordinates of the highest valued installations as the initial DGZ coordinates.
My concept of and algorithm for locating optimal aimpoints for multiple nuclear weapons in a target complex were a synthesis of three analyst's ideas. Captain Mark Orlicky, Air Force Studies and Analysis, Major Steve Sperry, Joint Strategic Target Planning Staff, and Mrs. Adelaide Bialek, Academy for Interscience Methodology all contributed to my strategic targeting education. I thank each of them for the valuable suggestions that they provided to me. I also extend special thanks to Mr. Carol Strom of the Computer Sciences Corporation; he shared his ideas with me and provided me with results from the model NUCWAVE.

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Finally, I recognize and extend thanks to my most valuable thesis contributor -- my typist, my editor, my best friend, and my wife Mary. I thank you for your encouragement and because you, and our children, Greg, Becky, and Leanne, have endured more hardships than I have during the past 18 months.
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ABSTRACT

Strategic nuclear targeting studies generally include more target installations than there are weapons. Hence, a weapon is not assigned to an installation, but rather, to a Desired Ground Zero (DGZ). The objective of this study was to investigate optimal DGZs for multiple nuclear weapons against installations in a target complex. To accomplish this, it was necessary to develop the target Complex Expected Damage Function (CEDF) maximization algorithm. The algorithm locates optimal DGZs by maximizing the CEDF; the CEDF is a nonlinear function of \(2m\) variables, the \((x_i, y_i)\) DGZ coordinates for each of the \(m\) weapons.

The algorithm uses two CEDF models and two optimization techniques. These models use DIA Physical Vulnerability System probability of damage models. The CEP-Included model includes each weapon's CEP; the simpler CEP-Excluded model assumes each weapon's CEP equals 0. An analytical expression for the gradient of the CEP-Excluded model was calculated; the algorithm maximizes this CEDF using a conjugate gradient with restarts search technique. The algorithm maximizes the CEP-Included CEDF using a direct search technique, Powell's method of conjugate directions.

This investigation characterized three factors that affect the optimal DGZ locations for multiple nuclear weapons in a target complex. The first factor was gradient symmetry; this symmetry resulted from either a geographically symmetric target complex or collocated weapons. The second factor was weapon CEP. Maximization of the two CEDF models
produced slightly different optimal DGZs; this difference depended on a weapon's CEP and the CEDF model. The third factor was the initial DGZ location prior to CEDF maximization. The algorithm located different CEDF local maximums depending on the initial DGZ condition. However, the investigation revealed that the most successful initial DGZ condition is to use the coordinates of the highest valued installations as the initial DGZ coordinates.
I. Introduction

Effective U. S. targeting of an enemy's resources is an important part of U. S. military air power. One of the fundamental objectives of U. S. military forces is to sustain deterrence (Ref 8: para 1-6). Deterrence is an enemy's state of mind brought about by the existence of U. S. military power or the enemy's perception of U. S. resolve to use that power. Strategic nuclear targeting, an assignment process, is a key element of nuclear deterrence. The nuclear weapons planner must assign a weapon system to a specific target. Targeting consists of three interacting processes: the target intelligence process, the threat estimate process, and the operational planning process (Ref 7:2-2). This study investigates an important phase of the target intelligence process, nuclear weaponeering, and a weaponeering problem. Lee defines weaponeering as "the process that determines the physical vulnerabilities of targets, the optimum weapon type, the number of weapons, and sometimes the best system required to achieve a desired level of damage on a target or a target system" (Ref 18:122).

Background

Weapons planners allocate weapons to Desired Ground Zeros (DGZs) to achieve damage to installations within a target complex. A target
complex is a geographical area that includes different types and numbers of target elements or installations. Nuclear detonations within a complex will cause insignificant damage to installations within all adjacent complexes (Ref 20:6). A complex may contain one installation or a few hundred installations. For example, a 50-square-mile Air Force base may be a target complex. Similarly, the runway, the maintenance facility, the parked aircraft, and the headquarters command post are installations of this target complex. However, weapons planners do not allocate weapons to each installation. Instead, they allocate weapons to DGZs and plan to damage more than one installation with one weapon.

A DGZ is a point on the surface of the earth at or vertically below the center of a planned weapon explosion (Ref 7:5-6). In this study, a DGZ refers to a nuclear weapon detonation at a specific geographical location. A DGZ may be located directly on an installation; or, if one weapon will sufficiently damage two or more installations, then the DGZ may be located between the installations.

Weapons analysts use the concept of a lethal aimpoint region (LAIR) to locate DGZs within a target complex (Ref 20:10; 21:4; and 25:2-6). The LAIR is a circular area whose center is the target installation. It represents a geographical region within which a weapon can detonate and achieve at least a minimum probability of damage (Pd) to a target. Pd is the probability that a desired level of damage (severe, moderate, light) will occur to a target (Ref 7:5-6). The general definitions of the three damage levels are: (1) severe damage -- a level which requires essentially complete reconstruction or replacement of one or more critical major elements of the target, plus reconstruction, repair, or replacement of associated structures or equipment. Severe damage
precludes utilization of the target for any purpose, (2) moderate damage -- a level which requires major repairs to one or more critical major elements of the target, plus major reconstruction, repair, or replacement of associated structures or equipment. Moderate damage precludes effective utilization of the target for its intended purpose, (3) light damage -- a level which does not significantly impair the target function, but requires some repairs to restore the target to complete usefulness" (Ref 19:1-7).

The radius of the LAIR depends on specific weapon system and target parameters. The accuracy of the missile or aircraft system that delivers the weapon to the DGZ affects the LAIR. Also, the yield of the nuclear weapon affects the LAIR. Yield is a numeric value measured in kilotons (kt) and is a relative indicator of the explosive energy the weapon releases when it detonates. This explosive energy causes damage to installations. A nuclear weapon distributes its damage energy in several ways through damage mechanisms or weapon effects. For ground targets, the most prominent mechanism is the blast effect. The primary elements of blast are overpressure and dynamic pressure. Overpressure creates a force that crushes an installation; dynamic pressure creates a force from the resulting high wind velocity (Ref 11:80-82). But thermal effects, cratering, and impulse are other nuclear weapon effects that may contribute to target damage. The occurrence and intensity of these weapon effects vary for different weapon yields.

The LAIR also depends upon target characteristics, specifically, the vulnerability of the target to blast effects. The Defense Intelligence Agency (DIA) uses a Physical Vulnerability coding system to quantify a target's susceptibility to blast damage. Each installation is
characterized by a three-part Vulnerability Number (VN). The first part consists of a two-digit integer reflecting the target's relative hardness in terms of a 20-kt weapon and a specified damage level (severe, moderate, light). The second part is a letter indicating whether the target is predominantly sensitive to either overpressure (L,M,N,O,P) or dynamic pressure (Q,R,S,T,U). The third part is a K factor. This factor adjusts the target's relative hardness for weapon yields other than 20-kt (Ref 6:34 and 19:1-7).

In this paper, four factors characterize a nuclear weapon -- yield, accuracy, height of burst, and probability of arrival (Pa). Circular error probable (CEP) is a numeric value measured in units of length that represents a weapon's delivery accuracy. A 500-foot CEP indicates a weapon has a 50% chance of being delivered within 500 feet of the target. Similarly, height of burst is the weapon's distance above the ground when the weapon detonates. Pa is the probability that a delivery vehicle (bomber, missile) and its weapon arrive at the target and the weapon detonates as planned. Pa depends upon the delivery vehicle's pre-launch survivability (PLS), weapon and weapon system reliability (WSR), and probability to penetrate (PTP). Each of these factors is a probability (Ref 7:5-7).

The weapons analyst plans to damage installations within a target complex by assigning weapons to a prioritized list of DGZs. In addition to Pd and Pa, which are multiplied together to calculate an installation's Damage Expectancy (DE), the value of each installation is needed to develop the prioritized list. The value of an installation is a number that represents the value of the installation relative to all other
installations. Most value systems cardinally order targets over a range from the most valued target (highest value number) to the least valued target (Ref 7:6-19). The total complex expected target value damage is the sum of each installation's value multiplied by the installation's cumulative DE.

There is a shortfall in the nuclear weaponeering process. The prioritized target list generally has more DGZs than there are weapons available to assign to the DGZs. The weapons analyst must determine not only the best weapon-DGZ combination to achieve the desired attack objectives, but also alternative combinations (Ref 7:5-6).

The method that strategic nuclear weapon targeting models use to address this problem depends on the specific objective of the model. One objective is to minimize the number of weapons required to achieve at least a minimum acceptable probability of damage to all installations in the complex. This method determines the minimum number of weapons when installation Pds are prespecified. A different objective is to achieve the maximum total expected target value damage for the complex. This method determines the Pd to each installation when the number of weapons available is prespecified.

As an example, suppose a preliminary target analysis indicates five DGZs are necessary to achieve a minimum acceptable Pd for each installation in a complex. However, after allocating weapons to the entire prioritized DGZ list (all complexes), only four weapons are actually available to this complex. Should the four weapons be targeted against the four highest expected target value DGZs or should an attempt be made to locate four new DGZs, perhaps unrelated to the five potential DGZs? The former choice will achieve a minimum acceptable Pd on some,
but not all, of the installations in the complex. The installations that would have been damaged by a weapon allocation to the unassigned DGZ probably will receive insignificant damage. Conversely, the latter choice may increase the total expected target value damage to the complex with either no decrease or a minimal decrease in the minimum acceptable Pd for each installation.

According to a 15 September 1983 literature review, AF Studies and Analysis, Command and Control Technical Center (CCTC), and the Single Integrated Operational Plan (SIOP) Simulation Branch, Joint Strategic Target Planning Staff (JSTPS) use different models for DGZ optimization studies (Ref 4; 13; 20; 22; and 28). Each of these mathematical models has a limitation. Initially, the algorithms generate a DGZ list for a complex using an unlimited supply of weapons. Then the algorithms assign weapons either to the minimum number of DGZs required to achieve an acceptable level of damage on all targets or, when the numbers of weapons are constrained, to the DGZs that achieve the best total expected target value damage for the preplanned DGZs. The second situation, limited weapon supply, is more realistic. However, the development of new DGZs in the constrained weapons case to maximize total expected target value damage is not attempted. In some algorithms, DGZs are relocated, but relocation is sequential. One DGZ is moved until its contribution to the total expected target value damage is maximized, then that DGZ is assigned, and a second DGZ is sequentially moved.

**Problem Statement**

After a weapon allocation for all target complexes is completed, not all complexes may be allocated enough weapons to achieve an
acceptable Pd for all installations.

An algorithm is needed that will optimally locate DGZs in a target complex for a fixed number of weapons, while maximizing the total expected target value damage to installations within the complex.

DGZ Models

Multiweapon Optimizer for Strategic Targets (MOST), Seiler, and NUCWAVE are mathematical models that Air Force agencies use for strategic targeting studies (Ref 20; 21; and 25). The models locate DGZs within a target complex. Each of these models depends on the LAIR concept and uses either partial enumeration, or linear programming allocation, or sequential allocation to determine a set of DGZs for a complex.

MOST determines a DGZ list in two phases. Each phase satisfies an associated criteria. These phases allow MOST to achieve its objective — determining the fewest number of weapons (DGZs) required to achieve at least a minimum acceptable Pd for each installation in a complex (Ref 21). There are several steps in the first phase. Initially, MOST generates a LAIR for each installation. These LAIRs satisfy the criteria to achieve a minimum acceptable Pd on all installations. Next, MOST compiles subsets of DGZs through a partial enumeration process; each subset contains a list of LAIR intersections. For one subset, all installations in the complex must be included in at least one intersection. Then MOST selects the subset that contains the fewest number of LAIR intersections; if several equivalent subsets require the fewest number of aimpoints, then the subset with the highest total expected target value damage is selected. As an example, consider the target complex
in Figure 1. The algorithm would select the DGZ subset that contains the LAIR intersection of installations B, C, and D as one DGZ and the LAIR intersection of installations A and E as the second DGZ of this target complex (the shaded regions in Figure 1).

In the second phase, MOST adjusts the final DGZ locations within the LAIR intersection regions using a weighted installation value system. This process maximizes the total expected target value damage for all DGZs. If installation D was more valuable than B and C in Figure 1, then the actual DGZ would be moved proportionately closer to installation D. These adjustments to final DGZs are accomplished sequentially. First, the DGZ associated with the greatest number of LAIR intersections would be maximized (the DGZ associated with the intersection of target LAIRs B, C, and D). Then the DGZ associated with the second greatest number of LAIR intersections, etc. MOST was designed to find the minimum number of DGZs for Poseidon re-entry vehicles, irrespective of weapon supply constraints. If there are not enough weapons to
allocate to the complex, then the least valued DGZs would not be assigned weapons.

The objectives and purpose of the Seiler model are similar to MOST. Seiler was designed to study the assignment of nuclear weapon missile systems (ICBM and SLBM) to installations within many target complexes (Ref 20). Seiler also uses two phases to assign weapons to a prioritized list of DGZs. In the first phase, generation of aim-points, Seiler creates DGZs using the LAIR concept and a tiered DGZ system. The primary tier consists of the minimum number of DGZs that are required to achieve a minimum acceptable Pd to all installations when only the largest yield weapon is considered. For each subset of installations contained in a primary tier DGZ, supplementary DGZs are created for the next largest yield weapon. Supplementary DGZs are always subsets of a primary tier DGZ or a higher-tiered supplementary DGZ. Each DGZ, supplementary or primary, achieves a minimum acceptable Pd on a subset of the installations in a target complex and has an associated DGZ value. This value depends on the cumulative total expected target value damage of the associated installations.

In the second phase, Seiler uses a linear programming (LP) algorithm to determine an optimal (or near optimal) assignment of weapons. The LP objective is to maximize the total complex expected target value damage. Seiler accomplishes this assignment using the primary and supplementary tiered DGZs, missile delivery vehicle range capabilities, and constraints on the number of primary and supplementary tier weapons available. If there are not enough weapons available to allocate to the installations in the complex, then lower value DGZs (and hence
installations) remain untargeted just as in MOST.

NUCWAVE determines the number and the location of DGZs using a different approach (Ref 25). It is a one-sided nuclear weapons allocation war gaming model. The user can select one of two strategies -- (1) allocate a limited number of weapons to DGZs in order to maximize the total expected target value damage to all target complexes considered, or (2) determine the minimum number of DGZs required to achieve a minimum acceptable Pd to installations within all target complexes. NUCWAVE generates DGZs using the LAIR concept, similar to MOST and Seiler, for allocation strategy 2. Allocation strategy 1 is accomplished using a sequential algorithm and will be discussed later.

The NUCWAVE algorithm consists of three phases, irrespective of the allocation strategy chosen. The first phase, potential allocation, uses an unlimited supply of weapons to maximize the damage attained by each weapon until a sufficient number of potential DGZs are located to satisfy the strategy objective. In the second phase, an LP weapon selection program uses these potential DGZs and weapon supply constraints to select the number and the type of weapons to be assigned to each complex. In the final phase, real allocation, the specific number and types of weapons are "optimally allocated" against the installations in each complex (Ref 25:2-8). If allocation strategy 2 has been selected and the weapon selection program allocated fewer than the required number of weapons to a complex, then the lower valued DGZs and their associated installations would not be targeted. No relocation of the DGZs is attempted with strategy 2.

When allocation strategy 1, maximize total expected target value
damage, is selected, NUCWAVE determines DGZs sequentially in both the potential and real allocation phases. NUCWAVE starts by locating the first DGZ at the highest-valued installation in the complex. The algorithm then moves the DGZ to a location that maximizes the total complex expected target value damage. The algorithm may move the DGZ closer to several installations, thus increasing the Pd and expected target value damage for these installations. Similarly, the algorithm may move the DGZ farther away from other installations, thus decreasing the Pd and expected target value damage for these installations. When the optimal location is determined, two steps occur. First, a weapon-DGZ location, having been determined, is stored. Next, the surviving value of all installations is calculated by multiplying the previous value of the installation by the Pd of the installation from the current weapon-DGZ combination. NUCWAVE then selects the installation with highest surviving value and the entire process is repeated. DGZs are sequentially determined in this manner until the specified stopping condition is reached. In the potential allocation phase, the stopping condition is that a user-specified percent of the total expected target value damage has been achieved; in the real phase, the condition is no more remaining weapons.

After a weapon allocation is made for the entire target list, only a finite number of weapons may be assigned to a target complex. Only NUCWAVE attempts to locate new DGZs, but it uses a sequential optimization algorithm. When less than the desired number of weapons are allocated to a target complex, a simultaneous optimal solution specifying the location of the final DGZs should exist. In this study,
Objectives

The primary objective of this study is to investigate the optimal DGZ locations within a target complex. In order to accomplish this, it is necessary to develop an algorithm. This algorithm will optimally locate the DGZs for fixed numbers of weapons in a target complex by maximizing the expected target value damage to all installations. The algorithm will not be restricted to one type of weapon; that is, different weapons may be included in the fixed number of weapons.

Also, it will be necessary to determine the sensitivity of the algorithm to two factors -- first, the mathematical technique used to locate the optimal DGZs; second, the initial starting conditions (latitude and longitude coordinates) for the DGZs.

The algorithm will consist of two elements. The first element is a mathematical model of the total complex expected target value damage. The second element is an optimization technique to determine the maximum total complex expected value damage and to locate the corresponding optimal DGZs. The following steps are an outline of the algorithm:

1. Specify target installation parameters. These include installation coordinates, VN numbers, and values.
2. Specify weapon parameters. These include yield, quantity, CEP, Pa, and height of burst.
3. Specify either the mathematical form or an acceptable approximation of the probability of damage function for an installation.
4. Determine the mathematical form of the Installation Expected
Damage Function (IEDF). This function represents the total expected target value damage to an installation from all weapons.

5. Specify the form of the Complex Expected Damage Function (CEDF). This function is a summation of all of the IEDFs.

6. Select a nonlinear optimization technique to maximize the CEDF and to locate the final coordinates of the DGZs.

Scope and Assumptions

This study will develop an algorithm subject to certain restrictions that optimally locates DGZs in a target complex. Secondary damage will be assumed within the target complex; however, secondary damage from weapons detonated in adjacent complexes will not be considered. Also, the algorithm will not consider target avoidance areas.

Only military/industrial installations that can be modeled as point targets will be considered. Also, since blast is the primary damage mechanism for ground targets, other nuclear weapon effects will not be considered. Each installation's susceptibility to overpressure and to dynamic pressure will be specified with VN numbers. Also, the mathematical model of the probability of damage function developed by the Defense Intelligence Agency (DIA) will be used to specify the installation expected damage function (Ref 6).

The algorithm will consider weapon system delivery methods and accuracy since they will affect the expected target value damage. Delivery methods will be characterized by a specified $P_a$ for each weapon. However, feasible delivery constraints will not be considered, for
example, Multiple Independent Reentry Vehicle (MIRV) footprinting and a weapon delivery system's range capability. A circular normal distribution will be assumed for weapon system accuracy, and a CEP will be specified for each weapon. Pa and CEP are a function of range for some weapons, but in this study they are prespecified numbers.

Different optimization techniques and initial DGZ conditions will be evaluated. These evaluations will characterize the properties, capabilities, and any limitations of the algorithm.

Overview

This paper reports the methods and findings of a study that investigated the location of optimal aimpoints for multiple nuclear weapons against installations in a target complex. A CEDF maximization algorithm was developed to optimally locate DGZs for these weapons by maximizing the Complex Expected Damage Function (CEDF). The algorithm consists of two elements -- a mathematical model of the CEDF and an optimization technique. Chapter II presents the mathematical formulation of two CEDF models and the gradient for one of these models. Chapter III presents an overview of numerical search techniques; it also discusses the two techniques that are used to maximize the two CEDF models. Chapter IV contains the computerization of the algorithm and the verification and validation processes. Chapter V discusses the algorithm's convergence criteria, and specific properties of symmetric target complexes, and symmetric CEDF gradient elements. Chapter VI is an analysis of optimal DGZs for three, four, and seven installation target complexes. It also discusses and summarizes the three conclusions of this study. Finally, Chapter VII presents concluding remarks and recommendations.
II. Mathematical Formulation of the Complex

Expected Damage Function and Gradient

The algorithm determines the optimal DGZ locations within a target complex by maximizing the Complex Expected Damage Function (CEDF). Initially, the CEDF is developed from prespecified weapon and installation parameters; then the CEDF is maximized with an unconstrained, nonlinear optimization technique. One replication of the algorithm determines the optimal DGZ locations in a finite number of iterative steps. Each iterative step finds improved DGZ locations and an associated larger CEDF value as compared to the previous locations and CEDF value. The algorithm iterates until no significant increase in the CEDF is possible. This chapter explains the mathematical formulation of the CEDF and its gradient. Chapter III presents the unconstrained, nonlinear optimization techniques used to maximize the CEDF.

Conceptual Model

The CEDF is a function of weapon and installation parameters and the coordinates of the DGZs. The conceptual model of the CEDF is shown in Figure 2. The $i$ subscript of either a variable or a parameter refers to one of the $m$ weapons; the $j$ subscript refers to one of the $n$ installations in the target complex. All parameters are constants (either prespecified or calculated values) except the $(X_i,Y_i)$ DGZ coordinates for each of the $m$ weapons. The basic element of the the CEDF is the $P_{d_{ij}}$ -- the probability of achieving a specified level of damage to installation $j$ from weapon $i$. Similarly, the expected damage to installation $j$ from weapon $i$ is $DE_{ij}$ -- the product of $P_{d_{ij}}$ and
for each installation $j$, $j = 1, \ldots, n$

installation $j$ prespecified or calculated parameters

for each weapon $i$, $i = 1, \ldots, m$

weapon $i$ prespecified or calculated parameters

$P_{d_{i,j}}$

weapon $i$ Pa

$DE_{i,j} = P_{d_{i,j}} \times Pa_i$

$DE_j = 1 - \prod_{i=1}^{m} (1 - DE_{i,j})$

installation value $v_j$

$IEDF_j = v_j \times DE_j$

$CEDF = \sum_{j=1}^{n} IEDF_j$

Figure 2. Conceptual Model of the CEDF
the probability of arrival for weapon $i$, $P_{a_i}$. The cumulative expected damage to installation $j$ from all $m$ weapons is $DE_j$. This formulation for determining the cumulative damage to an installation from multiple bursts is similar to the formulation used in SIDAC and NUCWAVE (Ref 19:A-4 and 25:A-2). For each installation, the Installation Expected Damage Function, $IEDF$, is the product of its value $v_j$ and $DE_j$. The $CEDF$ is an unconstrained, nonlinear function; it is the summation of $n$ $IEDFs$. The function is nonlinear because $P_{d_i,j}$ is a nonlinear function. Again, all $CEDF$ parameters are constants except the $(X_i,Y_i)$ DGZ coordinates for each of the $m$ weapons.

In order to determine the $CEDF$, the algorithm requires scenario dependent inputs, installation and weapon parameters. The minimum necessary installation parameters include:

1. The number of installations in the target complex - $n$
2. The coordinates of each installation - $(x_j,y_j)$
3. A VNTK code for each installation indicating the installation's susceptibility to blast damage
4. A value from a relative installation value system - $v_j$

The minimum necessary weapon parameters include:

1. The number of weapons available - $m$
2. The height of burst for each weapon - $HOB_i$
3. The yield in kilotons for each weapon - $Y_i$
4. The CEP for each weapon - $CEP_i$
5. The probability of arrival for each weapon - $P_{a_i}$
6. The initial DGZ locations prior to optimization - $(X_i,Y_i)$

The assignment of specific numeric values to these parameters was not a critical element of the study. Consequently, several hypothetical
target complexes were used. These complexes are described in Chapter IV.

Two mathematical forms of the CEDF are used; hence, there are two parallel algorithms, one for each form of the CEDF. The first CEDF is the CEP-Excluded version; the second CEDF is the CEP-Included version. These two versions are explained in the next section. The only difference between the two CEDFs is their respective forms of the probability of damage function. The Defense Intelligence Agency (DIA) developed these Pd models to provide analytical approximations to actual blast damage data. The CEP-Excluded CEDF uses a closed form analytical expression of an installation's Pd function that is independent of weapon delivery system accuracy, that is, weapon CEP = 0. The CEP-Included CEDF uses a more complicated analytical expression of an installation's Pd function that includes weapon CEP.

The two CEDF forms are used for three reasons. First, a closed form analytical expression for the gradient of the CEP-Included CEDF expression was not available; hence, this CEDF could only be maximized with an optimization technique that used function values. However, since gradient optimization techniques are generally more efficient than function value techniques (Ref 2:152; 5:321; and 10:286), a second form of the CEDF is desired. Therefore, an analytical expression for the gradient of the CEP-Excluded CEDF is calculated. This CEDF is maximized using gradient optimization techniques. Chapter III explains different optimization schemes and the optimization techniques used to maximize the two CEDFs. The second reason for using two CEDFs is verification. The results of the algorithm are compared to insure that they provide the same DGZ locations and complex expected damage value. The last reason for using two CEDFs is to investigate the
effect that the assumption of CEP = 0 has on the location of the optimal aimpoints.

Probability of Damage Models

The CEDF is an unconstrained, nonlinear function of 2m variables — the \((X_i, Y_i)\) DGZ coordinates for each of the m weapons.

\[
\text{CEDF} = \sum_{j=1}^{n} \text{LEDJ}_j = \sum_{j=1}^{n} v_j \left[ \sum_{i=1}^{m} (1 - P_{d_{i,j}} \cdot P_{a_i}) \right] \quad (1)
\]

The basic element of the CEDF is \(P_{d_{i,j}}\) -- the probability of achieving a specified level of damage to installation \(j\) from weapon \(i\). \(P_{d_{i,j}}\) is a function of two independent variables, the \((X_i, Y_i)\) coordinates of weapon \(i\). Two forms of the \(P_{d_{i,j}}\) function used in this study are part of the DIA Physical Vulnerability (PV) System. They are not independent formulations. These formulations are described in *Mathematical Background and Programming Aids for the Physical Vulnerability System for Nuclear Weapons* (Ref 6). Therefore, only a usable, but limited, description will be presented here.

The \(P_{d_{i,j}}\) depends on the known distance \(s\) between DGZ \(i\) and installation \(j\). The coordinates of installation \(j\) are \((x_j, y_j)\). The geometry of the installation-weapon interaction is shown in Figure 3. The algorithm uses a flat earth approximation to calculate this distance, that is,

\[
s = \left[ (X_i - x_j)^2 + (Y_i - y_j)^2 \right]^{1/2} \quad (2)
\]
Figure 3. Geometry of the installation-weapon interaction (Ref 6:20).

The $(X_i,Y_i)$ and $(x_j,y_j)$ coordinates are measured in feet from a common origin. Comparatively, the distance $r$ is the distance between installation $j$ and the actual weapon impact point. When $CEP = 0$, the impact point and $DGZ_i$ may not be the same point, and $s$ will not equal $r$.

**CEP-Excluded Model.** If the CEP of the weapon delivery system can be assumed to equal 0, that is, the actual weapon impact point is the $DGZ_i$, then the distance $r$ from installation $j$ to the impact point is known. The distance $r$ can be calculated from the $(X_i,Y_i)$ $DGZ$ coordinates, the $(x_j,y_j)$ installation coordinates, and Eq (2). The distance damage function, $P_d(r)$, is the DIA analytical approximation for the probability of damage function when weapon $CEP = 0$. It is based upon actual blast damage data. $P_d(r)$ is the complement of the cumulative log normal distribution function. For this CEDF version, CEP-Excluded, the probability of damage function, $P_{d_{i,j}}$, is the distance damage function, $P_d(r)$. However, it will be referred as the distance damage function $P_d(r)$ to parallel the DIA development. The shape of a $P_d(r)$ function is shown.
Figure 4. A probability of damage function $P_d(r)$

in Figure 4. The independent variable is the distance $r$ between the installation and the impact point.

The distance damage function is a nonlinear expression in integral form; it is specified by the location and dispersion parameters, $\alpha$ and $\beta$.

$$
P_d(r) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z(r)} e^{-\frac{1}{2}t^2} dt
$$

$$
z(r) = \frac{1}{\phi} \ln\left(\frac{\alpha}{r}\right)
$$

The parameter $t$ is a dummy variable of integration for the normal probability distribution. "The parameter $\alpha$, which is the median of the log normal density function, is the distance from ground zero at which there is a 50% chance of achieving a specified level of damage. The parameter $\phi$ is the standard deviation of $\ln(r)$" (Ref 6:6,7). These parameters depend on the weapon radius (WR) and distance damage sigma ($\sigma_d$). If a weapon is detonated within a uniform distribution of targets, then the WR is
the radius of a circle centered at the weapon impact point. The circle contains as many targets undamaged to a specified level inside the circle as there are targets damaged to a specified level outside the circle. "\( \sigma^2 \) is a measure of the variance of the density function. A small \( \sigma^2 \) indicates a relatively rapid fall off of the damage function; a large \( \sigma^2 \) indicates a more gradual fall off" (Ref 6:11).

