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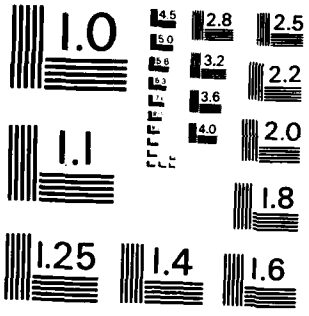

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ON THE FACIAL STRUCTURE OF SCHEDULING POLYHEDRA

by

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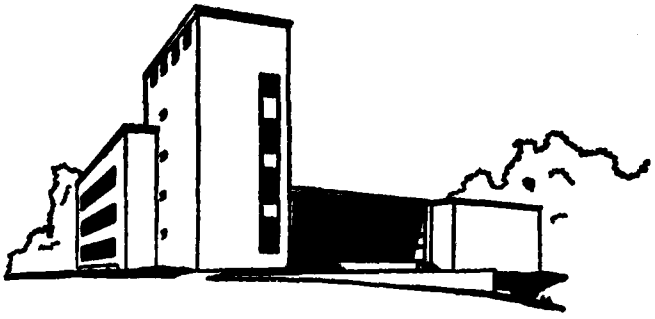

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August 1983

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Abstract

A well-known job shop scheduling problem can be formulated as follows. Given a graph G with node set N and with directed and undirected arcs, find an orientation of the undirected arcs that minimizes the length of a longest path in G. ~~We~~ treats the problem as a disjunctive program, without recourse to integer variables, and give a partial characterization of the scheduling polyhedron P(N), i.e., the convex hull of feasible schedules. In particular, ~~we~~ he derives all the facet inducing inequalities for the scheduling polyhedron P(K) defined on some clique with node set K, and give a sufficient condition for such inequalities to also induce facets of P(N). One of our results is that any inequality that induces a facet of P(H) for some H properly included in K, also induces a facet of P(K). Another one is a recursive formula for deriving a facet inducing inequality with p positive coefficients from one with p-1 positive coefficients. ~~We~~ also addresses the constraint identification problem, and gives a procedure for finding an inequality that cuts off a given solution to a subset of the constraints.

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APPENDIX.....A.1

1. Introduction

We consider the following machine sequencing problem which is a special case of resource-constrained scheduling (for background material see [1, 2], [8,...,12]). A number of items have to be processed by performing a sequence of operations on each of them on specified machines. There are n operations to be performed, including a fictitious "stop" (operation n), the objective being to minimize total completion time subject to (i) precedence constraints between the operations, and (ii) the condition that a machine can process only one item at a time, and operations cannot be interrupted. The problem can be stated as

$$
(P) \qquad
\begin{aligned}
&\min t_n \\
&t_j - t_i \geq d_{ij} && (i,j) \in A \\
&t_i \geq 0, && i \in N \\
&t_j - t_i \geq d_{ij} \vee t_i - t_j \geq d_{ji}, && (i,j) \in E
\end{aligned}
$$

where t_i is the starting time of operation i, d_{ij} is the minimum required time lapse between starting operation i and starting operation j (for instance, completion time of operation i, plus set-up time for operation j), A indexes the pairs of operations constrained by precedence relations, E the pairs that use the same machine and therefore cannot overlap in time, and "$\vee$" is the logical "or". It is useful to represent the problem by a disjunctive graph [1, 2, 10, 12] $G = (N^o, A^o, E)$, where $N^o = \{0\} \cup N$ is a set of nodes, one for each operation, plus a source node 0; $A^o = A \cup \{(0,j) \mid$ is not preceded by an operation$\}$ is a set of (conjunctive) directed arcs; E is a set of undirected arcs, one for every pair of operations to be performed

on the same machine. Solving the problem involves orienting the undirected arcs, i.e., choosing for each of them one of the two possible directions. It is therefore convenient to represent each undirected arc by a disjunctive pair of directed arcs, i.e., a pair of which one member needs to be selected: hence the name disjunctive graph. We will use this latter representation, and consider E to consist of pairs of directed arcs $(i,j),(j,i)$, with $E^+ = \{(i,j) \in E \mid i < j\}$, $E^- = \{(i,j) \in E \mid i > j\}$, and $E = E^+ \cup E^-$. The arcs of E occur in disjoint maximal cliques (by a clique we mean a complete digraph), of which there is one for every machine. Thus if M indexes the set of maximal cliques (machines), and for $V \subset N^o$, $< V >$ denotes the subgraph of G induced by V, then for every $r \in M$, the node set K_r of the rth maximal clique $< K_r^{\cdot} >$ corresponds to the set of operations to be performed on the same machine (r).

Every directed arc $(i,j) \in A \cup E$ has a positive length d_{ij}, while the arcs $(0,j) \in A^o \setminus A$ have length $d_{oj} = 0$. For a pair $\{(i,j), (j,i)\} \in E$, $d_{ij} \neq d_{ji}$ is not only possible, but typical. We will assume that the arc lengths are integers satisfying the <u>triangle inequality</u> $d_{ij} + d_{jk} \geq d_{ik}$, $\forall$ i,j,k. Though this assumption involves some loss of generality, it is realistic for the machine sequencing problem. The disjunctive graph G is illustrated in Figure 1

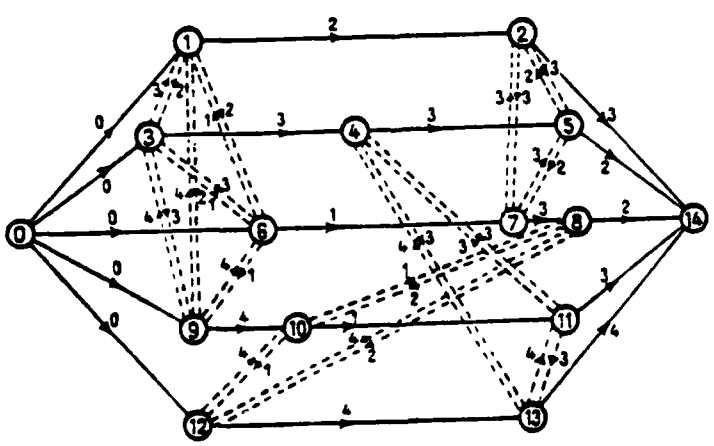


Figure 1

on a problem with 5 items (directed source-sink paths), 4 machines (maximal cliques, whose arcs are shown in dotted lines), and 14 operations (nodes other than the source). The numbers on the arcs are the lengths d_{ij}.

The subgraph obtained from G by deleting the disjunctive arc set E is the ordinary digraph $D = (N^o, A^o)$, in which node 0 has indegree zero and outdegree the number of items, node n has indegree the number of items and outdegree zero, while all remaining nodes have indegree and outdegree one. In fact D is the union of as many disjoint (except for their end nodes) paths from 0 to n, as there are items.

A <u>selection</u> in G consists of exactly one member of each pair of disjunctive arcs in E. Thus, if $\alpha = \frac{1}{2}|E|$, there are 2^α possible selections in G. In the undirected representation of E, a selection in G corresponds to an orientation of all the undirected arcs of G.

For every selection S in G, $D_S = (N^o, A^o \cup S)$ is an ordinary digraph; and the problem obtained from (P) by replacing the set of disjunctive constraints indexed by E with the set of conjunctive constraints indexed by S is the dual of a longest path (critical path) problem in D_S. Thus solving (P) amounts to finding a minimaximal path in the disjunctive graph G, i.e., finding a selection (orientation) S that minimizes the length of a critical path in D_S over the set of all possible selections.

Problem (P) stated at the beginning of this section has a variable t_j associated with every node of G except for 0. One can of course introduce a variable t_o for node 0, but then the problem does not change if t_o is constrained by $t_o = 0$, which leads to the elimination of the variable just introduced. We therefore prefer to work with vectors $t \in \mathbb{R}^n$ that don't have a component t_o constrained to be 0.

Problem (P) is a disjunctive program. It can also be represented as a mixed integer program by introducing a binary variable for every disjunctive constraint, but there are advantages to not doing that and using instead the disjunctive programming approach (for background see [3, 5]). In this paper we investigate the properties of the scheduling polyhedron P, the closed convex hull of all vectors $t \in \mathbb{R}^n$ satisfying the constraints of (P). Section 2 introduces the polyhedron P, states some of its basic properties, and discusses the relationship of P to polyhedra defined by subsets of the constraint set. Section 3 deals with scheduling polyhedra P(K) defined on a clique with node set K, and characterizes the vertices of P(K). Section 4 gives a complete characterization of the facets of P(K). One of the results is that any inequality that defines a facet of P(H) for some $H \subset K$ also defines a facet of P(K). Another result is a procedure for deriving a facet defining inequality for P(K) with p nonzero coefficients from a facet defining inequality with p-1 nonzero coefficients. This section also lists all the facets of P(K), for K of arbitrary size, having one, two or three nonzero coefficients. Section 5 gives a sufficient condition for an inequality that defines a facet of P(K) to also define a facet of P. The condition is verifiable in $O(|E|)$ time. Finally, section 6 addresses the constraint identification problem and gives a procedure for identifying facet defining inequalities that cut off a given $t \in \mathbb{R}^n$ that violates some of the disjunctions of (P). Some of our results were presented in [4].

2. Some Properties of the Scheduling Polyhedron

Any $t \in \mathbb{R}^n$ satisfying the constraints of (P) will be called a schedule for G. The feasible set of (P), or the set of schedules for G, can be written as

$$T = \left\{ t \in \mathbb{R}^n \;\middle|\; \begin{array}{ll} t_j - t_i \geq d_{ij}, & (i,j) \in A \\ t_i \geq 0, & i \in N \\ t_j - t_i \geq d_{ij} \vee t_i - t_j \geq d_{ji}, & i,j \in K_\ell,\ i \neq j,\ \ell \in M \end{array} \right\}.$$

The closed convex hull of T, clconv T, will be called the <u>scheduling polyhedron</u>, and denoted P(N), or simply P.

T is a disjunctive set, and its convex hull is easiest to describe when T is in disjunctive normal form [3, 4], i.e., in the form $T = \bigcup_{S \in Q} T_S$, where Q is the index set of all selections in G, and T_S is the (polyhedral) set of schedules for the digraph D_S defined by the selection S in G:

$$T_S = \left\{ t \in \mathbb{R}^n \;\middle|\; \begin{array}{ll} t_j - t_i \geq d_{ij} & (i,j) \in A \cup S \\ t_i \geq 0\ , & i \in N \end{array} \right\}.$$

If D_S contains a cycle, $T_S = \emptyset$. So the only selections of interest are those for which the associated digraph D_S has no cycles, i.e., those indexed by $Q^* = \{S \in Q \mid D_S \text{ is acyclic}\}$, since $T = \bigcup_{S \in Q^*} T_S$. In the sequel we assume that $Q^* \neq \emptyset$. For any $S \in Q^*$, we will denote by $\ell(i,j)_S$ the length of a longest path from i to j in D_S. The length of the (unique) path from i to j in D will be denoted by L(i,j).

<u>Theorem 2.1</u>. For every $S \in Q^*$, T_S has dimension n.

<u>Proof</u>. We define n + 1 vectors $t^i \in \mathbb{R}^n$, $i = 0,1,\ldots,n$, as follows. Let t^0 be defined by $t^0_j = \ell(0,j)_S$, $j = 1,\ldots,n$; and for $i = 1,\ldots,n$, let t^i be defined by

$$t_j^i = \begin{cases} 2t_j^o + \epsilon & j = i \\ 2t_j^o & j \neq i, \end{cases}$$

where $0 < \epsilon < 1/2$.

Clearly, $t^o \epsilon T_S$. For $i=1,\ldots,n$, $t_j^i \geq 0$, $\forall j$, and for $(h,j) \epsilon A \cup S$, one can easily check that $t_j^i - t_h^i \geq d_{hj}$. Thus for $i = 0,1,\ldots,n$, $t^i \epsilon T_S$. Also, the n+1 points $t^i \epsilon \mathbb{R}^n$ are affinely independent, since the $n \times n$ matrix whose i^{th} row is $t^i - 2t^o$, $i = 1,\ldots,n$, is ϵ times the identity matrix of order n.||

<u>Corollary 2.2</u>. P is full dimensional.

Next we turn to the extreme points of P. First we characterize the extreme points of T_S for an arbitrary $S \epsilon Q^*$.

<u>Theorem 2.3</u>. A schedule t for D_S is an extreme point of T_S if and only if $t_n = \ell(0,n)_S$, and for all $j \epsilon N \backslash \{n\}$, $t_j = \ell(0,j)_S$ or $t_j = \ell(0,n)_S - \ell(j,n)_S$, or both.

<u>Proof</u>. Necessity. Let $t^* \epsilon T_S$ be such that $t_n^* > \ell(0,n)_S$. Define t^1 and t^2 by $t_n^1 = t_n^* + \epsilon$, $t_j^1 = t_j^*$, $j \neq n$; and $t_n^2 = t_n^* - \epsilon$, $t_j^2 = t_j^*$, $j \neq n$. For, suitably small $\epsilon > 0$, $t^1, t^2 \epsilon T_S$, and $t^* = \frac{1}{2}(t^1 + t^2)$, with $t^1 \neq t \neq t_2$; hence t^* is not extreme. Thus the condition $t_n = \ell(0,n)_S$ is necessary.

Now let $t^o \epsilon T_S$ be such that $t_n^o = \ell(0,n)_S$, but the remaining conditions of the Theorem are violated for $j \epsilon N^* \subseteq N \backslash \{n\}$; i.e., let $N^* := \left\{ j \epsilon N \backslash \{n\} \middle| \ell(0,j)_S < t_j^o < \ell(0,n)_S - \ell(j,n)_S \right\}$. Define t' and t'' by $t_j' = t_j^o + \epsilon$, $j \epsilon N^*$, $t_j' = t_j^o$, $j \epsilon N \backslash N^*$; and $t_j'' = t_j^o - \epsilon$, $j \epsilon N^*$, $t_j'' = t_j^o$, $j \epsilon N \backslash N^*$. Then for suitably small ϵ, $t', t'' \epsilon T_S$, $t' \neq t^o \neq t''$, and $t^o = \frac{1}{2}(t' + t'')$. Thus the condition $t_j = \ell(0,j)_S$ or $t_j = t_n - \ell(j,n)_S$ or both, for all $j \epsilon N \backslash \{n\}$, is also necessary.

