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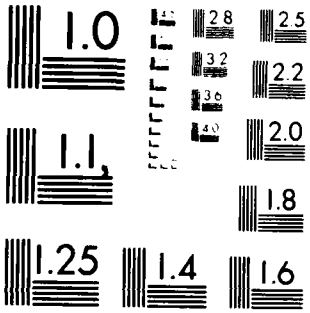

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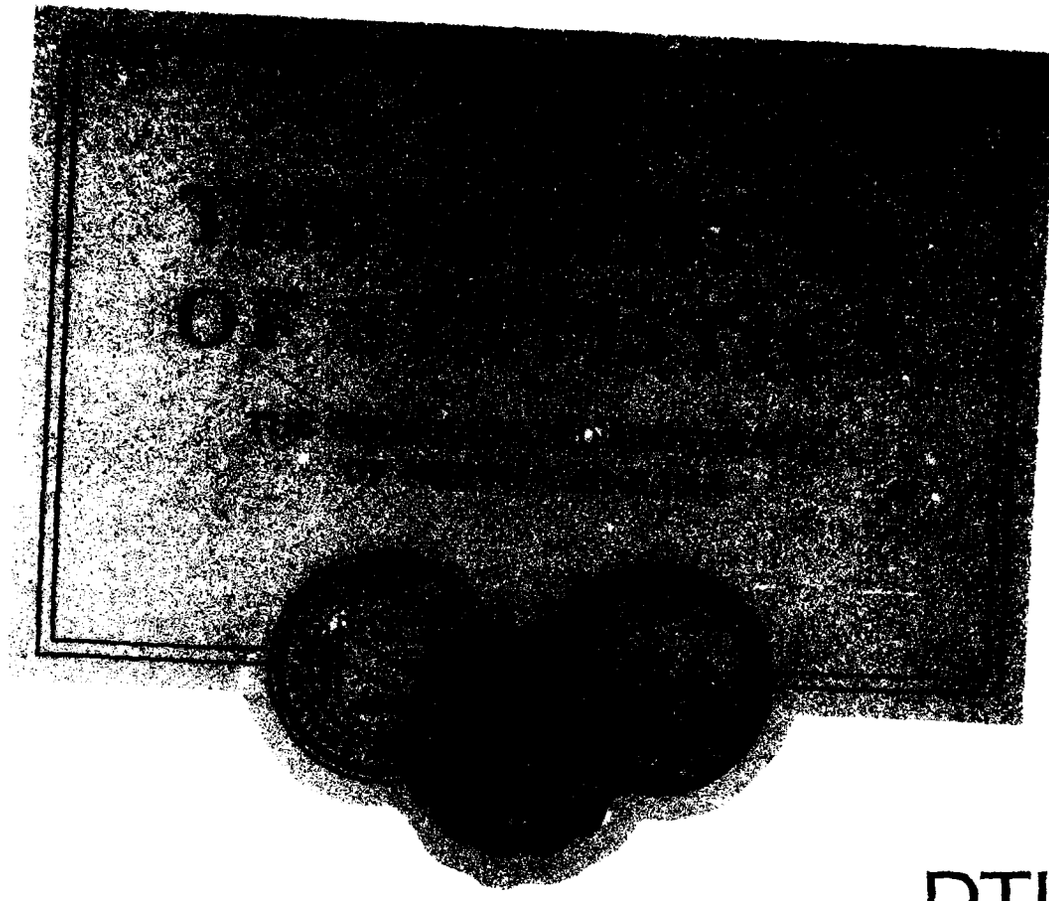

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ASYMPTOTIC MAXIMAL DEVIATION OF M-SMOOTHERS

by

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ASYMPTOTIC MAXIMAL DEVIATION OF M-SMOOTHERS*

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ABSTRACT

Let $(X_1,Y_1),\ldots,(X_n,Y_n)$ be iid rv's with pdf $f(x,y)$ and let $m(x) = E(Y|X = x) = \int yf(x,y)dy/f_X(x)$ be the regression function of Y on X. The function $m(x)$ is estimated by $m_n(x)$ a solution of $(nh)^{-1}\sum_{i=1}^{n} K((x-X_i)/h)\Psi(Y_i-\cdot) = 0$ for some odd and bounded Ψ-function making $m_n(x)$ a robust estimate of $m(x)$. Probabilities of maximal deviation of $|m_n(x) - m(x)|$ are computed in a similar way as in Bickel and Rosenblatt (1973) for density estimation and in Johnston (1982) for nonparametric regression function estimation.