Prior to 1 September 1972, the analytical approximation of actual blast damage data was the circular coverage function with parameters \( WR \) and \( r \). However, before that date, DIA decided that the distance damage function with parameters \( \alpha \) and \( \beta \) provided a better fit to actual blast damage data. Since previously measured and calculated target vulnerability data depended on \( WR \) and \( \sigma^2 \), DIA developed mathematical transformations to determine \( \alpha \) and \( \beta \) from \( WR \) and \( \sigma^2 \).

\[
\beta = \sqrt{-\ln(1 - \sigma^2)}
\]

\[
\alpha = WR e^{-\beta^2}
\]

With these transformations, the distance damage function could specify the \( P_d \) for targets characterized by the Physical Vulnerability (PV) coding system.

Consequently, the probability of damage to installation \( j \) from weapon \( i \), \( P_{dij} \), can be calculated using Eqs (3) and (4), after \( \sigma^2 \), \( WR \), and \( r \) have been determined. \( WR \) and \( \sigma^2 \) are parameters that are calculated using prespecified user values. \( WR \) depends on the weapon's yield and HOB and the installation's VNIK code. Hence, there is a unique \( WR \) for each weapon \( i \)-installation \( j \) interaction -- \( WR_{ij} \).
Likewise, $\sigma_d$ and $\beta$ depend only on the installation's VNTK code. Hence, there is a unique $\sigma_d$ and $\beta$ for each installation -- $\sigma_{d_j}$ and $\beta_j$. Appendix A presents the calculation of $\sigma_d$ and WR. The independent variable $r$ is actually a function of two independent variables, the $(X_i,Y_i)$ DGZ coordinates, and two constants, the $(x_j,y_j)$ installation coordinates.

The $P_d(r)$ cannot be expressed in closed form in terms of elementary functions; however, it can be calculated by use of the error function, erf(u) (Ref 6:21 and 1:298). The erf(u) specifies the probability that a standard normal random variable observation is within $\pm u$ of the mean value.

\[ P_d(r) = P_{d_i,j}(X_i,Y_i) = 0.5 + 0.5 \text{erf} \left( \frac{|z(r)|}{\sqrt{2}} \right) \quad \text{for} \quad z(r) > 0 \]

\[ = 0.5 - 0.5 \text{erf} \left( \frac{|z(r)|}{\sqrt{2}} \right) \quad \text{for} \quad z(r) < 0 \quad (5) \]

where

\[ z(r) = z(X_i,Y_i) = \frac{1}{\beta_j} \ln \left( \frac{\alpha_i,j}{r(X_i,Y_i)} \right) \]

\[ = \frac{1}{\beta_j} \ln \left( \frac{WR_{i,j}}{r(X_i,Y_i)} \right) - \beta_j \quad (6) \]

A polynomial function of the independent variable $u$ can approximate erf(u) (Ref 14:185).

In summary, if the CEP of the weapon can be assumed to equal 0, then the probability of damage $P_{d_i,j}$ to installation $j$ from weapon $i$ can be calculated using Eqs (5) and (6). Prespecified target and weapon parameters are necessary to calculate $WR_{i,j}$, $\beta_j$, and $r(X_i,Y_i)$.  

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CUP-Included Model: If the CEP of the weapon delivery system cannot be assumed to equal 0, that is, the actual impact point of the weapon is unknown, then the distance r from the impact point to installation j is unknown. The geometry of the installation-weapon interaction is shown in Figure 3. The distance s from DGZ$_i$ to installation j can be calculated from the $(X_i,Y_i)$ DGZ coordinates, the $(x_j,y_j)$ installation coordinates, and Eq (2). The unknown distance r from DGZ$_i$ to the actual impact point is a function of s and the independent variables r and $\theta$.

The DIA model determines the probability of damage to installation j in the following way. First, for each possible impact point, the probability of damage is multiplied by the probability that the weapon arrives and detonates at that point (Ref 6:19). The sum of these products for all possible impact points specifies the probability of achieving the desired level of damage to installation j from weapon i, $P_{di,j}$. This summation is a multiple integral over the area that contains all possible impact points.

$$P_{di,j} = \int_0^{2\pi} \int_0^\infty P_d(r) \frac{1}{2\pi \sigma^2} e^{-\frac{\rho^2(r,\theta)}{2\sigma^2}} r dr d\theta$$

where $P_d(r) = $ distance damage function, Eq (3)

$$\sigma^2 = \text{CEP}/1.1774$$

$$\rho(r,\theta) = [r^2 + s^2 - 2rs \cos\theta]^{1/2}$$

For this CEDF model, $P_{di,j}$ and $P_d(r)$ are not the same function. $P_{di,j}$ has two distinct, yet dependent, terms: $P_d(r)$ and $\frac{1}{2\pi \sigma^2} e^{-\frac{\rho^2(r,\theta)}{2\sigma^2}}$. 

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(1) $P_d(r)$ specifies the probability of damage to an installation from an impact point, and

(2) $\frac{1}{2\pi\sigma^2} e^{\frac{-\sigma^2}{2\pi}}$ specifies the circular normal probability of the weapon arriving and detonating at the DGZ.

A closed form solution to Eq (7) does not exist; however, an analytical approximation does (Ref 6:23).

$$P_{d_1,j} = \int_{a}^{b} f(r)dr$$

The limits of integration, $a$ and $b$, are selected such that when $r < a$ or $r > b$, $f(r) = 0$. They are functions of $a$, CEP, WR, and $\sigma$.

Appendix B presents the development of $f(r)$, the determination of $a$ and $b$, and the calculation of $P_{d_1,j}$. The function $f(r)$ has two different forms. Each form depends on the distance $a$ between DGZ$_i$ and installation $j$, the distance $r$, and the weapon's CEP.

This integral can be evaluated using Gauss-Legendre quadrature, a numerical integration technique. This technique approximates a definite integral as a finite series. Each term in the series is a weighted function value.

$$P_{d_1,j} = \frac{(b - a)}{2} \sum_{k=1}^{10} w_k * f(r_k)$$

where

$$r_k = \frac{(b - a)}{2} \zeta_k + \frac{(a + b)}{2}$$

Gauss-Legendre quadrature differs from the more common trapezoidal numerical integration. In Gauss-Legendre, the distances between
<table>
<thead>
<tr>
<th>Step</th>
<th>Given Parameters or values</th>
<th>Results</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>VNTK</td>
<td>σ_{ij}</td>
</tr>
<tr>
<td>2</td>
<td>ω_{ij}</td>
<td>β_{ij}</td>
</tr>
<tr>
<td>3</td>
<td>VNTK, Yield_{i}, h_{ob_{i}}</td>
<td>WR_{i,j}</td>
</tr>
<tr>
<td>4</td>
<td>X_{i}, Y_{i}, x_{j}, y_{j}</td>
<td>r(X_{i},Y_{i}), s(X_{i},Y_{i})</td>
</tr>
<tr>
<td>5</td>
<td>β_{j}, WR_{i,j}, r</td>
<td>Pd_{i,j} (CEP-Excluded)</td>
</tr>
<tr>
<td>5</td>
<td>σ_{ij}, WR_{i,j}, s, CEP_{i}</td>
<td>a, b</td>
</tr>
<tr>
<td>6</td>
<td>β_{j}, WR_{i,j}, r, s, a, b, CEP_{i}</td>
<td>Pd_{i,j} (CEP-Included)</td>
</tr>
</tbody>
</table>

The $r_{k}$ values along the abscissa are not equal. The values of the quadrature coefficients, $w_{k}$, and the base points, $z_{k}$, can be determined from the $N$th Lagendre polynomial. Gauss-Legendre quadrature is discussed in more detail in Appendix C.

In summary, if the CEP of the weapon cannot be assumed to equal 0, then the probability of damage to installation $J$ from weapon $i$, $Pd_{i,j}$, can be calculated using Eqs (8) and (9). Specific target and weapon parameters are necessary to calculate $a$, $b$, and $f(r_{k})$.

Each $Pd_{i,j}$ is an integral part of a Complex Expected Damage Function (CEDF). Eqs (5) and (8) are used to calculate $Pd_{i,j}$ for the CEP-Excluded and for the CEP-Included CEDF models. Table I lists a summary of the steps necessary to calculate $Pd_{i,j}$.
The CEDF Model

The CEDF is an unconstrained, nonlinear function of the \((X_i,Y_i)\) DGZ coordinates for each of the \(m\) weapons.

\[
CEDF = \sum_{j=1}^{n} v_j \left[ 1 - \prod_{i=1}^{m} \left( 1 - P_{d_i,j} \times P_{a_i} \right) \right] 
\]  

(1)

The \(i\) subscript refers to one of the \(m\) weapons; the \(j\) subscript refers to one of the \(n\) installations. The \(2m\) independent variables of the CEDF are

\[(X_1, Y_1, X_2, Y_2, \ldots, X_m, Y_m)\]

The \(2m\) elements of the DGZ coordinate vector, \(\mathbf{x}\), are these \(2m\) variables in a revised order.

\[\mathbf{x} = (X_1, X_2, \ldots, X_m, Y_1, Y_2, \ldots, Y_m)\]

The \((X_i, Y_i)\) DGZ coordinates of weapon \(i\) are \((x_i, x_{i+m})\). Similarly, the \(2n\) parameters specifying the \((x_j, y_j)\) coordinates of the \(n\) installations are

\[(x_1, y_1, x_2, y_2, \ldots, x_n, y_n)\]

The \(2n\) elements of the installation coordinate vector, \(\mathbf{y}\), are these \(2n\) parameters in a revised order.

\[\mathbf{y} = (x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n)\]
Figure 5. A representative CEDF with three installations and two weapons.

The \((x_j, y_j)\) coordinates of installation \(j\) are \((xx_j, xx_{j+n})\).

Like the \(Pd_{i,j}\), the damage expectancy for installation \(j\) from weapon \(i\) is a function of two independent variables \(DE_{i,j}(x_{i,x_{i+m}})\). Similarly, like the CEDF, the cumulative damage expectancy for installation \(j\) from all weapons is a function of \(2m\) independent variables \(DE_j(x_j)\).

A pictorial description of a representative CEDF is shown in Figure 5. The target complex consists of three installations and two weapons. Each installation is characterized by a value -- \(v_j\), a VNTK code, and \((xx_j, xx_{j+3})\) coordinates. Each weapon is characterized
by a yield — $Y_i$, height of burst — $HOB_i$, probability of arrival — $P_{ai}$, and $(x_i, x_{i+2})$. In order to determine all of the $DE_{i,j}$s, six $Pd_{i,j}$s are calculated according to the steps in Table I, one for each combination of $i = 1, 2$ and $j = 1, 2, 3$. For the weapon coordinates $(x_i, x_{i+2})$, $i = 1, 2$

\[ CEDF(x) = v_1 \times DE_1 + v_2 \times DE_2 + v_3 \times DE_3 \]  

(1)

However, if the weapon coordinates are changed to $(x'_i, x'_{i+2})$, $i = 1, 2$, then $DE_j$ may change for each of the $j$ installations. That is, $DE_j(x)$ may not equal $DE_j(x')$ for all $j$. If this is true, then the $CEDF(x')$ may be either greater than, equal, or less than $CEDF(x)$.

In order to maximize the $CEDF$, it is necessary to locate the $x^*$ DGZ coordinates such that

\[ CEDF(x^*) > CEDF(x) \quad \text{for all } x \]

One of the optimization techniques used to maximize the $CEDF(x)$ and to locate the optimal $x^*$ DGZ coordinates required the gradient of the $CEDF(x)$.

Gradient of the CEP-Excluded Model

The Complex Expected Damage Function (CEDF) is a nonlinear function of $2m$ variables — the $(x_i, x_{i+m})$ DGZ coordinates of the $m$ weapons.

\[ CEDF(x) = \sum_{j=1}^{n} v_j^* \left[ 1 - \prod_{i=1}^{m} \left( 1 - Pd_{i,j}(x_i, x_{i+m}) \times P_{ai} \right) \right] \]  

(1)

All parameters of the $CEDF(x)$ are constants except the probability of
achieving a specified level of damage to installation $j$ from weapon $i$, $P_{d_{i,j}}$. Each $P_{d_{i,j}}$ is a function of two independent variables, the $(x_i, x_{i+m})$ DGZ coordinates of weapon $i$. Therefore, to calculate the gradient of the CEDF, the $P_{d_{i,j}}$ must be differentiable with respect to the two independent variables. A closed form analytical expression for the gradient of the CEP-Included $P_{d_{i,j}}$, Eq (7), was not available. This was one of the reasons for formulating the second version of the CEDF, the CEP-Excluded model. This section presents the calculation of the gradient of the CEP-Excluded CEDF.

The gradient of the CEDF($x$) is a vector of $2m$ elements:

$$\text{grad}(\text{CEDF}) = \frac{\partial \text{CEDF}(x)}{\partial x} = \left( \frac{\partial \text{CEDF}}{\partial x_1}, \frac{\partial \text{CEDF}}{\partial x_2}, \ldots, \frac{\partial \text{CEDF}}{\partial x_{2m}} \right)$$ (10)

where $\text{CEDF}(x) = v_1 * \text{DE}_1(x) + v_2 * \text{DE}_2(x) + \ldots + v_n * \text{DE}_n(x)$

and $\text{DE}_j(x) = 1 - (1 - P_{a_1} P_{d_{1,j}})(1 - P_{a_2} P_{d_{2,j}}) \ldots (1 - P_{a_m} P_{d_{m,j}})$

Since $v_j$ is a constant, the $k^{th}$ element of grad(CEDF) is of the form

$$\frac{\partial \text{CEDF}}{\partial x_k} = v_1 \frac{\partial \text{DE}_1}{\partial x_k} + v_2 \frac{\partial \text{DE}_2}{\partial x_k} + \ldots + v_n \frac{\partial \text{DE}_n}{\partial x_k}$$

$$= \sum_{j=1}^{n} v_j \frac{\partial \text{DE}_j}{\partial x_k}$$ (11)
Each of the \( n \) terms of Eq (11) is of the form

\[
v_j \frac{\partial \Phi_1}{\partial x_k} = -v_j \left[ \frac{\partial (1 - P_{1,1,1,1})}{\partial x_k} \right] \ldots (1 - P_{1,1,1,1}) \ldots (1 - P_{m,1,1,1}) + \ldots (1 - P_{1,1,1,1}) \ldots (1 - P_{m,1,1,1}) \ldots (1 - P_{m,1,1,1}) \ldots \]

However, since each \( P_{1,1,1,1} \) is a function of only two variables \((x_i,x_{i+m})\),

for all \( i \), all \( \frac{\partial (1 - P_{1,1,1,1})}{\partial x_k} \) terms equal 0 except for \( k = i \) and \( i + m \).

Hence, for \( k = i \) and \( i + m \)

\[
v_j \frac{\partial \Phi_1}{\partial x_k} = -v_j (1 - P_{1,1,1,1}) \ldots \frac{\partial (1 - P_{1,1,1,1})}{\partial x_k} \ldots (1 - P_{m,1,1,1}) \ldots \frac{\partial (1 - P_{m,1,1,1})}{\partial x_k} \ldots \]

Now define

\[
\text{factor}(j) = v_j \prod_{i=1}^{m} (1 - P_{1,1,1,1})
\]

and rewrite Eq (12) as

\[
v_j \frac{\partial \Phi_1}{\partial x_k} = \frac{\text{factor}(j)}{(1 - P_{1,1,1,1})} \cdot P_{1,1,1,1} \cdot \frac{\partial P_{1,1,1,1}}{\partial x_k} \quad \text{for } k = i \text{ and } i + m \quad \text{(13)}
\]
The gradient of the CEDF is a vector of \(2m\) elements. Eq (11) is the form of the \(k\)th element. Similarly, each element is a summation of \(n\) terms. Eq (13) is the form of each of these \(n\) terms. Analytical expressions of \(\frac{\partial P_{d_{1,i+1}}}{\partial x_k}\) for \(k = i\) and \(i + m\) are needed to completely specify the gradient of the CEDF(x). The CEP-Excluded version of the \(P_{d_{i,j}}\) is

\[
P_{d_{i,j}}(x_i, x_{i+m}) = 0.5 + 0.5 \text{erf}\left(\frac{|z|}{\sqrt{2}}\right) \quad \text{for } z > 0
\]

\[
= 0.5 - 0.5 \text{erf}\left(\frac{|z|}{\sqrt{2}}\right) \quad \text{for } z < 0
\]

where

\[
z(r) = \frac{1}{\beta_j} \ln \frac{WR_{i,j} e^{-\beta_j^2}}{r} = \frac{1}{\beta_j} \ln(WR_{i,j} e^{-\beta_j^2}) - \ln(r)
\]

First, \(\frac{\partial P_{d_{i,i+1}}}{\partial x_1}\) will be calculated for \(z > 0\), that is, for \(r < WR_{i,j} e^{-\beta_j^2}\).

Let \(u = \frac{|z|}{\sqrt{2}}\) and use the chain rule

\[
\frac{\partial P_{d_{i,i+1}}}{\partial x_1} = 0.5 \frac{\partial \text{erf}(u)}{\partial u} \left(\frac{\partial u}{\partial x_1}\right)
\]

where

\[
\frac{\partial u}{\partial x_1} = \frac{1}{\sqrt{2}} \frac{\partial |z|}{\partial x_1}
\]
Now since \[ r = \left[ \left( x_i - xx_j \right)^2 + \left( x_{i+m} - xx_{j+n} \right)^2 \right]^{1/2} \] (2)

and \( z(r) = z(x_i, x_{i+m}) \) from Eq (6), and again using the chain rule

\[
\frac{\partial |z|}{\partial x_i} = \frac{\partial |z|}{\partial r} \frac{\partial r}{\partial x_i} \tag{16}
\]

where

\[
\frac{\partial |z|}{\partial r} = \frac{1}{\beta_j} \left( \frac{-1}{r} \right) - \frac{1}{\beta_j r}
\]

and

\[
\frac{\partial r}{\partial x_i} = \frac{x_i - xx_j}{r} \tag{17}
\]

hence

\[
\frac{\partial u}{\partial x_i} = \frac{1}{\sqrt{2}} \left( \frac{-1}{\beta_j r} \right) \left( \frac{x_i - xx_j}{r} \right) \tag{18}
\]

The derivative of erf(u) from the Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables (Ref 1:298,801) is

\[
\frac{\partial \text{erf}(u)}{\partial u} = \frac{2}{\sqrt{\pi}} e^{-u^2} \tag{19}
\]

Hence, combining Eqs (14), (18), and (19) specifies

\[
\frac{\partial P_{d_{i+1}}}{\partial x_i} = \frac{1}{\sqrt{2}} \frac{2}{\sqrt{\pi}} \frac{e^{-u^2}}{\sqrt{2}} \left( \frac{1}{\beta_j r} \right) \left( \frac{x_i - xx_j}{r} \right) \text{ for } z > 0
\]

\[
= \frac{e^{-u^2}}{\sqrt{2\pi} \beta_j r^2} (xx_j - x_i) \tag{20}
\]
where \[ u = \frac{1}{\sqrt{2}} \left| \frac{1}{\beta_j} \ln \left( \frac{WR_{i,j}}{r} \right) - \beta_j \right| \] (21)

A similar mathematical development was used to calculate

\[ \frac{\partial P_{d_{i,j}^+}}{\partial x_{i+m}} = \frac{e^{-u^2}}{\sqrt{2\pi \beta_j r^2}} (x_{j+n} - x_{i+m}) \] (22)

For \( z < 0 \), that is \( r > WR_{i,j} - \beta_j^2 \), then

\[ \frac{\partial P_{d_{i,j}^+}}{\partial x_i} = -0.5 \frac{\partial \text{erf}(u)}{\partial u} \frac{\partial u}{\partial x_i} \] (23)

The only difference between this development and the previous development for \( z > 0 \) is the sign of \( \frac{\partial |z|}{\partial r} \). This partial derivative is positive because \( r > WR_{i,j} e^{-\beta_j^2} \) and \( |z| \) is

\[ |z| = \frac{1}{\beta_j} \left[ \ln(r) - \ln(WR_{i,j} e^{-\beta_j^2}) \right] \]

hence,

\[ \frac{\partial |z|}{\partial r} = \frac{1}{\beta_j r} \] (24)

Combining Eqs (15), (16), (17), and (24) yields

\[ \frac{\partial u}{\partial x_i} = \frac{1}{\sqrt{2} \left( \frac{1}{\beta_j r} \right)} \left( \frac{x_i - xx_{j,n}}{r} \right) \] (25)
Now Eqs (19), (23), and (25) specify

\[
\frac{\partial P_{d_{i+1}}}{\partial x_1} = -\frac{1}{2} \frac{2}{\sqrt{\pi}} e^{-u^2} \frac{1}{\sqrt{2}} \left( \frac{1}{\beta_j r} \right) \left( x_{i} - x_{i} \right) \quad \text{for } z < 0
\]

\[
= \frac{e^{-u^2}}{\sqrt{2\pi} \beta_j r^2} (x_{i} - x_{i})
\]

(26)

Hence, comparing Eqs (20) and (26), \( \frac{\partial P_{d_{i+1}}}{\partial x_1} \) is the same for all \( z \).

A similar development indicates \( \frac{\partial P_{d_{i+1}}}{\partial x_{i+m}} \) is also the same for all \( z \).

In summary, the gradient of the CEP-Excluded CEDF(\( x \)) is a vector of \( 2m \) elements. The \( k \)th element of the gradient is

\[
\frac{\partial \text{CEDF}}{\partial x_k} = \sum_{j=1}^{n} v_j \frac{\partial \text{DE}_i}{\partial x_k}
\]

(11)

where

\[
v_j \frac{\partial \text{DE}_i}{\partial x_k} = \frac{\text{factor}(i)}{(1 - P_a P_{d_{i,j}})} \frac{\partial P_{d_{i+1}}}{\partial x_k} \quad \text{for } k = i \text{ and } i + m
\]

(13)
Also, for $k = i$

$$
\frac{\partial P_{d_{i+1,i}}}{\partial x_k} = \frac{e^{-u^2}}{\sqrt{2\pi} \beta_j r^2} (x_j - x_i)
$$

(20)

and for $k = i + m$

$$
\frac{\partial P_{d_{i+1,i}}}{\partial x_k} = \frac{e^{-u^2}}{\sqrt{2\pi} \beta_j r^2} (x_{j+m} - x_{i+m})
$$

(22)

where

$$
u = \frac{|z|}{\sqrt{2}} = \frac{1}{\sqrt{2}} \left| \frac{1}{\beta_j} \ln \left( \frac{WR_{i+1,i}}{r} \right) - \beta_j \right|
$$

(21)

and

$$
r = \left[ (x_i - x_j)^2 + (x_{i+m} - x_{j+m})^2 \right]^{1/2}
$$

(2)
III. Optimization of the Complex Expected Damage Function

Two approaches are available to maximize a function of n variables—analytical and numerical search. An analytical approach is preferred if the roots of the n equations defining the critical points of the function are easily determined and solved. These equations are the first partial derivatives of the function set equal to 0. However, if these analytical expressions are not easily determined or solved, then numerical search techniques are necessary to determine the maximum of the function. Numerical search techniques require an organized, efficient exploration of the solution space.

Numerical search techniques were used to maximize the Complex Expected Damage Function (CEDF) because of the complexity of the CEDF. This chapter presents a general methodology and overview of numerical search techniques that are used to maximize unconstrained functions. Also, this chapter discusses the two related techniques that were used to maximize the two versions of the CEDF -- the CEP-Included and the CEP-Excluded models. The primary difference between the two CEDF models is that there was a closed form expression for the gradient of the CEP Excluded model. Therefore, gradient search techniques could be used to maximize the CEP Excluded model. The CEP-Included model was maximized using Powell's conjugate directions method (Ref 23). This method maximizes a function using only function values. The CEP-Excluded model was maximized using a conjugate gradient with restarts method (Ref 24).
Optimization

Optimization is a process that attains the best or most effective results for a problem, while satisfying any given conditions or constraints. Optimization can be either maximization or minimization. One part of this study was to maximize the CEDF(\( \chi \)), a nonlinear function of 2m independent variables -- the \((x_i, x_{i+1})\) DGZ coordinates of each weapon. The CEDF(\( \chi \)) is an unconstrained function. It can be maximized by minimizing \(-\text{CEDF}(\chi)\). That is, the point \( \chi^* \) in 2m space, such that CEDF(\( \chi \)) is a maximum, is the same point where \(-\text{CEDF}(\chi)\) is a minimum.

In this chapter, direct references to maximizing the CEDF are not made. Instead, all references concerning optimization reference minimizing an unconstrained, nonlinear function of n variables, \( f(\chi) \); \( \chi \) is an n element vector in n-dimensional space, \( \mathbb{R}^n \). The gradient of \( f(\chi) \) is \( \nabla f(\chi) \); the Hessian matrix of \( f(\chi) \) is \( H(\chi) \). A base point in \( \mathbb{R}^n \) is \( \chi^0 \); the optimal point in \( \mathbb{R}^n \) is \( \chi^* \).

There is an important difference between a strict local minimum and the global minimum of \( f(\chi) \). The following two definitions are extracted from Avriel (Ref 3:10). A real valued function \( f(\chi) \) with domain \( D \) in \( \mathbb{R}^n \) has a strict local minimum at \( \chi^* \), if there exists a number \( \delta \) such that

\[
f(\chi^*) < f(\chi) \quad \text{for all } \chi \in D
\]

such that \(|\chi - \chi^*| < \delta\). This definition states that \( \chi^* \) is a local minimum over a region bounded by a number \( \delta \). If Eq (27) holds for all \( \chi \in D \), that is, \( \chi \) not contained within a bounded region, then \( \chi^* \) is the global minimum. Optimization techniques locate the global minimum only.
under special conditions. That is, the function is known to be unimodal. Generally, if a function is not known to contain a global minimum, an accepted procedure is to search D from a number of initial, separated base points to determine all local minimums. Beveridge and Schechter state "the only method of determining the global optimum is the direct comparison of the function values at various local optima" (Ref 5:357).

**Numerical Search Techniques**

The numerical search for the minimum value of an unconstrained function \( f(x) \) with domain D in \( \mathbb{R}^n \) is a sequential, iterative process. It includes the successive calculation of new objective function values, \( f(x_i) \), and the comparison of these values with the best value that has been obtained so far. It is necessary to determine \( x^* \) by

\[
f(x_1) > f(x_2) > \ldots > f(x_i) > \ldots > f(x^*) \quad \text{for all } x \in D
\]

While generating the sequence of \( x_i \), each unconstrained numerical search technique must consider three important elements -- the search direction, the distance to move, and the stopping criteria. From a base point \( x^1 \), a search technique must select (1) a direction of movement \( d \) and (2) a distance to move \( t \). These values specify the next point in \( \mathbb{R}^n \)

\[
x_{i+1} = x_i + td
\]

If \( f(x_{i+1}) < f(x_i) \), then \( x_{i+1} \) is a better estimate of the local minimum.
than $x_i$. The stopping criteria for a search technique depends upon the values of either $x_i^{i+1}, f(x_i^{i+1})$, or $\nabla f(x_i^{i+1})$. That is, if either $|x_i^{i+1} - x_i^i| < \delta_1$, or $|f(x_i^{i+1}) - f(x_i^i)| < \delta_2$, or $|\nabla f(x_i^{i+1})| < \delta_3$, then the technique stops iterating, and $x_i^* = x_i^{i+1}$ is the optimal point in $\mathbb{R}^n$ such that $f(x_i^{i+1})$ is a minimum. Numerical search techniques use different methods to determine $\delta$ and $t$.

There are three categories of numerical search techniques (Ref 2:101). The first category includes direct search techniques. These techniques use only functional values to locate $x_i^{i+1}$ from $x_i^i$. The second category includes gradient or first-order search techniques. These techniques use $f(x_i^i)$ and $\nabla f(x_i^i)$ to determine $x_i^{i+1}$. Generally, gradient methods are more efficient and preferred to direct techniques (Ref 2:152; 5:321; and 10:386). However, when the gradient is not easily obtained, direct searches are more appropriate. The last category includes second-order techniques. These techniques use $f(x_i^i)$, $\nabla f(x_i^i)$, and the Hessian, $H(x_i^i)$, to locate $x_i^{i+1}$. Detailed explanations of the following techniques can be found in most optimization books (Ref 2; 3; 5; and 10). Hence, only a brief explanation is presented here.

If an unconstrained objective function $f(x)$ is not easily differentiated, then a direct search technique is necessary to minimize $f(x)$. These techniques use two stages, an exploratory and a pattern, to move from $x_i^i$ to $x_i^{i+1}$. Two older techniques are the Hooke-Jeeves pattern search and Rosenbrock's method of rotating directions. In the exploratory stages, Hooke-Jeeves only searches along the axial coordinate directions; Rosenbrock searches along a set of mutually orthogonal directions that are determined from $x_i^i$ and $x_i^{i+1}$. Both of these techniques
use a fixed step length when exploring around $x_i^0$. The exploratory function evaluations specify the direction $d$ of the pattern move. A more efficient technique is Powell's method of conjugate directions (Ref 23). In the exploratory stage, Powell's method searches along conjugate directions that are determined from $x_i^0$ and $n-1$ of the previous $n$ exploratory search directions. Conjugate directions are a generalization of orthogonal directions. Also, Powell's method does not use a fixed step length. Rather, this method conducts a one-dimensional search in each of the conjugate directions from $x_i^0$. A more complete description of conjugate directions and Powell's method of conjugate directions is presented later in this chapter.