Sufficiency. Suppose $t \in T_S$ is not extreme. If $t_n > \ell(0,n)_S$, we are done. So let $t_n = \ell(0,n)_S$. By assumption, there exist $t^1, t^2 \in T_S$, $t^1 \neq t \neq t^2$, such that $t = \frac{1}{2}(t^1+t^2)$. Now $t_n = \ell(0,n)_S$ implies $t^1_n = t^2_n = \ell(0,n)_S$, and since $t^1 \neq t \neq t^2$, there exists $j \in N\setminus\{n\}$ such that either $t^1_j < t_j < t^2_j$ or $t^2_j < t_j < t^1_j$. In the first case

$$\ell(0,j)_S \leq t^1_j < t_j < t^2_j \leq \ell(0,n)_S - \ell(j,n)_S ,$$

and in the second case the same condition holds with the roles of t^1 and t^2 reversed. In both cases t violates the condition that $t_j = \ell(0,j)_S$ or $t_j = \ell(0,n)_S - \ell(j,n)_S$ or both.‖

<u>Corollary 2.4</u>. If t is an extreme point of P, then $t_n = \ell(0,n)_S$, and $t_j = \ell(0,j)_S$ or $t_j = \ell(0,n)_S - \ell(j,n)_S$ or both, $\forall j \in N\setminus\{n\}$, for some $S \in Q^*$.

<u>Proof</u>. Every extreme point of P is an extreme point of T_S for some $S \in Q^*$.‖

For every (not necessarily maximal) clique $< K >$, we define a <u>schedule for $< K >$</u> as a vector $t \in T(K)$, where

$$T(K) = \left\{ t \in \mathbb{R}^p \;\middle|\; \begin{array}{ll} t_i \geq L(0,i), & i \in K \\ t_j - t_i \geq d_{ij} \vee t_i - t_j \geq d_{ji}, & i,j \in K,\ i \neq j \end{array} \right\},$$

where $p = |K|$, and $L(0,i)$ is the length of the (unique) path from 0 to i in $D = (N^o, A^o)$. The closed convex hull of $T(K)$, clconv $T(K)$, will be called the <u>scheduling polyhedron on $< K >$</u>, and denoted $P(K)$.

For any $V \subset N$, we denote by $S(V)$ a <u>selection in $< V >$</u>, i.e., a set of arcs containing exactly one member of each disjunctive pair of arcs with both ends in V. For $V' \subset V \subseteq N$, we say that the selection $S(V)$ is an <u>extension to $< V >$</u> of the selection $S(V')$ (the selection $S(V')$ is a restriction to $< V' >$ of the selection $S(V)$) if the arcs of $S(V)$ with both ends in V' are precisely those of $S(V')$.

versely, we say that a schedule t' for $< K >$ is a <u>restriction to $< K >$</u> of the schedule t for G, if t is an extension of t'. By the choice of the lower bounds $L(0,i)$, $i \in K$, every schedule for G can be restricted to any of the cliques of G. Therefore, for every clique $< K >$ of G,

$$P \subseteq P(K) . \tag{2.2}$$

The more interesting question, of course, is when can a schedule for some clique $< K >$ be extended to a schedule for G. This question is intimately related to the problem of facet lifting, i.e., to the connection between facet inducing inequalities for $P(K)$ and for P. It will be investigated in section 5, where we will give a sufficient condition for an inequality that defines a facet of $P(K)$ to also define a facet of P. This condition is always satisfied for some of the cliques of G, so at least some of the facet inducing inequalities for $P(K)$ are always facet inducing for P itself. This provides the main, though not the only, motivation for focusing in the next 2 sections on the polyhedra $P(K)$.

3. <u>The Scheduling Polyhedron on a Clique</u>

In this section we study the properties of the scheduling polyhedron on a clique, or briefly the <u>clique polyhedron</u> $P(K) = \text{clconv } T(K)$. If $|K| = p$ and if we denote $L_i = L(0,i)$, $i \in K$, then

$$T(K) = \left\{ t \in \mathbb{R}^p \;\middle|\; \begin{array}{c} t_i \geq L_i , \quad i \in K \\ t_j - t_i \geq d_{ij} \vee t_i - t_j \geq d_{ji}, \; \forall\, i,j \in K,\; i \neq j \end{array} \right\} .$$

As before, a vector $t \in T(K)$ will be called a schedule for $< K >$.

Apart from its connection with machine sequencing, and more generally with the resource constrained scheduling problem, the polyhedron $T(K)$ is an

interesting object in its own right. A selection $S(K)$ in $< K >$ is the arc set of a tournament in $< K >$. Every tournament is known to have a directed Hamilton path (i.e., a directed path containing all the vertices), and for an acyclic tournament this path is unique. In fact, every acyclic tournament is the transitive closure of its unique directed Hamilton path. A selection $S(K)$ is therefore uniquely determined by the sequence of the nodes of K in its directed Hamilton path, and conversely, every selection $S(K)$ defines a unique sequence of the nodes of K. Thus the scheduling problem on a clique, namely the problem of finding

$$\min_{t \in T(K)} \max_{i \in K} t_i \,, \tag{3.1}$$

with $L_i = 0$, $i \in K$, is a "dual" formulation of the problem of finding a shortest Hamilton path in $< K >$, using node rather than arc variables. The latter problem in turn is polynomially equivalent to the traveling salesman problem (TSP). Indeed, an optimal tour for the TSP yields a shortest Hamilton path by deletion of the largest arc. Conversely, finding for each $i \in K$ a shortest Hamilton path originating in i (which is problem (3.1)) with the extra condition that $t_i = L_i = 0$), then adding to each path the unique arc that closes it, and choosing the shortest of the p resulting tours, yields an optimal solution to TSP.

The scheduling polyhedron $P(K)$ on a clique $< K >$ is related to the linear ordering polyhedron P_{LO} on $< K >$ studied recently by Grötschel, Jünger and Reinelt [5]. P_{LO} is the convex hull of the incidence vectors of acyclic tournaments in $< K >$. It is a bounded polytope in $\mathbb{R}^{p(p-1)}$, the space spanned by the arcs of the complete digraph $< K >$, whereas $P(K)$ is an unbounded polyhedron in $\mathbb{R}^p$. When $P(K)$ is specialized to the case where $L_i = 0$, $i \in K$, there

is a one to one correspondence between its vertices and acyclic tournaments in $< K >$, as will be shown later in this section. Hence there is a one to one correspondence between the vertices of $P(K)$ (in the case $L_i = 0$, $i \in K$) and those of P_{LO}. One might therefore expect a similarly close relationship between facets of P_{LO} and those of $P(K)$. In fact, however, the facets of $P(K)$ are rather different from, and seemingly unrelated to, those of P_{LO}. A set of p vertices that lie on a facet of P_{LO} may not lie on a facet of $P(K)$, and vice versa. While the facets of P_{LO} are independent of the arc lengths, the facets of $P(K)$ strongly depend on the arc lengths d_{ij}.

Whenever possible without risking confusion, the notation $S(K)$ for a selection in $< K >$ will be abbreviated to S. Every selection S in $< K >$ defines a polyhedron

$$T(K)_S = \left\{ t \in \mathbb{R}^p \;\middle|\; \begin{array}{ll} t_i \geq L_i \;, & i \in K \\ t_j - t_i \geq d_{ij}, & (i,j) \in S \end{array} \right\},$$

which is nonempty if and only if S is acyclic. Let $Q(K)$ be the set of selections in $< K >$, and $Q(K)^* = \{S \in Q(K) \mid S \text{ is acyclic}\}$. Then the disjunctive normal form of $T(K)$ becomes

$$T(K) = \bigcup_{S \in Q(K)^*} T(K)_S \;.$$

For every $S \in Q(K)^*$, $T(K)_S$ is obviously full-dimensional; hence so is $P(K)$.

For $i \in K$ and an acyclic selection S in $< K >$, we define the <u>rank</u> of i in S as the position (rank) of i in the sequence associated with S.

<u>Theorem 3.1</u>. Let S be an acyclic selection in $< K >$, and let $j(1),\ldots,j(p)$ be the sequence defined by S on K. Then $T(K)_S$ is a displaced polyhedral cone with vertex t^o defined by

$$(3.2)\qquad \begin{aligned} t^o_{j(1)} &= L_{j(1)} \\ t^o_{j(k)} &= \max\{L_{j(k)},\ t^o_{j(k-1)} + d_{j(k-1),j(k)}\}, \quad k = 2,\ldots,p, \end{aligned}$$

and extreme rays given by w_i, $i = 1,\ldots,p$, where

$$(3.3)\qquad w^i_{j(k)} = \begin{cases} 1 & k = p - i + 1,\ldots,p \\ 0 & \text{otherwise.} \end{cases}$$

<u>Proof</u>. The vector t^o satisfies all the inequalities of $T(K)_S$, and any $t \in T(K)_S$ satisfies $t_j \geq t^o_j$, $\forall\ j \in K$. Now define new variables $t'_j = t_j - t^o_j$, $j \in K$, and let $T(K)'_S$ be the polyhedral cone

$$T(K)'_S = \left\{ t' \in \mathbb{R}^p \,\middle|\, \begin{array}{ll} t'_i \geq 0, & i \in K \\ t'_j - t'_i \geq 0 \quad , & (i,j) \in S \end{array} \right\} .$$

We will show that $t \in T(K)_S$ if and only if $t' \in T(K)'_S$, where $t' = t - t^o$.

Let $t' \in T(K)'_S$. Then $t'_i \geq 0$ implies $t_i \geq t^o_i \geq L_i$, and $t'_j - t'_i \geq 0$ implies $t_j - t_i \geq t^o_j - t^o_i \geq d_{ij}$, hence $t \in T(K)_S$.

Conversely, let $t \in T(K)_S$. Then $t \geq t^o$, which implies $t'_j \geq 0$, $j \in K$. Further, for every $(i,j) \in S$, $t_j - t_i \geq \max\{d_{ij},\ L_j - t^o_i\}$; and hence

$$t'_j - t'_i \geq \max\{d_{ij},\ L_j - t^o_i\} - (t^o_j - t^o_i) = 0,$$

since

$$\begin{aligned} t^o_j - t^o_i &= \max\{L_j,\ t^o_i + d_{ij}\} - t^o_i \\ &= \max\{d_{ij},\ L_j - t^o_i\}. \end{aligned}$$

Finally, the vertex $t' = 0$ of $T(K)'_S$ corresponds to the point $t = t^o$ of $T(K)_S$. This proves that $T(K)_S$ is a displaced polyhedral cone with vertex t^o.

In view of the correspondence between $T(K)_S$ and $T(K)'_S$, the extreme directions of $T(K)_S$ are precisely those of $T(K)'_S$. Since the vertex of $T(K)'_S$ is the zero vector, w is a direction vector of $T(K)'_S$ (hence of $T(K)_S$) if and only if $w \epsilon T(K)'_S$. For $i = 1,\ldots,p$, it is easy to see that $w^i \epsilon T(K)'_S$. Further, each w^i satisfies $w^i_j = 0$ for $j = j(1),\ldots,j(p-i)$, and $w^i_{j(k)} - w^i_{j(k-1)} = 0$ for all $k \epsilon \{1,\ldots,p\} \setminus \{p - 1 + 1\}$. Since each $w^i \epsilon \mathbb{R}^p$ satisfies with equality p-1 inequalities whose coefficient matrix has full row rank, each w^i is extreme.

It remains to be shown that w^i, $i = 1,\ldots,p$ are the only extreme direction vectors of $T(K)'_S$. This we do by expressing an arbitrary $t' \epsilon T(K)'_S$ as a positive linear combination of the w^i, $i = 1,\ldots,p$.

Let $t' \epsilon T(K)'_S$, and consider the p vectors $t^{(i)}$, $i = 1,\ldots,p$, defined by $t^{(1)}_{j(k)} = t'_{j(1)}$, $k = 1,\ldots,p$, and for $i = 2,\ldots,p$,

$$t^{(i)}_{j(k)} = \begin{cases} 0 & k = 1,\ldots,i-1 \\ t'_{j(i)} - t'_{j(i-1)} & k = i,\ldots,p. \end{cases}$$

Then each $t^{(i)}$ is either the zero vector, of a positive multiple of the extreme direction vector w^{p-i+1}, and $t' = \sum_{i=1}^{p} t^{(i)}$.||

Next we turn to the extreme points and extreme direction vectors of P(K). Naturally, every extreme point of P(K) is an extreme point of $T(K)_S$ for some $S \epsilon Q(K)^*$; but the converse will be shown to be true only if P(K) satisfies a regularity condition. Also, every extreme direction of P(K) is an extreme direction of $T(K)_S$ for some $S \epsilon Q(K)^*$, but the converse is never true.

In order to prove some properties of the vertices of P(K) we need a characterization of the extreme direction vectors of P(K), so we start with the latter.

Theorem 3.2. The extreme direction vectors of $P(K)$ are precisely the unit vectors e_i, $i = 1,\ldots,p$.

Proof. For $i = 1,\ldots,n$, the unit vector e_i is an extreme direction vector of every cone $T(K)_S$ such that i is the last node of the sequence defined by S. Hence every e_i is a direction vector of $P(K)$, and since e_i is a unit vector and $T(K)$ is contained in the positive orthant, each e_i is extreme for $P(K)$. Every other extreme direction vector of $T(K)_S$, for every $S \in Q(K)^*$, is the sum of unit vectors; hence none of them is extreme for $P(K)$. Since every extreme direction of $P(K)$ is an extreme direction of $T(K)_S$ for some $S \in Q(K)^*$, it follows that $P(K)$ has no extreme direction vectors other than the p unit vectors e_i.||

Next we turn to the extreme points of $P(K)$. We will say that the disjunctive set $T(K)$ (as well as the polyhedron $P(K)$ and the clique $< K >$) is regular, if

$$L_j - L_i < d_{ij} \quad , \quad \forall\, i,j \in K \;,\; i \neq j \tag{3.4}$$

and

$$d_{ij} + d_{jk} > d_{ik} \quad , \quad \forall\, i,j,k \in K,\; i \neq j \neq k \neq i. \tag{3.5}$$

Condition (3.4) implies that for every $S \in Q(K)^*$, the vertex t^o of the cone $T(K)_S$ satisfies $t^o_{j(1)} = L_{j(1)}$ and

$$t^o_{j(k)} = t^o_{j(k-1)} + d_{j(k-1),j(k)} > L_{j(k)}, \; k = 2,\ldots,p. \tag{3.6}$$

i.e., the second term of the bracketed expression in (3.2) is strictly greater than the first term for $k = 2,\ldots,p$, where, as before, $p = |K|$. By recursively substituting for $t^o_{j(k-1)}$, (3.6) can also be written as

$$t^o_{j(k)} = L_{j(1)} + \sum_{h=2}^{k} d_{j(h-1),j(h)} > L_{j(k)} \quad , \quad k = 2,\ldots,p. \tag{3.7}$$

Later we will show that regularity plays a crucial role in the facial structure of P(K): certain facets exist only if P(K) is regular. Here we prove that regularity is a necessary and sufficient condition for the vertex of every cone $T(K)_S$ to be a vertex of P(K).

