Rough Surface Scattering via the Smoothing Method

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Abstract. The smoothing method is used to find the first two moments, i.e., the mean and the two-point two-time correlation function, of the field scattered by a rough surface. The results are expressed in terms of a reflection coefficient and a differential scattering coefficient. They are compared with those found by several other methods.

1. Introduction

An acoustic wave which hits a rough surface produces a mean or coherent reflected field and a fluctuating or partially coherent scattered field. These fields can be calculated by the regular perturbation or Born expansion method, see e.g., [1], but the results diverge at grazing incidence. However they can be modified or renormalised to remain finite [1] by writing them in forms suggested by Twersky's self-consistent field method [2, 3]. We shall show that these same non-divergent results can be obtained directly by a different perturbation procedure, known as the smoothing method.

The smoothing method has been used to find the coherent field for certain kinds of rough surfaces by Wenzel [4], DeSanto, and others. (See DeSanto's review [5].) Here we apply it to more kinds of rough surfaces, which can be either moving or at rest, and we calculate both the coherent field and the two-point two-time correlation function of the field. Then we show that the results are the same as the renormalised ones in [1], which were compared there with Twersky's [2, 3] results for embossed surfaces. Thus the present calculation elucidates the relationship among the results of the regular perturbation or Born expansion method, the renormalised regular perturbation method, Twersky's self-consistent field method and the smoothing method.

2. The Integral Equation for the Scattered Field

Suppose that a wave with the acoustic velocity potential \( \psi(x, y, z, t) \) is incident upon a rough surface \( S \), and that it produces a scattered wave with potential \( \phi(x, y, z, t) \). Both potentials must satisfy the wave equation with constant sound speed \( c \) above the surface \( S \), and \( \psi \) must be outgoing. In addition \( \psi + \phi \) must satisfy on \( S \) boundary condition which depends upon the nature of the surface. We shall consider the following four boundary conditions:

\[
\begin{align*}
\partial_1 \psi + \phi &= 0 \quad \text{on} \quad z = c \chi \\
(\partial_z - \epsilon h_0 \psi_z - c \phi_z)(\psi + \phi) &= \epsilon \chi h_0 \quad \text{on} \quad z = c \chi \\
(c \partial_t + \epsilon c h_0 \partial_z)(\psi + \phi) &= 0 \quad \text{on} \quad z = 0 \\
(\partial_z + \epsilon \partial h_0 \partial_z)(\psi + \phi) &= 0 \quad \text{on} \quad z = 0
\end{align*}
\]

In all cases \( \epsilon \) is a small parameter and \( h(x, y, z) \) is a given function. Case a represents a soft surface, \( z = c \chi \), on which the pressure vanishes; case b represents a hard surface \( z = c \chi \) on which the acoustic normal velocity
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equals the surface normal velocity, case $c$ represents a flat surface $z = 0$ with the admittance $\varepsilon h$, and case $d$ represents a flat surface $z = 0$ with the impedance $\varepsilon h$. In case $d$ we shall not consider the acoustic radiation produced by the motion of the surface, so we shall omit the term $\varepsilon h_i$ on the right side of (1b).

Before proceeding we shall simplify (1a') and the homogeneous form of (1b') by expanding $\psi + \varphi$ in a Taylor series about $z = 0$ to obtain

$$[1 + \varepsilon h + \frac{\varepsilon^2 h^2}{2} - \partial_s^2 + O(\varepsilon^3)]\partial_s(\psi + \varphi) = 0 \quad \text{on } s = 0 \quad (1a)$$

$$[\partial_s + \varepsilon(\partial_s^2 - h_s - h_y) + \varepsilon^2(\frac{\partial_s^2}{2} - hh_s - h_y - h_y + O(\varepsilon^3))(\varphi + \psi) = 0 \quad \text{on } s = 0 \quad (1b)$$

Now all four boundary conditions (1a)–(1d) apply on the plane $s = 0$. This makes it convenient to introduce the Fourier transform pairs:

$$F(s) = (2\pi)^{-1} \int f(p)e^{-ip\cdot p}dp \quad (2)$$

$$f(p) = \int F(s)e^{ip\cdot s}ds \quad (3)$$

Here $p = (x, y, t)$ and $s = (\alpha, \beta, \omega)$ are three component vectors in physical space-time and in wavenumber-frequency space, respectively. We shall always use the corresponding capital letter to denote the transform of a function denoted by a small letter.