Gradient search techniques are separated into two categories, either those techniques that follow the gradient as closely as possible (the methods of steepest descent) or those techniques that use the gradient to guide the search (the conjugate gradient methods). Cauchy's steepest descent method uses the gradient to find the direction of greatest functional decrease from a base point. The greatest decrease in $f(x)$ is in the direction of the largest negative gradient. That is,

$$d = -\nabla f(x_i^0)$$

$$x_i^{i+1} = x_i^0 - t \nabla f(x_i^0)$$

The steepest descent method uses a one-dimensional minimization search in the direction of $-\nabla f(x_i^0)$ to determine the step length $t$ and to subsequently locate $x_i^{i+1}$.

$$\min_t f(x_i^0 - t \nabla f(x_i^0))$$

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Conversely, the method of conjugate gradients locates a new base point $x_{i+1}$ by searching along a mutually conjugate direction $d$ (Ref 9 and 24). The direction $d$ is determined using the gradient at the current base point and the previous search direction. From $x_i$, the method uses a one-dimensional minimization search in this direction to determine the step distance $t$.

$$\min_t f(x_i - td)$$

This one-dimensional search establishes a new base point $x_{i+1}$. A more complete description of the conjugate gradient method is presented later in this chapter.

If first and second partial derivatives of $f(x)$ are available, then Newton's method could be used to minimize the function. This technique uses the function's gradient and Hessian to specify the direction and the distance of the maximum decrease in $f(x)$.

$$x_{i+1} = x_i - H(x_i)^{-1} \nabla f(x_i)$$

Avriel states, "If there are a large number of variables, the function and derivative evaluations and especially the matrix inversions, are time-consuming and expensive operations" (Ref 3:288).

These are not the only techniques available to minimize unconstrained, nonlinear functions. However, they are representative of the three categories of techniques -- direct, gradient, and second-order. A detailed presentation and summary of numerical search techniques for each category is provided by Gill, Murray, and
Wright (Ref 10). In addition to the above techniques, several variations are available to minimize unconstrained nonlinear functions. The most powerful is the "variable metric" or quasi-Newton method. This algorithm differs from Newton's method. It does not use the Hessian matrix. Instead of calculating $H(x^i)$, the technique approximates the inverse of $H(x^i)$ by using the gradient and the previous estimate of the inverse. There are other variations of Newton's method. Similarly, finite difference techniques are variations of gradient methods; they use function values to approximate $\nabla f(x^i)$. Generally, it is not possible to single out a method as the one to be used in every case.

Each form of the Complex Expected Damage Function (CEDF) was maximized using only one technique. Since an analytical expression for the gradient of the CEP-Included model was not available, it was maximized using a direct search technique — Powell's method of conjugate directions. Conversely, since an analytical expression for the gradient of the CEP-Excluded model was calculated, it was maximized using a gradient search technique — a conjugate gradient with restarts method.

Conjugate Directions and Quadratic Termination

Conjugate directions are a generalization of orthogonal directions. A set of $n$ vectors $d^1, d^2, \ldots, d^n$ in $\mathbb{R}^n$ are orthogonal if their inner product is 0, that is,

$$d_i^T d_j = 0 \quad \text{for all } i \neq j \quad (28)$$

A set of $n$ vectors $d^i$ is mutually conjugate with respect to the $n \times n$
symmetric, positive definite matrix $A$ if

$$d_i^T A d_j = 0 \quad \text{for all } i \neq j \quad (29)$$

Thus, for every $n \times n$ symmetric, positive definite matrix there is at least one set of $n$ mutually conjugate directions. If the matrix $A$ is the identity matrix, then Eq (29) becomes Eq (28), the definition of orthogonal directions.

Powell's method of conjugate directions and conjugate gradient methods depend upon the concept of quadratic termination. Powell proved the following theorem on quadratic termination (Ref 23).

Theorem: If $g_1, g_2, \ldots, g_m$, $m < n$ are mutually conjugate directions, then the minimum of the quadratic function $f(x)$ is a point in $m$-dimensional space, $\mathbb{R}^m$, containing $x^0$, the initial point, and the directions $g_1, g_2, \ldots, g_m$, and the minimum of $f(x)$ may be found by searching along each of the directions only once. The required minimum is the point

$$x^* = x^0 + \sum_{i=1}^{m} t_i g_i$$

The distances $t_i$ are determined by minimizing $f(x)$ in each direction $g_i$

$$\min_t f(x^0 + \sum_{i=1}^{m} t_i g_i)$$

where

$$f(x) = x^T A x + b^T x + c$$

and, $A$ is a symmetric, positive definite Hessian matrix.
Powell's theorem proved that the minimum of a quadratic function $f(x)$ with domain $D$ in $\mathbb{R}^n$ and a symmetric, positive definite Hessian could be located in $n$ steps. Each step is a search along one of the $n$ mutually conjugate directions $d_i$. However, since each direction $d_i$ has $n$ component directions in $\mathbb{R}^n$, each step requires $n$ one-dimensional searches to minimize $f(x)$.

CEDF Optimization Methods

Powell's theorem is the basis for the method of conjugate directions and conjugate gradient methods. If $f(x)$ is quadratic, then the minimum can be located in a finite ($< n$) number of steps. However, even if $f(x)$ is not quadratic, the concept of quadratic termination can still be used to locate the minimum. When the method is applied to non-quadratic functions, it becomes iterative and a test of convergence is necessary to determine the minimum of $f(x)$. This section presents a brief explanation of these two optimization techniques. Detailed explanations of them are available in Refs 9, 23, and 24; also, most optimization books provide complete explanations of these techniques.

Powell's Method of Conjugate Directions. This section presents an algorithm for Powell's method of conjugate directions (Ref 17 and 23). This method assumes quadratic convergence of $f(x)$; the method will not locate the local minimum in $n$ steps unless the $f(x)$ is quadratic. Instead, the method iterates from $x_i$ to $x_{i+1}$ until $|x_{i+1} - x_i| < \varepsilon$. In this development, the superscript $i$ refers to the iteration and the subscript $j$ refers to one of the $n$-dimensional component directions of $\mathbb{R}^n$. The starting point in $\mathbb{R}^n$ is $x^0$; the initial search directions $d^i_j$ are the $\mathbb{R}^n$ coordinate directions.
An iteration process is used to locate $x^*$ such that $f(x^*)$ is a local minimum. For the $i^{th}$ iteration,

1. From $x^i$, search each of the $n$ directions $d_j^i$ where $j=1,\ldots,n$. These one-dimensional searches use functional values to locate a minimum in each direction. A quadratic approximation and unimodal behavior of $f(x)$ is assumed.

2. These searches locate three specific points in $R^n$ -- $x^i$, the last point; $x^i_t$, an expanded point; and $x^i_m$, the point where the greatest decrease in $f(x^i)$ occurred.

3. The convergence test checks $x^i_n$ to determine if $f(x^i_n)$ is a local minimum. If $x^i_n$ passes the convergence condition $|x^i_n - x^{i-1}_n| < \delta$, then $x^* = x^i_n$. If not, the algorithm continues.

4. The modification test checks the decrease in $f(x)$ from $x^i$ to $x^i_n$. These functional changes specify the set of directions $d_j^{i+1}$ for the next iteration. The same mutually conjugate directions may be used again or a new set of mutually conjugate directions may be determined.

A Conjugate Gradient Method. This section presents an algorithm for a conjugate gradient with restarts method (Ref 9 and 24). Again, for functions which are not quadratic, the method will not locate the local minimum in $n$ steps. Instead, the method iterates from $x^i$ to $x^{i+1}$ until $|\nabla f(x^{i+1})| < \delta$.

$$x^{i+1} = x^i + t_i d^i$$
The direction $d^i$ is mutually conjugate to the previous $i-1$ search directions; it is determined using the previous direction $d^{i-1}$ and $\nabla f(x^i)$. The starting point in $\mathbb{R}^n$ is $x^0$; the initial search direction is the negative of the gradient, $-\nabla f(x^0)$.

An iterative process is used to locate $x^*$ such that $f(x^*)$ is a local minimum. For the $i$th iteration,

1. Calculate $\nabla f(x^i)$

2. From $x^i$, use a one-dimensional minimization search in the direction $d^i$ to determine the step length $t_i$ and to subsequently locate the point $x^{i+1}$.

$$x^{i+1} = x^i + t_i d^i$$

$$\min_{t_i} f(x^i + t_i d^i)$$

3. Calculate $\nabla f(x^{i+1})$

4. The convergence test checks $x^{i+1}$ to determine if $f(x^{i+1})$ is a local minimum. If $\| \nabla f(x^{i+1}) \| < \delta$, then $x^* = x^{i+1}$. If not, the algorithm continues.

5. Compute $\beta_i = \frac{\| \nabla f(x^{i+1}) \|^2}{\| \nabla f(x^i) \|^2}$

6. Determine the next mutually conjugate search direction.

$$d^{i+1} = -\nabla f(x^{i+1}) + \beta_i d^i$$

This algorithm locates the minimum of a quadratic function with a
symmetric, positive definite Hessian matrix in n or less iterations. However, for functions that are not quadratic, the minimum will generally not be determined in n steps. After the n steps, n mutually orthogonal directions have been searched. \( x_n \) may or may not have converged rapidly towards \( x^* \).

For functions with slow rates of convergence, because of nearly parallel \( g_i \) and \( g_{i+1} \), Fletcher and Reeves developed the restart procedure (Ref 9). After every \( n + 1 \) steps, the method reverts to the direction of steepest descent, the largest negative gradient, for the next search direction. That is, following iteration \( i = n + 1 \), which located \( x_{n+2} \), the direction \( g_{n+2} \) would not be specified as in step 6 above, but rather \( g_{n+2} = -\nabla f(x_{n+2}) \). "Thus the whole procedure is restarted from the current \( x \), discarding all previous experience that would normally be transmitted in the calculation of \( g^i \). The process remains quadratically convergent provided such restarts are not more frequent than every n steps" (Ref 9).

The CEDF models developed in Chapter II are maximized with these two techniques. Powell's method of conjugate directions maximizes the CEP-Included CEDF model; a conjugate gradient with restarts method maximizes the CEP-Excluded CEDF model. These methods require the vector of the 2m independent variables \( x \) and the function \( \text{CEDF}(x) \); the conjugate gradient technique also requires the gradient of \( \text{CEDF}(x) \). The computerization of the algorithm that maximizes the two \( \text{CEDF}(x) \) models is presented in Chapter IV.

Greerwood developed a similar version of the \( \text{CEDF}(x) \) (Ref 12). His algorithm uses a different, yet related approach to determine
optimal DGZ locations. His function $G(x)$ depends on the total expected target value undamaged. The $2m$ first partial derivatives of $G(x)$ are set equal to $0$. Then these $2m$ nonlinear equations are solved iteratively to yield a $x^*$ such that $G(x^*)$ is a minimum. NUCWAVE uses a modified Greenwood technique to determine optimal DGZ locations (Ref 25:4-3). It optimizes one weapon at a time. Hence, it iteratively solves $2$ nonlinear equations to determine $(X_i, Y_i)$ -- the coordinates of weapon $i$. Then it repeats the process for the next weapon.
IV. Computerization, Verification, and Validation of the CEDF Maximization Algorithm

The Complex Expected Damage Function (CEDF) maximization algorithm includes the CEDF models, CEP-Included and CEP-Excluded, and the optimization techniques, Powell's method of conjugate directions and the conjugate gradient with restarts method. The algorithm determines optimal DGZ locations for a finite number of nuclear weapons against installations in a target complex by maximizing the CEDF.

Evaluation of the algorithm consisted of three related stages -- construction, verification, and validation. These stages formed an iterative process that was necessary to develop user confidence in the capability of the algorithm. Construction is the formulation and computerization of a model. The computerization of the CEDF maximization algorithm used a modular approach. Smaller segments of the CEDF model were developed to accomplish lower level procedures. These segments became subprograms in the final computer code. Verification of the CEDF maximization algorithm used example problems to insure that the results of each subprogram were correct. Validation measures the relative agreement between the model and the system modeled (Ref 26:215). Validation of the CEDF maximization algorithm was a comparison of the results from the algorithm with the results from a current DGZ model. This chapter presents the evaluation of the CEDF maximization algorithm with respect to these three stages.

Computerization

A flow chart of the CEDF maximization algorithm is shown in Figure 6.
Input weapon and installation data
INITLZ

Calculate weapon and installation parameters
WRADS

Distance damage function \( P_d(r) \)
PDR

CEP-Excluded
\( P_{d,1,j} = P_d(r) \)
PDAM

Calculate CEDF(\( \xi \))
and grad (CEDF)
GFUNCT

Conjugate gradient
Optimization ZXCGR

CEDF MODELS

Conjugate directions
Optimization PWMIN

OPTIMIZATION

Mixed Technique
ZXCGR and PWMIN

Output final DGZ coordinates
OUTIDGZ

Optimal DGZ locations (\( \xi^* \))
and maximum CEDF(\( \xi^* \)) value

Figure 6. Flowchart of the CEDF Maximization Algorithm
The symmetry of Figure 6 illustrates several characteristics of the algorithm. The blocks above the dashed line correspond to the procedures that use weapon and installation parameters to develop the $CEDF(\chi)$ and the gradient of the $CEDF(\chi)$. The lower blocks correspond to the optimization techniques that were used to maximize the respective CEDF. The left blocks correspond to the CEP-Excluded model; alternately, the right blocks correspond to the CEP-Included model. The upper three and lower three blocks are common to both CEDF models. Each block is a smaller segment of the CEDF maximization algorithm.

The computer code of the CEDF maximization algorithm was written using FORTRAN V. Appendix D contains a listing of the code and a glossary of the FORTRAN variables. The computer code includes a driver module, seven subroutines and two functions. The hierarchy of the algorithm's subprograms is shown in Figure 7. All program variables, including weapon and installation parameters, that are used in more than one subprogram, are stored in six named common blocks. Only the DGZ coordinate vector $\chi$ is transferred between subprograms by the subprograms' argument lists.
The driver module OPTMZ controls the CEDF maximization algorithm. OPTMZ calls the five highest level subroutines. The primary functions of these subroutines are: (1) INITLZ inputs user-specified weapon and installation parameters, (2) WRADS calculates additional installation and weapon parameters, (3) ZXCGR is a conjugate gradient with restarts subroutine that maximizes the CEP-Excluded CEDF, (4) PWMIN is a conjugate directions subroutine that maximizes the CEP-Included CEDF, and (5) OUTDGZ outputs the final DGZ coordinates. ZXCGR calls GFUNCT, a subroutine that calculates the value of the CEDF(ξ) and the gradient of the CEDF(ξ). PWMIN calls FUNCT, a subroutine that calculates the value of the CEDF(ξ). GFUNCT and FUNCT call PDAM, a function that calculates the probability of achieving a specified level of damage to installation j from weapon i, Pd_i,j. PDAM, in turn, calls PDR, a function that calculates the distance damage function, Pd(r). More complete descriptions of these subroutines and functions are given below. WRADS, PDAM, and PDR are modifications of subprograms from Mathematical Background and Programming Aids for the Physical Vulnerability System for Nuclear Weapons (Ref 6).

INITLZ. This subroutine has four primary functions. First, it reads user-specified weapon and installation parameters from an external file, INDATA. Appendix E contains samples of an input file and an output data file. For each weapon i, the user specifies a yield, a hob, and a Pa; the user may specify initial DGZ coordinates. Also, for each installation j, the user specifies coordinates, a VNTK code, and a value. The user inputs the weapon and installation latitude and longitude in degrees-minutes-seconds and the direction from either the prime meridian or the equator. Positive coordinates are east of the prime meridian or the equator.
meridian and north of the equator.

Second, INITLZ assigns the initial coordinates of the m weapons, prior to maximization. The user has three options: (1) provide independent estimates of the weapon coordinates; (2) let INITLZ assign the coordinates of the m highest valued installations to be the coordinates of the m weapons in decreasing order of yield, that is, the largest yield weapon is initially located at the highest valued installation; or (3) let INITLZ assign the coordinates of the m hardest installations to be the coordinates of the m weapons in decreasing order of yield, that is, the largest yield weapon is initially located at the installation with the largest VN number.

Third, INITLZ transforms all weapon and installation degrees-minutes-seconds into coordinates measured in feet and relative to a common origin in an XY coordinate system. The CEDF maximization algorithm assumes a flat earth model to locate all coordinates. Each minute of latitude equals 6080 feet. However, one minute of longitude equals 6080 feet only at the equator. When the latitude is not the equator, one minute of longitude is less than 6080 feet because of the merging of the meridian lines. The scale factor is the cosine of the latitude.

Lastly, INITLZ specifies accuracy requirements for ZXGGR and PWMIN. These subroutines need prespecified values to test for the convergence of $\gamma$ to the maximum value of $CEDF(x^*)$.

WRADS. This subroutine calculates additional weapon and installation parameters from the user-specified parameters. For each installation $j$, it determines a distance damage sigma ($\sigma_d^j$) and a $\beta_j$. Also, for each weapon $i$-installation $j$ combination, it calculates a weapon radius, $WR_{i,j}$. These calculations are described in Appendix A.

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FDR. This function calculates the distance damage function $P_d(r)$, the probability of achieving a specified level of damage to installation $j$ from weapon $i$ when the distance $r$ between installation $j$ and weapon $i$ is known. See Eq (5).

PDAM. This function calculates the probability of achieving a specified level of damage to installation $j$ from weapon $i$, $P_{d_{ij}}$. For the CEP-Excluded CEDF model, $P_{d_{ij}}$ is the distance damage function from FDR. For the CEP-Included CEDF model, the distance $r$ between installation $j$ and the impact point is unknown, and $P_{d_{ij}}$ is calculated using Gauss-Legendre quadrature and the distance damage function. See Eq (8).

GFUNCT. This subroutine calculates the CEDF($x$) and the gradient of the CEDF($x$). One function and gradient evaluation requires $m \times n$ calls to function PDAM. These calls specify $P_{d_{ij}}$ for each weapon $i$-installation $j$ combination using the CEP-Excluded model. CEDF($x$) is calculated using Eq (1). Each element of the gradient is calculated using Eq (11).

FUNCT. This subroutine also calculates the CEDF($x$) using Eq (1), and one function evaluation requires $m \times n$ calls to function PDAM. However, these calls specify $P_{d_{ij}}$ for each weapon $i$-installation $j$ combination using the CEP-Included CEDF model.

ZXCSR. This subroutine minimizes $-\text{CEDF}(x)$ for the CEP-Excluded model. It is a conjugate gradient with restarts routine from the International Mathematical and Statistical Libraries, Inc. (Ref 16:ZXCSR). ZXCSR requires function and gradient evaluations from GFUNCT and the DGZ coordinate vector $x$. It uses two control parameters -- $\text{DFPRED}$ and $\text{ACC}$. $\text{DFPRED}$ specifies an estimate of the expected increase in the CEDF; $\text{ACC}$ specifies the desired accuracy of the convergence check. This check
requires the sum of the squares of the gradient elements to be less than ACC.

**FWMIN.** This subroutine minimizes \(-CEDF(x)\) for the CEP-Included model. It is a conjugate directions routine from *Optimization Techniques with FORTRAN* (Ref 17:331-343). FWMIN requires function evaluations from FUNCT and the DGZ coordinate vector \(\mathbf{z}\). It also uses two control parameters -- ESCALE and E. ESCALE specifies the maximum step size multiplier for a single step of any \(x_k\); E specifies the accuracy of the convergence check. This check requires the absolute value of the differences between each element of \(x_i^j\) and \(x_i^{j+1}\) to be less than E.

**OUTDGZ.** This subroutine translates the XY coordinates of the final DGZs from feet into degrees-minutes-seconds and the direction from either the prime meridian or the equator. Then it outputs these coordinates to the external data file, TAPE6.

The CEDF maximization algorithm provides three sets of optimal DGZ locations. The first set is from the CEP-Excluded CEDF model and ZXCGR maximization; the second set is from the CEP-Included CEDF model and FWMIN maximization. The last set of DGZ locations is from both CEDF models and ZXCGR and FWMIN maximization -- a mixed technique.

**Verification**

The verification of the CEDF maximization algorithm included four phases. Each phase verified the subprograms of the algorithm using example problems. For each computer program, the results of each example problem, including the values of intermediate variables, were calculated independently of the respective computer program. Then the computer program solved the example problem. PRINT statements in the
program printed values of most FORTRAN variables. These values were compared with the values calculated by pencil and paper to verify that the computer program calculated the correct values.

The first phase verified five lower level subprograms — PDR, PDAM, WRADS, INITLZ, and OUTDGZ. Each of these modules was coded and debugged as a small FORTRAN program. Mathematical Background and Programming Aids for the Physical Vulnerability System for Nuclear Weapons includes example problems. Fifteen of these problems were used to check PDR, PDAM, and WRADS. These programs calculated the same values as the example problems. The outputs of INITLZ for several example problems were compared with results that were calculated independently of the computer program. These comparisons indicated INITLZ was properly forming the XY coordinate system and the installation and DGZ coordinate vectors. Similarly, the outputs of OUTDGZ for several test cases were compared with pencil and paper calculated results. These comparisons indicated OUTDGZ was correctly translating the final DGZ coordinate vector from feet into degrees-minutes-seconds and the direction from either the prime meridian or the equator. These five subprograms were merged into one program and became the foundation of the next verification phase.

The second phase verified the subroutine FUNCT. The small programs, PDAM and PDR, became FORTRAN functions; the programs WRADS, INITLZ, and OUTDGZ, became FORTRAN subroutines. FUNCT calculates the value of the CEDF(χ) using Eq (1). The pencil and paper calculated results from several example problems were compared with the results from FUNCT. One example included two identical installations. Each installation's value and VNTK code were 15.0 and 15P2. The distance between the two installations was 6000 feet. Two identical weapons
were collocated halfway between the two installations. Each weapon's yield, height of burst, and CEP were 100-kt, 1000 feet, and 600 feet. The independent calculation of the CEDF for this complex was 29.15; the CEDF value from FUNCT was 29.1492. These example problems indicated FUNCT was properly calculating the CEDF(\chi).

This two weapon-two installation complex was used to investigate the results of moving the two collocated DGZs. The geometry of this complex is shown in Figure 8. Initially, the two 100-kt weapons were collocated at point 0, and the CEDF value was 29.15. The CEDF value
decreased as the two DGZs were moved in opposite directions a distance \(d\) from point 0. A graph of CEDF versus \(d\) for this problem is also included in Figure 8. This example indicated the existence of a CEDF maximum, on the line considered, as the distances between the DGZs and the installations varied.

The third phase verified the gradient calculation of the CEP-Excluded CEDF in the subroutine GFUNCT. GFUNCT also calculates the CEDF(\(x\)) using Eq (1). The gradient vector from GFUNCT was checked using two weapon-installation geometries. Appendix F includes the table and calculations used to verify the gradient of the CEDF(\(x\)) for these two examples.

The first example included one weapon and two installations. The first installation's value and VNMX code were 5000 and 11P2; the second installation's value and VNIX code were 12000 and 15P2. The weapon's yield and height of burst were 100-kt and 1000 feet, and the CEP was 0 feet. Forty values of the CEDF(x) were calculated for different DGZ locations. The \(x\) direction was along the line connecting the two installations. These 40 values were then plotted. Figure 9 is a plot of CEDF(x) versus \(x\) for this example. A DGZ between the two installations was selected (\(x = 63500\)) and the gradient calculated using two methods. In this example, the gradient had only one element because the \(y\) variable was constant, and only the \(x\) variable was allowed to vary. The gradient values for the two calculation methods were compared with the gradient from GFUNCT. The first method used a difference equation \(\frac{\Delta \text{CEDF}}{\Delta x}\) to approximate the gradient. For the DGZ selected, the difference equation approximation of the gradient was 3.939. The second method was pencil and paper calculations of all the steps necessary to determine
the gradient. Chapter II presented these steps. For the DGZ selected, the pencil and paper calculation of the gradient was 3.9791. The value of the gradient from GFUNCT for the DGZ selected was 3.9791. These comparisons indicated the subprogram GFUNCT was properly calculating the gradient of the CEDF(x).

The second example included two weapons and three installations. The gradient of the CEDF(x) in this example has 2m or four elements. Only one element was completely checked by pencil and paper calculations. A location for each DGZ within the three-installation complex was selected.
Installation 1
(60000,11162)

Installation 2
(68000,11162)

Figure 10. The one weapon-two installation geometry.

Then \( \frac{\partial \text{CEDF}(x)}{\partial x_2} \) was calculated to be 0.192548; the subsequent value from GFUNCT was 0.19255136. These comparisons indicated GFUNCT was correctly forming the gradient of the \( \text{CEDF}(x) \).

The last phase verified the CEDF maximization algorithm's ability to locate a local maximum of the CEDF. All subprograms, the subroutines ZXGR and PWMIN, and the driver module OPTMZ were merged into one program -- the CEDF maximization algorithm. The two installation-one weapon complex described above to verify GFUNCT was also used to verify the algorithm. Figure 10 presents this complex, several initial starting points for the DGZ, and the mean location \( \bar{x}^* \) for the local maximum of the \( \text{CEDF}(x) \). In this simple example, the local maximum is also the global maximum. The graph of the \( \text{CEDF}(x) \) versus \( x \) in Figure 9 indicated the maximum \( \text{CEDF} \) value was approximately 15000 for \( 65000 < x < 66000 \).

The CEDF maximization algorithm was run with seven different
initial DGZ. From the initial locations 1, 2, 3, 6, and 7, the algorithm converged to a maximum CEDF value and an optimal DGZ. The algorithm did not move the DGZ from the initial locations 4 and 5. For these locations, the ZXGR and PWMIN convergence criteria were satisfied, and the CEDF value was 4950. The algorithm did not move the DGZ from these locations because there were no indications of a CEDF increase. Chapter VI explains this result in more detail.

The mean optimal DGZ location and CEDF value were calculated for the other five initial DGZ locations. For the CEP-Excluded model using ZXGR, the mean location of the optimal DGZ was (65319,11184). The standard deviation for x was 8 feet; for y it was 15 feet. The mean value of the CEDF(x) was 15019; the standard deviation was 4.3. For the CEP-Included model using PWMIN, the mean location of the optimal DGZ was (65300,11172). The standard deviation for x was 58 feet; for y it was 98 feet. The mean value of the CEDF(x) was 15006; the standard deviation was 16.6. PWMIN is a slower optimization routine; hence, less restrictive convergence criteria were established for PWMIN. This could account for PWMIN's smaller CEDF value and larger standard deviations for x and y.

The results from both CEDF models were compared to the values from Figure 9. These comparisons indicated that the CEDF maximization algorithm located an optimal DGZ location by maximizing the CEDF(x) for this simple two installation-one weapon complex. More detailed complexes are considered in the next section and the next two chapters.

Validation

Validation measures the relative agreement between the model and
the system modeled. It is not possible to make comparisons between the
CEDF maximization algorithm and the real world. Similarly, validation
is not a yes or a no answer; it is a qualitative, relative measure.
The CEDF maximization algorithm results for two example problems were
compared with the results from NUCWAVE (Ref 29). NUCWAVE is a one-sided
nuclear weapons allocation war gaming model. It optimizes the damage
to a set of targets using a preselected set of weapons.

Two of the primary differences between the CEDF maximization algo-
rithm and NUCWAVE are NUCWAVE's starting solution and optimization
technique. It optimizes sequentially by starting the largest yield
weapon at the highest valued installation. It optimizes over the \((X,Y)\)
coordinates of this weapon. Then it stores the final coordinates of
this weapon, calculates the damage of all affected installations, and
determines the remaining values for all installations. Then it opti-
mizes the next largest yield weapon by starting it at the highest
remaining valued installation. NUCWAVE continues to iterate through
the entire weapon set until no further movement of a DGZ results in an
increase in the total expected target value damage. Chapter I includes
a description of NUCWAVE methodology. For these comparisons, the CEDF
maximization algorithm assigned the initial DGZ locations to the highest
valued installations.

The first problem included one weapon and two installations. This
complex was very similar to the complex in Figure 10. The first instal-
lation's coordinates were \(46^\circ 03'15'' \ N - 45^\circ 10'00'' \ E\); its VNTK code and
value were 11P2 and 5000. The second installation's coordinates were
\(46^\circ 03'25'' \ N - 45^\circ 11'20'' \ E\); its VNTK code and value were 15P2 and 12000.
The weapon's yield, height of burst, and CEP were 100-kt, 1000 feet,
and 0 feet. The coordinates of NUCWAVE's optimal DGZ were \(46^\circ 03'22'' N - 45^\circ 10'58'' E\); the total expected target value damage was 16579 or 97.52% of the complex value. The coordinates of the CEDF maximization algorithm's optimal DGZ were \(46^\circ 03'21'' N - 45^\circ 10'50'' E\); the CEDF was 16788 or 98.75% of the complex value. This represents a difference of approximately 575 feet and an increase in CEDF value of approximately 1%.

Then the final DGZ coordinates from NUCWAVE were used as starting coordinates for the CEDF maximization algorithm. The CEDF value at these coordinates was 16647. The final coordinates for this run of the algorithm were also \(46^\circ 03'21'' N - 45^\circ 10'50'' E\). These results indicate that the CEDF maximization algorithm achieves comparable results with an existing model, NUCWAVE.