Lemma 3.3. Let $t^o \in \mathbb{R}^p$ be a schedule for $< K >$. If there exists a schedule t^* such that $t^* \leq t^o$ and $t^*_i < t^o_i$ for some $i \in K$, then t^o is not a vertex of P(K).

Proof. If there exists t^* as described, then t^o can be expressed as the sum of t^* and a positive combination of unit vectors. Since the latter are direction vectors of P(K), t^o is not an extreme point of P(K).||

Lemma 3.4. Let $j \in K$ be such that $L_j - L_i \geq d_{ij}$ for some $i \in K \setminus \{j\}$. Then for every $S \in Q(K)^*$ in which j has rank 1, the vertex of $T(K)_S$ is not a vertex of P(K).

Proof. Let S be any acyclic selection with associated sequence $j(1),\ldots,j(p)$ such that $j = j(1)$ and $i = j(q)$ for some $1 < q \leq p$; and let t^o be the vertex of $T(K)_S$. Then t^o satisfies (3.2). Now let $S^* \in Q(K)^*$ be the selection associated with the sequence $\ell(1),\ldots,\ell(p)$, where $\ell(1) = i = j(q)$, and

$$\ell(k) = \begin{cases} j(k-1) & k = 2,\ldots,q \\ j(k) & k = q+1,\ldots,p; \end{cases}$$

and let t^* be the vertex of $T(K)_{S^*}$. Then $t^*_{\ell(1)} = L_{j(q)} < t^o_{j(q)} = t^o_{\ell(1)}$, since the positivity of $d_{k\ell}$ for all $k,\ell \in K$ implies that $t^o_{j(q)} > L_{j(1)} = L_j > L_i = L_{j(q)}$. For $2 \leq k \leq q$, we show by induction that $t^*_{j(k)} = t^o_{j(k)}$. For $k = 2$, we have $t^*_{\ell(2)} = L_j = t^o_{j(1)} = t^o_{\ell(2)}$. Suppose the equality holds for $k = 2,\ldots,r-1$, and let $k = r \leq q$. Then by the induction hypothesis

$$(3.8)\qquad t^*_{\ell(r)} = \max\{L_{j(r-1)},\ t^o_{j(r-2)} + d_{j(r-1),j(r)}\}$$
$$= t^o_{j(r-1)} = t^o_{\ell(r)}.$$

For $k = q + 1$, we have

$$(3.9)\qquad t^*_{\ell(q+1)} = \max\{L_{j(q+1)},\ t^o_{j(q-1)} + d_{j(q-1),j(q+1)}\}$$
$$\leq \max\{L_{j(q+1)},\ t^o_{j(q)} + d_{j(q),j(q+1)}\}$$
$$= t^o_{j(q+1)} = t^o_{\ell(q+1)},$$

where we have used the triangle inequality $d_{j(q-1),j(q+1)} \leq d_{j(q-1),j(q)} + d_{j(q),j(q+1)}$.

Finally, for $q + 2 \leq k \leq p$ (if $p \geq q + 2$), we have again by induction on k,

$$(3.10)\qquad t^*_{\ell(k)} \leq \max\{L_{j(k)},\ t^o_{j(k-1)} + d_{j(k-1),j(k)}\}$$
$$= t^o_{j(k)} = t^o_{\ell(k)}.$$

We have shown that $t^*_h \leq t^o_h$ for all $h \in K$, with $t^*_h < t^o_h$ for $h = j(q) = \ell(1)$. From Lemma 3.3, it then follows that t^o is not a vertex of $P(K)$.||

<u>Lemma 3.5</u>. Let the ordered triple $\{i,j,h\}$ be such tnat $d_{ij} + d_{jh} = d_{ih}$. Then for any $S \in Q(K)^*$ in which i and h have rank 1 and 2 respectively, the vertex of $T(K)_S$ is not a vertex of $P(K)$.

<u>Proof</u>. Let S be any acyclic selection with associated sequence $j(1),\ldots,j(p)$ such that $i = j(1)$, $h = j(2)$, and $j = j(q)$ for some $2 < q \leq p$; and let t^o be the vertex of $T(K)_S$. Then $t^o_{j(1)} = L_{j(1)}$ and

$$t^o_{j(k)} = \max\{L_{j(k)},\ t^o_{j(k-1)} + d_{j(k-1),j(k)}\},\ k = 2,\ldots,p.$$

Now let S^* be the selection associated with the sequence $\ell(1),\ldots,\ell(p)$, where $\ell(1) = i = j(1)$, $\ell(2) = j = j(q)$, and

$$\ell(k) = \begin{cases} j(k-1) & k = 3,\ldots,q \\ j(k) & k = q+1,\ldots,p, \end{cases}$$

and let t^* be the vertex of $T(K)_{S^*}$. Then $t^*_{\ell(1)} = L_{j(1)} = t^o_{j(1)} = t^o_{\ell(1)}$. For $2 \le k \le q$, we show by induction that $t^*_{\ell(k)} \le t^o_{\ell(k)}$. For $k = 2$, $t^*_{\ell(2)} = \max\{L_{j(q)}, L_{j(1)} + d_{j(1),j(q)}\} \le t^o_{j(q)} = t^o_{\ell(2)}$. Suppose $t^*_{\ell(k)} \le t^o_{\ell(k)}$ for $k = 2,\ldots,r-1$, and let $k = r \le q$. Then by the induction hypothesis

$$\begin{aligned} t^*_{\ell(r)} &\le \max\{L_{j(r-1)}, t^o_{j(r-2)} + d_{j(r-2),j(r-1)}\} \\ &= t^o_{j(r-1)} = t^o_{\ell(r)}. \end{aligned}$$

For $k = q+1$, (3.9) holds and for $q+2 \le k \le p$, (3.10) holds for the same reasons as in the proof of Lemma 3.4. This proves that $t^*_{\ell(k)} \le t^o_{\ell(k)}$ for $k = 1,\ldots,p$.

Further, from the positivity of $d_{k\ell}$ for all $k,\ell \in K$, it follows that $t^*_{\ell(2)} < t^*_{\ell(3)}$ and $t^o_{\ell(2)} = t^o_{j(q)} > t^o_{\ell(3)} = t^o_{j(2)}$. Therefore at least one of the two inequalities $t^*_{\ell(2)} \le t^o_{\ell(2)}$ and $t^*_{\ell(3)} \le t^o_{\ell(3)}$ holds strictly.

Thus from Lemma 3.3 it follows that t^o is not a vertex of $P(K)$.||

<u>Theorem 3.6</u>. The vertices of $P(K)$ are precisely the vertices of the cones $T(K)_S$, $S \in Q(K)^*$, if and only if $T(K)$ is regular.

<u>Proof</u>. The "only if" part follows from Lemmas 3.4 and 3.5. To prove the "if" part, suppose $T(K)$ is regular, and let t^o be the vertex of the cone $T(K)_S$ for some $S \in Q(K)^*$ with associated sequence $j(1),\ldots,j(p)$, where $p = |K|$. Suppose now that $t^o = \frac{1}{2}(t^1 + t^2)$ for some $t^1, t^2 \in T(K)$. We will show by induction on k that for $i = 1,2$, $t^i_{j(k)} = t^o_{j(k)}$ for $k = 1,\ldots,p$, and $t^i_{j(k)} < t^i_{j(\ell)}$ for $\ell > k$, for $k = 1,\ldots,p-1$. For $k = 1$, the inequalities

$t^i_{j(1)} \geq L_{j(1)}$, $i = 1,2$, and the equation $t^o_{j(1)} = L_{j(1)}$ imply that $t^i_{j(1)} = L_{j(1)}$, $i = 1,2$. Further, $t^i_{j(1)} < t^i_{j(\ell)}$ for all $\ell > 1$, or else for some $\ell \in \{2,\ldots,p\}$ we have

$$L_{j(\ell)} + d_{j(\ell),j(1)} \leq t^i_{j(\ell)} + d_{j(\ell),j(1)} \leq t^i_{j(1)} = L_{j(1)},$$

contrary to the first regularity condition, (3.4).

Suppose for $i = 1,2$, $k = 1,\ldots,r-1$, $t^i_{j(k)} = t^o_{j(k)}$, and $t^i_{j(k)} < t^i_{j(\ell)}$ for all $\ell > k$; and let $k = r$. Then $t^i_{j(r)} = t^o_{j(r)}$, or else from (3.4) and the induction hypothesis

$$\begin{aligned} t^i_{j(r)} < t^o_{j(r)} &= t^o_{j(r-1)} + d_{j(r-1),j(r)} \\ &= t^i_{j(r-1)} + d_{j(r-1),j(r)}, \end{aligned}$$

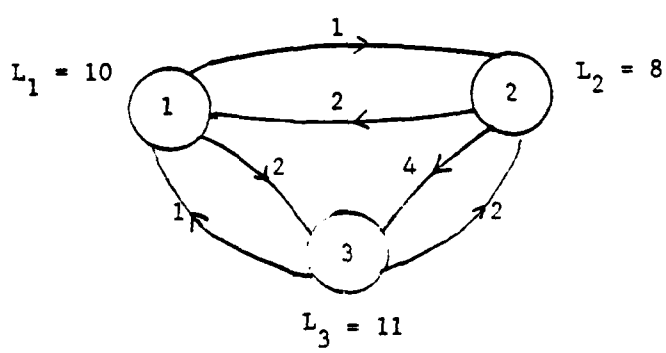
a contradiction. Further if $r < p$, then for $i = 1,2$, $t^i_{j(r)} < t^i_{j(\ell)}$ for all $\ell > r$; or else there exists $\ell > r$ with $t^i_{j(r-1)} < t^i_{j(\ell)} < t^i_{j(r)}$ for $i = 1$ or 2, which implies

$$t^i_{j(r-1)} + d_{j(r-1),j(\ell)} + d_{j(\ell),j(r)} \leq t^i_{j(r-1)} + d_{j(r-1),j(r)} \leq t^i_{j(r)}$$

contrary to the second regularity condition, (3.5).

This completes the induction and proves the "if" part of the Theorem.||

<u>Example 3.1</u>. Consider the clique $K = \{1,2,3\}$ shown in Fig. 2, with $L_1 = 10$, $L_2 = 8$, $L_3 = 11$; $d_{12} = 1$, $d_{13} = 2$, $d_{21} = 2$, $d_{23} = 4$, $d_{31} = 1$, $d_{32} = 2$.



<u>Fig. 2</u>

Condition (3.4) is violated for the ordered pair $\{i,j\} = \{2,1\}$ and condition (3.5) for the ordered triples $\{2,1,3\}$ and $\{3,1,2\}$. Table 1 lists

Table 1

Sequences associated with S	Vertices t of $T(K)_S$		
	all d_{ij} as specified	$d_{21} = 3$	$d_{21} = 3$, $d_{31} = 2$
1,2,3	(10,11,15)	(10,11,15)*	(10,11,15)*
3,1,2	(13, 8,12)	(13, 8,12)*	(14, 8,12)*
2,3,1	(12,13,11)*	(12,13,11)*	(13,14,11)*
1,3,2	(10,14,12)	(10,14,12)*	(10,14,12)*
2,1,3	(10, 8,12)*	(11, 8,13)*	(11, 8,13)*
3,2,1	(15,13,11)	(16,13,11)	(16,13,11)*

*Vertices of P(K).

the sequences associated with the $3! = 6$ acyclic selections $S \in Q(K)^*$ and the vertices of the corresponding cones $T(K)_S$. Because the regularity conditions are violated, only 2 of the 6 vertices of the cones $T(K)_S$ are vertices of P(K): (12,13,11) and (10,8,12). For every other t, there exists some t' such that $t' \leq t$. If we replace $d_{21} = 2$ by $d_{21} = 3$, condition (3.4) is satisfied for all $i,j \in K$, $i \neq j$, and condition (3.5) is violated only for the triplet $\{3,1,2\}$. As a result, all but one of the vertices of the cones $T(K)_S$ become vertices of P(K), the exception being (16,13,11) (since there exists a vertex (12,13,11)). If we also replace $d_{31} = 1$ by $d_{31} = 2$, T(K) becomes regular, and as a result all 6 vertices of the cones $T(K)_S$ become vertices of P(K).||

Next we turn to the facets of P(K).

4. Facets of the Clique Polyhedron

Given a convex polyhedron $C \subset \mathbb{R}^n$, an inequality $\alpha x \geq \alpha_o$ is said to define (or induce) a <u>k-dimensional face</u> of C, if $\alpha x \geq \alpha_o$ for every $x \in C$

and $\alpha x = \alpha_o$ for $k + 1$ affinely independent points $x \in C$. Thus the inequality $\alpha x \geq \alpha_o$ defines a <u>facet</u> of C, if $\alpha x \geq \alpha_o$, for all $x \in C$, and $\alpha x = \alpha_o$ for n affinely independent points $x \in C$.

Let $|K| = p$. For $i = 1,\ldots,p!$, let S^i be the ith acyclic selection in $< K >$, and let $j_i(1),\ldots,j_i(p)$ be the sequence associated with S^i. Further, let v^i be the vertex of the cone $T(K)_{S^i}$, i.e., let $v^i \in \mathbb{R}^p$ be the vector whose components are defined recursively by

$$(4.1) \qquad v^i_{j_i(k)} = \begin{cases} L_{j_i(1)}, & k = 1 \\ \max\{L_{j_i(k)},\ v^i_{j_i(k-1)} + d_{j_i(k-1),j_i(k)}\}, & k = 2,\ldots,p. \end{cases}$$

Finally, let V be the $p! \times p$ matrix whose i^{th} row is v^i, and let $e = (1,\ldots,1)^T$ have $p!$ components.