We now apply the transform (2) to the wave equation for $\varphi$ and find that $\Phi(s, z)$ satisfies the ordinary differential equation $\partial_s^2 + k^2(s) \Phi = 0$ where $k^2(s) = \omega^2/c^2 - \alpha^2 - \beta^2$. Since $\Phi$ must be outgoing at $s = +\infty$ when $k$ is real, and must vanish there when $k$ is imaginary, $\Phi$ must be of the form

$$\Phi(s, z) = A_s(s)e^{-ik(s)z} \quad (4)$$

Here $k(s)$ is defined by

$$k(s) = (\omega/c)[1 - (\alpha^2 + \beta^2)(\omega/c)^{-2}]^{\frac{1}{2}}, \quad (\omega/c)^2 \geq \alpha^2 + \beta^2$$

$$k(s) = -i[(\alpha^2 + \beta^2 - (\omega/c)^2)^{\frac{1}{2}}, \quad (\omega/c)^2 \leq \alpha^2 + \beta^2$$

The scattered amplitude $A_s(s)$ is unknown, and is to be found. To find it we first write $\Psi(s, z)$, the transform of the incident field, as

$$\Psi(s, z) = A_i(s)e^{ik(s)z} \quad (6)$$

The incident amplitude $A_i(s)$ is assumed to be known. We now apply the transform (2) to each of the boundary conditions (1a)–(1d) and use (4) and (6) in the resulting equations. In each case we obtain an integral equation for $A_s(s)$ of the form

$$K(-z)A_s = \pm K(e)A_s \quad (7)$$

The linear operator $K(e)$ in (7), which depends upon $\varepsilon$, can be written as

$$K(e) = I + \varepsilon K_1 + \varepsilon^2 K_s + O(\varepsilon^3)$$

Here $I$ is the identity while $K_1$ and $K_2$ are integral operators of the forms

$$K_1A = \int H(s-s')A(s')ds' \quad (8)$$

$$K_2A = -\frac{1}{2} \int H(s-s')H(s'-s'')[\omega^2]k^2(s')A(s')ds'ds' \quad (10a)$$
\[
K_2 A = \frac{1}{2} \int H(s - s') H(s'' - s''') \frac{\hat{k}(s)}{k(s)} \left( k^2(s') - 2\left(\omega''/c\right)^2 - \alpha\alpha'' - \beta\beta''\right) A(s'') ds'' ds'
\]  

(10b)

We recall that \( H \) is the transform of \( h \), while \( m \) and the choice of sign in (7) are given in Table I. For the admittance and impedance boundary conditions, cases c and d, the operator \( K_2 \) and the \( O(\varepsilon^3) \) term in (8) are both zero.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>(a) Soft</th>
<th>(b) Hard</th>
<th>(c) Admittance</th>
<th>(d) Impedance</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \pm )</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>( m(s, s') )</td>
<td>( i\omega k(s')/\omega )</td>
<td>( ik^{-1}(s) \left( \left( \frac{\omega'}{\omega} \right)^2 - \alpha\alpha' - \beta\beta' \right) )</td>
<td>( \omega'/ck(s) )</td>
<td>( \omega/ck(s') )</td>
</tr>
</tbody>
</table>

Table I. The sign and the function \( m(s, s') \).