The second problem included two weapons and five installations. The installations' VNTK codes ranged from 14P3 to 20P3; the installations' values ranged from 3000 to 12000. The total complex value was 33000. Both weapons' yield, height of burst, and CEP were 100-kt, 1000 feet, and 0 feet. The coordinates of NUCWAVE's optimal DGZs were \(46^\circ 01'58'' N - 45^\circ 09'55'' E\) and \(46^\circ 00'48'' N - 45^\circ 09'42'' E\); the total expected target value damage was 28730 or 87.06% of the complex value. The coordinates of the CEDF maximization algorithm's optimal DGZs were \(46^\circ 01'58'' N - 45^\circ 09'54'' E\) and \(46^\circ 00'45'' N - 45^\circ 09'38'' E\); the CEDF value was 29543 or 89.52% of the complex value. This represents a difference of approximately 70 feet in the first DGZ and 415 feet in the second DGZ and an increase in CEDF value of approximately 3%.

Again, the final DGZ coordinates from NUCWAVE were used as starting coordinates for the CEDF maximization algorithm. The CEDF value at these coordinates was 29018. The final coordinates from this run
were also 46°01'58'' N - 45°09'54'' E and 46°00'45'' N - 45°09'38'' E; the CEDF value was also 29543.

The comparisons between the results from the CEDF maximization algorithm and the results from NUCWAVE for the two examples indicate that the algorithm correctly determines the same local maximum as NUCWAVE. The results from these two examples do not validate the algorithm, but because the DGZ locations were consistent between NUCWAVE and the CEDF maximization algorithm, the algorithm's results are not invalid. These results provide the user confidence in the capability of the CEDF maximization algorithm.
V. CEDF Maximization Algorithm Properties

The Complex Expected Damage Function (CEDF) maximization algorithm determines optimal DGZ coordinates for multiple nuclear weapons against installations in a target complex by maximizing the CEDF. The previous chapter presented the computerization and evaluation of the algorithm. The flowchart in Figure 6 summarizes the algorithm's modules. It also presents the algorithm's three CEDF maximization techniques: (1) ZXCGR, a conjugate gradient with restarts optimization method that maximizes the CEP-Excluded CEDF model; (2) PWMIN, Powell's method of conjugate directions that maximizes the CEP-Included CEDF model; and (3) a mixed technique that uses both CEDF models.

The ZXCGR and the mixed techniques each consist of two stages. The first stage of the ZXCGR technique has a less restrictive convergence criteria than the second stage, and its DGZ coordinates are used as initial DGZ coordinates for the second stage of the ZXCGR maximization algorithm. These optimal DGZ coordinates from the first stage of the ZXCGR algorithm also are used as the initial DGZ coordinates for the second stage of the mixed technique. Mixed maximization has an initial ZXCGR stage and then a PWMIN stage. For brevity and completeness, the following nomenclature will be used throughout this report. ZXCGR conjugate gradient maximization will be referred to as ZXM. Powell's method of conjugate directions will be referred to as PWM. Finally, the mixed technique will be referred to as MXM.

This chapter contains two sections. Each section presents characteristics of the algorithm's three CEDF maximization techniques. Differences and similarities between ZXM, PWM, and MXM are discussed.
Similarly, some capabilities and limitations are presented. The first section discusses convergence criteria, installation value scaling, user guidelines and user cautions. Also, it presents comparisons of CEDF results for different convergence values. The second section presents characteristics and optimization results for specific, geometrically symmetric, two and four-installation target complexes. It also discusses the effects of symmetric gradient elements on ZXM maximization.

**Convergence Criteria**

The optimal DGZ coordinates and CEDF values from the CEDF maximization algorithms, ZXCGR using the CEP-Excluded model and PWMIN using the CEP-Included model, were sensitive to the convergence control parameters.

The ZXCGR control parameters are ACC and DFPRED. ACC specifies the desired accuracy of the convergence check. This check requires the norm of the gradient to be less than ACC. The norm of the gradient, \(|\nabla F|\), is the sum of the squares of the gradient elements. When ZXCGR locates a point \(x^*\) in 2m space, such that the norm of the gradient is less than the prespecified value of ACC, the optimization routine stops iterating. DFPRED is an estimate of the expected increase in the CEDF. ZXCGR uses it to determine the size of the initial change in each \(x\).

The values of the installations affected the choice of values for ACC and DFPRED. This is because the installation values directly scale the CEDF and the magnitude of the gradient elements. Eqs (1) and (11) in Chapter II present this relationship. Most of the example problems in this report used installation values between 0 and 10000 \((10^4)\). If
installation values are within this range, then the most versatile
parameter values for second stage ZXM maximization of two optimal DGZs
are ACC = 0.001 and DFPRED = 1000. If the installation values are
not within this range, then a heuristic guideline is suggested to assist
the user in estimating reasonable parameter values. For either case,
the first stage value of ACC = 0.01 determined acceptable DGZ coor-
dinates for a wide range of installation values and number of weapons.

A general guideline to determine ACC for two weapons depends on the
highest valued installation in the target complex. The highest value
is rounded up to the largest power of the base 10. Then ACC equals $10^{-7}$
times this adjusted value. For example, for a four-installation complex
with installation values between 2500 and 7000, the adjusted value would
be the 7000 rounded up to 10000 ($10^4$). Then ACC would equal $10^{-3}$. This
heuristic implies that a smaller ACC is needed for complexes with over-
all lower valued installations. Intuitively, this makes sense because
the scaling effect of smaller installation values decreases the CEDF and
the magnitudes of the gradient elements. When more than two weapons are
used, a larger ACC value is needed to account for the additional gradient
elements.

As an example, Table II presents the results of two ZXM optimiza-
tions for a two weapon-four installation target complex. Weapon and
installation parameters, except the installation values, were the same for
both optimizations. In the original problem, the most valuable instal-
lation's value was 7000, and an ACC of 0.001 was used. The CEDF maximum
value was 15436. In the 1/10 value scaled problem, the most valuable
installation's value was reduced by a factor of 10 to 700, and a smaller
ACC of 0.0001 was used. The CEDF maximum value was 1543. Each of the
TABLE II
CEDF Comparison between an Original Problem and a 1/10 Value Scaled Problem

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Original Problem</th>
<th>Scaled Problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>Target</td>
<td>VNTX Value</td>
<td>Value</td>
</tr>
<tr>
<td>1</td>
<td>16P2</td>
<td>3500</td>
</tr>
<tr>
<td>2</td>
<td>22P2</td>
<td>2500</td>
</tr>
<tr>
<td>3</td>
<td>21P4</td>
<td>5000</td>
</tr>
<tr>
<td>4</td>
<td>19Q3</td>
<td>7000</td>
</tr>
</tbody>
</table>

| ACC        | 0.0010           | 0.00010       |
| || \nabla F || at convergence | 0.0004 | 0.00002 |
| CEDF at convergence | 15436 | 1543 |

ZKM CEDF maximizations located essentially the same coordinates for both DGZs. Comparing the two optimization results, the coordinates of the first DGZ were within one foot of each other, and the coordinates of the second DGZ were within five feet of each other. Hence, the original and the scaled optimization problems located the same DGZ coordinates without regard for the magnitude of each installation's value.

Another heuristic is suggested for estimating the value of DFPRED. For the m weapons, sum the values of the m highest valued installations. If the m weapons were assigned to these m highest valued installations, then this sum would be an approximate value for the CEDF. Next determine the total value of all the installations. Then subtract the value sum of the m highest installations from the complex's total value. This difference is the maximum possible CEDF increase. An estimate for DFPRED
### TABLE III
Comparison of ACC Convergence Criteria

<table>
<thead>
<tr>
<th>Parameter</th>
<th>ACC</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.0100</td>
</tr>
<tr>
<td>| | | F | | at convergence</td>
<td>0.0095</td>
</tr>
<tr>
<td>Number of function evaluations</td>
<td>14</td>
</tr>
<tr>
<td>CEDF at convergence</td>
<td>15433.</td>
</tr>
<tr>
<td>DGZ 2 final coordinates</td>
<td>(44840,23075)</td>
</tr>
</tbody>
</table>

is one-half of this difference. Again, this guideline implies that for lower installation values DFPRED, the estimated increase in the CEDF, should be smaller.

Generally, by decreasing ACC, ZXCSR can determine better estimates of the CEDF maximum and its respective optimal DGZ coordinates. For the original two weapon-four installation complex of Table II, ZXCSR was used to compare the CEDF maximum value and \| \| F \| \| for three values of ACC. DFPRED equaled 1000 for these three examples. Table III presents the results and the DGZ 2 optimal coordinates for these examples. The results of these ZXCSR maximizations indicated that, by decreasing ACC, ZXCSR can determine a better estimate of a CEDF local maximum. That is, ZXCSR can achieve a larger CEDF value and a smaller \| \| F \| \|. The final
coordinates for DGZ 1 were the same for the three cases. Only DGZ 2 coordinates were different; Table III indicates this difference was barely noticeable. Since the final DGZ 2 coordinates were within 17 feet of each other, a less restrictive ACC is acceptable. That is, a value of ACC smaller than $10^{-7}$ times the adjusted highest installation value is unnecessary.

The results of a similar experiment using the same two weapon-four installation target complex indicated that the value of DFPRED also did not significantly affect the CEDF maximum value or the optimal DGZ coordinates. Five values of DFPRED, 100, 1000, 2500, 5000, and 6000, were compared using a constant ACC of 0.001.

Occasionally, ZXCGR will not converge satisfactorily and locate an optimal point in 2m space. The IMSL subroutine will return an IER = 129 error message. This message indicates that the subroutine abandoned a line search; this was probably because of conflicting information. The gradient may indicate that a point is not optimal; that is, $\| \nabla F \| > \text{ACC}$. However, each additional iteration may be on either side of the optimal point and the algorithm is unable to terminate satisfactorily. For most of the occurrences of this error message, the point located by the subroutine actually was a good estimate to the local CEDF maximum. Three options are available to the user when the algorithm terminates with this error message. First, select another ACC value and rerun the same problem. Second, select another DFPRED value and rerun the problem. Third, compare the DGZ locations and the CEDF maximum value with the results of PWM and MXM. Again, for most occurrences of this message, the third option indicated that the point located was a good estimate of
the CEDF maximum value and the respective optimal DGZ coordinates.

Just as ZXCGR maximization results depended on the values of the convergence control parameters, ACC and DFPRED, PWMIN maximization results depended on the values of E and ESCALE. E specifies the desired accuracy of the convergence check. This check requires the absolute value of the differences between each element of \( \mathbf{x} \) for iteration \( i \) and each element of \( \mathbf{x} \) for iteration \( i - 1 \) to be less than E. When PWM locates a point \( \mathbf{x}^* \) in 2m space, such that all element differences are less than the prespecified value of E, the optimization routine stops iterating. ESCALE is the maximum step size multiplier for a one-dimensional search. PWMIN will not increment each \( \mathbf{x} \) by more than ESCALE*E.

The effect of E and ESCALE on the maximum CEDF value and the optimal DGZ locations was not as evident as the ZXCGR convergence control variables. Accordingly, an indepth sensitivity analysis of these parameters was not accomplished. Preliminary investigations indicated that ESCALE/E values of 10000/0.1 were the most effective in maximizing the CEF-Included CEDF model. E values of 1, 5, and 10 often resulted in computer runs that exceeded 60 seconds of computer processing (CP) time. These incomplete runs generally aborted after the third or fourth PWM iteration. Also, ESCALE values of 1000 and 5000 were examined.

The most promising values of ESCALE/E were 5000/0.1 and 10000/0.1. These two combinations were used for more than 143 CEDF maximization algorithm evaluations using 3, 4, 5, and 7-installation target complexes. Eighty of 83 runs (96%) using the ESCALE/E values of 5000/0.1 converged to a solution; similarly, 54 of 60 runs (90%) using the ESCALE/E values of 10000/0.1 converged to a solution. The other nine runs were terminated because of excessive CP time. Twenty-eight CEDF maximization runs
were identical except that 14 used the ESCLAE/E values of 5000/0.1 and
14 used the values of 10000/0.1. Differences between the two parameter
pairs for two criteria, CEDF maximum value and CP time, were evaluated.
The results of a sign test indicated that there was no difference be-
tween the parameter pairs.

The four convergence control parameters need to be specified prior
to a CEDF maximization algorithm run. The subroutine INITLZ initializes
the ACC value for the first stage of ZXM to 0.01. The user provides the
ACC value for the second stage of ZXM through the external file, INDATA.
Similarly, the user provides the DFPRED value through INDATA. Appendix E
discusses the necessary input procedures. The subroutine INITLZ also
initializes the values of ESCLAE/E to 5000/0.1. If the user desires
different PWM convergence control parameters, then only two lines of the
code need to be changed.

The norm of the gradient, $\| \nabla F \|$, will be used as a relative indi-
cator of convergence for all ZXM maximizations. For a two-weapon complex
with a maximum installation value of 10000, $\| \nabla F \| = 0.001$ implies that
the mean value for each of the four gradient elements is approximately
0.015. That is, a change of 1000 feet in any of the 4 spatial direc-
tions would change the CEDF by only 15 value points.

**Symmetry Characteristics**

Two simple target complexes were investigated to characterize the
CEDF models and their respective maximization techniques. Initial DGZ
locations were selected to emphasize special features of ZXM and PWM
optimization techniques. The examples included either symmetric target
CASE I.
\[
\begin{align*}
\text{1} & \triangle \rightarrow \text{1,2} \\
\triangle & \rightarrow \text{1,2} \\
\end{align*}
\]

CASE II.
\[
\begin{align*}
\text{1} & \triangle \rightarrow \text{1,2} \\
\triangle & \rightarrow \text{1,2} \\
\end{align*}
\]

CASE III.
\[
\begin{align*}
\text{1} & \triangle \rightarrow \text{1,2} \\
\triangle & \rightarrow \text{1,2} \\
\end{align*}
\]

for each installation:
- value = 5000
- VNTK = 15P2

for each weapon:
- yield = 100 kt
- HOB = 1000 feet
- CEP = 0 feet
- Pa = 0.99

Figure 11. A symmetric two-installation complex.

complexes or a CEDF with symmetric gradient elements.

A two weapon-two installation complex was analyzed to determine the consequences of symmetric gradient elements on the ability of ZXM to locate a CEDF maximum value and optimal DGZ coordinates. Figure 11 presents the complex geometry, weapon and installation parameters, initial DGZ locations, and the ZXM direction of DGZ movement for five cases. Each of these cases had either a geometrically symmetric weapon-installation complex or a CEDF with symmetric gradient elements. The \(x_1\) and \(x_2\) directions were along the line segment connecting the two installations; the \(x_3\) and \(x_4\) directions were perpendicular to this line segment. For
each of these cases, PWM converged to a maximum CEDF value of 9900. This was each weapon's Pa times the total value of both installations. PWM also separated collocated DGZs. The results of the five cases provided further insight into the capability of the ZXM maximization technique.

**CASE I.** The initial DGZ locations for two identical weapons were collocated halfway between two identical targets. ZXM neither separated nor moved the two weapons. This was because all four of the CEDF gradient elements equaled 0. The $x_3$ and $x_4$ gradient elements were 0 because all $y$ values were equal; the $x_1$ and $x_2$ elements were 0 because weapons 1 and 2 were halfway between the installations. That is, one installation's contributions to the $x_1$ and $x_2$ gradient elements cancelled the other installation's contributions. ZXM made the initial DGZ coordinates the optimal DGZ coordinates with a CEDF value of 3465.

**CASE II.** The collocated identical weapons in CASE I were separated. Weapon 1 was moved one minute of longitude west (approximately 70 feet). With this move from the complex's geometric center, the $x_1$ gradient element was no longer 0; ZXM separated the two DGZs and moved weapon 1 towards installation 1 and weapon 2 towards installation 2. ZXM converged to a maximum CEDF value of 9900 and two optimal DGZs with a $\| \nabla F \| = 10^{-9}$.

**CASE III.** The two identical weapons were again collocated. However, the initial DGZ location was neither halfway between the two identical installations nor along the line segment connecting the installations. The gradient elements were all non-zero. The $x_1$ and $x_2$ elements were equal and positive; the $x_3$ and $x_4$ elements were equal and also positive. In this case, ZXM did not separate the two weapons because their respective gradient elements were the same; however, it did move...
the two weapons together towards installation 1. That is, ZXM kept the two weapons collocated with a CEDF value of 5737 and $\| \nabla F \| = 7.7 \times 10^{-5}$.

Next, a variation of this case was examined. The same initial DGZ location was used, but the yield of weapon 2 was reduced from 100-kt to 95-kt. This yield reduction altered the gradient elements. The $x_1$ and $x_2$ elements were no longer equal; similarly, the $x_3$ and $x_4$ gradient elements were no longer equal. The two different weapons separated from the same initial DGZ location. ZXM moved weapon 1 towards installation 2 and weapon 2 towards installation 1. ZXM converged to a maximum CEDF value of 9900 and two optimal DGZs with $\| \nabla F \| = 1.2 \times 10^{-8}$.

CASE IV. The two identical weapons were separated along a line segment that was perpendicular to the line segment connecting the two identical installations. Figure 11 displays this geometry. The two weapons were each equidistant from the two installations. Again, because of the symmetry of the target complex, the identical weapons, and the identical installations, the gradient elements were symmetric. The $x_1$ and $x_2$ gradient elements were 0. That is, each installation's contributions to the $x_1$ and $x_2$ gradient elements negated each other. The $x_3$ and $x_4$ gradient elements were equal in magnitude, but opposite in direction; the $x_3$ element was positive and the $x_4$ element was negative. Because the $x_1$ and $x_2$ gradient elements were equal to 0, this restricted the DGZ movements to only the $x_3$ and $x_4$ directions. With the $x_3$ and $x_4$ gradient elements with equal magnitude but opposite direction, ZXM moved the DGZs directly towards each other to point P in Figure 11. This point was along the line segment connecting the two installations and halfway between the installations. ZXM converged to a local maximum CEDF value of 5729 and one collocated DGZ with $\| \nabla F \| = 0.00028$. 

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CASE V. A variation of CASE IV was examined. The initial DGZ location of weapon 1 was moved one minute of longitude east (approximately 70 feet). The $x_1$ gradient element was no longer 0; ZXM moved weapon 1 towards installation 2 and weapon 2 towards installation 1. ZXM converged to a maximum CEDF value of 9895 and two optimal DGZs with $\| \nabla F \| = 0.00085$.

The algorithm located CEDF local maximums for this two weapon-two installation complex. However, the algorithm's optimal DGZ coordinates were not exactly the coordinates of the two installations. For this complex, the optimal DGZ locations would be one weapon on each installation, since the weapons and the installations were identical. Nevertheless, these examples demonstrate two important features, an operational characteristic and a limitation, of the CEP-Excluded CEDF model using ZXM maximization. Both of these are a result of the symmetry of the CEDF gradient elements. These cases indicated that there are two types of gradient symmetry. There is symmetry from weapons at symmetrical, initial DGZ locations (CASES I and IV) and from identical weapons at the same initial DGZ location (CASE III). The first type of symmetry is an operational characteristic; the second type, collocation of initial DGZ locations, is a minor limitation. This limitation means that the ZXM maximization algorithm cannot use collocated initial DGZ locations. The PMW maximization algorithm did not have this limitation.

A unique one weapon-four installation target complex also exhibited symmetric gradient properties. Figure 12 presents the complex geometry, weapon and installation parameters, and CEDF values for different potential DGZ locations.
for each installation:  
value = 5000  
VNIK = 15P2

for each weapon:  
yield = 200 kt  
HOB = 1000 feet  
CEP = 0 feet  
Pa = 0.99

CASE VI.

CASE VII.

Initial DGZ

Figure 12. A symmetric four-installation complex.
The geometric shape of the complex was not a square, but rather, a rectangle. The north-south distance between the installations was 8511 feet; the east-west distance was 8448 feet. There were four local maximums or potential DGZ locations for this target complex; one at each midpoint of the four line segments of the rectangular perimeter. The CEDF value at each local maximum was 7776. The CEDF value for a DGZ located at the center of the target complex was 3390. The CEDF value for a DGZ located at one of the installations was 4990. Lastly, for a DGZ located halfway between an installation and the complex center along one of the complex's two diagonals at one of the points L in Figure 12, the CEDF value was 5660.

CASE VI. The initial DGZ was the geometric center of the target complex. Figure 12 also presents this location and the optimal DGZ location for PWM. At the initial DGZ location, the damage expectancy (DE) for each installation was less than 0.17. PWM moved the DGZ in the +x direction and converged to a maximum CEDF value of 7776 and an optimal DGZ location between installations 3 and 4. The DE for these installations from this optimal DGZ was approximately 0.78; the DE for installations 1 and 2 was approximately 0.001. ZXM did not move the DGZ; the two gradient elements were 0 because the initial DGZ was at the geometric center of the complex.

Next, a variation of this example was examined. The initial DGZ was moved one minute of longitude west and one minute of latitude north (approximately 120 feet). PWM moved the DGZ in the +x direction as before. However, the gradient elements were no longer 0 because the initial DGZ was not at the geometric center of the complex. ZXM
converged to a CEDF maximum value of 7776 and an optimal DGZ location between installations 1 and 3 with \( \| \nabla F \| = 0.0008 \). A possible explanation as to why ZXM moved to this optimal DGZ is presented in the next case.

**CASE VII.** The initial DGZ location was installation 1. Figure 12 also presents this location and the optimal DGZ locations. PWM again moved the DGZ in the \(+x\) direction to a CEDF maximum value of 7776 and an optimal DGZ location between installations 1 and 3. ZXM did not move the DGZ towards the closest local maximum as it did in CASE VI. Instead, ZXM moved the DGZ to a CEDF maximum value of 7643 at an optimal DGZ location between installations 3 and 4 with \( \| \nabla F \| = 0.0017 \).

Investigation of the first 20 iterations of ZXM for this complex provided a plausible explanation as to why ZXM converged to this optimal DGZ instead of the closest DGZ. The geometry of the complex was not a square, but rather, a rectangle. Hence at the initial DGZ, the two gradient elements were not exactly equal. The \( x_1 \) gradient element was 0.0352. This was larger than the \( x_2 \) gradient element which was 0.0310. Hence, the first iteration's search direction was above the diagonal of the complex along the line \( U-U' \) in Figure 12. Figure 13 shows an approximate curve of CEDF values, using seven known points, along \( U-U' \). ZXM located the point \( U^* \). ZXM next searched along the line segment \( V-V' \) through the point \( U^* \) and the two potential local maximums \( V_1^* \) and \( V_2^* \).

Again, Figure 13 shows an approximate curve of CEDF values, using five known points, along \( V-V' \). From \( U^* \), ZXM located the optimal DGZ \( V_2^* \).

In summary, ZXM does not always move the initial DGZ towards the closest local maximum.
These seven cases demonstrated two important ZXM maximization characteristics. These characteristics depend on symmetry of the CEDF gradient elements. This symmetry is a result of either the geometrical symmetry of the target complex or the collocation of two or more similar weapon types. This second characteristic, a minor limitation, prohibits the CEDF maximization algorithm from using collocated initial DGZs. The next chapter provides a description of initial DGZ locations for more typical, nonsymmetric target complexes.
VI. Algorithm Results for Different Initial DGZ Locations

This chapter presents results of the CEDF maximization algorithm using different initial DGZ conditions. The important results are the maximum CEDF value and the optimal DGZ coordinates. The three algorithm maximization techniques are: ZXM, conjugate gradient optimization of the CEP-Excluded model; FWM, conjugate directions optimization of the CEP-Included model; and MDM, a mixed technique.

Four initial DGZ conditions were evaluated using three target complexes. However, all four conditions were not matched with each of the complexes. The four initial DGZ conditions for m weapons against a target complex were: (1) locating the weapons at the m highest valued (HV) installations, (2) locating the weapons at the m hardest installations, (3) locating the weapons at the complex's centroid, and (4) locating the weapons at m pseudo-random points.

Intuitively, the most logical initial DGZ condition was the highest valued installations, and the least logical condition was random locations. The HV condition was a greedy condition; it started with the maximum damage on the m most valuable installations and then searched for other DGZs that provided an increase in the CEDF value. The random locations condition was not completely evaluated. Instead, for a two-weapon complex, six pairs of initial DGZs were evaluated, and the six CEDF values and optimal DGZs were compared to each other. The coordinates of one of the initial DGZs in each pair were fixed and common to all pairs. The coordinates of the other initial DGZ were changed for each of the six pairs and the respective six runs of the CEDF maximization
algorithm. The results of these pseudo-random initial DGZ locations provided additional insight concerning different initial DGZ locations.

The three target complexes included three, four, and seven installations. The CEDF maximization algorithm located optimal DGZs for one, two, or three weapons against these complexes. However, each complex was not matched with each of these number of weapons. That is, the three-target complex was only evaluated using one and two weapons, not three. The highest valued installation in any of the complexes was 9000. Hence, for the convergence control parameters, the algorithm generally used values of ACC = 0.001, DFPRED = 1000, E = 0.1, and ESCALE = 5000.

Three conclusions were made from the results of these examples. First, the algorithm requires some indication of a potential increase in CEDF value in order to move a DGZ. Second, there is a difference between the optimal DGZ coordinates from the CEP-Excluded model using ZXM maximization and those from the CEP-Included CEDF model using PNM maximization. This difference depends on a weapon's CEP and the CEDF model and not on the optimization technique. Third, the initial DGZ coordinates that the algorithm uses can affect the maximum CFDF value and the optimal DGZ coordinates. Statistical evidence of these conclusions is not presented. Rather, specific examples are presented that indicate the conclusions are not invalid.

A Three-Installation Complex

CEDF maximization algorithm results were analyzed for one and two weapons against a three-installation complex using different initial DGZ conditions. Figure 14 shows the geometry and specific parameters of the
target complex. The total available target value for the complex, reduced by each weapon's Pa, was 10890. Figure 14 also shows the optimal DGZ coordinates for the highest valued (HV) initial DGZ condition using ZXM and PWM maximization. DGZ 2 was installation 3. The ZXM algorithm converged to a maximum CEDF value of 9812 or 90% of the complex value and to optimal DGZ coordinates with $\|\nabla F\| = 0.00036$. Similarly, PWM converged to a maximum CEDF value of 9223 or 85% of the complex value. The damage expectancy (DE) for installations 1, 2, and 3 from ZXM maximization were 0.96, 0.53, and 0.99. The algorithm did not move DGZ 2 from installation 3 and moved DGZ 1 from installation 1 towards installation 2. However, the two algorithms located the optimal DGZ 1 coordinates 480 feet apart. This difference was less than the CEP of 600 feet and initially appeared
insignificant. However, the differences between ZXM and PWM maximum CEDF values and DGZ 1 coordinates were important; these differences do not indicate ZXM is a better algorithm. These differences depended on the CEDF model and are discussed in more detail in the next subsection.

The CEDF maximization algorithm also converged to a local CEDF maximum for the centroid initial DGZ condition. ZXM converged to a CEDF value of 6932 and to optimal DGZ coordinates with $\| \nabla F \| = 0.00003$; PWM converged to a CEDF value of 6910. However, these optimal DGZ locations were not the same locations as determined using the HV initial DGZ condition. Instead, ZXM and PWM moved the DGZs towards installations 1 and 2 until the DE for each installation were greater than 0.99. The final DE for installation 3, the second most valuable installation, was less than 0.001. The total available target value for installations 1 and 2, reduced by each weapon's $P_a$, was 6930; this was the same CEDF value as determined by ZXM maximization. These optimal coordinates, which were different and less valuable than the HV initial DGZ condition's optimal coordinates, were also identified by three pairs of the pseudo-random initial DGZ condition.

Six pairs of the pseudo-random initial DGZ condition were also evaluated. Figure 15 shows the initial DGZ locations and the respective optimal DGZ locations for ZXM maximization for two cases. It also shows the approximate weapons radius (WR) for each installation. For all six pairs, PWM maximization, using the CEP-Included CEDF model, converged to the same optimal DGZ locations as determined by the highest valued initial DGZ condition. The pictorial results in Figure 15 are for ZXM maximization using the CEP-Excluded CEDF model.
Case I. For the initial DGZ locations -- 1-1', 2-2', and 6-6' -- ZXM converged to a maximum CEDF value of 981.0 or 90% of the complex value and to optimal DGZ coordinates with $\|\nabla F\| < 0.004$. This was the same CEDF local maximum that the HV initial DGZ condition located.

Case II. For the initial DGZ locations -- 3-3', 4-4', and 5-5' -- ZXM converged to a maximum CEDF value of 6932 or 64% of the complex value and to optimal DGZ coordinates with $\|\nabla F\| < 0.0004$. This was the same CEDF local maximum that the centric initial DGZ condition located. Each of the optimal DGZ locations are slightly different locations; however, each of the locations are equivalent. ZXM moved from 3-3', 4-4', and 5-5' towards installations 1 and 2 until the DE for these installations was greater than 0.99.