<u>Theorem 4.1</u>. The inequality $\alpha t \geq 1$, where α, $t \in \mathbb{R}^p$, defines a facet of P(K) if and only if α is a vertex of the polyhedron

$$F = \left\{ \alpha \in \mathbb{R}^p \;\middle|\; \begin{array}{l} V\alpha \geq e \\ \alpha \geq 0 \end{array} \right\}.$$

<u>Proof</u>. $\alpha t \geq 1$ defines a facet of P(K) if and only if (i) $\alpha t \geq 1$ for all $t \in P(K)$, and (ii) $\alpha t = 1$ for p affinely independent points $t \in P(K)$.

Condition (i) holds if and only if $\alpha \in F$. Indeed, every vertex of P(K) is present among the row vectors v^i of V; and the extreme direction vectors of P(K) are the rows of the identity matrix associated with the constraint $\alpha \geq 0$. Furthermore, every row v^i that is not a vertex of P(K), is nevertheless contained in P(K). Hence $\alpha t \geq 1$ is satisfied by all $t \in P(K)$, if and only if $V\alpha \geq e$ and $\alpha \geq 0$, i.e., if and only if $\alpha \in F$.

Further, condition (ii) holds if and only if for some integer $k \in \{1,\ldots,p\}$, P(K) has k extreme points $v^{i(h)}$, $h = 1,\ldots,k$, and p-k extreme direction vectors

$e_{j(h)}$, $h = k+1,\ldots,p$, such that $v^{i(h)}\alpha = 1$, $h = 1,\ldots,k$ and $e_{j(h)}\alpha = \alpha_{j(h)} = 0$, $h = k+1,\ldots,p$. The "if" part of this statement holds since $v^{i(1)}\alpha = 1$ and $\alpha_{j(h)} = 0$ imply $(v^{i(1)} + e_{j(h)})\,\alpha = 1$, $k = k+1,\ldots,p$, and the p points $v^{i(1)},\ldots,v^{i(k)}$, $v^{i(1)} + e_{j(k+1)},\ldots,v^{i(1)} + e_{j(p)}$ are affinely independent. The "only if" part follows from the fact that any $t\epsilon T(K)$ that is not a vertex of P(K) and satisfies $\alpha t = 1$, can be represented as a positive linear combination of extreme points v^i of P(K) that satisfy $\alpha v^i = 1$, and extreme direction vectors e_j of P(K) that satisfy $e_j\alpha = 0$, where the weights of the v^i sum to 1. Thus (ii) holds if and only if for some $k\epsilon\{1,\ldots,p\}$, α satisfies with equality k of the inequalities $v^i\alpha \geq 1$ and p-k of the inequalities $\alpha_j \geq 0$, such that the p inequalities in question form a system of rank p; i.e., if and only if α is a vertex of F.||

Of course, Theorem 4.1 remains true if all redundant inequalities are removed from the system defining F. Because of the large number of constraints that define F, Theorem 4.1 by itself does not seem to offer a practical way of generating facets of P(K). When combined with the next Theorem however, it provides an efficient way of obtaining those facet inducing inequalities with few positive coefficients.

<u>Theorem 4.2</u>. Let $< H >$ and $< K >$ be cliques, with $H \subset K$, $|H| = \ell$ and $|K| = p$, $2 \leq \ell \leq p$. The inequality $\alpha y \geq 1$, where α, $y\epsilon\,\mathbb{R}^{\ell}$, defines a facet of P(H), if and only if the inequality $(\alpha,0)t \geq 1$, where $(\alpha,0)$, $t\epsilon\,\mathbb{R}^{p}$, dedefines a facet of P(K).

<u>Proof</u>. Necessity. Suppose $\alpha y \geq 1$ defines a facet of P(H). Then there exist ℓ affinely independent points $y^i\epsilon P(H)$, $i = 1,\ldots,\ell$, such that each y^i is a schedule for $< H >$, and $\alpha y^i = 1$, $i = 1,\ldots,\ell$. Each y^i can be extended to a schedule t^i for $< K >$ as follows. If $S(H)^i$ is the selection in $< H >$ de-

that of K. Thus there are large classes of facets of P(K) that can be generated at a fixed computational cost, whatever the size of K. More generally, the work needed to derive a facet inducing inequality for P(K) grows with the number of positive coefficients of the inequality; and facets defined by inequalities with few positive coefficients are easy to generate.

Next we address the question of how one can derive a facet inducing inequality with p positive coefficients from one with p-1 positive coefficients. Let $< K >$ be a clique with $|K| = p$, let $H \subset K$ with $|H| = p-1$, say $H = \{1,\ldots,p-1\}$, and let V and W be the matrices whose rows are the vertices of P(K) and P(H), respectively. Note that every row w^i of W corresponds to some row v^k of V, where $v^k = (w^i, v_{kp})$, and the sequence associated with v^k assigns rank p to node p. For all the remaining rows of V, the associated sequence assigns rank p to some node $j \in \{1,\ldots,p-1\}$. Let R(V) and R(W) denote the row index sets of V and W, where every row of V that corresponds to a row of W preserves the index of the latter, i.e., the first $|R(W)|$ elements of R(V) are those of R(W).

For any matrix M, let det(M) denote the determinant of M, let M_S denote the matrix whose rows are those rows of M indexed by S, and let M^j be the matrix obtained from M by substituting a column of 1's for the jth column.

<u>Theorem 4.3</u>. Let W_S be a $(p-1)\times(p-1)$ submatrix of W such that the inequality $\alpha t \geq 1$, where the components of α are

$$(4.2) \qquad \alpha_j = \begin{cases} \dfrac{\det(W_S^j)}{\det(W_S)} > 0 & , \quad j = 1,\ldots,p-1 \\ 0 & , \quad j = p, \end{cases}$$

induces a facet of P(K). Further, let

$$(4.3) \qquad \frac{\det(V^p_{S\cup\{k\}})}{\det(V_{S\cup\{k\}})} = \min_{i\in R(V)\setminus S} \frac{\det(V^p_{S\cup\{i\}})}{\det(V_{S\cup\{i\}})}$$

Then the inequality $\beta t \geq 1$, where the components of β are

$$(4.4) \qquad \beta_j = \frac{\det(V^j_{S\cup\{k\}})}{\det(V_{S\cup\{k\}})}, \qquad j = 1,\ldots,p,$$

also induces a facet of $P(K)$; and if the minimum in (4.3) is positive and unique, then $\beta_j > 0$, $j = 1,\ldots,n$.

<u>Proof</u>. Since the inequality $\alpha t \geq 1$ induces a facet of $P(K)$, it also induces a facet of $P(H)$ (Theorem 4.2), hence the vector $\alpha = (\alpha_1,\ldots,\alpha_{p-1})$ is a vertex of the polyhedron $F^W = \{\alpha | W\alpha \geq e,\ \alpha \geq 0\}$ (Theorem 4.1). We have to show that if (4.3) holds, then β defined by (4.4) is a vertex of $F^V = \{B | V\beta \geq e,\ \beta \geq 0\}$. Then by Theorem 4.1, the inequality $\beta t \geq 1$ induces a facet of $P(K)$.

Consider the system of equations

$$(4.5) \qquad v^i\beta = 1 \quad , \quad i\in S$$

where v^i is the ith row of V. Since $S\subset R(W)$, each v^i is of the form (w^i, v_{ip}). There are two possible cases.

Case 1. There exists no $\beta\in F^V$ satisfying (4.5) with $\beta_p > 0$. Then there exists some $k\in R(V)\setminus S$ such that (4.5) together with $v^k\beta = 1$ implies $\beta_p = 0$ and has the unique solution $\beta = \alpha$. Hence the minimum in (4.3) is 0 and $\beta = \alpha$ is a vertex of F^V.

Case 2. The minimum in (4.3) is positive, i.e., there exists $\beta\in F^V$ satisfying (4.5) with $\beta_p > 0$. Then (4.5) defines an edge of F^V, one of whose endpoints is $\hat{\beta} = \alpha$, whereas the other endpoint is given by the smallest

value of β_p for which either (i) some inequality $\beta_j \geq 0$, $j\epsilon\{1,\ldots,p-1\}$, becomes tight; or (ii) some inequality $v^i\beta \geq 1$, $i\epsilon R(V)\backslash S$, becomes tight. Let β_p^1 and β_p^2 be the values of β_p for which (i) and (ii), respectively, occur. We claim that $\beta_p^1 \geq \beta_p^2$. For suppose $\beta_p^1 < \beta_p^2$, i.e., there exists a vector $\beta^o \epsilon \mathbb{R}^p$ that satisfies (4.5) and $\beta_{j_*}^o = 0$ for some $j_*\epsilon\{1,\ldots,p-1\}$, and such that $v^i\beta^o > 1$, $\forall$ $i\epsilon R(V)\backslash S$. Then $(\beta_1^o,\ldots,\beta_{j_*-1}^o, \beta_{j_*+1}^o,\ldots,\beta_p^o)$ is a vertex of $F^V \cap \{\beta|\beta_{j_*} = 0\}$, hence we have $v^i\beta^o = 1$ for p-1 of those inequalities indexed by $i\epsilon R(V)\backslash S$, for which j_* has rank p in the sequence defined by v^i. But this contradicts the assumption that $\beta_p^1 < \beta_p^2$.

Now β_p^2 is the value defined by (4.3), namely the pth component of the solution β, as defined by (4.4), of the system $v^i\beta = 1$, $i\epsilon S \cup \{k\}$, where $k\epsilon R(V)\backslash S$ is the index of the inequality that becomes tight for $\beta_p = \beta_p^2$. Hence β is a vertex of F^V, i.e., $\beta t \geq 1$ induces a facet of P(K).

Further, if the minimum in (4.3) is both positive (as in case 2 above) and unique, then $\beta_j > 0$ for all j, since otherwise, as shown above, the minimum in (4.3) is not unique.||

In the following we will list all facet inducing inequalities for P(K) with 2 or 3 positive coefficients. But first we examine the trivial facet inducing inequalities, i.e., those having a single positive coefficient.

<u>Proposition 4.4</u>. For all $j\epsilon K$, the inequality $t_j \geq L_j$ induces a facet of P(K).

<u>Proof</u>. W.l.o.g, we assume that $L_j > 0$ for all j. This can always be guaranteed by shifting the origin of the coordinate system, which does not affect the facial structure of P(K). Then the vector α defined by $\alpha_j = 1/L_j$, $\alpha_i = 0$, $\forall$ $i \neq j$, is a vertex of $F = \{\alpha|V\alpha \geq e,\ \alpha \geq 0\}$, where the rows of V are the vectors v^i defined by (4.1). Hence from Theorem 4.1, the inequality $\alpha t \geq 1$, that is $t_j \geq L_j$, induces a facet of P(K).||

Next we turn to facet defining inequalities with two nonzero coefficients.

Theorem 4.5. Let $< K >$ be a clique. For any $i,j \in K$, $i \neq j$,

$$(d_{ij} + L_i - L_j)t_i + (d_{ji} + L_j - L_i)t_j \geq d_{ij}d_{ji} + L_i d_{ji} + L_j d_{ij} \tag{4.6}$$

is a nontrivial facet inducing inequality for $P(K)$ if and only if

$$-d_{ji} < L_j - L_i < d_{ij}. \tag{4.7}$$

Proof. From Theorem 4.2, (4.6) defines a facet of $P(K)$ if and only if it defines a facet of $P(\{i,j\})$. From Theorem 4.1, this is the case if and only if the point $\alpha^o = (\alpha_i^o, \alpha_j^o)$, where

$$\alpha_i^o = \frac{d_{ij} + L_i - L_j}{d_{ij}d_{ji} + L_i d_{ji} + L_j d_{ij}}, \qquad \alpha_j^o = \frac{d_{ji} + L_j - L_i}{d_{ij}d_{ji} + L_i d_{ji} + L_j d_{ij}},$$

is a vertex of the polyhedron $F(\{i,j\})$ defined by the inequalities

$$\begin{aligned} L_i\alpha_i + \max\{L_j, L_i + d_{ij}\}\alpha_j &\geq 1 \\ \max\{L_i, L_j + d_{ji}\}\alpha_i \qquad + \quad L_j\alpha_j &\geq 1 \\ \alpha_i \geq 0, \quad \alpha_j &\geq 0 \end{aligned} \tag{4.8}$$

If (4.7) holds, then the maximum in the first and second inequalities of (4.8) is attained for $L_i + d_{ij}$ and $L_j + d_{ji}$, respectively, and α_o is the unique solution to the system obtained by requiring these two inequalities to be tight. Since α^o also satisfies the remaining two inequalities of (4.8), it is a vertex of $F(\{i,j\})$ and hence the inequality (4.6) defines a facet of $P(K)$. Further, if (4.7) holds, then $\alpha_i^o > 0$ and $\alpha_j^o > 0$, i.e., the facet is nontrivial.

On the other hand, if $L_i - L_j \geq d_{ij}$ or $L_j - L_i \geq d_{ji}$ (both inequalities cannot hold at the same time), then the maximum in the first or second in-

equality of (4.8) is attained for L_i or L_j, respectively, and the solution to the system of two equations is $\alpha_i = 0$, $\alpha_j = 1/L_j$ in the first case, $\alpha_j = 0$, $\alpha_i = 1/L_i$ in the second; hence in these cases $\alpha t \geq 1$ coincides with one of the two trivial facet defining inequalities associated with the indices i,j, and (4.6) does not induce a facet.||

Note that (4.7) is the regularity condition (3.4) for the clique $< \{i,j\} >$. Since $|\{i,j\}| = 2$, condition (3.5) does not apply. Thus regularity of the clique $< \{i,j\} >$ is a necessary and sufficient condition for the polyhedron P(K) (where $< K >$ is any clique containing $\{i,j\}$) to have a facet inducing inequality $\alpha t \geq 1$ with $\alpha_i > 0$, $\alpha_j > 0$ and $\alpha_k = 0$, $\forall\, k \neq i,j$.