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1. BACKGROUND AND INTRODUCTION

Nadaraya (1964) and Watson (1964) independently proposed the following kernel estimator

$$(1.1) \qquad m_n^*(x) = (nh_n)^{-1} \sum_{i=1}^{n} K((x-X_i)/h_n)Y_i/[(nh_n)^{-1} \sum_{j=1}^{n} K((x-X_j)/h_n)]$$

of the regression function $m(x) = \int y\, f(x,y)dy/f_X(x)$ where $f_X(x)$ denotes the marginal density of X, $K(\cdot)$ is a kernel and $\{h_n\}$ is a sequence of positive constants ("bandwidth"). Basically this estimator averages the Y's around $X = x$ motivated from the integral formula for $m(x)$ above. The numerator is a weighted local average of the Y's while the denominator is a density estimate of $f_X(x)$.

It is clear that occasional outliers generated by heavy tailed conditional densities $f(y|x)$ introduce smooth peaks and troughs in the estimated curve $m_n^*(x)$. Such outliers occur quite often in practice. (Ruppert et al., 1982 Figure 7 or Bussian et al., 1982). To avoid this misleading property of $m_n^*(x)$ due to spiky Y-observations we introduce a robust estimate, the M-smoother, $m_n(x)$ as the solution of

$$(1.2) \qquad (nh_n)^{-1} \sum_{i=1}^{n} K((x-X_i)/h_n)\Psi(Y_i - \cdot) = 0,$$

where Ψ denotes a bounded, odd and continuous function. Note that if $\Psi(u) = u$, then m_n is the Nadaraya-Watson estimator m_n^*. Bias and variance rates for $m_n(x)$ with K as the uniform window where obtained by Stuetzle and Mittal (1979), robustness properties, consistency and asymptotic normality of $m_n(x)$ were considered by Härdle (1982). For the case of nonrandom design, i.e. X_i attains fixed values, we may refer to Härdle and Gasser (1982). In this paper we show that

$$(1.3) \qquad P\{(2\delta \log n)^{\frac{1}{2}}[\sup_{0\le t\le 1} |(m_n(t) - m(t))\cdot r(t)|/\lambda(K)^{\frac{1}{2}} - d_n] < x\} \to \exp(-2\exp(-x)) \quad ,$$

where δ, $r(t)$, $\lambda(K)$, d_n are suitable scaling parameters.

The result (1.3) improves upon that of Johnston (1982) in a number of ways. First, Johnston obtains results like (1.3), but for estimates different from the Nadaraya-Watson estimator (1.1); our result (1.3) of course applies to the Nadaraya-Watson estimator as a special case. Secondly, (1.3) holds for a much broader class of estimators. Finally, we obtain (1.3) under assumptions weaker than those needed by Johnston.

2. ASSUMPTIONS AND RESULTS

We write h for the bandwidth h_n from here on unless there is no need to do so. We make use of the following assumptions.

(A1) the kernel $K(\cdot)$ is positive has compact support $[-A,A]$ and is continuously differentiable.

(A2) $(nh)^{-\frac{1}{2}}(\log n)^{3/2} \to 0 \qquad (n \log n)^{\frac{1}{2}} h^{5/2} \to 0$
$(nh^3)^{-1}(\log n)^2 \le M$, M a constant .

(A3) $h^{-3}(\log n) \int_{|y|> a_n} f_y(y)dy = O(1)$, $f_y(y)$ the marginal density of Y, $\{a_n\}_{n=1}^{\infty}$ a sequence of constants tending to infinity as $n \to \infty$.

(A4) $\inf_{0<t\le 1} |q(t)| \ge q_0 > 0$, where $q(t) = E(\Psi'(Y-m(t))|X=t)-f_X(t)$

(A5) the regression function $m(x)$ is twice continuously differentiable, the conditional densities $f(y|x)$ are symmetric for all x, Ψ is piecewise twice continuously differentiable.

We need some more definitions before we discuss the assumptions.

Define

$$\sigma^2(t) = E(\Psi^2(Y-m(t))|X=t)$$

$$H_n(t) = (nh)^{-1} \sum_{i=1}^{n} K((x-X_i)/h)\Psi(Y_i-m(t))$$

$$D_n(t) = (nh)^{-1} \sum_{i=1}^{n} K((x-X_i)/h)\Psi'(Y_i-m(t)).$$

We further assume that $\sigma^2(t)$ and $f_X(t)$ are differentiable.