A possible explanation exists for the difference between the two CEDF local maximums for the two cases. The Case I initial DGZ locations
each had one initial DGZ (1', 2', and 6') within the WR of one of the two highest valued installations. The other initial DGZ (1, 2, and 6) was outside the WR of all installations. Alternately, the Case II initial DGZ locations had neither of the initial DGZs within the WR of the two highest valued installations. Hence, for Case II, the algorithm moved one initial DGZ location (3', 4', and 5') towards installation 2 and the other initial DGZ location (3, 4, and 5) towards installation 1, the most valuable one. Figure 4 shows that the probability of achieving a specified level of damage to an installation at the WR is less than 0.5. Using WR to interpret CEDF local maximums is not a definitive technique. However, the relationship between the location of an initial DGZ and an installation's WR does provide insight and a possible explanation for the two CEDF local maximums.

In summary, these results point out the first of three conclusions of this study. The CEDF maximization algorithm requires some indication of a potential increase in the CEDF in order to move a DGZ. That is, if there is no indication of a CEDF increase in the direction of a valued installation and there is an indication of a CEDF increase in the direction of a lesser valued installation, then the algorithm may move the DGZ towards the lesser valued installation. Eventually, the algorithm will converge to a less valuable CEDF local maximum.

Using the CEDF maximization algorithm to evaluate one weapon against this three-installation complex produced similar results for three initial DGZ conditions. For the highest valued condition, ZXM started from installation 1 and converged to a CEDF maximum value and an optimal DGZ with \( \| \nabla F \| = 0.00030 \). The coordinates of this DGZ coincided with the
coordinates of one of the two-weapon HV condition DGZs. Similarly, PWM converged to the same optimal DGZ location as one of the two-weapon HV condition DGZs. This result indicated that the two weapons against the three installations were not dependent but rather, unrelated DGZs. The fact that DGZ 2 never moved from installation 3 for the two-weapon example also indicated that the two DGZs were independent.

A comparison between the one-weapon centroid initial DGZ condition results and the two-weapon HV initial DGZ condition results indicated the sensitivity of the gradient. Both of these examples located an optimal DGZ at 10°09' E - 03°27' N. However, when the optimal coordinates in feet were compared, the two DGZs were approximately 30 feet apart. For one DGZ, the \( \| \nabla F \| = 0.00031 \); for the other DGZ, only 30 feet away, \( \| \nabla F \| = 0.00760 \).

For the hardest initial DGZ condition, neither ZXM nor PWM moved the one-weapon initial DGZ. The DGZ started at installation 3 and remained there. The \( \| \nabla F \| = 10^{-5} \) at this point. The CEDF value for ZXM was 3962 or 99% of the value of installation 3; the CEDF value for PWM was 3918 or 98% of the value of installation 3. The difference in these CEDF values depended on the CEDF model and are discussed next.

The CEP Effect

CEDF maximization algorithm results for the three-installation complex using different initial DGZ conditions point out the second conclusion of this study. Both a weapon's CEP and the CEDF model used to maximize the CEDF affect the maximum CEDF value and the optimal DGZ coordinates. Specific results from three previous three installation examples provide evidence to support this conclusion.
First, the results from the one weapon-three installation complex using the hardest installation initial DGZ condition highlight this difference between the two CEDF models. The CEP-Excluded CEDF model, using ZXM maximization, converged to a CEDF value of 3962; the CEP-Included CEDF model, using PWM maximization, converged to a CEDF value of 3918. The optimal DGZ coordinates for these algorithms were within 1 foot of each other. The difference in CEDF values was attributed to the CEDF models. The Pd_{i,j} for the CEP-Excluded model does not include weapon accuracy or CEP. This probability is the distance damage function value, P_{d}(r). The Pd_{i,j} for the CEP-Included model does include weapon CEP. Hence, this probability is less than P_{d}(r). Consequently, the PWM damage expectancy for an installation is less than the ZXM damage expectancy for the same installation.

The second example that supports the conclusion was the two weapon-three installation complex using the highest valued initial DGZ condition. Analysis of this example's results provided an explanation for the CEDF differences between ZXM and PWM. Figure 14 shows the optimal DGZ coordinates for these algorithms. Only DGZ 1 coordinates are considered; DGZ 2 coordinates were the same for both algorithms. PWM converged to optimal DGZ 1 coordinates approximately 480 feet closer to installation 1 than ZXM. Two additional initial DGZ conditions were necessary to further investigate this difference. For the initial DGZ coordinates, the first condition used the PWM optimal coordinates; the second condition used the ZXM optimal coordinates. Table IV presents the final coordinates and CEDF values from these initial DGZ conditions. The coordinates are in feet and, even though they appear different within the three optimization categories, they are not. The ZXM coordinates were
TABLE IV
Comparison of ZXM, PWM, and MXM Optimal DGZs

<table>
<thead>
<tr>
<th>Initial DGZ Condition</th>
<th>ZXM CEDF Values</th>
<th>PWM CEDF Values</th>
<th>MXM CEDF Values</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Start</td>
<td>End</td>
<td>Final Location</td>
</tr>
<tr>
<td>Highest Value</td>
<td>8985</td>
<td>9812</td>
<td>858,1039</td>
</tr>
<tr>
<td>PWM Optimal</td>
<td>9480</td>
<td>9813</td>
<td>837,1055</td>
</tr>
<tr>
<td>ZXM Optimal</td>
<td>9812</td>
<td>9812</td>
<td>824,976</td>
</tr>
<tr>
<td>Mean Values</td>
<td>9812</td>
<td>840,1023</td>
<td>9222</td>
</tr>
</tbody>
</table>

10'09" E - 10'27" N and the PWM and MXM coordinates were 10'06" E - 03'23" N.

Using PWM optimal DGZ coordinates as the initial DGZ coordinates, the algorithm produced three results. First, the PWM optimal DGZ location was the initial DGZ location. Second, the PWM optimal DGZ coordinates, the initial coordinates, were not optimal for ZXM. At these initial DGZ coordinates, the ZXM CEDF value was 9480. ZXM maximization moved the DGZ from 10'06" E - 03'23" N back to 10'09" E - 03'27" N and a maximum CEDF value of 9813. ZXM optimal coordinates are initial DGZ coordinates for the mixed technique, MXM. Third, these initial MXM coordinates were not optimal for MXM. At these DGZ coordinates, the MXM CEDF was 9008. MXM maximization moved the DGZ from 10'09" E - 03'27" N back to 10'06" E - 03'23" N, the PWM optimal DGZ coordinates, and a maximum CEDF value of 9218.

The algorithm produced three similar results when it used the ZXM 90°
optimal DGZ coordinates as the initial DGZ coordinates. First, the ZXM maximization optimal DGZ location was the initial DGZ location. Second, the ZXM optimal DGZ coordinates, the initial coordinates, were not optimal for PWM. At these initial DGZ coordinates, the PWM CEDF value was 9073. PWM maximization moved the DGZ from this initial DGZ back to 10'06" E - 03'23" N and a maximum CEDF value of 9223. Third, the initial coordinates, 10'09" E - 03'27" N, again were not optimal for MXM. At these coordinates, the MXM CEDF value was 9009. MXM maximization moved the DGZ from this initial DGZ location back to 10'06" E - 03'23" N and a maximum CEDF value of 9223.

Finally, analysis of a third three-installation complex provided further insight into the capability of the CEDF models. In the previous example, the mixed technique moved the DGZs from the ZXM optimal DGZ coordinates to the PWM optimal DGZ coordinates. However, this readjustment did not occur in all examples. For instance, ZXM maximization for the pseudo-random initial DGZ pairs, 3-3', 4-4', and 5-5', converged to optimal coordinates that were different from the PWM optimal coordinates. The PWM optimal coordinates were the HV coordinates; the ZXM optimal locations were near installations 1 and 2. The mixed technique was unable to move the DGZs from these ZXM optimal coordinates to the PWM optimal coordinates. The $\| \nabla F \| < 0.004$ for each ZXM local maximum.

These CEDF maximization algorithm results indicated that each CEDF model located a unique set of optimal DGZs. This occurred because of the difference in $P_{d_{i,j}}$ for the two models. $P_{d_{i,j}}$ is larger for the CEP-Excluded model than it is for the CEP-Included model.
Figure 16. Multiple local optimal DGZs for a two weapon four installation complex.

**Larger Complexes**

For a four-installation complex using the four initial DGZ conditions, the CEDF maximization algorithm produced results similar to the results for the three-installation complex. Figure 16 shows the geometry and specific parameters of the target complex. The total available target value for the complex, reduced by each weapon’s Pa, was 17820. Figure 16 also shows the optimal DGZ locations for several initial DGZ conditions. Four local CEDF maxima and their respective optimal DGZ pairs were located for this complex: the HV (highest valued) DGZs, 1-1', 2-2', and 3-3'.

Again, ZXM and PWM located their highest valued DGZ 1 at slightly
different coordinates. The difference between the two locations was 90 feet. ZXM converged to a CEDF maximum value of 15434 or 87% of the complex value and to optimal DGZ coordinates with $\| \nabla F \| = 0.00056$. As with the three-installation complex, the ZXM optimal DGZ coordinates were not optimal for PWM. The mixed technique moved DGZ 1 from the ZXM optimal DGZ to the PWM optimal DGZ and a CEDF maximum value of 15142 or 85% of the complex value.

The three remaining local CEDF maxima appear to be related. When the initial DGZ condition was two weapons at the complex's centroid, ZXM converged to a local CEDF maximum value of 10921 and to the optimal DGZ pair 3-3' with $\| \nabla F \| = 0.0007$. Similarly, when the initial DGZ condition was the two hardest installations, PWM converged to a local CEDF maximum value of 10798 and to the optimal DGZ pair 2-2'. ZXM and the same initial DGZ condition produced a third local CEDF maximum.

When the initial DGZ condition was the hardest installations, ZXM converged to a CEDF maximum value of 11431 and to the optimal DGZ pair 1-1' with $\| \nabla F \| = 0.0027$. This optimal DGZ pair had a larger CEDF value than the pairs 2-2' and 3-3'. Yet, it had a smaller CEDF value than the pair of highest valued DGZs.

The local CEDF maximum for PWM, when the initial DGZ condition was two weapons at the complex's centroid, was the same local maximum as determined using the highest valued initial condition. Additionally, this local maximum was located by all eight of the pseudo-random initial DGZ conditions.

Results of the CEDF maximization algorithm using only one weapon against this four-installation complex were examined. ZXM and PWM, using the centroid initial DGZ condition, converged to the highest
valued DGZ 1 in Figure 16. This is one of the two optimal DGZ locations determined by the two weapon evaluation. ZXM converged to a CEDF maximum value of 8547 or 78% of the value of installations 1, 2, and 3 and to an optimal DGZ location with \( \| \nabla F \| = 0.0015 \). PWM achieved a CEDF maximum value of 8252. Next, the algorithms used the highest valued initial DGZ condition for one weapon. Neither algorithm moved the optimal DGZ from the initial DGZ, installation 4. ZXM and PWM terminated with a CEDF maximum value of 6930.

The CEDF maximization algorithm's results for the three and four-installation complexes point out the last conclusion of this study. The most likely to succeed initial DGZ condition is to use the coordinates of the m highest valued installations as the initial DGZ coordinates. For all examples considered, the other three initial DGZ conditions located at least one local CEDF maximum that was less valuable than the local CEDF maximum determined from the highest valued initial DGZ condition. However, there is always an exception. The CEDF results of the simple one weapon-four installation complex indicated the HV initial DGZ is not always the best. For this reason, the CEDF maximization algorithm does not include a decision structure to determine the initial DGZ condition to use. Sometimes, one condition may be more likely than another to succeed and to achieve the most valuable local CEDF maximum.

The results of a three weapon-seven installation complex were analyzed to further define the CEP effects of the two CEDF models. Only three CEDF maximization algorithm runs were made with this complex. For the three runs, all weapon and installation parameters remained constant except each weapon's CEP, and the algorithm used the highest valued initial DGZ condition. Using each weapon's CEP = 0 feet, the algorithm
converged to a CEDF maximum value and optimal DGZ coordinates for the three weapons. Each weapon's CEP equaled 250 feet for the algorithm's second run. For this example, the ZXM optimal DGZ coordinates remained the same, as they should have. The PWM optimal DGZs were along line segments between the highest valued initial DGZs and the optimal DGZs from the first run when each CEP was 0 feet. However, because each weapon's CEP > 0, each of the second run optimal DGZs were slightly closer to their respective highest valued initial DGZ. Similarly, each weapon's CEP was 400 feet for the algorithm's final run. Again, the PWM optimal DGZs were along the same line segments as the optimal DGZs of examples one and two. These optimal DGZs were even closer to their respective highest valued initial DGZ. Thus, the effects of each weapon's CEP need to be included in locating optimal DGZ coordinates in a target complex.
The primary objective of this study was to investigate optimal DGZ locations within a target complex. In order to accomplish this, it was necessary to develop the Complex Expected Damage Function (CEDF) maximization algorithm. The algorithm locates optimal DGZ coordinates for multiple nuclear weapons against installations in a target complex. It does this by maximizing the expected target value damage for all installations. The two subobjectives of this study were to determine the sensitivity of the algorithm's results, the maximum CEDF value and the optimal DGZ coordinates, to two factors: first, the mathematical technique used to locate the optimal DGZs; second, the initial DGZ locations prior to CEDF maximization. This chapter discusses these objectives and their associated conclusions.

The CEDF maximization algorithm contains two related algorithms, and both of these include two elements. The first element is the CEDF, a mathematical model of the total complex expected target value damage. The CEDF is an unconstrained, nonlinear function of $2m$ variables -- the $(X_i, Y_i)$ DGZ coordinates for each of the $m$ weapons. There is a CEDF model for each of the related algorithms. The basic element of each model is $P_{d_{ij}}$ -- the probability of achieving a specified level of damage to installation $j$ from weapon $i$. This study used two forms of the $P_{d_{ij}}$ function; both forms depend on the DIA Physical Vulnerability system.

The major difference between the two CEDF models is their respective $P_{d_{ij}}$ expressions. The CEP-Excluded CEDF model assumes each weapon's CEP is 0 feet. This simplifying assumption results in two conditions: a less complicated mathematical expression for the CEDF and a closed-
form analytical expression for the gradient of the CEP-Excluded CEDF. The CEP-Included CEDF model includes each weapon's CEP; it is a more complicated expression.

The second element of the algorithm is a nonlinear optimization technique that maximizes the CEDF models and locates the corresponding optimal DGZs. Since an analytical expression for the gradient of the CEP-Included CEDF model was not available, the algorithm maximizes this CEDF using a direct search technique -- Powell's method of conjugate directions, PWM. This numerical search technique requires only function evaluations to locate a local maximum. Conversely, an analytical expression for the gradient of the CEP-Excluded model was calculated. The algorithm maximizes this CEDF using a gradient search technique -- a conjugate gradient with restarts method, ZXM. The algorithm also contains a third CEDF maximization technique -- a mixed technique, MiM. This technique consists of two stages, an initial ZXM stage and a PWM stage. The optimal DGZ coordinates from the ZXM stage become the initial DGZ coordinates for the PWM stage.

The CEDF maximization algorithm was verified and validated using two, three, and five-installation target complexes. The CEDF maximum value and optimal DGZ locations for two example problems were also compared with results from NUCWAVE. NUCWAVE is a one-sided nuclear weapons allocation war gaming model. It also optimizes the damage to a set of installations using a finite number of weapons. However, NUCWAVE determines a sequential optimal solution; it optimizes one weapon at a time until no increase in complex damage is possible. The comparisons between the results from the CEDF maximization algorithm and from NUCWAVE for two and five-installation target complexes indicated that the
algorithm determines the same local maximum as NUCWAVE.

A symmetric one weapon-four installation complex was designed to have four local maximums. CEDF results from this complex were analyzed and pointed out two ZXM maximization features, an operational characteristic and a limitation. These features depend on the two types of gradient symmetry. There is gradient symmetry from either geometrical symmetry of the target complex or collocation of two or more similar weapon types. The second type of symmetry is a limitation and prohibits ZXM maximization from using collocated initial DGZs.

Further analysis of three and four-installation target complexes indicated the presence of multiple local CEDF maximums. A two weapon-three installation target complex was analyzed using CEDF algorithm results. There were two distinct local CEDF maximums and two corresponding pairs of optimal DGZs. Similarly, a two weapon-four installation complex was analyzed. There were three distinct local CEDF maximums and three corresponding pairs of optimal DGZs. The CEDF maximization algorithm located these local maximums using different initial DGZ conditions. For both complexes analyzed, one local maximum was definitely the highest valued local maximum for the complex.

Conclusions

The first subobjective was to determine the sensitivity of the results of the CEDF maximization algorithm to the mathematical technique used to locate the optimal DGZs. Two conclusions of the study emphasize the differences in CEDF results for the two CEDF models and their respective optimization techniques -- ZXM and PWM.

First, the algorithm requires some indication of a potential
increase in CEDF value in order to move a DGZ. That is, if there is no indication of a CEDF increase in the direction of a valued installation and there is an indication of a CEDF increase in the direction of a lesser valued installation, then the algorithm will move the DGZ towards the lesser valued installation. Eventually, the algorithm may converge to a less valuable CEDF local maximum.

The second conclusion is that a weapon's CEP and the CEDF model affect the CEDF maximum value and the respective optimal DGZ coordinates. All three, four, five, and seven-installation target complexes analyzed, that used weapons with $\text{CEP} > 0$, confirmed this conclusion. ZXM optimal DGZ coordinates were not optimal for PWM; similarly, PWM optimal DGZ coordinates were not optimal for ZXM. Results indicated that each CEDF model located a unique set of optimal DGZs; however, the distance difference between a ZXM and a PWM optimal DGZ was less than the weapon's CEP. This difference occurred because of the difference in $P_{d_i,j}$ for the two models. For a weapon $i$-installation $j$ interaction, $P_{d_i,j}$ is larger for the CEP-Excluded model than it is for the CEP-Included model. This is because the CEP-Included $P_{d_i,j}$ is reduced by a factor that depends on the weapon's CEP.

The second subobjective was to determine the sensitivity of the results of the CEDF maximization algorithm to the initial DGZ locations prior to optimization. Four initial DGZ conditions were evaluated using three and four-installation target complexes. The four initial conditions for an $m$-weapon complex were: (1) locating the weapons at the $m$ highest valued installations, (2) locating the weapons at the $m$ hardest installations, (3) locating the weapons at the complex's centroid, and (4) locating the weapons at $m$ pseudo-random points. The algorithm
using these initial DGZ conditions located more than one local CEDF maximum for three and four-installation complexes. Thus, the last conclusion of the study emphasizes that no single initial DGZ condition always locates the most valuable local CEDF maximum. Hence, the algorithm does not include a decision structure to determine the correct initial DGZ condition. However, this conclusion also indicates that the most likely to succeed initial DGZ condition is to use the coordinates of the m highest valued installations as the initial DGZ coordinates.

This investigation characterized three factors that affect the optimal DGZ locations for multiple nuclear weapons in a target complex. The first factor was gradient symmetry; this symmetry resulted from either a geographically symmetric target complex or collocated weapons. The second factor was weapon CEP. Maximization of the two CEDF models produced slightly different optimal DGZs; this difference depended on a weapon's CEP and the CEDF model. The third factor was the initial DGZ location prior to CEDF maximization. The algorithm located different CEDF local maximums depending on the initial DGZ condition.

Recommendations

The weapons analyst can use the algorithm to solve large targeting problems that include many complexes and different types and numbers of weapons. The algorithm can be a valuable sensitivity analysis tool to investigate weapon allocation tradeoffs. The analyst can evaluate the changes in total complex expected target value damage as a result of an increase or decrease in the number of weapons available to a target complex. Similarly, the analyst can estimate the effects that changes in a weapon's yield, CEP, or Pa can cause to the optimal DGZs.
The CEDF maximization algorithm does have some limitations. However, there is a specific improvement or recommendation associated with each limitation. The following recommendations would provide a more capable algorithm for strategic targeting studies:

1. Currently, the algorithm accomplishes only two-dimensional location of optimal DGZs; the user provides each weapon's height of burst. Optimization of each weapon's height of burst could be added to the algorithm.

2. Currently, the algorithm only allows military/industrial targets that are modeled as point targets. The algorithm could be modified to include area targets, equivalent area targets, and target avoidance areas.

3. In an analogous manner, the algorithm only includes blast damage effects for these point targets. Other nuclear weapon damage effects could be added to the model.

4. Similarly, other optimization techniques could be used to further investigate and characterize the CEDF local maximums for a target complex.

5. User-specified constraints that establish a minimum acceptable Pd for some or all installations could be included. This addition would provide a new initial DGZ condition. That is, locate the weapons at the installations with the largest minimum Pd.
Appendix A: Determination of the Distance Damage Sigma ($\sigma_j$) and the Weapon Radius (WR)

The parameters $\sigma_j$ and WR are necessary to calculate the probability of achieving a specified level of damage to installation $j$ from weapon $i$.

1. Distance damage sigma, $\sigma_j$. The value of $\sigma_j$ depends on the T factor of an installation's VNIK code. Table A-1 lists the T factors and their associated $\sigma_j$ values. This table was extracted from NUCWAVE Model Methodology Analysis (Ref 25:3-4).

2. Weapon Radius, $WR = f$(weapon and installation parameters). The calculation of a WR depends on the concept of yield scaling. The following information on yield scaling is based upon Glasstone and Dolan's presentation in The Effects of Nuclear Weapons (Ref 11:100).

"In order to calculate the characteristic properties of the blast wave from an explosion of any given energy if those of another energy are known, appropriate scaling laws are applied" (Ref 11:100). A 1-kiloton nuclear explosion is the reference explosion for nuclear weapon calculations. Pressure vs range data have been tabulated and graphed for the 1-kt reference explosion. Also, the distance scaling laws use the cube root of the weapon's yield as the scaling factor.

That is, if a pressure is experienced at a ground distance $d_1$ from a 1-kt reference explosion, then this same pressure value will be experienced at a distance $d_w$ from a $w$-kiloton explosion.

$$d_w = d_1(w)^{1/3}$$

The needed pressure vs range ($d_w$) data for a $w$-kiloton explosion can
be determined using the scaled distance $d_1$. Therefore, to determine either the overpressure or dynamic pressure from an explosion of $w$-kt, all distances need to be transformed to the 1-kt scaled reference frame. The amount of pressure an installation experiences is the primary determinant of the installation's probability of damage.

Weapon and installation parameters are needed to determine the WR. The weapon parameters needed are the yield ($Y$) and the height of burst (HOB). The HOB and the subsequently calculated scaled weapon radius (SWR) are the two distances that need to be yield scaled. HOB is scaled to start the formulation; after the SWR is calculated, it is inversely scaled to specify the WR. The only installation parameter necessary to calculate the WR is the VNIK code. The following presentation is based upon the material in *Mathematical Background and Programming Aids for the Physical Vulnerability System for Nuclear Weapons* (Ref 6:57-61).

$$WR = SWR \frac{c}{1 - \sigma_d^2} (Y)^{1/3}$$  \hspace{1cm} (A.1)
The parameter c is a constant that equals either 0.96 for overpressure sensitive installations or 0.91 for dynamic pressure sensitive installations. The SWR is calculated from the scaled height of burst (SHOB) and the adjusted VN number (VN_{adj}).

\[ \text{SHOB} = \left( \frac{\text{OB}}{\text{Y}} \right)^{1/3} \]

\[ \text{VN}_{\text{adj}} = \text{VN} + \Delta \text{VN} \]

\[ \Delta \text{VN} = 5.485 \ln(R) \text{ for over-pressure sensitive installations} \]
\[ = 2.742 \ln(R) \text{ for dynamic pressure sensitive installations} \]

An iterative procedure is used to calculate the VN adjustment factor R

\[ R = 1 - \frac{K}{10} + \frac{20}{Y} \left( \frac{R}{10} \right)^{1/3} \]

K is the installation's K factor and the exponent e equals either 1/2 for overpressure sensitive installations or 1/3 for dynamic pressure sensitive installations.

\[ \text{SWR} = \exp f(\text{VN}_{\text{adj}}, \text{SHOB}) \]

The function, \( f(\text{VN}_{\text{adj}}, \text{SHOB}) \), is a polynomial expression whose coefficients are available at 100-foot increments of the SHOB between 0 and 900 feet. Hence, the SWR is derived by linearly interpolating between a low SWR, that is calculated from a low SHOB, and a high SWR, that is calculated from a high SHOB. For example, for a SHOB of 632 feet, a SWR low is calculated for a SHOB of 600 feet and a SWR high is calculated for a SHOB of 700 feet. The actual SWR is a linear interpolation of the high and
low SWRs. Other algorithms use a table look-up with parameters, SHOB and VN_{adj} to specify the SWR (Ref 19 and 25). The SWR is inversely yield scaled using Eq (A.1) to determine the WR.
Appendix B: Formulation of $f(r)$ and Calculation of the Integration Limits

This information is based on material in Mathematical Background and Programming Aids for the Physical Vulnerability System for Nuclear Weapons (Ref 6:69-75).

The probability of achieving a specified level of damage to installation j from weapon i depends on weapon and installation parameters.

$$P_{d_{i,j}} = \int_{0}^{2\pi} \int_{0}^{\infty} P_{d}(r) \frac{1}{2\pi\sigma^2} e^{-\frac{r^2}{2\sigma^2}} r \, dr \, d\theta$$  \hspace{1cm} (B.1)

where $P_{d}(r) = \text{distance damage function}$

$$\sigma = \text{CEP}/1.1774$$

$$p(r,\theta) = (r^2 + s^2 - 2rs \cos\theta)^{1/2}$$

A closed form solution to Eq(B.1) does not exist; however, an analytical approximation does.

$$P_{d_{i,j}} = \int_{0}^{\infty} f(r) \, dr$$  \hspace{1cm} (B.2)

where $f(r) = P_{d}(r)r \frac{1}{\pi\sigma^2} \int_{0}^{2\pi} e^{-\frac{(r^2 + s^2 - 2rs \cos\theta)}{2\sigma^2}} \, d\theta$  \hspace{1cm} (B.3)

Eq(B.3) can be rewritten using a zeroth order modified Bessel Function, $I_0(x)$ (Ref 19:378).

$$I_0\left(\frac{rs}{\sigma^2}\right) = \frac{1}{\pi} \int_{0}^{2\pi} e^{-\frac{rs \cos\theta}{\sigma^2}} \, d\theta$$
\[ f(r) = P_d(r) \frac{r}{\sigma^2} e^{-\frac{(r^2 + s^2)}{2\sigma^2}} I_0 \left( \frac{rs}{\sigma^2} \right) \]  

(B.4)

\( P_{d_{i,j}} \) is calculated using normalized distance variables and Eq (B.1). That is, \( r, s, \) and \( WR \) are divided by \( \sigma \), the standard deviation of the circular normal distribution.

Let \( r' = \frac{r}{\sigma} \) = the normalized distance between the installation and the impact point

\( dr' = \frac{1}{\sigma} \) = the normalized differential element of \( r \)

\( s' = \frac{s}{\sigma} \) = the normalized distance between the installation and the DGZ

\( WR' = \frac{WR}{\sigma} \) = the normalized weapon radius

Then Eqs (B.2) and (B.4) become

\[ P_{d_{i,j}} = \int_0^{\infty} f(r') dr' \]  

(B.5)

\[ -\frac{(r')^2 + (s')^2}{2} \]

where \( f(r') = P_d(r') r' e^{-\frac{(r')^2 + (s')^2}{2}} I_0 (r's') \)  

(B.6)

A polynomial approximation of the zeroth order modified Bessel function, \( I_0 (r's') \), specifies a value of \( f(r') \) for a given \( r' \) (Ref 1:378).

Eq (B.5) can be rewritten as a definite integral with limits of integration, \( a \) and \( b \), such that \( f(r') = 0 \) for \( r' < a \) or \( r' > b \).

\[ P_{d_{i,j}} = \int_a^b f(r') dr' \]  

(B.7)

Therefore, it is necessary to determine \( a \), the smallest possible value
of \( r' \), and \( b \), the largest possible value of \( r' \). These limits depend on weapon accuracy limitations and distance damage limitations. The weapon-installation geometry necessary to determine the limits of integration, \( a \) and \( b \), is shown in Figure B-1. Two cases are examined.

Case I. The normalized distance between the installation and the DGZ is less than 4 (0 \( \leq s' \leq 4 \)). The distance between the closest possible impact point and the installation is \( r' = 0 \). The distance between the farthest possible impact point and the installation is the minimum of either \( r' = s' + 4 \) or \( r' = R' = 1.06 \times W R' \exp (2.86 \times \sigma_d) \). The point at \( r' = s' + 4 \) corresponds to
the maximum distance from the DGZ, \( \varphi' \), that a weapon could be expected to impact. The probability that a weapon would impact farther than \( \varphi = 4\sigma \) is less than 0.00005. Similarly, the point at \( r' = R' \) corresponds to the maximum distance from the installation, \( r' \), that the weapon could detonate and expect to damage the installation. The \( P_d(r) \) for \( r' > R' \) is less than 0.0005. \( R' \) may be either greater than, equal, or less than \( s' + 4 \).

Case II. The normalized distance between the installation and the DGZ is greater than 4 \( (s' > 4) \). The distance between the closest possible impact point and the installation is \( r' = s' - 4 \). The distance between the farthest possible impact point and the installation is again the minimum of either \( r' = s' + 4 \) or \( r' = R' \).