Next we characterize the facet inducing inequalities with 3 nonzero coefficients for an arbitrary clique $< K >$ with $|K| = p$. From Theorem 4.2, an inequality of the form $\alpha_{j_1} t_{j_1} + \alpha_{j_2} t_{j_2} + \alpha_{j_3} t_{j_3} \geq 1$ induces a facet of P(K) if and only if it induces a facet of $P(\{j_1, j_2, j_3\})$, the clique polyhedron defined on the vertex set $\{j_1, j_2, j_3\}$. From Theorem 4.1, this is the case if and only if $\alpha = (\alpha_{j_1}, \alpha_{j_2}, \alpha_{j_3})$ is a vertex of the polyhedron

$$F = \{\alpha \in \mathbb{R}^3 \,|\, V\alpha \geq e,\ \alpha \geq 0\},$$

where $e \in \mathbb{R}^6$ and V is the 6 x 3 matrix whose rows are defined by (4.1) for p = 3. To simplify the notation, we assume that $\{j_1, j_2, j_3\} = \{1,2,3\}$. Denoting by p_i the sequence (permutation) associated with row v^i of V, we will assume that the rows of V, indexed by R(V), are ordered so that

$$p_1 = (1,2,3) \qquad p_4 = (1,3,2)$$
$$p_2 = (2,3,1) \qquad p_5 = (2,1,3)$$
$$p_3 = (3,1,2) \qquad p_6 = (3,2,1).$$

Further, we will assume that $< \{1,2,3\} >$ is regular; which implies that the matrix V is of the form

$$V = \begin{pmatrix} L_1 & L_1 + d_{12} & L_1 + d_{12} + d_{23} \\ L_2 + d_{23} + d_{31} & L_2 & L_2 + d_{23} \\ L_3 + d_{31} & L_3 + d_{31} + d_{12} & L_3 \\ L_1 & L_1 + d_{13} + d_{32} & L_1 + d_{13} \\ L_2 + d_{21} & L_2 & L_2 + d_{21} + d_{13} \\ L_3 \quad d_{32} + d_{21} & L_3 + d_{32} & L_3 \end{pmatrix} .$$

As in Theorem 4.3, we let $V_{\{i,k,\ell\}}$ denote the 3×3 matrix consisting of rows i,k,ℓ of V, and let $V^j_{\{i,k,\ell\}}$ be the matrix obtained from $V_{\{i,k,\ell\}}$ by substituing 1 for every entry of column j.

<u>Theorem 4.6</u>. Let $K = \{1,\ldots,p\}$, let $< \{1,2,3\} >$ be regular, and let every 4×4 submatrix of (V,e) be nonsingular. Then P(K) has exactly four facets induced by inequalities $\alpha t \geq 1$ with $\alpha_j > 0$ for $j = 1,2,3$, $\alpha_j = 0$ for $j = 4,5,\ldots,p$. In particular the coefficients of the four inequalities are defined by

$$\alpha_j = \det\left(V^j_{\{i,k,\ell\}}\right) \Big/ \det\left(V_{\{i,k,\ell\}}\right) , \quad j = 1,2,3, \tag{4.9}$$

$\alpha_j = 0$, $j = 4,5,\ldots,p$, where the four triplets $i,k,\ell \in R(V)$ are $\{1,5,r\}$, $\{2,6,s\}$, $\{3,4,t\}$ and $\{r,s,t\}$, with $\{r,s,t\} = \{2,3,1\}$ or $\{4,5,6\}$.

<u>Proof</u>. From Theorem 4.2, an inequality $\alpha t \geq 1$ with $\alpha_j = 0$, $j = 4,5,\ldots,p$, induces a facet of P(K), if and only if the inequality $\alpha_1 t_1 + \alpha_2 t_2 + \alpha_3 t_3 \geq 1$ induces a facet of $P(\{1,2,3\})$. From Theorem 4.1, this is the case if and only if α is a vertex of the polyhedron $F = \{\alpha \in \mathbb{R}^3 | V\alpha \geq e,\ \alpha \geq 0\}$.

According to a classical result of Steinitz, the number of vertices of a polytope (bounded polyhedron) in $\mathbb{R}^3$ is bounded by $2f - 4$, where f is the number of facets; and this bound is attained when the polytope is simple (totally nondegenerate), i.e., when each vertex lies on exactly 3 facets, or, equivalently, on exactly 3 edges (see for instance Grünbaum [5], p. 190). Now F is never simple, since $v_{11} = v_{41}$, $v_{22} = v_{52}$ and $v_{33} = v_{63}$, and as a result each of the 3 vertices having a single positive component (namely: $\alpha_1 = 1/L_1$, $\alpha_2 = \alpha_3 = 0$; $\alpha_2 = 1/L_2$, $\alpha_1 = \alpha_3 = 0$; and $\alpha_3 = 1/L_3$, $\alpha_1 = \alpha_2 = 0$) lies on 4 facets, i.e., is degenerate, if it exists at all (i.e., if $L_j \neq 0$). Furthermore, F is unbounded. We therefore define a polytope (bounded polyhedron) F*, obtained from F by

(i) assuming $L_j > 0$, $j = 1,2,3$ (this guarantees the existence of the 3 vertices with one positive component);

(ii) replacing L_j by $L_j + \epsilon > L_j$, $j = 1,2,3$, in rows 4,5,6 (this makes those same 3 vertices nondegenerate); and

(iii) adding the inequality $\alpha_1 + \alpha_2 + \alpha_3 \leq M$, where $M > 1/L_j$, $j = 1,2,3$ (this makes F* bounded).

Given the regularity of $< \{1,2,3\} >$ and the assumption that every 4×4 submatrix of (V,e) is nonsingular, F* is simple; and listing its vertices allows us to list those of F.

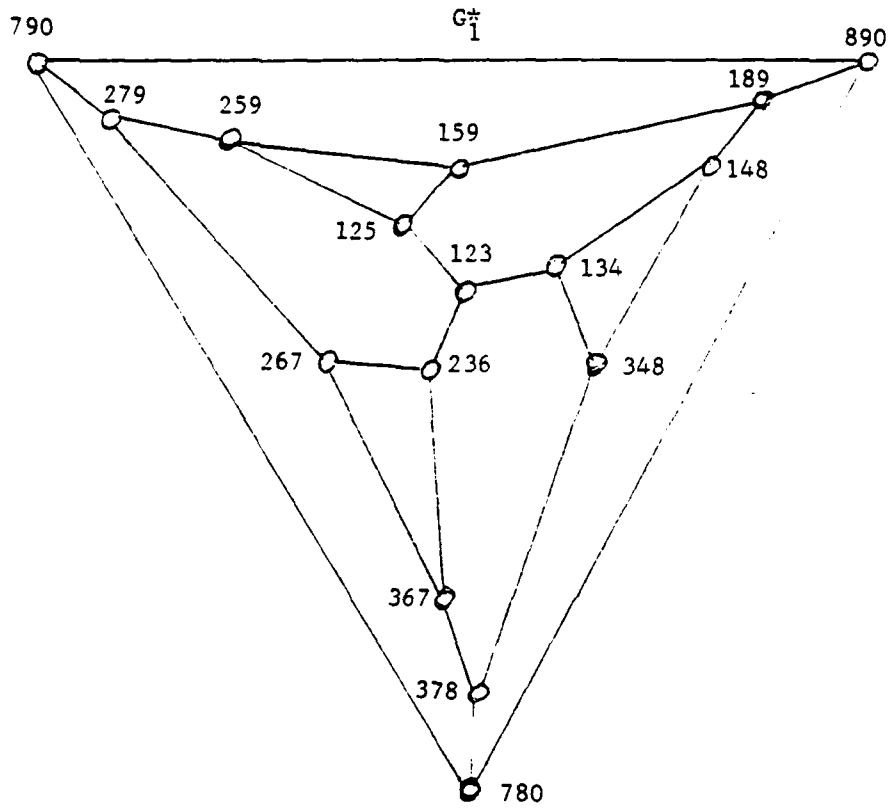
Since F* has 10 facets (defined by the 6 inequalities $v^i\alpha \geq 1$, the 3 inequalities $\alpha_j \geq 0$, and the inequality introduced in (iii)), it has $2f - 4 = 16$ vertices. Of these, 3 lie on the plane $\alpha_1 + \alpha_2 + \alpha_3 = M$ and are therefore not vertices of F. Another triplet consists of the 3 vertices with exactly one positive component; these are also vertices of F. A third triplet of vertices of F*, also shared with F, are those with exactly two positive

components, that give rise to the facet defining inequalities (4.6) for the corresponding 2-clique polyhedra. A fourth triplet consists of those vertices of F* having two positive components, whose counterparts in F have a single positive component (because of the degeneracy caused by $v_{11} = v_{41}$, $v_{22} = v_{52}$, $v_{33} = v_{63}$). This is a total of 12 vertices of F* (6 vertices of F) with one or two positive components (see Table 2, in which the facets are numbered from 1 to 6 for $v^i\alpha \geq 1$, $i = 1,\ldots,6$; 7,8,9 for $\alpha_j \geq 0$, $j = 1,2,3$; and 0 for $\alpha_1 + \alpha_2 + \alpha_3 \leq M$). Thus there are 4 facets left, each with 3 positive components.

From Theorem 4.3, there is a vertex with 3 positive components adjacent to every vertex with 2 positive components. Two vertices (of a 3-dimensional polytope) are of course adjacent if and only if they share

Table 2

Vertex of F*	Positive components	Lies on facets	Vertex of F	Positive components	Lies on facets
1	j = 1	0,8,9	-	j = 1	1,4,8,9
2	j = 2	0,7,9	-		
3	j = 3	0,7,8	-		
4	j = 1	1,8,9	1	j = 1	1,4,8,9
5	j = 2	2,7,9	2	j = 2	2,5,7,9
6	j = 3	3,7,8	3	j = 3	3,6,7,8
7	j = 1,2	1,5,9	4	j = 1,2	1,5,9
8	j = 2,3	2,6,7	5	j = 2,3	2,6,7
9	j = 1,3	3,4,8	6	j = 1,3	3,4,8
10	j = 1,2	2,5,9	2	j = 2	2,5,7,9
11	j = 2,3	3,6,7	3	j = 3	3,6,7,8
12	j = 1,3	1,4,8	1	j = 1	1,4,8,9


G_1^*
790
890
279
259
189
159
148
125
123
134
267
236
348
367
378
780

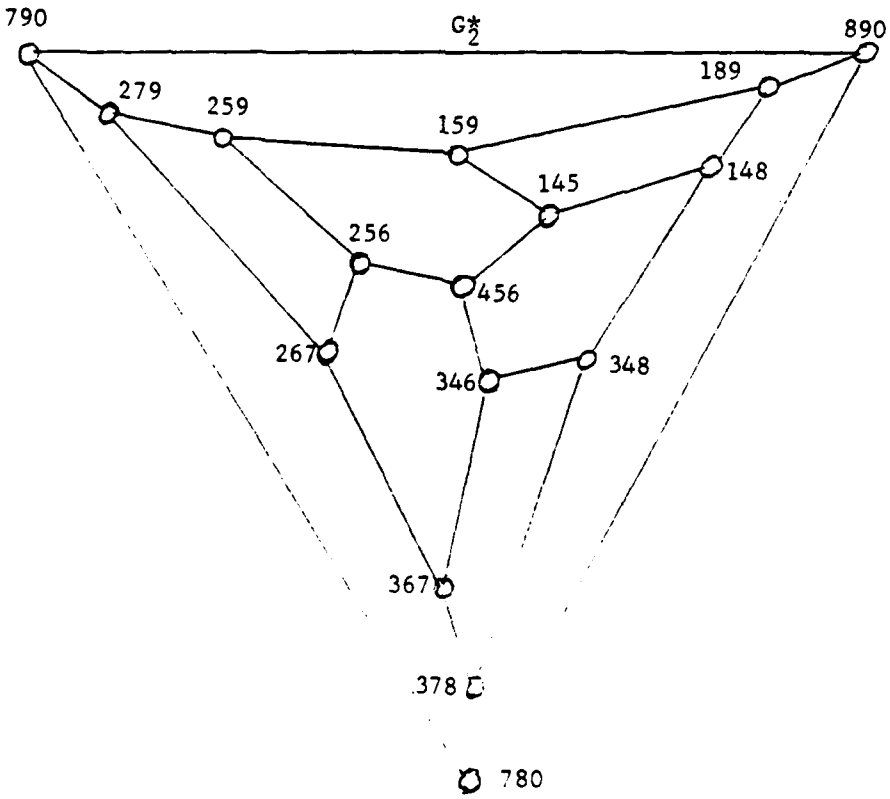
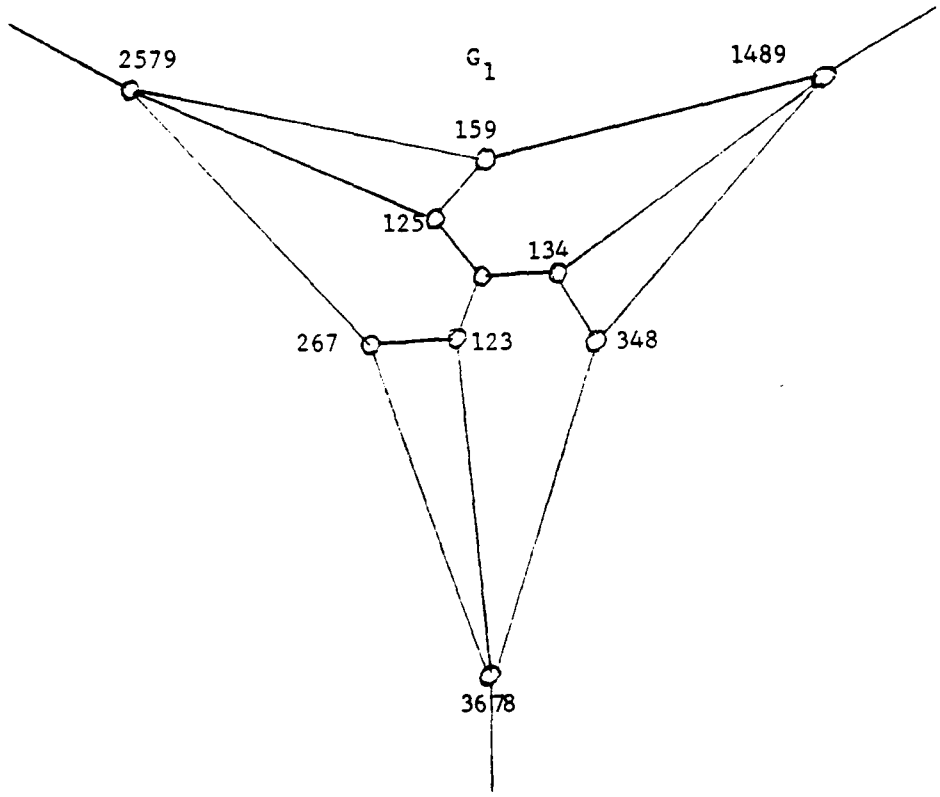