Assumption (A1) on the compact support of the kernel could possibly be relaxed introducing a cutoff technique as Csörgö and Hall (1982) for density estimators. Assumption (A2) has purely technical reasons: to keep the bias down and to ensure the vanishing of the nonlinear remainder terms. Assumption (A3) appears in a somewhat modified form also in Johnston's paper (1982). When we want to apply the following theorem to the Nadaraya-Watson estimator $m_n^*(x)$ we have actually to restate (A2) as $h^{-3}(\log n)\int_{|y|>a_n} y^2 f_y(y)dy$ (which is assumption A1 in Johnston (1982)). Assumption (A5) stating the symmetry of the conditional densities is common in robustness considerations (Huber, 1981). It guarantees that the only solution of $\int\Psi(y-\cdot)f(y|x)dy = 0$ is $m(x) = E(Y|X=x)$. If we had skew distributions then we would no longer estimate the conditional mean but rather a conditional quantile such as the median.

Theorem

Let $h = n^{-\delta}$, $1/5 < \delta < 1/3$ and $\lambda(K) = \int_{-A}^{A} K^2(u)du$ and

$$d_n = (2\delta \log n)^{\frac{1}{2}} + (2\delta \log n)^{-\frac{1}{2}}\{\log(c_1(K)/\Pi^{\frac{1}{2}}) + \tfrac{1}{2}[\log \delta + \log\log n]\},$$

$$\text{if } c_1(K) = K^2(A) + K^2(-A)/[2\lambda(K)] > 0$$

$$d_n = (2\delta \log n)^{\frac{1}{2}} + (2\delta \log n)^{-\frac{1}{2}}\{\log (c_2(K)/2\Pi)\}$$

$$\text{otherwise with } c_2(K) = \int_{-A}^{A}[K'(u)]^2du/[2\lambda(K)] .$$

Then (1.3) holds with

$$r(t) = (nh)^{\frac{1}{2}} q(t)[\sigma^2(t) f_X(t)]^{-\frac{1}{2}} .$$

This theorem can be used to construct uniform confidence intervals for the regression function as stated in the following corollary.

Corollary: Assuming the theorem above holds, an approximate $(1-\alpha)\times 100\%$ confidence band over $[0,1]$ is

$$m_n(t) \pm (nh)^{-\frac{1}{2}}[\sigma^2(t) f_X(t)\lambda(K)]^{\frac{1}{2}}\, q^{-1}(t)[d_n + c(\alpha)(2\delta \log n)^{-\frac{1}{2}}] \cdot [\lambda(K)]^{\frac{1}{2}}$$

where $c(\alpha) = \log 2 - \log|\log(1-\alpha)|$.

The proof is essentially based on a linearization argument due to Taylor series expansion. The leading linear term will then be approximated in a similar way as in Johnston (1982), Bickel and Rosenblatt (1973). The main idea behind the proof is a strong approximation of the empirical process of $\{(X_i, Y_i)\}_{i=1}^n$ by a sequence of Brownian bridges (with two dimensional time) as provided Tusnady (1977).

It follows by Taylor expansions applied to the defining equation (1.2) that

$$m_n(t) - m(t) = (H_n(t) - EH_n(t))/q(t) + R_n(t) \tag{2.1}$$

where $[H_n(t) - EH_n(t)]/q(t)$ is the leading linear term and

$$\begin{aligned} R_n(t) = {} & H_n(t)[q(t) - D_n(t)]/[D_n(t)\cdot q(t)] + EH_n(t)/q(t) \\ & + \tfrac{1}{2}(m_n(t) - m(t))^2 \cdot [D_n(t)]^{-1} \cdot (nh)^{-1} \sum_{i=1}^{n} K((x-X_i)/h)\Psi''(Y_i - m(t) + r_n^{(i)}(t)), \\ & |r_n^{(i)}(t)| < |m_n(t) - m(t)|. \end{aligned} \tag{2.2}$$

is the remainder term. In the third section it is shown (Lemma 3.1) that $\|R_n\| = \sup_{0\le t\le 1} |R_n(t)| = o_p((nh \log n)^{-\frac{1}{2}})$.

Furthermore the rescaled linear part

$$Y_n(t) = (nh)^{\frac{1}{2}}[\sigma^2(t) f_X(t)]^{-\frac{1}{2}}(H_n(t) - EH_n(t))$$

is approximated by a sequence of Gaussian processes, leading finally to the following process

$$Y_{5,n}(t) = h^{-\frac{1}{2}} \int K((t-x)/h)\, dW(x),$$

as in Bickel and Rosenblatt (1973).