Therefore, \( a = \max (0, s' - 4) \) and \( b = \min (s' + 4, R') \) (Ref 6:73). Eq (B.7) can be evaluated using Gauss-Legendre Quadrature between the limits, \( a \) and \( b \),

$$P_{d_{i,j}} = \frac{(b - a)}{2} \frac{10}{\sum_{k=1}^{10} w_k f(r'_k)}$$

(B.8)

where \( r'_k = \frac{(b - a)}{2} z_k + \frac{(a + b)}{2} \)

and \( f(r'_k) \) is Eq (B.6). Gauss-Legendre, the quadrature points, \( z_k \), and the coefficients, \( w_k \), are explained in Appendix C.

Eq (B.8) is evaluated to determine the probability of achieving a specified level of damage to installation \( j \) from weapon \( i \).
Appendix C: Gauss-Legendre

Quadrature to Integrate \( f(r') \)

Gauss-Legendre quadrature is a numerical integration technique that approximates a definite integral as a finite series (Ref 27:125). Each term is a weighted function value.

The series to approximate a definite integral along the interval \([-1,1]\) by Gauss-Legendre is

\[
\int_{-1}^{1} f(y) dy = \sum_{k=1}^{n} w_k f(y_k) + R_n \quad (C.1)
\]

where
- \( w_k \) = quadrature coefficients
- \( y_k \) = quadrature base points
- \( R_n \) = remainder (negligible)
- \( n \) = number of quadrature points

Gauss-Legendre integration differs from symmetric, trapezoidal numerical integration. In Gauss-Legendre, the distances between the \( y_k \) base points along the abscissa are not equal. The points are spaced symmetrically, yet unequally, with respect to the midpoint of the interval \([-1,1]\), the origin. See Figure C-1. This method is more efficient than equal spacing trapezoidal methods because it requires fewer function evaluations to achieve comparable accuracy (Ref 15:378). The quadrature coefficients \( w_k \) are positive numbers between 0 and 1; they are weights for the \( f(y_k) \) values.

The quadrature coefficients and weights are calculated from the nth
Legendre polynomial, $P_n$. The base points, $y_k$, are the $k = 1, ..., 10$ zeros of $P_n(y)$. Similarly, the coefficients, $w_k$, are calculated from $P_n(y)$ and the $y_k$ (Ref 1:888). Table C-1 lists the base points and coefficients for $n = 10$ (Ref 6:74 and 27:131).

However, to calculate $P_{i,j}$ the interval of integration is not $[-1,1]$, but rather, it is $[a,b]$. For this interval of integration, the quadrature base points, coefficients, and limits of integration specify the transformed variables, $w_k$ and $r_k$ (Ref 1:887). Eq (C.1) becomes

\[
\int_a^b f(y) dy = \sum_{k=1}^{10} w_k f(r_k)
\]
TABLE C-1

Quadrature Base Points and Coefficients

<table>
<thead>
<tr>
<th>n</th>
<th>Y_k</th>
<th>WW_k</th>
</tr>
</thead>
<tbody>
<tr>
<td>1, 10</td>
<td>0.9739065285</td>
<td>0.0666713433</td>
</tr>
<tr>
<td>2, 9</td>
<td>0.8650633667</td>
<td>0.1494513442</td>
</tr>
<tr>
<td>3, 8</td>
<td>0.6794095683</td>
<td>0.2190863625</td>
</tr>
<tr>
<td>4, 7</td>
<td>0.4333953941</td>
<td>0.2692667193</td>
</tr>
<tr>
<td>5, 6</td>
<td>0.148874339</td>
<td>0.2955242247</td>
</tr>
</tbody>
</table>

where \( w_k = \frac{(b - a)}{2} \cdot WW_k \)

\[ r_k = \frac{(b - a)}{2} y_k + \frac{(a + b)}{2} \]

This finite series is used to calculate \( P_{d_{i,j}} \). For each \( k = 1, \ldots, 10 \), an \( r_k \) and associated \( f(r_k) \) is calculated.

\[ f(r_k) = \frac{(r')^2 + (s')^2}{2} \]

Where \( f(r_k) \) is weighted by \( w_k \) and summed to form \( P_{d_{i,j}} \).
Appendix D: Computer Code of the CEDF Maximization Algorithm

This appendix presents a glossary of the FORTRAN variables and lists the FORTRAN V source code of the CEDF maximization algorithm. The algorithm has the capacity to handle a target complex with up to ten weapons and ten installations. These capacities can be increased by changing the array dimension variables, MXM, MXN, MTM, MTN, MSQ, and M12. Parameter statements assign values to these variables; the glossary describes these variables.

The algorithm has several options. The user can specify one of three initial DGZ conditions and convergence parameters for ZXGCR. Chapters IV and V discuss user guidelines in selecting a particular option. Also, appendix E provides instructions in how to create an input data file.

The subroutine INITLZ assigns values to the two PWMIN maximization convergence control parameters, E and ESCALE. Chapter V discusses these parameters. Minor code changes would be required to modify either of the parameters.

A typical two weapon-four installation CEDF maximization problem requires approximately 3.0 seconds of execution time. The current program requires 50,000 words of core memory on a Control Data Corporation (CDC) 6600 Cyber computer.

**Glossary of Variables**

A[WRADS] - a coefficient that is used to adjust the VN number.

AA - the difference between weapon i and installation j x coordinates in feet.

ACC - the ZXGOR convergence control parameter.

ACC2 - the second stage ZXGOR convergence control parameter.

B - the upper limit of integration for the Gauss-Legendre quadrature.

BB - the difference between weapon i and installation j y coordinates in feet.

BESB and BESC - polynomial approximations of the modified zeroth order Bessel function that are used to determine $P_{d_i,j}$.

BETA(j) - the beta value for installation j.

CC(j) - an intermediate value that is used to calculate the gradient of the CEDF.

CEP(i) - the circular error probable of weapon i in feet.

CEPA(i) - the adjusted circular error probable of weapon i in feet.

CEPS(i) - intermediate storage of the circular error probable of weapon i in feet.

DFPRED - an estimate of the expected increase in the CEDF.

DGLN(i,3) - the degree-minute-second longitude coordinate for weapon i.

DGLT(i,3) - the degree-minute-second latitude coordinate for weapon i.

DLNC(i) - the east/west direction from the prime meridian for weapon i.

DLTC(i) - the north/south direction from the equator for weapon i.

DNCD(i) - the longitude coordinate in degrees for weapon i.

DTCD(i) - the latitude coordinate in degrees for weapon i.

E(i) - the PWMIN convergence control parameter.

EC - the value of CEDF.

EF - the value of $-CEDF$ from PWMIN.

ERF - a polynomial approximation of the error function that is used to determine a distance damage function value.
ESCALE - the maximum step size multiplier for a single step of each x.

EV(j) - the expected target value damage contribution to EC from installation j.

EX - an exponent that is used to calculate the VN reduction factor.

F - the value of -CEDF from FUNCT and GFUNCT.

F(5,2) - the ten Gauss-Legendre function evaluations.

FACTOR(j) - PS(j) * value(j).

FN - the sum of the ten Gauss-Legendre function evaluations.

FV - the value of -CEDF from ZXCGR.

G(2) - a polynomial expression that is used to determine an SWR.

GR(i) - the 2m gradient elements of the CEP-Excluded CEDF model.

GRAD(i) - same as GR(i).

H - an intermediate value that is used to determine the appropriate Bessel Function approximation, BESB or BESC.

HOB(i) - the height of burst for weapon i in feet.

I - generally, the subscript of a weapon array.

IER - an error code from ZXCGR.

ILNC(j) - the east/west direction from the prime meridian for installation j.

ILTC(j) - the north/south direction from the equator for installation j.

INASG(j) - a logical decision variable that indicates whether an installation's coordinates have been assigned to a DGZ.

INC - the user-specified indicator variable that controls the assignment of initial DGZ locations.

INCD(J) - the longitude coordinate in degrees for installation j.

ISHOB(i,2) - the low and high scaled heights of burst for weapon i in feet.

ISLN(j,3) - the degree-minute-second longitude coordinate for installation j.
ISLT(j,3) - the degree-minute-second latitude coordinate for installation j.

ITCD(j) - the latitude coordinate in degrees for installation j.

ITERS - the number of ZXGR calls of function GFUNCT.

J - generally, the subscript of an installation array.

JJ - the argument of the modified zeroth order Bessel functions, BESB and BESC.

K(j) - the K factors for installation j.

KK - the incremental contribution to grad(i) or grad(i+m) from installation j.

KO thru K7 - the eight coefficients of the polynomial G(2).

LN - the length of one minute of longitude in feet.

M - the number of weapons for the target complex.

MAXFN - the maximum number of function evaluations ZXGR is authorized.

MAXIT - the maximum number of iterations PWMIN is authorized.

MSQ - the dimension variable for a PWMIN work array [=2m * (2m + 3)].

MTM - the dimension variable for all 2m-element weapon arrays.

MTN - the dimension variable for all 2n-element installation arrays.

MKM - the dimension variable for all m-element weapon arrays.

MKN - the dimension variable for all n-element installation arrays.

ML2 - the dimension variable for a ZXGR work array [= 12 * m].

N - the number of installations in the target complex.

NOM - an intermediate value in calculating the gradient of the CEDF.

NS - the standard deviation scaled distance between weapon i and installation j.

NW - the dimension variable for a PWMIN work array.

NWR - the standard deviation scaled weapon radius.
N2 - the dimension of the xy weapon coordinate vector x.

ONCE - a logical decision variable that controls the algorithm so that ZXCGA runs a second time with a smaller ACC.

ORLN - the longitude coordinate in degrees for the origin of the XY coordinate system.

ORLT - the latitude coordinate in degrees for the origin of the XY coordinate system.

Pa(i) - the probability of arrival for weapon i.

PDAM - the probability of achieving a specified level of damage to installation j from weapon i.

PFMG(i,j) - Pd, - the probability of achieving a specified level of damage to installation j from weapon i.

PDR - the distance damage function value.

PP - the distance damage function value used to calculate Pd, for the CEP-Included CEDF model.

PS(j) - the probability of not achieving a specified level of damage to installation j.

R[P DAM] - the standard deviation scaled distance between the impact point and the installation.

R[PDR] - a distance, either S or R, from the subroutine PDAM.

REM - a remainder that is used to translate the final DGZ coordinates from feet into degree-minute-second coordinates.

RR - the flat earth distance between weapon i and installation j.

R1 - an intermediate value that is used to calculate the VN reduction factor.

R2 - the VN reduction factor.

S - the known distance between weapon i and installation j.

SHOB(i) - the scaled height of burst for weapon i in feet.

SIG - the square root of the quantity, (1 - σd^2).

SIGMA(j) - the distance damage sigma for installation j.

SWR - the scaled weapon radius.

T(j) - the T-factor for installation j.
TLAT - the sum of all installation latitude coordinates that is used to determine the number of feet per minute of longitude.

U - the argument of the error function.

UU - an intermediate exponent that is used to calculate the gradient of the CEDF.

V[P DAM] - the dummy argument of BESB and BESC.

V[P DR] - the dummy argument of ERF.

V[WRADS] - the change in an installation's VN number when it is subjected to yields other than 20-kt.

VAL - the current value of the highest valued installation.

VALUE(j) - the value of installation j.

VIND - the subscript of the current highest value/hardest installation.

VN(j) - the integer VN number for installation j.

VNA - the adjusted VN number.

VNI - the current VN number of the hardest installation.

W(5)[PDAM] - the Gauss-Legendre quadrature coefficients.

W[WRADS] - the low and high SWR that are linearly interpolated between to determine the actual SWR.

WR(i,j) - the weapon radius for the weapon i-installation j interaction.

WW(M12) - a ZXCR work array.

WWW(MSQ) - a PWMIN work array.

X(i) - the X coordinate of weapon i in feet.

X(i + m) - the Y coordinate of weapon i in feet.

XX(j) - the X coordinate of installation j in feet.

XX(j + n) - the Y coordinate of installation j in feet.

XXX - same as X(i) and X(i + m).

X4 - the (X1,Y1) coordinates of the m weapons after ZXCR maximization, but before PWMIN maximization. Used with the mixed CEDF maximization algorithm.
YIELD(i) - the yield of weapon i in kilotons.

Z5 - the Gauss-Legendre quadrature base points.

ZZ - a standard normal random variable.

**Source Code**

The next 27 pages list the FORTRAN V code of the CEDF maximization algorithm.
PROGRAM OPTMZ

C DRIVER MODULE FOR THE COMPLEX EXPECTED
C DAMAGE FUNCTION (CEOF) MAXIMIZATION ALGORITHM.  THE
C ALGORITHM DETERMINES THE OPTIMAL PGZ LOCATIONS FOR A
C FINITE NUMBER OF NUCLEAR WEAPONS AGAINST INSTALLATIONS
C IN A TARGET COMPLEX BY MAXIMIZING THE CEOF.

PARAMETER(MXM=10,MXN=10,MTM=20,MTN=20,MSQ=4,MSQ=M12=120)
EXTERNAL GFUNCT
INTEGER MN,VN(MXM),N2,MAXFN,IER,MAXIT,NW
REAL VALUE(MXM),YIELD(MXM),CEP(MXM),HOB(MXM),PA(MXM),SIGMA(MXM)
REAL X(MTM),XXX(MTN),GR(MTM),FW,BETA(MXM),MM(M12),ACC,DFPRED
REAL E(MTM),WM(MSQ),XXX(MTN),X4(MTM),ESCALE,EP,CEPS(MXM),ACC2
LOGICAL ONCE
CHARACTER T(MXM)
COMMON/INSTLN/ N,VALUE,VN,K,XX
COMMON/PQIND/T
COMMON/MPNS/ M,YIELD,CEP,HOB,PA
COMMON/PARMS/ WR,SIGMA,BETA,ITERS
COMMON/CNTRL/ N2,ACC,ACC2,DFPRED,ESCALE,E

WRITE(6,10)
10 FORMAT(1X,5(I1))  
WRITE(6,*)  'CEOF MAXIMIZATION ALGORITHM'
WRITE(6,10)
ONCE=.TRUE.

CALL INITLZ(X)
CALL WRADS

DO 10 I=1,N2
   XXX(I)=X(I)
   CONTINUE

CONJUGATE GRADIENT OPTIMIZATION OF THE CEP-EXCLUDED
CEOF MODEL

WRITE(6,100)
WRITE(6,*)  'ZXCGR MAXIMIZATION'
WRITE(6,100)
ITERS=
MAXFN=100
DO 20 I=1,M
   CEPS(I)=CEP(I)
   CEP(I)=0.
   CONTINUE

20 CONTINUE

WRITE(6,*)  'ACC=',ACC
CALL ZXCGR(GFUNCT,N2,ACC,MAXFN,DFPRED,X,GR,FW,WM,IER)
WRITE(6,*)  'IER=',IER
WRITE(6,*)  'FUNCTION=',FW
WRITE(6,*)  'FUNCTION EVALUATIONS=',ITERS
CONJUGATE GRADIENT OPTIMIZATION OF THE CEP-EXCLUDED CEDF MODEL USING A REDUCED CONVERGENCE CRITERIA

IF(ONCE) THEN
   DO 35 I=1,N2
      X4(I)=X(I)
   35 CONTINUE
   WRITE(6,100) WRITE(6,100) GRAD WITH ACC REDUCED ONCE=FALSE ACC=ACC2 ITERS=1 GO TO 25
ENDIF

POWELL'S CONJUGATE DIRECTIONS OPTIMIZATION OF THE CEP-INCLUDED CEDF MODEL

WRITE(6,100) WRITE(6,100) PWMIN MAXIMIZATION
WRITE(6,100) MAXIT=50 DO 40 I=1,M
   CEP(I)=CEPS(I)
  40 CONTINUE
   NW=N2*(N2+3)
   CALL PWMIN(XXX,E,N2,EF,ESCALE,MAXIT,WWW,NW)
   WRITE(6,100) FUNCTION= "*,-EF DO 50 I=1,N2
   WRITE(6,100) XXX(*,I,*)= *,XXX(I)
  50 CONTINUE
   CALL OUTDGZ(XXX)

MIXED OPTIMIZATION OF THE CEDF MODEL- THE OGZ COORDINATES FROM THE FIRST CONJUGATE GRADIENT OPTIMIZATION BECOME THE INITIAL OGZ COORDINATES FOR POWELL'S CONJUGATE DIRECTIONS OPTIMIZATION.

WRITE(6,100) WRITE(6,100) MIXED TECHNIQUE MAXIMIZATION
WRITE(6,100) CALL PWMIN(X4,E,N2,EF,ESCALE,MAXIT,WWW,NW)
WRITE(6,100) FUNCTION= "*,-EF DO 70 I=1,N2
   WRITE(6,100) X4(*,I,*)= *,X4(I)
  70 CONTINUE
   CALL OUTDGZ(X4)
  93 CONTINUE
END
C******************************************************************************

SUBROUTINE GFUNCT(N2,X,F,GRAD)
C
1. CALCULATES THE CEDF(X), THE COMPLEX EXPECTED
   DAMAGE FUNCTION, FOR M WEAPONS AND N INSTALLATIONS
   USING THE CEP-EXCLUDED CEDF MODEL.
C
2. CALCULATES THE 2M ELEMENTS OF THE GRADIENT OF
   THE CEDF(X).
C******************************************************************************

PARAMETER (MNX=10, MXN=10, MMT=20, MTN=30)
INTEGER M,N,VN(MNX),K(MXN),WR(NXN,MXN),N2
REAL VALUE(MXN),YIELD(HXM),CEP(MXM),HOB(MXM),PA(HXM)
REAL X(MTN),XX(MVN),EC,EV(MXN),PS(MXN),PDAM,SIGMA(MXN)
REAL PDMG(MXNMXN),BETA(MMN),F
REAL FACTOR(MMN),CC(MMN),GRAD(MMN),AA,BR,RR,UU,KK,NN
CHARACTER T(MMN)
COMMON/INSTLN/, N,VALUE,VN,K,XX
COMMON/PMIND/, T
COMMON/WINS/, V,YIELD,CEP,HOB,PA
COMMON/PARAMS/, WR,SIGMA,BETA,ITERS
C
110 FORMAT(25*WEAPON *F12.2 XY COORDINATES: (*.F7.3,*.F7.3))
EC=0.0
ITERS=ITERS+1
DO 60 L=I,M
   WRITE(6,100) L,X(L),X(L+M)
60 CONTINUE
C
CALCULATES THE CEDF(X).
C
DO 20 J=1,N
   PS(J)=1.0
   DO 10 I=1,M
      PDAM(I,J)=PDAM(I,J)+X
      PS(J)=PS(J)*(1.0-PA(I)*PDAM(I,J))
   CONTINUE

10 EV(J)=SIN(J)*VALUE(J)
WRITE(6,*) PS(J),=PS(J),=0,PS(J)
EC=EC+EV(J)
FACTOR(J)=SIN(J)*VALUE(J)
CC(J)=2.556242*BETA(J)
20 CONTINUE
WRITE(6,*) EC,=EC
F=-EC
C
125
CALCULATES THE GRADIENT OF THE CEDF(X).

DO 40 I=1,M
   GRAD(I) = D0
   GRAD(I+M) = D0
   DO 3 J=1,N
      AA = XX(J) - X(I)
      BB = XX(J+N) - X(I+M)
      RR = SQRT(AA**2 + BB**2)
      IF(RR .LT. 0.01) RR = 1.0
      UU = ABS(((1.0/ETA(J)) / LOG(WR(I,J)/RR) - ETA(J))/1.4142135)
      NCN = FACTOR(J) * PA(I) * EXP(-UU**2)
      KK = NOM / ((1.0 - PA(I) * PDMG(I,J)) * CC(J) * RR**2)
      GRAD(I) = GRAD(I) + KK * AA
      GRAD(I+M) = GRAD(I+M) + KK * BB
   CONTINUE
40 CONTINUE
   DO 50 L=1,M/2
      GRAD(L) = -GRAD(L)
   CONTINUE
50 CONTINUE
END

123
**SUBROUTINE FUNCT(N2, X, F)**

**C**  
CALCULATES THE CEDF(X) FOR M WEAPONS AND N INSTALLATIONS USING THE CEP-INCLUDED MODEL.

**C**  

PARAMETER (MXM=10, MXN=10, MTM=20, MTN=20)
INTEGER M, N, VN(MXN), K(MXN), WR(MXM, MXN), N2
REAL VALUE(MXM), YIELD(MXM), CEP(MXM), HOB(MXM), PA(MXM)
REAL X(MTM), XX(MTN), EC, EV(MXM), PS(MXM), PDAMF, BETA(MXM)
REAL PDGM(MXM, MXN), SIGMA(MXN)
CHARACTER T(MXM)
COMMON/INSTLN/ N, VALUE, VN, K, XX
COMMON/POIND/ T
COMMON/WPNS/ M, YIELD, CEP, HOB, PA
COMMON/PARAMS/ WR, SIGMA, BETA, ITERS

EC=0.0
DO 20 J=1, N
   PS(J)=1.0
   DO 10 I=1, M
      PDGM(I, J)=PDAM(I, J, X)
      PS(J)=PS(J)+(1.0 - PA(I) * PDGM(I, J))
   CONTINUE
   EV(J)=(1.0 - PS(J)) * VALUE(J)
   EC=EC+EV(J)
20 CONTINUE
C
F=EC
RETURN
END
SUBROUTINE INITLZ(X)

C 1. READS USER-SPECIFIED WEAPON AND INSTALLATION PARAMETERS FROM THE EXTERNAL FILE, INDATA.
C 2. ASSIGNS INITIAL DGZ COORDINATES ACCORDING TO THE USER OPTION VARIABLE, INC.
C 3. TRANSFORMS ALL WEAPON AND INSTALLATION COORDINATES INTO FEET RELATIVE TO A COMMON ORIGIN IN A XY COORDINATE SYSTEM.
C 4. Initializes accuracy and convergence criteria for the optimization subroutines, XMCGR and PMIN.

PARAMETER (MXN=10, MXM=11, MTM=20, MTN=20)
INTEGER DGLN(MXM,3),OGLT(MXM,3),ISLN(MXM,3),ISLT(MXM,3)
INTEGER VN(MXM),K(MXM),H,N,INC,N2,WIND,WINI
REAL DINC(MXM),DTCD(MXM),INCD(MXM),ITCD(MXM),ORLN,ORLT
REAL YIELD(MXM),CEP(MXM),MOB(MXM),PA(MXM),VALUE(MXM)
REAL X(MTM),XX(MTN),ACC,ACC2,DFPRED,ESCALE,E(MTM),VAL
REAL NGLN,NGLY,NISLN,NISLT,TLAT,LNGMN
LOGICAL INASG(MXM)
CHARACTER DLNC(MXM),DLTC(MXM),ILNC(MXM),ILTC(MXM),T(MXM)
CHARACTER PH*M
COMMON/INSTLN/ N,VALUE,VN,K,X
COMMON/PQIND/ T
COMMON/WPNS/ M,YIELD,CEP,MOB,PA
COMMON/CNTRL/ N,ACC,ACC2,DFPRED,ESCALE,E
COMMON/ORIGIN/ CRLN,ORLT,LNGMN

100 FORMAT(2X,I4,I2,I2,A1,4X,I3,I2,I2,A1,4X,*F5.2,2X,F5.2,2X,F6.2,1X,F4.2)
110 FORMAT(2X,I4,I2,I2,A1,4X,I3,I2,I2,A1,4X,I2,A1,I1,2X,F6.2)
115 FORMAT(/,*INITLZ*/*)
120 FORMAT(/,* THIS PROBLEM USES *I2* WEAPONS*/*)
130 FORMAT(/,* THIS COMPLEX CONTAINS *I2* INSTALLATIONS*/*)
140 FORMAT(/,* LONGITUDE LATITUDE VTNK VALUE*/*)
150 FORMAT(/,* THE XY COORDINATES OF THE INSTALLATIONS IN FEET*/*)
160 FORMAT(/,* INITIAL DGZ LOCATIONS ARE *A28*/*)

WRITE(6,I12)
OPEN(15,FILE="INDATA")
REWIND 15

READ THE USER INITIAL DGZ COORDINATE OPTION VARIABLE, INC.
     IF INC = 1, THEN USER-SPECIFIED COORDINATES
     IF INC = 2, THEN HIGHEST VALUE INSTALLATION COORDINATES
     IF INC = 3, THEN HARDEST INSTALLATION COORDINATES
READ(15,*)) INC

125
READ USER-SPECIFIED WEAPON PARAMETERS.

READ(15,*) M
WRITE(6,136) M
WRITE(6,140)
DO 10 I=1,M
   READ(15,10J) (DGLN(I,L),L=1,3),DLNC(I),(DGLT(I,L),L=1,3),
   *DLT/JYIELD(I),CEP(I),HOB(I),PA(I)
WRITE(6,105) I,YIELD(I),CEP(I),HOB(I),PA(I)
10 CONTINUE

READ USER-SPECIFIED INSTALLATION PARAMETERS.

READ(15,*) N
WRITE(6,150) N
WRITE(6,160)
DO 20 J=1,N
   READ(15,110) (ISLN(J,L),L=1,3),ILNC(J),(ISLT(J,L),L=1,3),
   *ILT/J,VN(J),T(J),K(J),VALUE(J)
   WRITE(6,110) (ISLN(J,L),L=1,3),ILNC(J),(ISLT(J,L),L=1,3),
   *ILT/J,VN(J),T(J),K(J),VALUE(J)
20 CONTINUE

SMNCO=180.0
SMNTO=180.0
TLAT=0.0

TRANSLATES INSTALLATION J DEGREE-MINUTE-SECOND
COORDINATES INTO DEGREES.

DO 30 J=1,N
   IF(ISLN(J)=ED.EQ.) THEN
      INCO(J)=REAL(ISLN(J))/360.0
      *(REAL(ISLN(J))/360.0)
   ELSE
      NISLN=REAL(ISLN(J))
      INCO(J)=NISLN-(REAL(ISLN(J))/60.0)-
      *(REAL(ISLN(J))/360.0)
   ENDIF
   IF(INCO(J).LT.SMNCO) SMNCO=INCO(J)
   IF(ILTC(J).EQ."N") THEN
      ITCO(J)=REAL(ISLT(J,1))+REAL(ISLT(J,2))/60.0
      *(REAL(ISLT(J))/60.0)
   ELSE
      NISLT=REAL(ISLT(J,1))
      ITCO(J)=NISLT-(REAL(ISLT(J,2))/60.0)-
      *(REAL(ISLT(J))/60.0)
   ENDIF
   IF(ITCO(J).LT.SMTCO) SMTCO=ITCO(J)
   TLAT=TLAT+ITCO(J)
30 CONTINUE
LNGMN=COS($0.017453292*TLAT/NI)*6.806
ORLN=INT($MNCD)
IF(IORLN*LT.0.5) ORLN=INT($MNCD-1.0)
ORLT=INT($MTCO)
IF(IORLT*LT.0.5) ORLT=INT($MTCO-1.0)

TRANSFORMS INSTALLATION J DEGREE COORDINATES INTO FEET RELATIVE TO A COMMON ORIGIN.

WRITE(6,173)
DO 40 J=1,N
   XX(J)=(INCJ(J)-ORLN)*60.0*LNGMN
   XX(J+N)=(ITCD(J)-ORLT)*3640.060
WRITE(6,173) XX(J),XX(J+N),XX(J+N)
CONTINUE

IF(INC.EQ.2).OR.(INC.EQ.3)) THEN
   DO 45 J=1,N
      INASG(J)=.FALSE.
   CONTINUE
ENDIF

ASSIGNS WEAPON I INITIAL DGZ COORDINATES ACCORDING TO THE USER OPTION VARIABLE, INC.

IF(INC.NE.2).AND.(INC.NE.3)) THEN
   IF(INC=1) THEN TRANSLATE USER-SPECIFIED WEAPON I DGZ DEGREE-MINUTE-SECOND COORDINATES INTO DEGREES.

   PH=USER SPECIFIED
   DO 91 I=1,M
      IF(DINC(I).EQ.0.E) THEN
         OINC=(REAL(DGLN(I,1))+(REAL(DGLN(I,2))/60.0)+
         *(REAL(DGLN(I,3))/360.0)
      ELSE
         DGLN=-REAL(DGLN(I,1))
         OINC=-DGLN-(REAL(DGLN(I,2))/60.0)+
         *(REAL(DGLN(I,3))/360.0)
      ENDIF
      IF(DLC(I).EQ.0.E) THEN
         OLC=(REAL(DGLT(I,1))+(REAL(DGLT(I,2))/60.0)+
         *(REAL(DGLT(I,3))/360.0)
      ELSE
         DGLT=-REAL(DGLT(I,1))
         OLC=-DGLT-(REAL(DGLT(I,2))/60.0)+
         *(REAL(DGLT(I,3))/360.0)
      ENDIF
   CONTINUE

   CONTINUE

91 CONTINUE
TRANSFORM WEAPON 1 DEGREE COORDINATES INTO FEET

RELATIVE TO A COMMON ORIGIN.

DO 60 I=1,M
   X(I)=(INCD(I)-D02*LN)*62.5*CN
   X(I+M)=(DTCD(I)-CRLT)*36480.
60 CONTINUE

ELSEIF(INC.EQ.2) THEN

IF INC = 2 THEN ASSIGN THE COORDINATES OF THE M
HIGHEST VALUED INSTALLATIONS AS THE INITIAL DGZ
COORDINATES OF THE M WEAPONS.