G_2^*
790
890
279
259
189
159
145
148
256
456
267
346
348
367
378
780

Fig. 3


G_1
2579
1489
159
125
134
267
123
348
3678

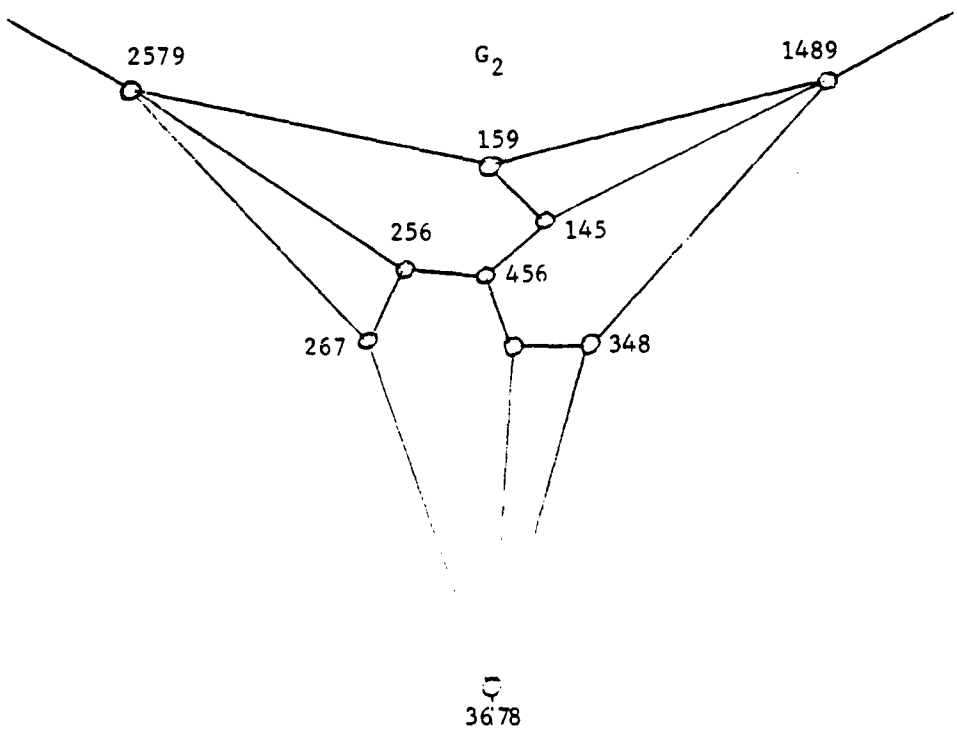

G_2
2579
1489
159
145
256
456
267
348
3678

Fig. 4

two facets. Thus the vertices with 3 positive components adjacent to $\{1,5,9\}$, $\{2,6,7\}$ and $\{3,4,8\}$ are of the form $\{1,5,r\}$, $\{2,6,s\}$ and $\{3,4,t\}$, respectively; whereas those adjacent to $\{2,5,9\}$, $\{3,6,7\}$ and $\{1,4,8\}$ are of the form $\{2,5,u\}$, $\{3,6,w\}$ and $\{1,4,z\}$, respectively. Clearly, at least 3 of these 6 potential vertices are distinct, and we know there exists a 4th vertex with 3 positive components. Finally, every vertex is adjacent to exactly 3 other vertices. Checking all possible combinations shows that there are only two ways of satisfying these requirements, namely if $\{r,s,t\} = \{2,3,1\}$ and $\{u,w,z\} = \{1,2,3\}$, or if $\{r,s,t\} = \{4,5,6\}$ and $\{u,w,z\} = \{6,4,5\}$. In the first case, there exists a vertex $\{1,2,5\}$, adjacent to $\{1,5,9\}$ and to $\{2,5,9\}$; a vertex $\{2,3,6\}$, adjacent to $\{2,6,7\}$ and $\{3,6,7\}$; and a vertex $\{1,3,4\}$, adjacent to $\{3,4,8\}$ and to $\{1,4,8\}$. The 4th vertex with 3 positive components is in this case $\{1,2,3\}$, adjacent to $\{1,2,5\}$, $\{2,3,6\}$ and $\{1,3,4\}$. In the second case, there exists a vertex $\{1,4,5\}$, adjacent to $\{1,5,9\}$ and $\{1,4,8\}$; a vertex $\{2,5,6\}$, adjacent to $\{2,6,7\}$ and $\{2,5,9\}$; and a vertex $\{3,4,6\}$, adjacent to $\{3,4,8\}$ and $\{3,6,7\}$. The fourth vertex in this case is $\{4,5,6\}$, adjacent to $\{1,4,5\}$, $\{2,5,6\}$ and $\{3,4,6\}$.

Thus the only two possible facial structures of F^* are those represented by the graphs G_1^* and G_2^* of Fig. 3.||

Note that the polytope F^*, which is bounded and totally nondegenerate (simple), has 16 vertices and 24 edges. The (unbounded) polyhedron F has at most (i.e., when the only degenerate vertices are those with 1 positive component) 10 vertices and 18 edges, as shown in Fig. 4, where G_1 and G_2 are the "graphs" of F (the 3 unbounded edges of F being represented by "half-edges" of G_1 and G_2).

Thus P(K) has at most 4 facets induced by inequalities $\alpha t \geq 1$ with $\alpha_j > 0$ for j = 1,2,3. The regularity of $< \{1,2,3\} >$ is a necessary condition for the existence of 4 distinct facets of this type, but is not by itself sufficient. For sufficiency we need, besides regularity, the absence of any singular 4 x 4 submatrices of (V,e), as assumed in the Theorem.

<u>Example 4.1</u>. Let G be the disjunctive graph shown in Fig. 5.

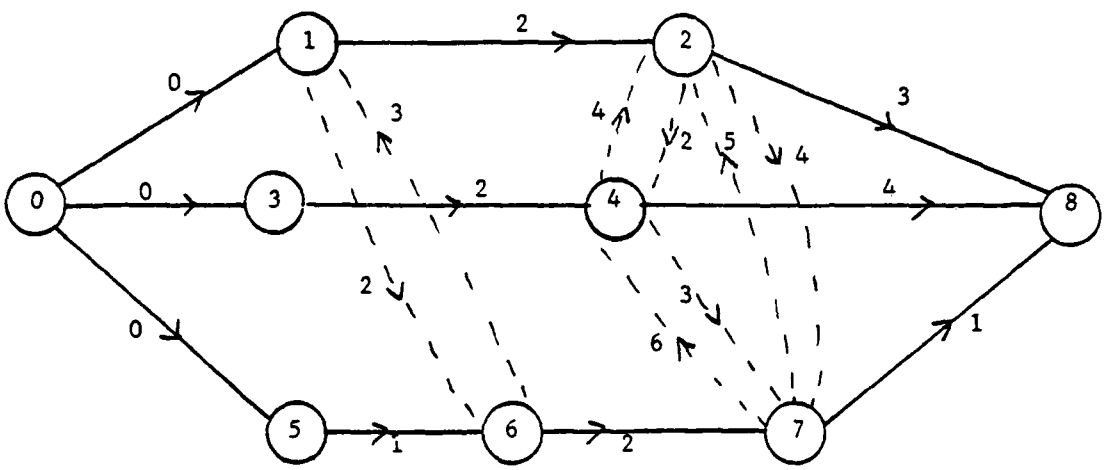


Fig. 5.

G has two disjunctive cliques, induced by the node sets $K_1 = \{1,6\}$ and $K_2 = \{2,4,7\}$, respectively. For $< K_1 >$ we have $L_1 = L(0,1) = 0$, $L_6 = L(0,6) = 1$, and $d_{16} = 2$, $d_{61} = 3$. $P(K_1)$ has 3 facets, defined by the inequalities $t_1 \geq 0$, $t_6 \geq 1$ (Proposition 4.4), and $t_1 + 4t_6 \geq 8$ (Theorem 4.5).

For $< K_2 >$, we have $L_2 = L(0,2) = 2$, $L_4 = L(0,4) = 2$, $L_7 = L(0,7) = 3$, and $d_{24} = 2$, $d_{27} = 4$, $d_{42} = 4$, $d_{47} = 3$, $d_{72} = 5$, $d_{74} = 6$. We see that $< K_2 >$ is regular, and the matrix defining the polyhedron F is

$$V = \begin{pmatrix} 2 & 4 & 7 \\ 10 & 2 & 5 \\ 8 & 10 & 3 \\ 2 & 12 & 6 \\ 6 & 2 & 10 \\ 13 & 9 & 3 \end{pmatrix}$$

$P(K_2)$ has 10 facets: 3 of them are defined by the trivial inequalities $t_2 \geq 2$, $t_4 \geq 2$, $t_7 \geq 3$ (Proposition 4.4); another 3 by the inequalities

$$\begin{aligned} t_2 + 2t_4 \qquad &\geq 10 \\ t_2 \qquad + 2t_7 &\geq 14 \\ 2t_4 + 7t_7 &\geq 39 \end{aligned}$$

with 2 positive coefficients (Theorem 4.5); and, finally, 4 facets are defined by inequalities with 3 positive coefficients (Theorem 4.6):

$$\begin{aligned} 5t_2 + 16t_4 + 4t_7 &\geq 102 \\ t_2 + 5t_4 + 19t_7 &\geq 115 \\ 13t_2 + 3t_4 + 24t_7 &\geq 206 \\ t_2 + t_4 + 3t_7 &\geq 27. \end{aligned}$$

These 4 inequalities correspond, in the notation of Theorem 4.6, to the vertices $\{1,2,5\}$, $\{2,3,6\}$, $\{1,3,4\}$ and $\{1,2,3\}$, respectively, of F. Here we have multiplied each inequality with the determinant in the denominator of the expression (4.9) in order to express them in integers.||

5. Lifting the Facets of the Clique Polyhedron

In this section we address the question as to how the results of the previous sections can be used to derive facet inducing inequalities for the general scheduling polyhedron P = clconv T introduced in section 1. In

define t^o by $t_j^o = \ell(0,j)_S$. Clearly, t^o is a schedule in G. Further, by the definition of S, $\ell(0,j)_S = L(0,j)$ for all $j \in \{i\} \cup B(i)$, hence t^o satisfies $t_i = L(0,i)$. The next n-q schedules t^h, $h = 1,\ldots,n-q$, are defined recursively by $t_j^h = t_j^{h-1}$ for $j \in N \setminus \{n-h+1\}$ and $t_j^h = t_j^{h-1} + 1$ for $j = n-h+1$. Each of these vectors is a schedule that satisfies $t_i = L(0,i)$. Then the $(n-q) \times n$ matrix whose rows are the vectors $t^h - t^o$, $h = 1,\ldots,n-q$, is of the form $M = (M_1, M_2)$, where M_1 is $(n-q) \times q$, while M_2 is the $(n-q) \times (n-q)$ nonsingular matrix

$$M_2 = \begin{pmatrix} 0 & . & . & . & 0 & 1 \\ 0 & . & . & . & 1 & 1 \\ \hline 1 & . & . & . & 1 & 1 \end{pmatrix}$$

Thus M has rank n-q, and the n-q+1 schedules t^h are affinely independent.||

<u>Corollary 5.2</u>. The inequality $t_i \geq L(0,i)$ defines a facet of P if and only if $B(i) = \emptyset$.

Next we address the question of lifting the facets of clique polyhedra. We need a couple of definitions and some auxiliary results.

Let $< K >$ be a clique, S(K) an arbitrary acyclic selection in K, and $< K_\ell >$ the maximal clique containing $< K >$. As before, let M be the index set of the maximal cliques of G. We will say that the selection

$$S = \bigcup_{r \in M} S(K_r)$$

is a <u>conformal</u> extension of S(K) to G, if it satisfies the following requirements:

(i) $S(K_\ell)$ is any acyclic extension of S(K) to $< K_\ell >$, such that, if $i \in K$ and $j \in K_\ell \setminus K$, the rank of i in $S(K_\ell)$ is less than that of j.

(ii) For $r \in M \setminus \{\ell\}$ such that $K_r \cap B(K) = \emptyset$, $S(K_r)$ is any acyclic selection in $< K_r >$.

(iii) For $r \in M \setminus \{\ell\}$ such that $K_r \cap B(K) \neq \emptyset$, $S(K_r)$ is any acyclic selection in $< K_r >$ such that

(α) if $i \in K_r \cap B(K)$ and $j \in K_r \setminus B(K)$, the rank of i in $S(K_r)$ is less than that of j;

(β) if $j \in K_r \cap B(i)$ for some $i \in K$, the rank of j in $S(K_r)$ is no greater than the rank of i in $S(K)$; and

(γ) if $i,h \in K$, $j(i) \in K_r \cap B(i)$, $j(h) \in K_r \cap B(h)$, and the rank in $S(K)$ of i is less than that of h, then the rank in $S(K_r)$ of $j(i)$ is less than that of $j(h)$.

For any $i \in N$, $B(i)$ is the set of nodes $j \in N \setminus \{i\}$ lying on the (unique) path $P(0,i)$ from 0 to i in D. Therefore every clique has at most one node in $B(i)$. Let $M(i)$ be the index set of cliques that have such a node, i.e., $M(i) = \{r \in M | K_r \cap B(i) \neq \emptyset\}$, and let $\{j_r(i)\} = K_r \cap B(i)$.

A (not necessarily maximal) clique $< K >$ of G will be called <u>dominant</u>, if for every $i,h \in K$ such that $M(i) \cap M(h) \neq \emptyset$, and every $r \in M(i) \cap M(h)$,

$$d_{j_r(i) j_r(h)} + L(j_r(h),h) < L(j_r(i),i) + d_{ih} \,. \tag{5.1}$$

The term "dominant" seems justified by the properties of these cliques.

<u>Lemma 5.3</u>. Let $< K >$ be a dominant clique in G, and $S(K)$ an acyclic selection in $< K >$. Then every conformal extension S of $S(K)$ to G has the property that, if $i \in K$, $j \in \{0\} \cup B(K)$ and i is reachable from j in the digraph $D_S = (N^o, A^o \cup S)$, every longest path from j to i in D_S contains only arcs of $A^o \cup S(K)$.