We also need the Rosenblatt transformation (Rosenblatt, 1952).

$$T(x,y) = (F_{X|y}(x|y),\ F_Y(y))$$

which transforms (X_i, Y_i) into $T(X_i, Y_i) = (X_i', Y_i')$ mutually independent uniform rv's. With the aid of this transformation Theorem 1 of Tusnády (1977) may be applied to obtain the following lemma.

Lemma 2.1: On a suitable probability space there exists a sequence of Brownian bridges B_n such that

$$\sup_{x,y} |Z_n(x,y) - B_n(T(x,y)| = O(n^{-\frac{1}{2}}(\log n)^2) \quad \text{a.s.},$$

where $Z_n(x,y) = n^{\frac{1}{2}}[F_n(x,y) - F(x,y)]$ denotes the empirical process of $\{(X_i, Y_i)\}_{i=1}^n$.

Before we define the different approximating processes let us first rewrite $Y_n(t)$ as a stochastic integral with respect to the empirical process $Z_n(x,y)$.

$$Y_n(t) = h^{-\frac{1}{2}} g'(t)^{-\frac{1}{2}} \iint K((t-x)/h)\Psi(y-m(t))\, dZ_n(x,y),\quad g'(t) = \sigma^2(t)\, f_X(t).$$

The approximating processes are now

$$Y_{o,n}(t) = (hg(t))^{-\frac{1}{2}} \iint_{\Gamma_n} K((t-x)/h)\Psi(y-m(t))\, dZ_n(x,y),$$

where $\Gamma_n = \{|y| \le a_n\}$, $g(t) = E(\Psi^2(y-m(t)) \cdot I(|y| \le a_n) | X=t) \cdot f_X(t)$

$$Y_{1,n}(t) = (hg(t))^{-\frac{1}{2}} \iint_{\Gamma_n} K((t-x)/h)\Psi(y-m(t))\, dB_n(T(x,y)),$$

$\{B_n\}$ being the sequence of Brownian bridges from Lemma 2.1.

$$Y_{2,n}(t) = (hg(t))^{-\frac{1}{2}} \iint_{\Gamma_n} K((t-x)/h)\Psi(y-m(t))\, dW_n(T(x,y))$$

$\{W_n\}$ being the sequence of Wiener processes satisfying

$$B_n(x',y') = W_n(x', y') - x'y'W_n(1,1)$$

$$Y_{3,n}(t) = (hg(t))^{-\frac{1}{2}}\iint_{\Gamma_n} K((t-x)/h)\Psi(y-m(x))\, dW_n(T(x,y))$$

$$Y_{4,n}(t) = (hg(t))^{-\frac{1}{2}}\int g(x)^{\frac{1}{2}}K((t-x)/h)dW(x)$$

$$Y_{5,n}(t) = h^{-\frac{1}{2}}\int K((t-x)/h)dW(x),$$

$\{W(\cdot)\}$ being the Wiener process on $(-\infty,\infty)$.

Lemmata 3.2 to 3.7 ensure that all these processes have the same limit distributions. The results then follows from the following lemma

Lemma 2.2 (Bickel and Rosenblatt (1973)). Let d_n, $\lambda(K)$, δ as in the theorem. Let

$$Y_{5,n}(t) = h^{-\frac{1}{2}}\int K((t-x)/h)\, dW(x).$$

Then

$$P((2\delta \log n)^{\frac{1}{2}}\{\sup_{0\le t\le 1} |Y_{5,n}(t)|/[\lambda(K)]^{\frac{1}{2}} - d_n\} < x) \to e^{-2e^{-x}} .$$

3. PROOFS

We show first that $\|R_n\| = \sup_{0\le t\le 1}|R_n(t)|$ vanishes asymptotically with the desired rate $(nh \log n)^{-\frac{1}{2}}$.

Lemma 3.1: For the remainder term $R_n(t)$ defined in (2.2) we have

$$\|R_n\| = o_p((nh \log n)^{-\frac{1}{2}}) \quad . \tag{3.1}$$

Proof: First we have by the positivity of the kernel K and $|\Psi''| \le C_1$

$$\|R_n\| \le [\inf_{0\le t\le 1}(|D_n(t)|\cdot q(t))]^{-1}\{\|H_n\| \cdot \|q-D_n\| + \|D_n\| \cdot \|EH_n\|\}$$

$$+ C_1 \cdot \|m_n-m\|^2 \cdot [\inf_{0\le t\le 1}|D_n(t)|]^{-1}\cdot \|f_n\| ,$$

where $f_n = (nh)^{-1}\sum_{i=1}^{n} K((x-X_i)/h)$.