PH="HIGHEST VALUED INSTALLATIONS"
DO 70 I=1,M
   VAL=0
   VIND=0
   DO 75 J=1,N
      IF(INASG(J)) GO TO 75
      IF(VALUE(J).LT.VAL) GO TO 75
      VAL=VALUE(J)
      VIND=J
75 CONTINUE
   X(I)=XX(VIND)
   X(I+M)=XX(VIND+N)
   INASG(VIND)=.TRUE.
70 CONTINUE

ELSE

IF INC = 3 THEN ASSIGN THE COORDINATES OF THE M
HARDEST INSTALLATIONS AS THE INITIAL DGZ COORDINATES
OF THE M WEAPONS.

PH="HARDEST INSTALLATIONS"
DO 80 I=1,M
   VNI=0
   VIND=0
   DO 85 J=1,N
      IF(INASG(J)) GO TO 85
      IF(VN(J).LT.VNI) GO TO 85
      VNI=VN(J)
      VIND=J
85 CONTINUE
   X(I)=XX(VIND)
   X(I+M)=XX(VIND+N)
   INASG(VIND)=.TRUE.
80 CONTINUE
ENDIF
WRITE(6,188) PH
DO 95 I=1,M
   WRITE(6,*) X(*,I+9)=.,X(I),.
   X(*,I+M,*)=.,X(I+M)
95 CONTINUE
INITIALIZE THE CONVERGENCE PARAMETERS OF THE SUBROUTINES ZXCGR AND PWMIN.

ACC=0.1
READ(15,*) ACC2
READ(15,*) DFPRED
N2=2*M
ESCALE=5000.*J
DO 90 I=1,N2
   E(I)=0.1
90 CONTINUE
CLOSE(15)
END
SUBROUTINE OUTDGZ(X)

C 1. TRANSLATES THE FINAL DGZ COORDINATES FROM FEET INTO DEGREE-MINUTE-SECOND LONGITUDE AND LATITUDE COORDINATES.

PARAMETER (MXM=10, MTM=20)
REAL X(MTM), ORLN, ORLT, REM, LNGMN
REAL YIELD(MXM), CEP(MXM), HOB(MXM), PA(MXM)
INTEGER DGLN(MXM, 3), DGLT(MXM, 3)
CHARACTER DLNC(MXM), OLTC(MXM)
COMMON/WPNS/ M, YIELD, CEP, HOB, PA
COMMON/ORIGIN/ ORLN, ORLT, LNGMN
10 FORMAT(5X, 12, 5X, 14, 12, 51, 14, I3, I2, I2, A1, I4, I3, I2, I2, A1)
110 FORMAT(/, * WEAPON LONGITUDE LATITUDE *)
120 FORMAT(/)
C
WRITE(6, 110)
DO 10 I=1, MTM
   IF(ORLN*GE.0.0) THEN
      DGLN(I, 1) = INT(ORLN)
      DLNC(I) = ''E''
      DGLN(I, 2) = INT(X(I)/LNGMN)
      REM = X(I) - LNGMN * DGLN(I, 2)
   ELSE
      DGLN(I, 1) = ABS(INT(ORLN + 1.0))
      DLNC(I) = ''W''
      DGLN(I, 2) = INT(6.0 - X(I)/LNGMN)
      REM = 60.0 * LNGMN - X(I) - LNGMN * DGLN(I, 2)
   ENDIF
   DGLN(I, 3) = INT(REM/60.0 / LNGMN)
   IF(ORLT*GE.0.0) THEN
      DGLT(I, 1) = INT(ORLT)
      DLTC(I) = ''N''
      DGLT(I, 2) = INT(X(I+M)/60.0 + .)
      REM = X(I+M) - 60.0 * DGLT(I, 2)
   ELSE
      DGLT(I, 1) = ABS(INT(ORLT + 1.0))
      DLTC(I) = ''S''
      DGLT(I, 2) = INT(6.0 - X(I+M)/60.0 + .)
      REM = 360.0 - X(I+M) - 60.0 * DGLT(I, 2)
   ENDIF
   DGLT(I, 3) = INT(REM/101.33333)
WRITE(6, 110) I, (DGLN(I, 1), DGLN(I, 2), DGLN(I, 3), DGLT(I, 1), DGLT(I, 2), DGLT(I, 3))
1 CONTINUE
WRITE(6, 120)
RETURN
END
SUBROUTINE WRADS

1. CALCULATES THE WEAPON RADIUS, WR(I,J), FOR EACH
WEAPON I - INSTALLATION J INTERACTION.

2. CALCULATES THE BETA(J) FOR EACH INSTALLATION.

PARAMETER(MX=M=10, MN=M=10, MT=M=20, TN=T=20)
REAL KK, YIELD(MX), HOB(MX), EX, SIGMA(MN), SIG, R1, R2
REAL VV, VNA, SHOB(MX), 0(2), M(2), SWR, K0, K1, K2, K3, K4, K5, K6, K7
REAL VALUE(MN), CEP(MX), PA(MX), BETA(MN), XX(MT)
INTEGER VN(MX), K(MN), I, SHOB(MX), WR(MX,MN), I, J, L, M, N
CHARACTER T(MX), T(5)
COMMON/INSTLN/ N, VALUE, VN, K, XX
COMMON/POINT/ T
COMMON/UPNS/ M, YIELD, HOB, PA
COMMON/PARMS/ WR, SIGMA, BETA, ITERS
FORMAT(/, "WRADS", /)

WRITE(*,10)

SPECIFIES THE DISTANCE DAMAGE SIGMA AND THE BETA FOR
EACH INSTALLATION J.

J=1
J=J+1
IF( (T(J).EQ. 'L') .OR. (T(J).EQ. 'R') ) THEN
  SIGMA(J) = 0.1
ELSEIF( (T(J).EQ. 'P') .OR. (T(J).EQ. 'S') ) THEN
  SIGMA(J) = 0.2
ELSEIF( (T(J).EQ. 'M') .OR. (T(J).EQ. 'Q') ) THEN
  SIGMA(J) = 0.3
ELSEIF( (T(J).EQ. 'N') .OR. (T(J).EQ. 'T') ) THEN
  SIGMA(J) = 0.4
ELSE
  SIGMA(J) = 0.5
ENDIF
IF( (T(J).EQ. 'L') .OR. (T(J).EQ. 'P') .OR. (T(J).EQ. 'M') .OR.
  (T(J).EQ. 'N') .OR. (T(J).EQ. 'T') ) THEN
  TT = 'PTYPE'
ELSE
  TT = 'QTYPE'
ENDIF
IF( (TT.EQ. 'PTYPE') ) THEN
  EX = 0.5
  SIG = 0.96
  A = 5.485
  R1 = 2.0
ELSE
  EX = (1.0/3.1)
  SIG = 0.91
  A = 2.742
  R1 = 3.0
ENDIF
BETA(J) = SQRT(-LCG(1.0-SIGMA(J)*2))
I=1
I=I+1
W(1)=0.0
W(2)=0.0
G(1)=0.0
G(2)=0.0

C CALCULATES THE VN REDUCTION FACTOR, R2, AND THE
C ADJUSTED VN NUMBER, VNA.

C KK=REAL(K(J))
R2=1.0-(KK/10.0)+(KK/10.0)*((2.0*YIELD(I))**(1.0/3.0))*R1**EX
IF(ABS(R2-R1)*GE.0.101) THEN
R1=R2
GO TO 10:
ENDIF
V=A+LOG(R2)
VNA=REAL(K(J))+V

C CALCULATES THE SCALED HEIGHT OF BURST (SHOB).

C SHOB(I)=HOB(I)/YIELD(I)*=(1.0/3.0)
IF(SHOB(I)*GE.90.0) THEN
WRITE(6,**) 'HOB TOO BIG'
SHOB(I)=90.0
ENDIF
ISHOB(I,1)=INT(SHOB(I)/100.0)*10
ISHOB(I,2)=ISHOB(I,1)+10
E=(SHOB(I)-REAL(ISHOB(I,1)))/100.0

C L=0
L=L+1

C DETERMINES THE POLYNOMIAL COEFFICIENTS TO CALCULATE
C G(VNA,ISHOB).

C IF(TT.EQ.*PTYPE*) GO TO 20:
GO TO 120
120 GL=K0+K1*VNA+K2*VNA**2+K3*VNA**3+K4*VNA**4
GO TO 140
130 GL=K0+K1*VNA+K2*VNA**2+K3*VNA**3+K4*VNA**4+
+K5*VNA**5+K6*VNA**6+K7*VNA**7

C CALCULATES THE SCALED WEAPON RADIUS (SWR).

C 140 W(L)=EXP(GL)
IF(E.LT.0.001) GO TO 150
IF(L.LT.2) GO TO 11:
150 SWR=W(1)+E*(W(2)-W(1))

C INVERSE YIELD SCALES THE SWR TO DETERMINE THE WEAPON
C RADIUS, WR(I,J), FOR THE WEAPON I-INSTALLATION J
C INTERACTION.

C WR(I,J)=INT((SWR*YIELD(I))**((1.0/3.0))SIG/(1.0-SIGMA(J)**2))+.5
WRITE(6,**) WR(I,J)
GO TO 170
C
160 WRITE(*,*) " VN TOO LARGE FOR HOB*
WP(I,J)=0
170 IF(I.LT.N) GO TO 31
IF(J.LT.N) GO TO 29
GO TO 220
200 IF(ISHOB(I,L).LT.10) THEN
C
C COEFFICIENTS OF G(VNA,ISHOB) FOR OVERPRESSURE (P-TYPE)
C TARGETS FOR ISHOB FROM 0 FEET TO 900 FEET.
C
IF(VNA.LE.7.5) THEN
   K0=8.206936
   K1=-9.8662222E-02
   K2=-6.270319E-03
   K3=4.67361E-03
   K4=0.0
ELSE
   K0=8.263243
   K1=-1.2109524E-01
   K2=12.74266E-04
   K3=-9.265496E-06
   K4=0.0
ENDIF
GO TO 12C
ELSE IF(ISHOB(I,L).LT.20) THEN
    IF(VNA.GT.51.0) GO TO 16C
    IF(VNA.LE.7.5) THEN
       K0=8.29123
       K1=-1.329339E-01
       K2=31.19908E-05
       K3=0.0
       K4=0.0
    ELSE
       K0=8.29259
       K1=-1.194333E-01
       K2=-4.6494E-05
       K3=65.8301E-06
       K4=-9.1680378E-7
    ENDIF
    GO TO 12C
ELSE IF(ISHOB(I,L).LT.30) THEN
    IF(VNA.GT.41.0) GO TO 16C
    K0=8.395223
    K1=-1.4717856E-01
    K2=12.74489E-03
    K3=-2.632771E-03
    K4=16.67591E-05
    K5=-6.84342E-06
    K6=1.23714E-02
    K7=-1.167515E-09
    GO TO 13C
ELSE IF (ISHOB(I), L) LT 40; THEN
  IF (VNA GT 34.0) GO TO 160
  K0 = 8.415584
  K1 = -9.9027816E-2
  K2 = -4.1872797E-3
  K3 = 54.49084E-05
  K4 = -3.759352E-1
  K5 = 14.11969E-07
  K6 = -2.03170989E-08
  K7 = 0.0
  GO TO 13v
ELSE IF (ISHOB(I), L) LT 50; THEN
  IF (VNA GT 30.0) GO TO 160
  K0 = 8.494689
  K1 = -1.955211E-2
  K2 = -3.4448747E-3
  K3 = 72.61706E-03
  K4 = -7.15905E-03
  K5 = 33.15013E-07
  K6 = 5.6685157E-03
  K7 = 0.0
  GO TO 13v
ELSE IF (ISHOB(I), L) LT 60; THEN
  IF (VNA GT 27.0) GO TO 160
  K0 = 8.329985
  K1 = -6.3120552E-2
  K2 = -2.5622131E-2
  K3 = 54.26447E-4
  K4 = -5.926339E-04
  K5 = 34.85504E-06
  K6 = 10.228646E-6
  K7 = 11.4432E-09
  GO TO 130
ELSE IF (ISHOB(I), L) LT 70; THEN
  IF (VNA GT 25.0) GO TO 160
  K0 = 8.586222
  K1 = -1.02711E-01
  K2 = -9.9171759E-3
  K3 = 26.0232E-04
  K4 = -3.6628224E-4
  K5 = 29.02515E-06
  K6 = -1.0826364E-6
  K7 = 15.41557E-09
  GO TO 130
ELSE IF (ISHOB(I), L) LT 80; THEN
  IF (VNA GT 22.0) GO TO 160
  K0 = 3.655962
  K1 = -1.3679886E-21
  K2 = 14.26281E-03
  K3 = -4.92993E-03
  K4 = 50.2125E-05
  K5 = -2.5712239E-05
  K6 = 43.79003E-08
  K7 = 0.0
  GO TO 130
ELSEIF(ISHOB(I,L) . LT. 90 ) THEN
  IF(VNA . GT. 21 ) GO TO 160
  IF(VNA . LE. 7.5) THEN
    K0 = 8.681285
    K1 = -1.1432664E-01
    K2 = -1.7888666E-03
    K3 = 15.35909E-05
    K4 = 0.0
  ELSE
    K0 = 12.51342
    K1 = -1.516344
    K2 = 17.69944E-02
    K3 = -3.06635E-03
    K4 = 14.07536E-05
  ENDIF
  GO TO 160
ELSE
  IF(VNA . GT. 20 ) GO TO 160
  IF(VNA . LE. 7.5) THEN
    K0 = 8.719654
    K1 = -1.218926E-01
    K2 = 12.6364E-04
    K3 = -1.3863251E-04
    K4 = 0.0
  ELSE
    K0 = 13.47289
    K1 = -1.971983
    K2 = 25.47267E-02
    K3 = -4.329119E-02
    K4 = 26.40371E-05
  ENDIF
  GO TO 120
ENDIF

C
C COEFFICIENTS OF G(VNA,ISHOB) FOR DYNAMIC PRESSURE
C (Q-TYPE) TARGETS FOR ISHOB FROM 1 FEET TO 900 FEET.
C
210 IF(ISHOB(I,L) . LT. 100 ) THEN
  IF(VNA . GT. 35.0 ) GO TO 160
  K0 = 3.315315
  K1 = -0.1686068
  K2 = 0.0005224
  K3 = -0.03313
  K4 = 3.22649E-05
  K5 = -1.23227E-06
  K6 = 1.96777E-08
  K7 = -1.5860E-11
  GO TO 130

135
ELSEIF(ISHOB(I,L) LT 2.) THEN
IF(VNA GT 35.0) GO TO 160

K0=8.376382
K1=0.1042945
K2=-0.0012014
K3=-3.91136E-05
K4=1.26757E-05
K5=-4.97979E-07
K6=5.77257E-09
K7=0.0
GO TO 130

ELSEIF(ISHOB(I,L) LT 3.) THEN
IF(VNA GT 35.0) GO TO 160

K0=8.42624
K1=0.09473E-01
K2=1.62288E-04
K3=-5.969792E-14
K4=6.97002E-06
K5=-3.149459E-06
K6=6.86229E-09
K7=-4.86613E-1
GO TO 130

ELSEIF(ISHOB(I,L) LT 4.) THEN
IF(VNA GT 35.0) GO TO 160

K0=8.49315
K1=-0.1031393
K2=0.0354114
K3=0.0313987
K4=-1.07267E-05
K5=3.15662E-07
K6=-5.55646E-09
K7=0.0
GO TO 130

ELSEIF(ISHOB(I,L) LT 5.) THEN
IF(VNA GT 35.0) GO TO 160

K0=8.57803
K1=0.103885
K2=0.0365786
K3=0.0012362
K4=0.0013333
K5=8.01397E-06
K6=-2.34684F-07
K7=2.51295E-09
GO TO 130

ELSEIF(ISHOB(I,L) LT 6.) THEN
IF(VNA GT 28.0) GO TO 160

K0=9.643504
K1=0.1110564
K2=0.0043904
K3=0.06644
K4=-7.76849E-05
K5=3.90695E-06
K6=-2.427079E-07
K7=3.36626F-09
GO TO 130

136
ELSE IF (ISHOB (I*L) .LT. 70) THEN
  IF (VNA .GT. 26.) GO TO 160
    K0 = 0.696697
    K1 = -0.1164822
    K2 = 0.003634
    K3 = -0.0006169
    K4 = 8.57541E-09
    K5 = -4.7263E-06
    K6 = 5.66402E-08
    K7 = 0.0
    GO TO 130
ELSE IF (ISHOB (I*L) .LT. 80) THEN
  IF (VNA .GT. 25.) GO TO 160
    K0 = 0.72449
    K1 = -0.1175502
    K2 = 0.023483
    K3 = -0.013054
    K4 = 0.001969
    K5 = -1.5200E-05
    K6 = 2.3379E-07
    K7 = -2.4474E-09
    GO TO 130
ELSE IF (ISHOB (I*L) .LT. 90) THEN
  IF (VNA .GT. 23.) GO TO 160
    K0 = 0.736328
    K1 = -0.1181635
    K2 = -0.021175
    K3 = 0.0015218
    K4 = 0.0062654
    K5 = -1.56750E-05
    K6 = 6.1815E-07
    K7 = -7.20962E-09
    GO TO 130
ELSE
  IF (VNA .GT. 22.4) GO TO 160
    K0 = 0.735042
    K1 = -0.1154885
    K2 = 0.0001871
    K3 = 0.0110000
    K4 = 0.0002357
    K5 = -2.01562E-05
    K6 = 6.97965E-07
    K7 = -8.48666E-09
    GO TO 130
ENDIF
200 CONTINUE
230 FORMAT(//1)
WRITE (6,230)
END
REAL FUNCTION PDAM(I,J,X)

C* CALCULATES THE PROBABILITY OF ACHIEVING A SPECIFIED LEVEL OF DAMAGE TO INSTALLATION J FROM WEAPON I.

PARAMETER (MXM=10, MXN=10, MTM=20, MTN=20)
PARAMETER (R95=0.0)

COEFFICIENTS OF THE POLYNOMIAL APPROXIMATION OF THE MODIFIED ZEROTH ORDER BESSEL FUNCTION.

PARAMETER (R1=3.156229, R2=3.0895424, R3=1.267432)
* R4=3.269732, R5=0.360768, R6=0.35813, C0=0.3989429,
* C1=0.01328332, C2=0.0225319, C3=0.0157565, C4=0.00916281,
* C5=0.0205776, C6=0.2635537, C7=0.1647633, C8=0.03392377)

INTEGER WR(MXM, MXN), Z(J*, K2, L2, M, N, VN(MXM), N(MXV))
REAL BESB1, BESC1, V, Z(5), W(5), S, X(MXM), R, BETA(MXM), R35
REAL SIGMA(MXM), CEP(MXM), CEPAP, PD, PP, PDR, WRS, NS, VALUE(MXM)
REAL A, B, F, H, J, F(5,2), YIELD(MXM), HOB(MXM), PA(MXM)
COMMON/INSTLN/N, VALUE, VN, K, XX
COMMON/UPNS/M, YIELD, CEP, HOB, PA
COMMON/PARAMS/W, SIGMA, BETA, ITS

QUADRATURE BASE POINTS Z(I) AND COEFFICIENTS W(I)

DATA Z(1), Z(2), Z(3), Z(4), Z(5), W(1), W(2), W(3), W(4), W(5)/
* 0.143876335, 0.4333953941, 0.579495683, 0.865633667,
* 1.373965265, 0.955242247, 0.2692667193, 0.2197863625,
* 0.1449313492, 0.866713437,
BESB(V)=1.0+.V*(B1+V*(B2+V*(B3+V*(B4+V*(B5+V+B6))))))
BESC(V)=C0+.V*(C1+V*(C2+V*(C3-V*(C4-V*(C5-V*(C6-V*(C7-V+C8)))))))

S=SQR((X(I)-XX(J)))*2+(X(I)+M)-XX(J+N))**2)
IF(S<LT.0.6) S=1.
CEPA=SQR((CEP(I))**2+.231*95**2)
IF(WR(I,J).EQ.1) THEN
PDAM=.0
GO TO 12C
ENDIF

138
CEP-EXCLUDED CEDF MODEL. THE PD(I,J) IS THE
DISTANCE DAMAGE FUNCTION.

IF(CEPA .LT. 0.001) THEN
  IF(S .GT. 0.0) THEN
    PDM = PD'R(REAL(UR(I,J)), S, BETA(J))
  ELSE
    PDM = 1.0
  ENDIF
ELSE

CEP-INCLUDED CEDF MODEL. DETERMINATION OF THE
INTEGRATION LIMITS A AND B

NWR = 1.1774*REAL(UR(I,J))/CEPA
NS = 1.1774*S/CEPA
b = 1.06*NWR*EXP(2.86*SIGMA(J))
IF(b .GT. (NS+4.0)) b = NS+4.0
A = NS-4.0
IF(A .LT. 0.0) A = 0.0
FN = 0.0

GAUSS-LEGENDRE TEN POINT QUADRATURE TO
DETERMINE PD(I,J).

DO 110 K2 = 1, 5
  DO 100 L2 = 1, 2
    R = 0.5*( (B-A)*Z(K2)*(-1)**L2*A+B)
    PP = PD'R(NWR, R, BETA(J))
    H = NS*R
    IF(H .EQ. 0.0) THEN
      F(K2, L2) = PP*R*EXP((-R**2)/2.0)
    ELSE IF(H .LE. 3.75) THEN
      JJ = (H/3.75)**2
      F(K2, L2) = PP*R*EXP((-(NS-R)**2/2.0)*BESB(JJ))
    ELSE IF(JJ .LE. 3.75) THEN
      JJ = 3.75/H
      F(K2, L2) = PP*R*EXP((-(NS-R)**2/2.0)*BESB(JJ)/SQRT(H))
    ENDIF
  FN = FN + W(K2)*F(K2, L2)
  CONTINUE
110 CONTINUE
  PDM = 0.5*(B-A)*FN
ENDIF
120 PDM = INT(100.0*PDM + 0.5)/10.0
END
REAL FUNCTION PDR(WR,R,BETA):

1. CALCULATES THE DISTANCE DAMAGE FUNCTION -- THE
PROBABILITY OF ACHIEVING A SPECIFIED LEVEL OF
DAMAGE TO INSTALLATION J FROM WEAPON I WHEN THEY
ARE SEPARATED A KNOWN DISTANCE R.

PARAMETER(E1=0.0,705230784,E2=0.422820123,E3=0.092795272,
*E4=0.010152 143,E5=9.0062765672,E6=9.0000432539)
REAL ERF,ZZ,BETA,WR,R,U,SIGN,V
ERF(V)=1.0-1.0/(1.0+V*(E1+V*(E2+V*(E3+V*(E4+V*(E5+V*E6))))))**1.0

CALCULATES ZZ, A STANDARD NORMAL RANDOM VARIABLE, AND
TESTS ZZ TO DETERMINE THE EXTREMES OF THE PROBABILITY
OF DAMAGE FUNCTION.

ZZ=(1.0/BETA)*LOG((WR*EXP(-BETA**2))/R)
IF(ZZ.GT.3.87) THEN
  PDR=1.0
  RETURN
ENDIF
IF(ABS(ZZ).LT.3.0E-6) THEN
  PDR=0.5
  RETURN
ENDIF
IF(ZZ.LT.-3.87) THEN
  PDR=0.0
  RETURN
ENDIF
U=ABS(ZZ)/SORT(2.0)
SIGN=1.0
IF(ZZ.LT.0.0) SIGN=-1.0

CALCULATES THE DISTANCE DAMAGE FUNCTION USING A
POLYNOMIAL APPROXIMATION OF THE ERROR FUNCTION.

PDR=0.5*SIGN*U*0.5*ERF(U)
RETURN
END
SUBROUTINE PWMIN(X,E,N,EF,ESCALE,MAXIT,W,NW)

C     POWELL'S METHOD OF CONJUGATE DIRECTIONS
C     DETERMINES THE MINIMUM OF A FUNCTION USING
C     ONLY FUNCTION EVALUATIONS.

REAL X(N),E(N),W(NW)

DDMAG=1.0*ESCALE
SCF=0.05/ESCALE
JJ=J=N*(N+1)
JJJ=JJ+N
K=N+1
NFC=1
IND=1
INI=1
WRITE(6,*), PWMIN
DO 4 I=1,N
   W(I)=ESCALE
DO 4 J=1,N
   W(K)=0.0
   IF(I.EQ.J) W(K)=ABS(E(I))
4   K=K+1
   ITERC=1
   ISGRAD=2
   CALL FUNCT(N,X,F)
   FKEEP=2.0*ABS(F)
   C
   C
   START THE NEXT ITERATION.
C
5   ITONE=1
   DO 20 I=1,N
      WRITE(6,*), X(*,I,*), F(*,I)
   20 CONTINUE
   WRITE(6,*), EF= *,F
   FP=F
   SUM=0.0
   IXP=JJ
   DO 6 I=1,N
      IXP=IXP+1
6    W(IXP)=X(I)
    IDIRN=N+1
    ILINE=1

C
C
   START THE NEXT ONE DIMENSIONAL SEARCH.
C
7   DMAX=W(ILINE)
   DAC=EMAX*SCF
   DDMAG=AMIN1(DDMAG,SCF*DMAX)
   DDMAG=AMAX1(DDMAG,SCF*DAC)
   DMAG=LM*DMAG
   GO TO (7J,7771), I-ONE

141
DL=0.0
G=OMAG
FPREV=F
IS=5
FA=FPREV
DA=DL
DD=D-DL
DL=D

C

SELECT THE NEXT SEARCH DIRECTION FOR ITERATION I.

K=IDIRN
DO 9 I=1,N
   X(I)=X(I)+DD*W(K)
9     K=K+1
CALL FUNCT(N,X,F)
NFCC=NFCC+1
GO TO (10,11,12,13,14,96), IS

IF(F.EQ.FA) THEN
   IF(ABS(D) .LE. DMAX) THEN
      D=D+D
      GO TO 8
   ENDIF
   WRITE(6,*)  ' MAX CHANGE DOES NOT ALTER FUNCTION'
   GO TO 20
ELSEIF(F.LT.FA) THEN
   FB=F
   DB=0
   ELSE
   FB=FA
   DB=DA
   FA=F
   DA=D
   ENDF
13 IF(F.GE.FA) GO TO 23
29 FC=FB
DC=DB
GO TO 30
12 IF(F.LE.FB) GO TO 23
FA=F
DA=DB
GO TO 30
11 IF(F.GE.FB) GO TO 10
FA=FB
DA=DB
GO TO 29
C
71 DL=1.5
DDMAX=3.0
FA=FP
DA=-1.0
FB=FM0LD
DB=1.0
D=1.0
10 FC=F
DC=D
33 A=(DB-DC)*(FA-FC)
B=(DC-DA)*(FB-FC)
IF(A+B)*(DA-DC).*GT.1.5) GO TO 34
FA=FB
DA=DB
FB=FC
DB=DC
GO TO 26
34 D=1.5*(A*(DB+DC)+B*(DA+DC))/(A+B)
DI=DB
FI=FB
IF(FB.LE.FC) GO TO 44
DI=DC
FI=FC
44 GO TO (86,86,85), ITONE
85 ITONE=2
GO TO 45
C
C CHECK THE ONE DIMENSIONAL MINIMIZATION SEARCH
C FOR CONVERGENCE.
C
86 IF(ABS(D-DI).LE.DACC) GO TO 41
IF(ABS(D-DI).LE.(C.3*ABS(D))) GO TO 41
45 IF((DA-DC).LE.(DC-D).*GE.0.0) THEN
FA=FB
DA=DB
FB=FC
DB=DC
GO TO 25
ELSE
IS=2
IF((DB-D)*(DC-DA) GE 0.) GO TO 8
IS=3
GO TO 8
ENDIF
F=F*1
D=DI-DL
RE=(DC-DB)*(DC-DA)/(DA-DB)/(A+B)
IF(RE LE 0.) THEN
WRITE(6,*), "ACCURACY LIMITED BY THE FUNCTION"
RETURN
ENDIF
DD=SQR(RE)

C C COMPLETES ONE OF THE N ONE DIMENSIONAL SEARCHES
C FOR ITERATION I. UPDATE X(I).

C DO 49 I=1,N
   X(I)=X(I)+D*W(IDIRN)
   W(IDIRN)=DD*W(IDIRN)
49 IDIRN=IDIRN+1
   W(ILINE)=W(ILINE)/DD
   ILINE=ILINE+1

C IF(ITONE EQ 0.2) GO TO 39
IF((FPREV-F-SUM) LT 0.) GO TO 94
   SUM=FPREV-F
   JIL=ILINE
94 IF(IDIRN LE JJ) GO TO 7

C ALL ONE DIMENSIONAL SEARCHES COMPLETED

C GO TO (92,72), IND
92 FHOLD=F
IS=6
IXP=JJ
DO 59 I=1,N
   IXP=IXP+1
59 W(IXP)=X(I)-W(IXP)
   DD=1.*)

C CALCULATES THE EXPANDED POINT.
C GO TO 53
96 GO TO (112,87), IND
THE MODIFICATION TEST

IF(FP.LE.F) GO TO 37
D=2.0*(FP+F-2.0*F HOLD)/(FP-F)**2
IF((D*(FP-F HOLD-SUM)**2-SUM)**.5*<.2) GO TO 37
J=J+1
IF(J.GT.JJ) GO TO 61
DO 62 I=J,JJ
   K=I-4
62   W(K)=W(I)
   DO 67 I=J,JJ
   W(I-1)=W(I)
   DO 67 :=1,N
   IXP=IXP+1
   W(K)=W(IXP)
   IF(AAA.LT.ABS(W(K)/E(I))) THEN
      AAA=ABS(W(K)/E(I))
   ENDIF
67   K=K+1
   DOIA=1.0
   W(N)=ESCALE/AAA
   ILINE=N
GO TO 7

SEARCH IN DIRECTION OF THE EXPANDED POINT.