3. Application of the Smoothing Method

Let \( h \) and \( A_0 \) be random functions statistically independent of one another. Then the solution \( A_0 \) of the integral equation (7) is random also. To calculate its first two moments we shall use the smoothing method, first used to treat waves in random media and similar problems by Bourret [6] and Keller [7]. It is convenient to write \( \langle f \rangle \) to denote the mean value with respect to \( h \) of any random function \( f \) that depends upon \( h \), and \( f' = f - \langle f \rangle \) to denote the corresponding fluctuating part of \( f \). Then \( \langle f' \rangle = 0 \). If \( f \) depends upon both \( h \) and \( A_0 \), then \( \langle f \rangle \) still depends upon \( A_0 \), so it is still random. We denote the average of \( f \) with respect to both \( h \) and \( A_0 \), by \( \langle f \rangle \).

We begin by using (8) in (7) to get

\[
(I - \varepsilon K_1 + \varepsilon^3 K_2) A_0 = \pm (I + \varepsilon K_1 + \varepsilon^3 K_2) A_0 + O(\varepsilon^3)
\]

(11)

Now we take the mean value of (11) with respect to \( h \) to obtain

\[
(I - \varepsilon K_1 + \varepsilon^3 K_2) A_0 - \varepsilon K_1 A_0' + \varepsilon^3 (K_2 A_0') = \pm (I + \varepsilon K_1 + \varepsilon^3 K_2) A_0 + O(\varepsilon^3)
\]

(12)

Next we subtract (12) from (11), keeping terms through \( O(\varepsilon) \), to get

\[
A_0' - \varepsilon K_1' (A_0) + (K_1 + K_2 A_0') = \pm \varepsilon K_1' A_0 + O(\varepsilon^3)
\]

(13)

Upon solving (13) for \( A_0' \) up to \( O(\varepsilon) \), we find

\[
A_0' = \pm \varepsilon K_1' (A_0) + O(\varepsilon^3)
\]

(14)

Finally we use (14) for \( A_0' \) in (12) and obtain

\[
[I - \varepsilon (K_1) - \varepsilon^3 (K_1') + \varepsilon^3 (K_2)](A_0) = \pm [I + \varepsilon (K_1) + \varepsilon^3 (K_1')] A_0 + O(\varepsilon^3)
\]

(15)

This is the desired smoothed or non-random equation for \( A_0 \), and (14) is an expression for \( A_0' \) in terms of \( (A_0) \). The operators \( (K_1) \) and \( (K_2) \) in (15) are given respectively by (9) with \( H \) replaced by \( (H) \) and by (10) with \( HH \) replaced by \( (HH) \). The operator \( (K_1') \) is given by

\[
(K_1') A = \int (H(s - s') H(s'' - s''')) m(s, s') m(s', s'') A(s'') ds' ds''
\]

(10)
The integral equation (15) simplifies considerably when the function \( h \) is statistically second order stationary. Then the mean of \( h \) is a constant \( h_{\mu} \), i.e., \( \langle h(p + p') \rangle = h(p) + h_4 \). Consequently the first two moments of the transform \( H \) are given by \( \langle H(s) \rangle = h_\mu \delta(s) \) and \( \langle H(s)H(s') \rangle = R(s)\delta(s + s') + h_2^2 \delta(s)\delta(s') \). Here \( R(s) \), the spectral power density of \( h(p) \), is the Fourier transform of the auto-correlation function \( r(p) \), so it is a positive, even, real-valued function of \( s \).

Consequently the first two moments of the transform \( H \) are given by

\[
\langle K_1 \rangle A = h_\mu m(s, s) A(s)
\]

\[
\langle K_1' K_1' \rangle A = \left[ \int R(s - s')m(s, s')m(s', s') ds' \right] A(s)
\]

\[
\langle K_2 \rangle A = -\frac{1}{2} k_2(s)[V + h_2^2] A(s)
\]

(19a, b)

Here \( V = \int R(s)ds \) is the variance of \( h \). Thus for stationary processes, these three operators are multiplicative, and the smoothed equation (15) is just a linear algebraic equation for \( A_\mu \). In cases c and d, \( \langle K_2 \rangle = 0 \).