The desired result (3.1) will then follow if we prove the following:

$$\|H_n\| = o_p(n^{-\frac{1}{4}}h^{-\frac{1}{4}}\cdot(\log n)^{-\frac{1}{2}}) \tag{3.2}$$

(3.3) $\|q-D_n\| = o_p(n^{-\frac{1}{4}}h^{-\frac{1}{4}}(\log n)^{-\frac{1}{2}})$

(3.4) $\|EH_n\| = O(h^2)$

(3.5) $\|m_n-m\|^2 = o_p((nh)^{-\frac{1}{2}}(\log n)^{-\frac{1}{2}})$.

Define $U_n(t) = n^{\frac{1}{2}}h^{\frac{1}{4}}(\log n)^{\frac{1}{2}}[H_n(t)-EH_n(t)]$.

We first show that $U_n(t) \xrightarrow{p} 0$ for all t. This follows from Markov's inequality since

$$U_n(t) = \sum_{i=1}^{n} U_{i,n}(t) ,$$

where $U_{i,n}(t) = n^{-3/4}h^{-3/4}(\log n)^{\frac{1}{2}}[K((t-X_i)/h)\Psi(Y_i-m(t))-EK((t-X)/h)\cdot\Psi(y-m(t))]$,

are iid rv's and thus

$$P(|U_n(t)|>\epsilon) \le \epsilon^{-2}n^{-\frac{1}{2}}h^{-\frac{1}{2}}(\log n)\cdot h^{-1}EK^2((t-X)/h)\Psi^2(Y-m(t)).$$

The RHS of this inequality tends to zero since

$$h^{-1}EK^2((t-X)/h)\Psi^2(Y-m(t)) = h^{-1}\int K^2((t-u)/h)E(\Psi^2(Y-m(t))|X=u)f_X(u)du$$
$$\sim \sigma^2(t)\cdot f_X(t)\cdot\int K^2(u)du$$

by continuity of $\sigma^2(t)$ and $f_X(t)$.

Next we show the tightness of $U_n(t)$ using the following moment condition (Billingsley, 1968, Th. 15.6)

$$E\{|U_n(t)-U_n(t_1)|\cdot|U_n(t_2)-U_n(t)|\} \le C_2\cdot(t_2-t_1)^2$$

where C_2 is a constant.

By the Schwarz inequality,

$$E\{|U_n(t)-U_n(t_1)|\cdot|U_n(t_2)-U_n(t)|\}$$
$$\le \{E[U_n(t)-U_n(t_1)]^2\cdot E[U_n(t_2)-U_n(t)]^2\}^{\frac{1}{2}} .$$

It suffices to consider only the term $E\{U_n(t)-U_n(t_1)]^2$.

Using the Lipschitz continuity of K,Ψ,m and assumption (A2) we have

$$\{E[U_n(t)-U_n(t_1)]^2\}^{\frac{1}{2}}$$

$$\cdot (\log n)(nh)^{-3/2} \cdot E[A+B]^2)^{1/2}$$

$$\cdot C_A(nh)^{-1/2}(\log n)^{1/2}\, |t-t_1| + C_B(n^{-1/2}h^{-3/4}(\log n)^{1/2} \cdot |t-t_1| \cdot C_3 \cdot |t-t_1|$$

where $A = \sum_{i=1}^{n} K((t-X_i)/h)[\psi(Y_i-m(t))-\psi(Y_i-m(t_1))]$

$$B = \sum_{i=1}^{n} (Y_i-m(t_1))[K((t_1-X_i)/h)-K((t-X_i)/h)],$$

and C_A, C_B are Lipschitz bounds for ψ, m, K.

Since (3.4) follows from the well-known bias calculation

$$EH_n(t) = h^{-1}\int K((t-u)/h)E(\psi(y-m(t))|X=u)f_X(u)du = O(h^2),$$

where $O(h^2)$ is independent of t (Parzen, 1962) we have from assumption (A2) that $\|EH_n\| = o((nh)^{-1/2}(\log n)^{-1/2})$.

Statement (3.2) thus follows using tightness of $U_n(t)$ and the inequality $\|H_n\| \le \|H_n-EH_n\| + \|EH_n\|$.