Proof. Let S be a conformal extension of $S(K)$ to G, and for some $i \in K$, $j \in \{0\} \cup B(K)$, let $P(j,i)_S$ be a longest path from j to i in D_S. Suppose now that $P(j,i)_S$ contains an arc of $S \setminus S(K)$; in particular, let (j_1,j_2) be the last such arc encountered when $P(j,i)_S$ is traversed in the direction of its arcs, and let $(j_1,j_2) \in S(K_r)$. Then from property (iii) of S, for $k = 1,2$, $j_k \in B(K)$; in particular, j_k lies on the unique path $P(0,i_k)$ in D for some $i_k \in K$, and $(i_1,i_2) \in S(K)$. Further, if $< K >$ is dominant, $d_{j_1 j_2} + L(j_2,i_2) < L(j_1,i_1) + d_{i_1 i_2}$, and replacing the segment of $P(j,i)_S$ from j_1 to i_2 by the path $P(j_1,i_1) \cup \{(i_1,i_2)\}$ yields a path from j to i in D_S longer than $P(j,i)_S$. This proves that $P(j,i)_S$ cannot contain any arc of $S \setminus S(K)$.||

Theorem 5.4. Let $< K >$ be a dominant clique in G, y^o a schedule for $< K >$ with associated selection $S(K)$, and S a conformal extension of $S(K)$ to G. Then the vector $t^o \in \mathbb{R}^n$ defined by

$$(5.2) \qquad t_j^o = \begin{cases} y_j^o & j \in K \\ \ell(0,j)_S & j \in B(K) \\ U - \ell(j,n)_S & j \in N \setminus K \cup B(K) \end{cases}$$

is a schedule for G if U is sufficiently large to satisfy, for any selection V in G, the condition

$$(5.3) \qquad U > \max \left\{ \ell(0,n)_V, \ \max_{j \in K} \{ y_j^o + \ell(j,n)_V \} \right\} .$$

Proof. We show that t^o is a schedule for G by showing that it is a schedule for D_S. For this purpose we examine all the arcs of D_S and show that t^o satisfies the associated inequalities. All pairs i,j considered below are such that $(i,j) \in A \cup S$.

If both i and j belong to any one of the three sets K, $B(K)$ or $N\backslash K \cup B(K)$, then substituting the values of t_i^o and t_j^o given by (5.2) into the inequality $t_j^o - t_i^o \geq d_{ij}$ shows the latter to be satisfied.

For $i \in B(K)$, $j \in N\backslash K \cup B(K)$, $t_j^o - t_i^o = U - \ell(j,n)_S - \ell(0,i)_S \geq d_{ij}$, since $U > \ell(0,n)_S \geq \ell(0,i)_S + d_{ij} + \ell(j,n)_S$.

For $i \in K$, $j \in N\backslash K \cup B(K)$, $t_j^o - t_i^o = U - \ell(j,n)_S - y_i^o \geq d_{ij}$, since $U > y_i^o + \ell(i,n)_S \geq y_i^o + d_{ij} + \ell(j,n)_S$.

It remains to be shown that the constraints are also satisfied for $i \in B(K)$, $j \in K$; for all remaining ordered pairings of the three index sets used in the definition of t^o, the corresponding arc sets are empty.

Now for $i \in B(K)$ and $j \in K$, $t_j^o - t_i^o = y_j^o - \ell(0,i)_S$. Let the rank of node j in $S(K)$ be k. The schedule y^o satisfies the inequalities $y_{j(h)}^o \geq L(0,j)h))$, $h = 1,\ldots,p$, and $y_{j(h)}^o - y_{j(h-1)}^o \geq d_{j(h-1),j(h)}$, $h = 2,\ldots,p$, where $p = |K|$ and h is the rank of $j(h)$ in $S(K)$.. It is not hard to see that these inequalities, plus the fact that $j = j(k)$, imply

$$(5.4) \qquad y_j^o \geq \max \Big\{ L(0,j(k)),\ L(0,j(k-1)) + d_{j(k-1),j(k)}, \ldots, L(0,j(1)) + \sum_{h=2}^{k} d_{j(h-1),j(h)} \Big\} .$$

The expression on the righthand side of (5.4) represents the length of a longest among those paths from 0 to j in D_S, which use only arcs in $A^o \cup S(K)$. Since $< K >$ is a dominant clique, from Lemma 5.3 this is equal to $\ell(0,j)_S$, the length of any longest path from 0 to j in D_S. Hence we have

$$t_j^o - t_i^o = y_j^o - \ell(0,i)_S \geq \ell(0,j)_S - \ell(0,i)S \geq d_{ij}.$$

Since t^o satisfies all the inequalities associated with the arcs of D_S, it is a schedule for D_S, hence for G.||

We are now ready to state the main result of this section.

Theorem 5.5. Let $< K >$ be a (not necessarily maximal) dominant clique of G, with $|K| = p > 1$. If the inequality $\alpha y \geq 1$, where $\alpha, y \in \mathbb{R}^p$, defines a facet of P(K), then the inequality $(\alpha,0)t \geq 1$, where $(\alpha,0), t \in \mathbb{R}^n$, defines a facet of P.

Outline of proof If the inequality $\alpha y \geq 1$ defines a facet of P(K), there exists a set of p extreme points y^i, $i = 1,\ldots,p$ of P(K), such that $\alpha y^i = 1$, $i = 1,\ldots,p$.

Since $< K >$ is dominant, from Theorem 5.4 every y^i has at least one conformal extension t^i to G. From each such schedule t^i for G, additional schedules can be constructed by adding a small positive scalar to certain components. Using this approach one can in fact construct n affinely independent schedules t^i for G, each of which is an extension of some schedule for $< K >$ and therefore satisfies $\alpha t^i = 1$. This proves that the inequality $(\alpha,0)t \geq 1$ induces a facet of P. Details are given in an Appendix.||

6. Identifying Violated Inequalities

For every clique $< K >$ of G, let $\mathcal{F}(K)$ be the set of all facet inducing inequalities for P(K) = clconv T(K), and let $\mathcal{F} = \cup\mathcal{F}(K)$, where the union is taken over all cliques of G. In order to be able to use the inequalities of $\mathcal{F}$ as cutting planes in an algorithm for solving (P), one needs a way to solve the following

Constraint Identification Problem (CIP). Given some $t^o \in \mathbb{R}^n$ that satisfies $t^o_j - t^o_i \geq t_{ij}$, $(i,j) \in A$, $t^o_i \geq 0$, $i \in N$, but violates some of the disjunctions defining T, find an inequality in $\mathcal{F}$ violated by t^o, or show that none exists.

Let $t^o \epsilon \mathbb{R}^n$ be as defined in CIP, let $< K >$ be a clique at least one of whose disjunctions is violated by t^o, let $F(K)$ be the polyhedron defined in Theorem 4.1 relative to $< K >$, and denote by t_K the vector whose components are t_j, $j \epsilon K$. Further, let α^o be defined by

$$(6.1) \qquad t_K^o \alpha^o = \min\{t_K^o \alpha \mid \alpha \epsilon F(K)\}.$$

Then if $t_K^o \alpha^o < 1$, the inequality $\alpha^o t_K \geq 1$ obviously cuts off t^o and CIP is solved. Otherwise we have

<u>Proposition 6.1</u>. If $t_K^o \alpha^o \geq 1$, $t_K^o \epsilon P(K)$, i.e., t^o satisfies all the inequalities of $\mathcal{F}(K)$.

<u>Proof</u>. If $t_K^o \alpha^o \geq 1$, then from the definition of α^o, $\alpha t_K^o \geq 1$ for every vertex α of $F(K)$.||

Thus the procedure that suggests itself for solving CIP is to choose some clique $< K >$ at least one of whose disjunctions is violated by t_K^o, and solve (6.1). However, in the absence of additional information we may well choose a clique $< K >$ for which $t_K^o \alpha^o \geq 1$. Also, if $< K >$ is large, solving (6.1) is expensive.

The next Theorem gives a sufficient condition for $\mathcal{F}(K)$ to contain an inequality violated by t^o. The condition occurs frequently and is easy to check. Furthermore, the Theorem restricts the size of $< K >$ to the minimum subject to the above condition.

<u>Theorem 6.2</u>. Let t^o be as defined in CIP. Let $< K >$ be a (not necessarily maximal) clique, with $|K| = p$ and $t_{j(1)}^o \leq \ldots \leq t_{j(p)}^o$, such that t^o satisfies

$$(6.2) \qquad t_{j(1)}^o = L(0, j(1)),$$

(6.3) $$t^o_{j(p)} < t^o_{j(p-1)} + d_{j(p-1),j(p)},$$

and, if $p \geq 3$,

(6.4) $$t^o_{j(k)} = t^o_{j(k-1)} + d_{j(k-1),j(k)} \quad , \quad k = 2,\ldots,p-1.$$

Further, let α^o be defined by (6.1). Then the inequality $\alpha^o t_K \geq 1$ cuts off t^o.

<u>Proof</u>. We prove by contradiction that $t^o_K \notin P(K)$. It then follows that $\mathcal{F}(K)$ contains an inequality that cuts off t^o, and from (6.1), $\alpha^o t_K \geq 1$ is such an inequality.

Suppose $t^o_K \epsilon P(K)$. Then there exist vectors $t^i_K \epsilon T(K)$ and scalars $\lambda_i \geq 0$, $i = 1,\ldots,p+1$, such that

$$t^o_K = \sum_{i=1}^{p+1} t^i_K \lambda_i \quad , \quad \sum_{i=1}^{\beta+1} \lambda_i = 1.$$

Since $t_{j(1)} \geq L(0,j(1))$ for any $t_K \epsilon T(K)$ and $t^o_{j(1)} = L(0,j(1))$, we have $t^i_{j(1)} = t^o_{j(1)}$, $i = 1,\ldots,p+1$. Similarly, since $t_{j(k)} \geq \max\{L(0,j(k)), t_{j(k-1)} + d_{j(k-1),j(k)}\}$ for all $t_K \epsilon T(K)$ and $t^o_{j(k)} = t^o_{j(k-1)} + d_{j(k-1),j(k)}$, $k = 2,\ldots,p-1$, it follows that $t^i_{j(k)} = t^o_{j(k)}$, $k = 2,\ldots,p-1$, whenever $p \geq 3$. But then from (6.3), for at least one $i \epsilon \{1,\ldots,p+1\}$, we have $t^i_{j(p)} < t^o_{j(p)} < t^o_{j(p-1)} + d_{j(p-1),j(p)} = t^i_{j(p-1)} + d_{j(p-1),j(p)}$, contrary to the assumption that $t^i_K \epsilon T(K)$, $i = 1,\ldots,p+1$. Thus $t^o_K \notin P(K)$.||

Condition (6.2) of Theorem 6.2 requires that the smallest component of t^o_K be equal to the lower bound on its value in any schedule. This condition is always met by a basic schedule t^o for those cliques $< K >$ such that no node of $B(K)$ is contained in any disjunctive clique. For other cliques, the condition may or may not be satisfied, but it is of course easy to check.

The remaining conditions simply state that a minimum size clique to be considered is the one with node set $K = \{j(1),\ldots,j(p)\}$, where $j(1)$ is the node for which $t^o_{j(1)} = L(0,j(1))$, and $j(p)$ is the first node in the sequence defined by t^o for which the condition $t_{j(p)} - t_{j(p-1)} \geq d_{j(p-1),j(p)}$ (and hence the corresponding disjunction) is violated.

When there is no clique for which the conditions of Theorem 6.2 are satisfied, there is no guarantee that α^o defined by (6.1) cuts off t^o. In such cases it is a reasonable heuristic to choose a clique for which (6.3) and (6.4) are satisfied, while $t_{j(1)} - L(0,j(1))$ is small (in comparison with other cliques), and which has not yet been used to derive a cut.

<u>Example 6.1</u>. Consider the disjunctive graph G of Example 4.1. Minimizing t_8 subject to $t_j - t_i \geq d_{ij}$, $(i,j) \in A$ and $t_i \geq 0$, $i \in N$, yields $t^o = (0,2,0,2,0,1,3,6)$. Since $t^o_1 = L(0,1) = 0$ and $t^o_6 = 1 < t^o_1 + d_{16} = 2$, the clique induced by $\{1,6\}$ satisfies the conditions of Theorem 6.2. Thus we solve

$$\begin{aligned} \min\ & 0\alpha_1 + 1\alpha_6 \\ \text{s.t.}\ & 0\alpha_1 + 2\alpha_6 \geq 1 \\ & 4\alpha_1 + 1\alpha_6 \geq 1 \\ & \alpha_1, \alpha_6 \geq 0 \end{aligned}$$

and find $(\alpha^1_1, \alpha^1_6) = (1/8, 1/2)$, which yields the inequality

$$t_1 + 4t_6 \geq 8$$

violated by t^o. Since $< \{1,6\} >$ is a dominant clique, this inequality induces a facet of P. Minimizing t_7 subject to the same constraints as before, plus $t_1 + 4t_6 \geq 8$, yields $t^1 = (0,2,0,2,0,2,4,6)$.

Since $t^1_2 = L(0,2) = 2$ and $t^1_4 < t^1_2 + d_{24} = 4$, the clique induced by $\{2,4\}$ satisfies the conditions of Theorem 6.2. Solving

$$\begin{aligned} \min\ & 2\alpha_2 + 2\alpha_4 \\ \text{s.t.}\ & 2\alpha_2 + 4\alpha_4 \geq 1 \\ & 6\alpha_2 + 2\alpha_4 \geq 1 \\ & \alpha_2,\ \alpha_4 \geq 0 \end{aligned}$$

yields $(\alpha_2^2, \alpha_4^2) = (1/10, 1/5)$, and the inequality

$$t_2 + 2t_4 \geq 10$$

violated by t^1. Again, $< \{2,4\} >$ is a dominant clique and hence the inequality induces a facet of P. Adding this inequality to the earlier constraint set on t and minimizing t_7 yields $t^2 = (0,4,0,3,0,2,4,7)$.