4. Solution for the Mean Amplitude

In general the solution of (15) for \( A_\mu \) can be written

\[
\langle A_\mu \rangle = MA_\mu + O(\varepsilon^3)
\]

The linear operator \( M \), which occurs in (20), is not random, and must be found by solving the integral equation (15). However, when \( h(p) \) is second order stationary, the solution of (15) is

\[
\langle A_\mu(s) \rangle = C(s)A_\mu(s) + O(\varepsilon^3)
\]

(21)

Here \( C(s) \) is just a scalar function which we call the reflection coefficient. It is given by

\[
C(s) = \pm \left[ 1 + \varepsilon(K_1) + \varepsilon^2(K_1'K_1') + \varepsilon^3(K_2)[1 - \varepsilon(K_1) - \varepsilon^2(K_1'K_1') + \varepsilon^3(K_2)] + O(\varepsilon^3) \right]
\]

\[
= \pm \left[ 1 + Q(s) \right] / \left[ 1 - Q(s) \right] + O(\varepsilon^3)
\]

(22)

The sign is shown in Table I and \( Q(s) \) is defined by

\[
Q(s) = \left[ eh_\mu m(s, s) + \varepsilon^2 \int R(s - s')m(s, s')m(s', s') ds' \right] \left[ 1 - \frac{\varepsilon^3}{2} k_2(s)[V + h_2^2] \right]^{-1}
\]

(23a, b)

\[
Q(s) = \varepsilon h_\mu m(s, s) + \varepsilon^3 \int R(s - s')m(s, s')m(s, s') ds'
\]

(23c, d)

When (21) is averaged with respect to \( A_\mu \), it yields

\[
\langle \langle A_\mu(s) \rangle \rangle = C(s) \langle A_\mu \rangle + O(\varepsilon^3)
\]

(24)

Here the average of \( A_\mu \) pertains to both \( h \) and \( A_\mu \), while \( \langle A_\mu \rangle \) means the average of \( A_\mu \), which is independent of \( h \).

The second form of \( C(s) \) in (22) is obtained from the first form by dividing numerator and denominator by \( 1 + \varepsilon^2(K_2) \), following Twerisky [8]. If the denominator in (23a, b) is replaced by unity, which corresponds to an \( O(\varepsilon^3) \) change in \( C(s) \), the resulting \( C(s) \) is identical with equation (42) of [1], which is the renormalised Born reflection coefficient.
5. The Second Moment of the Field

Let us consider the total acoustic potential \( \psi + \varphi \) at two points with separation \( z = (x, y, z, t) \) and midpoint \( X = (X, Y, Z, T) \). The second moment or two-point two-time correlation function of the acoustic potential at these points is \( \gamma(z, X) = \langle (\psi + \varphi)[X + \frac{z}{2}]\psi + \varphi[X - \frac{z}{2}] \rangle \). We denote its Fourier transform \( \Gamma(s, q, z, Z) \) where \( s = (\alpha, \beta, \omega) \) corresponds to \( (x, y, t) \) and \( q = (a, b, w) \) corresponds to \( (X, Y, T) \), and we write it as

\[
\Gamma(s, q, z, Z) = \langle (\psi + \Phi)(\frac{q}{2} + s, Z + \frac{z}{2})|\psi + \Phi|\frac{q}{2} - s, Z - \frac{z}{2}) \rangle
\]  

(25)

By using (4) and (6) for \( \Phi \) and \( \Psi \) we can express \( \Gamma \) in terms of the four correlations \( B_{\alpha\alpha}(s, q) = \langle (A_{s})(\frac{q}{2} + s)|A_{s}|(\frac{q}{2} - s) \rangle \), \( B_{\alpha\beta} = \langle (A_{s})(\frac{q}{2} + s)|A_{s}|(\frac{q}{2} - s) \rangle + \langle (A_{s})(\frac{q}{2} + s)|A_{s}|(\frac{q}{2} - s) \rangle + O(\varepsilon^2) \)

(26)

Each of the four correlations above can be evaluated by writing \( A_{s} \) in terms of the four correlations \( B_{\alpha\alpha} = \langle (A_{s})(\frac{q}{2} + s)|A_{s}|(\frac{q}{2} - s) \rangle = \langle (A_{s})(\frac{q}{2} + s)|A_{s}|(\frac{q}{2} - s) \rangle \). See [1].