IDIRN=IDIRN-N
ITONE=3
K=IDIRN

IXP=IXP
AAA=AAA
DO 67 :=1,N
   IXP=IXP+1
   W(K)=W(IXP)
   IF(AAA.LT.ABS(W(K)/E(I))) THEN
      AAA=ABS(W(K)/E(I))
   ENDIF
67   K=K+1
   DOIA=1.0
   W(N)=ESCALE/AAA
   ILINE=N
GO TO 7
UPDATE \( x(i) \) AND USE THE PREVIOUS SEARCH DIRECTIONS.

37 \( \text{IXP}=\text{JJ} \)  
\( \text{AAA}=0 \)  
\( F=F_{\text{HOLD}} \)  
DO 99 I=1,N  
\( \text{IXP} = \text{IXP} + 1 \)  
\( x(i) = x(i) - w(i) \)  
IF\( (\text{AAA}=\text{ABS}(e(i))) \cdot \text{LT} \cdot \text{ABS}(w(i)) ) \) THEN  
\( \text{AAA} = \text{ABS}(w(i)) / e(i) \)  
ENDIF  
99 CONTINUE  
GO TO 72  
36 \( \text{AAA} = \text{AAA} \cdot (1 + \text{DI}) \)  
GO TO (72,16), IND  
72 GO TO (109,B8), IND  
109 IF(\( \text{AAA} \leq 0.1 \)) GO TO 20  
C  
IF(\( F \lt F_{\text{P}} \)) GO TO 35  
WRITE(6,*)  ACCURACY LIMITED BY THE FUNCTION  
GO TO 20  
83 IND=1  
35 DDMAG=\( 0.4 \cdot \text{SQRT} (\text{ABS}(F-P)) \)  
IF(DDMAG.GE.1.0E+6) DDMAG=1.0E+6  
IGO1AD=1  
C  
ITERC=ITERC+1  
IF(\( \text{ITERC} \leq \text{MAXIT} \)) GO TO 5  
IF(\( F \leq F_{\text{KEEP}} \)) GO TO 20  
F=F_{\text{KEEP}}  
DO 111 I=1,N  
JJJ=JJJ+1  
111 \( x(i) = w(j) \)  
21 WRITE(6,*)  ITERATIONS = \( F_{\text{ITER}} \)  
RETURN  
106 IF(\( \text{AAA} \leq 0.1 \)) GO TO 21  
INN=1  
GO TO 35  
C  
END
Appendix E: User Guidelines and a Sample Problem

This appendix provides basic instructions for inputting user-specified weapon and installation parameters to the CEDF maximization algorithm. These instructions are presented using an example.

Initially, the user must determine the values of the convergence control parameters, ACC, DFPRED, E, and ESCALE. These values depend on installation values and the number of weapons and installations. User guidelines in Chapter V discuss specific considerations. The source code initializes the PWMIN convergence parameters to \( E(i) = 0.1 \) and \( \text{ESCALE} = 5000 \). Two minor code changes would be necessary to change either of these values. The user must input values for the ZXCSR convergence parameters, ACC and DFPRED.

Next, the user must decide on the type of initial DGZ coordinates to use. The user has three options. The user communicates the desired DGZ location option to the algorithm through the input variable -- INC. If \( \text{INC} = 1 \), then the algorithm uses the user-specified estimates of the initial DGZ locations. If \( \text{INC} = 2 \), then the algorithm assigns the coordinates of the \( m \) highest valued installations to be the initial coordinates of the \( m \) weapons in decreasing order of yield. Finally, if \( \text{INC} = 3 \), then the algorithm assigns the coordinates of the \( m \) hardest installations to be the initial coordinates of the \( m \) weapons in decreasing order of yield.

Then the user needs to compile the necessary input data in a FORTRAN external file, INDATA. Figure E-1 is the input data file, INDATA, for a two weapon-four installation example. The first line is the decision variable, INC. For this example, \( \text{INC} = 2 \) and the initial DGZ
Figure E-1. The CEDF maximization algorithm input file, INDATA, for a two weapon-four installation complex.

coordinates were the coordinates of the two highest valued installations. The second line in the file "INDATA is m, the number of weapons for the complex; for this example, m = 2. The next m lines contain the user-specified weapon parameters. The FORTRAN input format for these parameters is statement 100 in subroutine INITLZ of the source code. The order and units of the weapon parameters are: the longitude and latitude coordinates (these must be initialized to some value even if INC = 2 or 3), the yield in kilotons, the CEP in feet, the HOB in feet, and the Pa.

The line after the last weapon's parameters is n, the number of installations in the complex; for this example, n = 4. The next n lines contain the user-specified installation parameters. The FORTRAN input format statement for these parameters is statement 110 in subroutine INITLZ of the source code. The order of the installation parameters is: the longitude and latitude coordinates, a NIK code, and a value (a real number less than 100000.0).

Finally, the CONVERG convergence control parameters complete the
external file, INDATA. The line after the last installation's parameters contains the value of ACC for the second stage of ZXCGR maximization. The last line of the file contains the value of DFPRED. For this example, ACC = 0.001 and DFPRED = 1000.

The CEDF maximization algorithm outputs the results of a problem to another external file, TAPE6. The results of four maximizations are: (1) a ZXCGR maximization for ACC = 0.01; (2) a ZXCGR maximization for a user-specified value of ACC; (3) a PWMIN maximization; and (4) a mixed maximization.

The next six pages list the results of the two weapon-four installation problem.
****** CEDF MAXIMIZATION ALGORITHM ******

INITIZ

THIS PROBLEM USES 2 WEAPONS

<table>
<thead>
<tr>
<th>WEAPON</th>
<th>YIELD</th>
<th>CEP</th>
<th>MOB</th>
<th>PA</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>70</td>
<td>250</td>
<td>1000</td>
<td>.99</td>
</tr>
<tr>
<td>2</td>
<td>70</td>
<td>250</td>
<td>1000</td>
<td>.99</td>
</tr>
</tbody>
</table>

THIS COMPLEX CONTAINS 4 INSTALLATIONS

<table>
<thead>
<tr>
<th>LONGITUDE</th>
<th>LATITUDE</th>
<th>VNTK</th>
<th>VALUE</th>
</tr>
</thead>
<tbody>
<tr>
<td>4510°E</td>
<td>46°33'N</td>
<td>16P2</td>
<td>3500</td>
</tr>
<tr>
<td>451030E</td>
<td>46°4'30&quot;N</td>
<td>22P2</td>
<td>2500</td>
</tr>
<tr>
<td>4511°E</td>
<td>46°35'N</td>
<td>21P4</td>
<td>5000</td>
</tr>
<tr>
<td>451130E</td>
<td>46°3 5'N</td>
<td>1903</td>
<td>7000</td>
</tr>
</tbody>
</table>

THE XY COORDINATES OF THE INSTALLATIONS IN FEET

XX(1) = 42199.35367441 XX(5) = 21280.00000002
XX(2) = 44298.82135815 XX(6) = 24319.99999998
XX(3) = 46408.23904194 XX(7) = 23306.66666666
XX(4) = 48517.75672559 XX(8) = 18746.66666664

INITIAL O0Z LOCATIONS ARE HIGHEST VALUED INSTALLATIONS

X(1) = 48517.75672559 X(3) = 18746.66666664
X(2) = 46408.23904194 X(4) = 23306.66666666

WRADS

WR(1,1) = 3272
WR(2,1) = 3272
WR(1,2) = 1947
WR(2,2) = 1947
WR(1,3) = 2200
WR(2,3) = 2200
WR(1,4) = 2863
WR(2,4) = 2863
ZMGCR MAXIMIZATION

ACC = .71

WEAPON 1 XY COORDINATES: (48518., 16747.)
PS(1) = .97624
PS(2) = .08683
PS(3) = .0099999999998
PS(4) = .009841599999998
EC = 12293.4438

WEAPON 2 XY COORDINATES: (46408., 23307.)
PS(1) = .09760
PS(2) = .82833
PS(3) = .0099999999998
PS(4) = .009851499999998
EC = 12806.2345

WEAPON 1 XY COORDINATES: (40518., 18747.)
PS(1) = .97228
PS(2) = .82373
PS(3) = .0099999999998
PS(4) = .00999469999998
EC = 14192.0658

WEAPON 2 XY COORDINATES: (46333., 23317.)
PS(1) = .09760
PS(2) = .82833
PS(3) = .0099999999998
PS(4) = .009851499999998
EC = 12406.2345

WEAPON 1 XY COORDINATES: (48517., 18747.)
PS(1) = .86239
PS(2) = .01999
PS(3) = .01198
PS(4) = .01999639999999
EC = 14799.7182

WEAPON 2 XY COORDINATES: (4521., 23678.)
PS(1) = .83566
PS(2) = .01093
PS(3) = .0238600000001
PS(4) = .019997299999999
EC = 14858.6229

WEAPON 1 XY COORDINATES: (48516., 18747.)
PS(1) = .97624
PS(2) = .08683
PS(3) = .0099999999998
PS(4) = .009841599999998
EC = 15730.5129

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PS(2) = 0.2633
PS(3) = 1.0226
PS(4) = 0.0996399999999998
EC = 15253.722

WEAPON 1 XY COORDINATES: (48517°, 18747°)
WEAPON 2 XY COORDINATES: (44029°, 23019°)
PS(1) = 0.51787
PS(2) = 0.18321
PS(3) = 0.09613
PS(4) = 0.0994599999999998
EC = 15416.6958

WEAPON 1 XY COORDINATES: (48517°, 18747°)
WEAPON 2 XY COORDINATES: (44012°, 23034°)
PS(1) = 0.51391
PS(2) = 0.7931
PS(3) = 0.1306
PS(4) = 0.0994059999999998
EC = 15420.6558

WEAPON 1 XY COORDINATES: (48516°, 18747°)
WEAPON 2 XY COORDINATES: (44035°, 23069°)
PS(1) = 0.53767
PS(2) = 0.0674200000001
PS(3) = 0.0913000000001
PS(4) = 0.0994059999999998
EC = 15429.1708

WEAPON 1 XY COORDINATES: (48516°, 18747°)
WEAPON 2 XY COORDINATES: (45040°, 23346°)
PS(1) = 0.72775
PS(2) = 0.02287
PS(3) = 0.02584
PS(4) = 0.09956499999999999
EC = 15196.8465

WEAPON 1 XY COORDINATES: (48516°, 18747°)
WEAPON 2 XY COORDINATES: (44851°, 23194°)
PS(1) = 0.55351
PS(2) = 0.06149
PS(3) = 0.0128900000000001
PS(4) = 0.0994059999999998
EC = 15433.4338

IER = 0
FUNCTION = 15433.4338
FUNCTION EVALUATIONS: 18
X(1) = 48516.46037407
GRAD(1) = 0.0003386697914471
X(2) = 44850.73876378
GRAD(2) = 0.0194870170614
X(3) = 18747.19690665
GRAD(3) = 0.001474502035606
X(4) = 23093.65196727
GRAD(4) = 0.193911576447

<table>
<thead>
<tr>
<th>WEAPON</th>
<th>LONGITUDE</th>
<th>LATITUDE</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>451129E</td>
<td>46 5 5N</td>
</tr>
<tr>
<td>2</td>
<td>451137E</td>
<td>46 347N</td>
</tr>
</tbody>
</table>
ZXCGR WITH ACC REDUCED

ACC = 0.1
WEAPON 1 XY COORDINATES: (48516, 18747)
WEAPON 2 XY COORDINATES: (44951, 23194)

PS(1) = 0.55351
PS(2) = 0.6148
PS(3) = 0.0312300000001
PS(4) = 0.09940599999998
EC = 15433.0308
IER = 0
FUNCTION = 15433.0308
FUNCTION EVALUATIONS: 2

X(1) = 48516.46037407
GRAD(1) = 0.003246657914471
X(2) = 44951.73876378
GRAD(2) = 0.194870170614
X(3) = 18747.19690665
GRAD(3) = -0.30147450235606
X(4) = 23194.65196727
GRAD(4) = 0.13911975447

WEAPON LONGITUDE LATITUDE
1 451129E 46 3 5N
2 451037E 46 3 47N

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**PWMIN MAXIMIZATION**

PWMIN

\[ X(1) = 48517.75672559 \]
\[ X(2) = 46438.28904184 \]
\[ X(3) = 12746.66666666 \]
\[ X(4) = 23169.2299639 \]
\[ EF = 12358.3581 \]

\[ X(1) = 48517.75672559 \]
\[ X(2) = 44888.9449761 \]
\[ X(3) = 12746.66666666 \]
\[ X(4) = 23169.2299639 \]
\[ EF = 15137.3158 \]

ACCURACY LIMITED BY THE FUNCTION

FUNCTION: 15141.9708

FUNCTION: 15141.9708

\[ XXX(1) = 48517.75672559 \]
\[ XXX(2) = 44888.9449761 \]
\[ XXX(3) = 12746.66666666 \]
\[ XXX(4) = 23159.52991793 \]

WEAPON  LONGITUDE  LATITUDE
1  451130E  46 35N
2  451336E  46 348N
MIXED TECHNIQUE MAXIMIZATION

\[
P_{\text{MIN}}
\]

\[
\begin{align*}
X(1) &= 48516.46037407 \\
X(2) &= 44893.73876378 \\
X(3) &= 18747.19690665 \\
X(4) &= 23123.65196727 \\
EF &= 15133.608 \\
\end{align*}
\]

\[
\begin{align*}
X(1) &= 48516.46037407 \\
X(2) &= 44851.32790653 \\
X(3) &= 18747.19690665 \\
X(4) &= 23123.72924026 \\
EF &= 15136.308 \\
\end{align*}
\]

\[
\begin{align*}
X(1) &= 48516.46037407 \\
X(2) &= 44851.32790653 \\
X(3) &= 18747.19690665 \\
X(4) &= 23123.46811256 \\
EF &= 15139.4958 \\
\end{align*}
\]

ACCURACY LIMITED BY THE FUNCTION

ITERATIONS = 4

\[
\begin{align*}
\text{FUNCTION} &= 15132.676 \\
\chi(1) &= 48516.46037407 \\
\chi(2) &= 44851.35435102 \\
\chi(3) &= 18747.19690665 \\
\chi(4) &= 23105.5184597 \\
\end{align*}
\]

WEAPON | LONGITUDE | LATITUDE
---|---|---
1 | 451129E | 46 3 5N
2 | 451037E | 46 348N
Appendix F: Verification of the Gradient of the CEP-Excluded CEDF Model

The results of two example problems verified that the subroutine GFUNCT correctly calculates the gradient of the CEP-Excluded model. The pencil and paper results for each example were compared with the results from GFUNCT.

The first example included one weapon and two installations. The weapon and installation parameters are presented below. This verification example used a graph of CEDF(x) versus x. For 40 equally spaced DGZ locations, values of CEDF(x) were calculated. The x direction was along the line connecting the two installations. Table F-1 lists the 40 values of x and the corresponding function and gradient values. Figure 9 in Chapter IV is a plot of this data. A DGZ between the two installations was selected (x = 63500), and the gradient was calculated using two methods. In this example, the gradient had only one element because the y variable was constant; only the x variable was allowed to vary. The gradient values for the two calculation methods were compared with the gradient value from GFUNCT.

The first method used a difference equation, \( \frac{\Delta \text{CEDF}}{\Delta x} \), to approximate the gradient. The slope of the line segment connecting the CEDF values for the two DGZs on either side of the selected DGZ was an approximate gradient value. From Table F-1,

\[
\text{CEDF}(x = 63000) = 6257 \\
\text{CEDF}(x = 64000) = 10196
\]
<table>
<thead>
<tr>
<th>x(1)</th>
<th>x(2)</th>
<th>EC</th>
<th>GRAD(1)</th>
<th>GRAD(2)</th>
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</thead>
<tbody>
<tr>
<td>54000</td>
<td>20000</td>
<td>2384</td>
<td>1.6274</td>
<td>0.0000</td>
</tr>
<tr>
<td>54500</td>
<td>20500</td>
<td>3218</td>
<td>1.6494</td>
<td>0.0000</td>
</tr>
<tr>
<td>55000</td>
<td>21000</td>
<td>3985</td>
<td>1.3530</td>
<td>0.0000</td>
</tr>
<tr>
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<td>21500</td>
<td>4534</td>
<td>0.8355</td>
<td>0.0000</td>
</tr>
<tr>
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<td>22000</td>
<td>4826</td>
<td>0.3564</td>
<td>0.0000</td>
</tr>
<tr>
<td>56500</td>
<td>22500</td>
<td>4930</td>
<td>0.0891</td>
<td>0.0000</td>
</tr>
<tr>
<td>57000</td>
<td>23000</td>
<td>4951</td>
<td>-0.1064</td>
<td>0.0000</td>
</tr>
<tr>
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<td>23500</td>
<td>4955</td>
<td>-0.0058</td>
<td>0.0000</td>
</tr>
<tr>
<td>58000</td>
<td>24000</td>
<td>4956</td>
<td>-0.9212</td>
<td>0.0000</td>
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<td>58500</td>
<td>24500</td>
<td>4957</td>
<td>-0.0058</td>
<td>0.0000</td>
</tr>
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<td>25000</td>
<td>4953</td>
<td>-0.0017</td>
<td>0.0000</td>
</tr>
<tr>
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<td>4951</td>
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<td>0.0000</td>
</tr>
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<td>60000</td>
<td>26000</td>
<td>4952</td>
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<td>0.0000</td>
</tr>
<tr>
<td>60500</td>
<td>26500</td>
<td>4974</td>
<td>-0.3137</td>
<td>0.0000</td>
</tr>
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<td>61000</td>
<td>27000</td>
<td>5021</td>
<td>-1.4934</td>
<td>0.0000</td>
</tr>
<tr>
<td>61500</td>
<td>27500</td>
<td>5152</td>
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<td>0.0000</td>
</tr>
<tr>
<td>62000</td>
<td>28000</td>
<td>5485</td>
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<td>0.0000</td>
</tr>
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<td>6257</td>
<td>2.2036</td>
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</tr>
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<td>3.6791</td>
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</tr>
<tr>
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<td>29500</td>
<td>10196</td>
<td>5.4652</td>
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</tr>
<tr>
<td>64000</td>
<td>30000</td>
<td>12513</td>
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</tr>
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<td>0.0000</td>
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</tr>
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<td>0.0000</td>
</tr>
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<td>0.0000</td>
</tr>
<tr>
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<td>35000</td>
<td>11931</td>
<td>-0.0701</td>
<td>0.0000</td>
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<td>0.0000</td>
</tr>
<tr>
<td>71000</td>
<td>36000</td>
<td>11724</td>
<td>-0.8426</td>
<td>0.0000</td>
</tr>
<tr>
<td>71500</td>
<td>36500</td>
<td>10744</td>
<td>-3.3551</td>
<td>0.0000</td>
</tr>
<tr>
<td>72000</td>
<td>37000</td>
<td>9330</td>
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<td>0.0000</td>
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<td>-2.1445</td>
<td>0.0000</td>
</tr>
</tbody>
</table>
The difference equation approximation of the gradient at \( x = 63500 \) was

\[
\Delta \frac{\text{CEDF}}{\Delta x} = \frac{10196 - 6257}{1000} = 3.939
\]

The second method was pencil and paper calculations of all the steps necessary to determine the gradient. Chapter II presented these steps. Only a summary of the calculations are presented here.

<table>
<thead>
<tr>
<th>Given:</th>
<th>Weapon</th>
<th>Yield</th>
<th>CEP</th>
<th>HOB</th>
<th>Pa</th>
<th>(x,y) in feet</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>100-kt</td>
<td>0 feet</td>
<td>1000 feet</td>
<td>0.99</td>
<td>(63500,20000)</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Target</th>
<th>Value(v)</th>
<th>WR</th>
<th>(xx,yy) in feet</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>11P2</td>
<td>5000</td>
<td>(60000,20000)</td>
</tr>
<tr>
<td>2</td>
<td>15P2</td>
<td>12000</td>
<td>(68000,20000)</td>
</tr>
</tbody>
</table>

Note: In this example, \( y = x(2) = \text{constant} \). Hence, \( \frac{\partial \text{CEDF}}{\partial y} = 0 \) and \( \text{CEDF}(y) = \text{CEDF}(x) \).

From Eq (1),

\[
\text{CEDF}(x) = v_1*Pa_1*Pd(1,1) + v_2*Pa_1*Pd(1,2)
\]

However, \( v_j \) and \( Pa_1 \) are constants, so

\[
\text{CEDF}(x) = 4950*Pd(1,1) + 11880*Pd(1,2)
\]

and

\[
\frac{d \text{CEDF}(x)}{dx} = 4950* \frac{d \text{Pd}(1,1)}{dx} + 11880* \frac{d \text{Pd}(1,2)}{dx}
\]

where

\[
\frac{d \text{Pd}(1,j)}{dx} = \frac{e^{-u^2}}{\sqrt{2\pi} \beta \gamma^2} \cdot (xx_j - x)
\]

and

\[
u = \frac{1}{\sqrt{2}} \cdot \text{abs} \left[ - \beta + \frac{1}{\beta} \ln \left( \frac{\gamma(1,1)}{\gamma} \right) \right]
\]

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For overpressure (P-type) targets, \( \sigma_d = 0.2 \) and

\[
\beta = \sqrt{-\ln(1 - \sigma_d^2)} \tag{F-1}
\]

\[= 0.202045 \]

Let

\[
AA = xx_j - x_j \tag{F-2}
\]

\[
BB = xx_{j+n} - x_{i+m} \tag{F-3}
\]

\[
r = AA^2 + BB^2 \tag{F-4}
\]

For this verification example, \( BB = 0 \) for both targets and \( r = |AA| \), the absolute difference between the \( x \) coordinates of the weapon and installation \( j \).

For target 1:

\[
AA = 60000 - 63500 = -3500
\]

\[
r = 3500
\]

\[
u = 1.8548527
\]

\[
\frac{d Pd(1,1)}{dx} = -1.8080608 \times 10^{-5}
\]

For target 2:

\[
AA = 68000 - 63500 = 4500
\]

\[
r = 4500
\]

\[
u = 0.49780381
\]

\[
\frac{d Pd(1,2)}{dx} = 3.4247393 \times 10^{-4}
\]

Therefore,

\[
\frac{d \text{CDF}(x)}{dx} = 4950.0 \times (-1.8080608 \times 10^{-5}) + 11880.0 \times (3.4247393 \times 10^{-4})
\]

\[= 3.97909\]
The value of the gradient of the CEDF(x) from GFUNCT for the DGZ selected ($x = 63500$) was 3.9791. The gradient results from the difference equation approximation and the pencil and paper calculations were compared with the value from GFUNCT. These two comparisons indicated the subprogram GFUNCT was properly calculating the gradient of the CEDF(x).

The second gradient verification example included two weapons and three installations. The gradient of the CEDF(x) had 2m or four elements. However, only one element was completely checked by pencil and paper calculations.

**Given:**

<table>
<thead>
<tr>
<th>Weapon</th>
<th>Yield</th>
<th>CEP</th>
<th>HOB</th>
<th>Pa</th>
<th>(x,y) in feet</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>100 kt</td>
<td>0</td>
<td>1000</td>
<td>0.99</td>
<td>(61000,21500)</td>
</tr>
<tr>
<td>2</td>
<td>100 kt</td>
<td>0</td>
<td>1000</td>
<td>0.99</td>
<td>(62000,17500)</td>
</tr>
</tbody>
</table>

**Target**

<table>
<thead>
<tr>
<th>VNIK</th>
<th>Value(v)</th>
<th>WR</th>
<th>(x,y) in feet</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>12P2</td>
<td>5000</td>
<td>(60000,19500)</td>
</tr>
<tr>
<td>2</td>
<td>14P2</td>
<td>8000</td>
<td>(68000,20500)</td>
</tr>
<tr>
<td>3</td>
<td>12P2</td>
<td>4000</td>
<td>(63000,23500)</td>
</tr>
</tbody>
</table>

From Eq (1)

$$\frac{\partial CEDF(x)}{\partial x_2} = v_1 * Pa_2 * (1 - Pa_2 * Pd(1,1)) * \frac{\partial Pd(2,1)}{\partial x_2}$$

$$+ v_2 * Pa_2 * (1 - Pa_2 * Pd(1,2)) * \frac{\partial Pd(2,2)}{\partial x_2}$$

$$+ v_3 * Pa_2 * (1 - Pa_2 * Pd(1,3)) * \frac{\partial Pd(2,3)}{\partial x_2}$$

(F-5)
From the subprogram FUNCT,

\[
Pd(1,1) = 1.000 \\
Pd(1,2) = 0.007 \\
Pd(1,3) = 0.999
\]

Hence,

\[
V_1*Pa_2 = 4950 \\
V_2*Pa_2 = 7920 \\
V_3*Pa_2 = 3960
\]

and

\[
Pa_2*Pd(1,1) = 0.99000 \\
Pa_2*Pd(1,2) = 0.00693 \\
Pa_2*Pd(1,3) = 0.98901
\]

Eq (F-5) becomes

\[
\frac{\partial \text{CEDF}(x)}{\partial x_2} = 49.50* \frac{\partial \text{Pd}(2,1)}{\partial x_2} + 7865.1144* \frac{\partial \text{Pd}(2,2)}{\partial x_2} \\
+ 43.5204* \frac{\partial \text{Pd}(2,3)}{\partial x_2}
\]

where

\[
\frac{\partial \text{Pd}(2,1)}{\partial x_2} = \frac{e^{-u^2}}{\sqrt{2\pi} r^2} (xx_j - x_2)
\]

and

\[
u = \frac{i}{\sqrt{2}} \ast \text{abs} \left[ -\beta + \frac{1}{\bar{\beta}} \ln \left( \frac{WR(2,1)}{r} \right) \right]
\]

\(\beta, AA, BB, \) and \(r\) are calculated from Eqs (I.1) through (F-4).
For target 1: \[ AA = 60000 - 62000 = -2000 \]
\[ BB = 19500 - 17500 = 2000 \]
\[ r = 2828.427 \]
\[ \frac{\partial \text{pd}(x,1)}{\partial x_2} = -3.633135 \times 10^{-6} \]

For target 2: \[ AA = 68000 - 62000 = 6000 \]
\[ BB = 20500 - 17500 = 3000 \]
\[ r = 6708.204 \]
\[ \frac{\partial \text{pd}(2,2)}{\partial x_2} = 2.426656 \times 10^{-5} \]

For target 3: \[ AA = 63000 - 62000 = 1000 \]
\[ BB = 23500 - 17500 = 6000 \]
\[ r = 6082.763 \]
\[ \frac{\partial \text{pd}(2,3)}{\partial x_2} = 4.304235 \times 10^{-5} \]

Therefore,
\[ \frac{\partial \text{CEDF}(x)}{\partial x_2} = 49.50 \times (-3.633135 \times 10^{-6}) \]
\[ + 7865.1144 \times (2.426656 \times 10^{-5}) \]
\[ + 43.5204 \times (4.304235 \times 10^{-5}) \]
\[ = 0.19254823 \]

The value of the gradient element of the \text{CEDF}(x) for \( x_2 \) from \text{GFUNCT} was 0.19255136. This comparison also indicated that the routine \text{GFUNCT} was correctly forming the gradient of the \text{CEDF}(x).


12. Greenwood, Dr J. A. "Optimum Placement of DGZ's on a Target Complex - Successive Approximation Method". Report by AF Intelligence Center for the Assistant Chief of Staff Intelligence, HQ USAF. Washington DC, 16 May 1960.


Vita

Captain Edmund Glen Boy was born on 25 February 1949 in Tokyo, Japan, the son of Edmund G. and Jeannette C. Boy. He graduated from Nurnberg American High School in Nurnberg, Germany in 1967. After two years of college, Glen enlisted in the United States Air Force. In 1971, the Air Force selected him for the Airman’s Education and Commissioning Program. In 1973, he graduated from Colorado State University with a Bachelor of Science degree in Electrical Engineering, and on 4 September 1973 he received his Air Force commission. Next, he served for two years as an Electronic Warfare Test Engineer with the 3246th Test Wing at Eglin AFB, FL. Then he entered Undergraduate Pilot Training at Vance AFB, OK. On 1 February 1977, he pinned on his pilot’s wings. After completing B-52G Combat Crew Training at Castle AFB, CA, Glen was assigned to the 320th Bomb Wing at Mather AFB, CA. He served as a squadron copilot, a Stan/Eval copilot, and an aircraft commander. In August 1982, he entered the School of Engineering at the Air Force Institute of Technology.

He is married to the former Mary Beth Russell of North Platte, NE. They have three children, Gregory, Rebecca, and Leanne.

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