The conditions of Theorem 6.2 are no longer satisfied, since $t_j^o > L(0,j)$ for j = 2,4,7. However, each of the cliques not yet used to derive a cut, i.e., $\{4,7\}$, $\{2,7\}$ or $\{2,4,7\}$, provides an inequality that cuts off t^2 (this can be seen by checking the list of facet-inducing inequalities for $P(K_2)$ in Example 4.1). In particular, if we take the clique $\{2,4,7\}$, then solving

$$\begin{aligned} \min\ & 2\alpha_2 + 4\alpha_4 + 4\alpha_7 \\ \text{s.t.}\ & 2\alpha_2 + 4\alpha_4 + 7\alpha_7 \geq 1 \\ & 10\alpha_2 + 2\alpha_4 + 5\alpha_7 \geq 1 \\ & 8\alpha_2 + 10\alpha_4 + 3\alpha_7 \geq 1 \\ & 2\alpha_2 + 12\alpha_4 + 6\alpha_7 \geq 1 \\ & 6\alpha_2 + 2\alpha_4 + 10\alpha_7 \geq 1 \\ & 13\alpha_2 + 9\alpha_4 + 3\alpha_7 \geq 1 \\ & \alpha_2,\ \alpha_4,\ \alpha_7 \geq 0, \end{aligned}$$

yields $(\alpha_2^3, \alpha_4^3, \alpha_7^3) = (13/206, 3/206, 24/206)$ (with $s_i = 0$ for i = 1,3,4), and the (facet inducing) inequality

$$13t_2 + 3t_4 + 24t_7 \geq 206,$$

which cuts off t^2.||

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Appendix: Proof of Theorem 5.5.

We will make use of the following auxiliary result:

Lemma 5.6. Let $< K >$ be a dominant clique of G, y^o an extreme point of P(K) with associated selection S(K), and S a conformal extension of S(K) to G. Further, let t^o be the extension to G of y^o defined by (5.2), and let $k \in K$ be such that $y_k^o > L(0,k)$. Then every path $P(i,j)_S$ in D_S originating with some $i \in B(k)$ and such that $t_s - t_r = d_{rs}$ for all $(r,s) \in P(i,j)_S$, terminates in some $j \in B(K)$.

Proof. Let $P(i,j)_S$ be a path in D_S originating with some $i \in B(k)$ and such that $t_s - t_r = d_{rs}$ for all $(r,s) \in P(i,j)_S$. Since $t_i^o = \ell(0,i)_S$, there exists a (longest) path $P(0,i)_S$ from 0 to i in D_S such that $t_s^o - t_r^o = d_{rs}$ for all $(r,s) \in P(0,i)_S$. It then follows that the path $P(0,j)_S := P(0,i)_S \cup P(i,j)_S$ is a longest path from 0 to j in D_S, since $t_s^o - t_r^o = d_{rs}$ for all $(r,s) \in P(0,j)_S$. Now suppose $j \in K$. Since $< K >$ is a dominant clique of G, it then follows from Lemmma 5.3 that $P(0,j)_S$ contains only arcs of $A^o \cup S(K)$, i.e., is of the form $P(0,k) \cup P(k,j)_S$, where P(0,k) is the (unique) path from 0 to k in D. But then P(0,k) is a longest path from 0 to k in D_S and $t_k^o = L(0,k)$, contrary to $t_k^o = y_k^o > L(0,k)$, as assumed in the Lemma. Thus $j \notin K$.

Suppose next that $j \in N \setminus K \cup B(K)$, and let (r,s) be the (unique) arc of $P(i,,j)_S$ such that $r \in B(K)$, $s \in N \setminus K \cup B(K)$. Then from the definition of t^o, $t_s^o - t_r^o > d_{rs}$, contrary to our assumption about $P(i,j)_S$. This proves that $j \notin N \setminus K \cup B(K)$. Consequently $j \in B(K)$.‖

Proof of the Theorem. Let y^i, $i = 1,\ldots,p$, be extreme points of P(K), each of which satisfies $\alpha y = 1$. We will contract n schedules t^i for G, each of which is an extension of one of the p schedules y^i for $< K >$, and therefore satisfies $(\alpha,0)t = 1$. We will then prove that these n vectors $t^i \in \mathbb{R}^n$ are affinely independent, by showing that the $(n-1) \times n$ matrix whose rows are the vectors $t^i - t^1$, $i = 2,\ldots,n$, is of full row rank.

W.l.o.g., we assume that the numbering of the nodes of G is such that $K = \{1,\ldots,p\}$, $B(K) = \{p+1,\ldots,q\}$, and $N \setminus K \cup B(K) = \{q+1,\ldots,n\}$.

(i) First, we extend to G the p affinely independent schedules y^i, $i = 1,\ldots,p$, for $< K >$. To this end for $i = 1,\ldots,p$ we let $S(k)_i$ be the selection in $< K >$ associated with y^i, and S_i a conformal extension of $S(K)_i$ to G,

with the proviso that the arcs of S_i chosen freely under rule (ii) of the definition of a conformal extension (see section 5) are the same for all $i \in \{1,\ldots,p\}$. Next, for $i = 1,\ldots,p$, we let t^i be the extension of y^i to G defined by (5.2) for $S = S_i$, with the proviso that the scalar U used in the definition of t^i be the same for all $k \in \{1,\ldots,p\}$. The fact that the vectors t^i defined in this way are schedules for G follows from Theorem 5.4. Note that our specifications for S_i and t^i imply that $\ell(j,n)_{S_i} = \ell(j,n)_{S_1}$ and $t^i_j = t^1_j$, $j \in N \setminus K \cup B(K)$, $i = 2,\ldots,n$.

Subtracting the vector t^1 from each of the p-1 vectors t^i, $i = 2,\ldots,p$, yields the $(p-1) \times n$ matrix M_1 whose rows are $t^i - t^1$, $i = 2,\ldots,p$, and which is of the form $M_1 = (M_{11}, M_{12}, 0)$. Here M_{11} is the $(p-1) \times p$ full row rank matrix whose rows are the p-1 linearly independent vectors $y^i - y^1$, $i = 2,\ldots,p$, M_{12} is $(p-1) \times (q-p)$, and 0 is the $(p-1) \times (n-q)$ zero matrix.

(ii) The next q-p schedules t^i, $i = p+1,\ldots,q$, are generated as follows. For every node $k \in K$, there exists at least one among the p vectors y^i chosen at the beginning of this proof, say $y^{i(k)}$, such that $y^{i(k)}_k > L(0,k)$. To see why this is true, notice that if $y^i_k = L(0,k)$ for $i = 1,\ldots,p$, then the p vectors y^i lie in the (p-2)-dimensional subspace of $\mathbb{R}^p$ defined by the two equations $\alpha y = 1$ and $y_k = L(0,k)$, hence they cannot be affinely independent.

Now let $S(K)_{i(k)}$ be the selection in $< K >$ associated with $y^{i(k)}$, $S_{i(k)}$ a conformal extension to G of $S(K)_{i(k)}$, and $t^{i(k)}$ the extension to G of $y^{i(k)}$ defined by (5.2) for $S = S_{i(k)}$. For $i \in B(k)$, let $A(i)_{i(k)}$ be the set of nodes $j \in N$ reachable from i (including i itself) by a path $P(i,j)_{S_{i(k)}}$ in $D_{S_{i(k)}}$ such that for every $(r,s) \in P(i,j)_{S_{i(k)}}$, $t^{i(k)}_s - t^{i(k)}_r = d_{rs}$, and let

$$A(B(k))_{i(k)} = \bigcup_{i \in B(k)} A(i)_{i(k)}.$$

Then from Lemma 5.6, $A(B(k))_{i(k)} \subseteq B(K)$, $k = 1,\ldots,p$, and since for each $k \in \{1,\ldots,p\}$ by definition $A(B(k))_{i(k)}$ contains $B(k)$,

$$\bigcup_{k=1}^{p} A(B(k))_{i(k)} = B(K).$$

W.l.o.g., let the q-p nodes of B(K) be numbered in such a way that

$$A(B(1))_{i(1)} = \{p+1,\ldots,p+\beta_1\}$$

$$A(B(2))_{i(2)} \setminus A(B(1))_{i(1)} = \{p+\beta_1+1,\ldots,p+\beta_2\}$$

$$\text{- -}$$

$$A(B(p))_{i(p)} \setminus \bigcup_{r=1}^{p-1} A(B(r))_{i(r)} = \{p+\beta_{p-1}+1,\ldots,p+\beta_p\} ,$$

with $p+\beta_p = q$; and, in addition, if $i,j \in \{p+\beta_{k-1}+1,\ldots,p+\beta_k\}$ for some $k \in \{1,\ldots,p\}$ (where we define $\beta_o = 0$) and $(i,j) \in A \cup S(K)_{i(k)}$, then $i < j$.

We then define the vectors t^{p+h} for $h = 1,\ldots,\beta_1$ by

$$t_j^{p+h} = \begin{cases} t^{i(1)} + \epsilon_h & , \quad j = p+\beta_1-h+1,\ldots,p+\beta_1 \\ t^{i(1)} & , \quad \text{otherwise} \end{cases}$$

with $0 < \epsilon_h < 1$, $h = 1,\ldots,\beta_1$; and for $h = \beta_{k-1}+1,\ldots,\beta_k$, $k = 2,\ldots,p$, by

$$t_j^{p+h} = \begin{cases} t_j^{i(k)} + \epsilon_n , & j \in \{p+\beta_{k-1}+\beta_k-h+1,\ldots,p+\beta_k\} \cup \hat{A}(B(k))_{i(k)} \\ t_j^{i(k)} & \text{otherwise} , \end{cases}$$

where $0 < \epsilon_h < 1$, $\forall$ h, and

$$\hat{A}(B(k))_{i(k)} = A(B(k))_{i(k)} \cap \left(\bigcup_{r=1}^{k-1} A(B(r))_{i(r)} \right).$$

From Lemma 5.6 and the definition of $A(B(k))_{i(k)}$, each of the vectors t^{p+h} defined above is a schedule for $D_{S_{i(k)}}$, hence for G.

Renumbering the schedules t^{p+h}, $h = 1,\ldots,\beta_p(=q-p)$ as t^i, $i = p+1,\ldots,q$, and subtracting from each t^i the vector t^1, we obtain the $(q-p) \times n$ matrix M_2 whose rows are $t^i - t^1$, $i = p+1,\ldots,q$, and which is of the form $M_2 = (M_{21}, M_{22}, 0)$. Here M_{21} is $(q-p) \times p$, 0 is the $(q-p) \times (n-q)$ zero matrix, and M_{22} is a $(q-p) \times (q-p)$ lower block triangular matrix of the form

$$M_{22} = \begin{pmatrix} T_1 & 0 & \cdot & \cdot & \cdot & 0 \\ X_{21} & T_2 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ X_{s1} & X_{s2} & \cdot & \cdot & \cdot & T_s \end{pmatrix}$$

where $\dot{M}_{11}$ is (p-1) x p and has full row rank. Let $\hat{M}_{11}$ be a (p-1) x (p-1) nonsingular submatrix of M_{11}, and let $\hat{M}_{21}$ be the matrix obtained from M_{21} by removing the column corresponding to the one that was removed from M_{11}. Further, let us permute the blocks of columns of M_{22}, and the corresponding blocks of M_{12}, by reversing the order of the s blocks, and let $\hat{M}_{22}$ and $\hat{M}_{12}$ be the resulting matrices.

Then M is of full row rank if and only if the (n-1) x (n-1) matrix

$$\hat{M} = \begin{pmatrix} \hat{M}_{11} & \hat{M}_{12} & 0 \\ \hat{M}_{21} & \hat{M}_{22} & 0 \\ 0 & 0 & M_{33} \end{pmatrix}$$

is nonsingular. Since $\hat{M}_{11}$ and M_{33} are nonsingular, $\hat{M}$ is nonsingular if and only if the matrix $\hat{M}_o := \hat{M}_{22} - \hat{M}_{21}\hat{M}_{11}^{-1}\hat{M}_{12}$ is nonsingular. It is not hard to see that the numbers ϵ_j used in the construction of $\hat{M}_{22}$ can always be chosen in a way that makes $\hat{M}_o$ nonsingular. We show this by induction on q-p. For q-p = 1, the condition is $\epsilon_{\beta_s} \neq m_*$, where m_* is the first element of the last row of $\hat{M}_{21}\hat{M}_{11}^{-1}\hat{M}_{12}$. Such ϵ_{β_s} obviously exists. Suppose the condition can be satisfied for q-p = 1,2,...,t-1, and let q-p = t. Let A be the matrix consisting of the last t rows and first t columns of $\hat{M}_o$. Denoting by a_{ij} the elements of A and by A_{ij} the cofactor of a_{ij}, and using expansion by the last column of A, we have

$$\det(A) = a_{1t}A_{1t} + \sum_{i=2}^{t} a_{it}A_{it} .$$

By the induction hypothesis, there exist numbers $0 < \epsilon_j < 1$, $j = \beta_s$, $\beta_s - 1,\ldots,\beta_s - t + 1$, such that $A_{1t} \neq 0$. Since $a_{1t} = \epsilon_{\beta_s - t} - m_{1t}$, where m_{1t} is the element of $\hat{M}_{21}\hat{M}_{11}^{-1}\hat{M}_{12}$ in the position corresponding to a_{1t}, we have that det (A) $\neq$ 0 if and only if

$$\epsilon_{\beta_s - t} \neq m_{1t} - \sum_{i=2}^{t} a_{it}A_{it}/A_{1t} ,$$

a condition which can obviously be satisfied. This completes the induction.

Thus the n schedules t^i for G, i = 1,...,n, are affinely independent. In addition, each one of them is an extension of a schedule for < K >, hence satisfies $(\alpha,0)t = 1$. Therefore the inequality $(\alpha,0)t \geq 1$ defines a facet of P.||

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18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Job shop scheduling, machine sequencing, disjunctive graph, disjunctive programming, polyhedral combinatorics, facets, cutting planes

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A well-known job shop scheduling problem can be formulated as follows. Given a graph G with node set N and with directed and undirected arcs, find an orientation of the undirected arcs that minimizes the length of a longest path in G. We treat the problem as a disjunctive program, without recourse to integer variables, and give a partial characterization of the scheduling polyhedron P(N), i.e., the convex hull of feasible schedules. In particular, we derive all the facet inducing inequalities for the scheduling polyhedron P(K) defined on some clique with node set K, and give ~~a sufficient condition for~~ such inequalities

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to also induce facets of P(N). One of our results is that any inequality that induces a facet of P(H) for some $H \subset K$, also induces a facet of P(K). Another one is a recursive formula for deriving a facet inducing inequality with p positive coefficients from one with p-1 positive coefficients. We also address the constraint identification problem, and give a procedure for finding an inequality that cuts off a given solution to a subset of the constraints.

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