The auto-correlation \( B_{\alpha\alpha} \) becomes

\[
B_{\alpha\alpha}(s, q) = \langle (A_{s})(\frac{q}{2} + s)|A_{s}|(\frac{q}{2} - s) \rangle + \langle (A_{s})(\frac{q}{2} + s)|A_{s}|(\frac{q}{2} - s) \rangle + O(\varepsilon^2)
\]

(27)

When \( h \) is second order stationary, \( M \) becomes multiplication by \( C(s) \), and (26) simplifies to

\[
B_{\alpha\alpha}(s, q) = C(\frac{q}{2} + s)C(\frac{q}{2} - s)B_{\alpha\alpha}(s, q) + 4\varepsilon^2 \int \frac{R(s - s')m(\frac{q}{2} + s, \frac{q}{2} + s')m(\frac{q}{2} - s, \frac{q}{2} - s')B_{\alpha\alpha}(s', q)ds'}{[1 - Q(\frac{q}{2} + s)][1 - Q(\frac{q}{2} - s)]}
\]

(28)

The auto-correlation \( B_{\alpha\alpha} \) simplifies even more when the incident field \( \psi(z, y, z, t) \) is statistically second order stationary in \( x, y, t \). Then \( B_{\alpha\alpha} \) is just

\[
B_{\alpha\alpha}(s, q) = I_1(s)\delta(q)
\]

(29)

where \( I_1(s) \) is the intensity of the incident field. With (28) used in it, (27) becomes

\[
B_{\alpha\alpha}(s, q) = I_1(s)\delta(q) + O(\varepsilon^2)
\]

(30)

The function \( I_1(s) \) in (29), which is the intensity of the outgoing field in the direction \( s \), is defined by

\[
I_1(s) = [C(s)]^2I_0(s) + [h^{-2}(s)]\int_{s'}\sigma(s, s')I_1(s')ds' + O(\varepsilon^2)
\]

(31)

The coefficient \( \sigma(s, s') \) in (30) can be identified as the differential scattering coefficient of the surface. It is defined by

\[
\sigma(s, s') = \frac{4\pi^2|h^{-2}(s)|R(s - s')m^2(s, s')}{|1 - Q(s')|^2}
\]

(32)

With the denominator in (23a,b) replaced by unity, this is exactly the result for \( \sigma \) given in equation (44) of [1], and called there the renormalized Born approximation.

6. Conclusion

By using the smoothing method, we have calculated the first two moments of the field produced when a possibly random incident field hits a random slightly rough surface. The first moment of the field, called the coherent field, consists of an incident and a reflected field. The reflected field is determined by a reflection coefficient \( C(s) \), which depends upon the first two moments of the surface roughness \( h \). The second moment of the field involves a differential scattering coefficient \( \sigma(s, s') \) which also depends upon the second moment of \( h \).
Our results for $C(s)$ and $\sigma(s,a')$ are the same as those we obtained before [1] by renormalizing the divergent results given by the Born approximation. That renormalization was achieved by writing $C$ and $\sigma$ in the forms obtained by Twersky [2,3]. We also showed in [1] that for an embossed plane, our results are the same as his except for one difference. We had the second Born approximation to the differential scattering amplitude of a single boss instead of the exact amplitude which occurs in his theory. This same comparison applies to the results of the smoothing method, as used here.

We conclude that three different methods yield the same results for the $C$ and $\sigma$ of an embossed plane, provided that each boss is small enough, with a small enough slope, so that its differential scattering amplitude is well approximated by its second Born approximation. The three methods are Twersky's method, the renormalized Born approximation and the smoothing method as used here. When the bosses or their slopes are not small enough, Twersky's results are better than the other two. On the other hand, if the surface is not an embossed plane, Twersky's method does not apply but the other two methods do.

References