Statement (3.3) follows in the same way as (3.2) using assumption (A2) and the continuity properties of K, ψ', m.

Finally from Härdle and Luckhaus (1982), where uniform continuity of $m_n(t)-m(t)$ is shown, we have

$$\|m_n-m\| = O_p((nh)^{-1/2}(\log n)^{1/2}),$$

which implies (3.5) .

Now the assertion of the lemma follows since by tightness of $D_n(t)$, $\inf_{0\le t\le 1} |D_n(t)| \xrightarrow{p} q_0$ and thus

$$\|R_n\| = o_p((nh)^{-1/2}(\log n)^{-1/2})(1 + \|f_n\|).$$

Finally by Theorem 3.1 of Bickel and Rosenblatt (1973) $\|f_n\| = O_p(1)$, thus the desired result $\|R_n\| = o_p((nh)^{-1/2}(\log n)^{-1/2})$ follows. In the nonrobust case, i.e. $\psi(u) = u$, the remainder term R_n reads

$$R_n = [m_n^* - m][f_X-f_n]f_X^{-1} + E(\hat{m}_n-mf_n)/f_X \,, \tag{3.6}$$

where $\hat{m}_n(x) = (nh)^{-1} \sum_{i=1}^{n} K((x-X_i)/h)Y_i$.

Johnston (1982) proved that $(\hat{m}_n - E\,\hat{m}_n)/f$ has the desired asymptotic distribution as stated in our Theorem.

So if we apply the recent result of Mack and Silverman (1982) or Härdle and Luckhaus (1982) to $\|m_n^* - m\|$ and the well known result from Bickel and Rosenblatt (1973) to $\|f_X - f_n\|$ we may conclude that the first term on the RHS of (3.6) is $o_p((nh)^{-\frac{1}{2}}(\log n)^{-\frac{1}{2}})$. The second term in (3.6) is

$$[h^{-1}\int K((t-u)/h)\cdot m(u)f(u)du - m(t)h^{-1}\int K((t-u)/h)f(u)du]/f_X(t)$$

which is by the same calculations as mentioned above (Parzen, 1962) of the order $O(h^2)$. This shows that our result generalizes Johnston's paper. Our theorem says also that the confidence bounds are smaller. Johnston had $s^2(t) = E(Y^2|X=t)$ as a factor for the asymptotic confidence bound, we have $\sigma^2(t) = \mathrm{var}(Y|X=t)$ which is in general smaller than $s^2(t)$. We now begin with the subsequent approximations of the processes $Y_{0,n}$ to $Y_{5,n}$.

<u>Lemma 3.2</u>:

$$\|Y_{0,n} - Y_{1,n}\| = O((nh)^{-\frac{1}{2}}(\log n)^2) \qquad \text{a.s.}$$

Proof: Let t be fixed and put $L(y) = \Psi(y-m(t))$ still depending on t.

Use integration by parts and obtain:

$$\iint_{\Gamma_n} L(y)K((t-x)/h)dZ_n(x,y) =$$

$$= \int_{u=-A}^{A}\int_{y=-a_n}^{a_n} L(y)K(u)dZ_n(t-h\cdot u,y) =$$

$$= \int_{-A}^{A}\int_{-a_n}^{a_n} Z_n(t-h\cdot u,y)d[L(y)K(u)] + L(a_n)\int_{-A}^{A} Z_n(t-h\cdot u,a_n)dK(u)$$

$$-L(-a_n)\int_{-A}^{A} Z_n(t-h\cdot u,-a_n)dK(u) + K(A)[\int_{-a_n}^{a_n} Z_n(t-h\cdot u,y)dL(y)$$

$$+L(a_n)Z_n(t-h\cdot A,a_n) - L(-a_n)Z_n(t-h\cdot A,-a_n)]$$

$$-K(-A)[\int_{-a_n}^{a_n} Z_n(t+h\cdot A,y)dL(y) + L(a_n)Z_n(t+h\cdot A,a_n)$$

$$-L(-a_n)Z_n(t+h\cdot A,-a_n)] .$$

If we apply the same operations to $Y_{1,n}$ with $B_n(T(x,y))$ instead of $Z_n(x,y)$ and use Lemma 2.1 we finally obtain

$$\sup_{0\le t\le 1} h^{\frac{1}{2}} q(t)^{\frac{1}{2}} |Y_{0,n}^{(t)} - Y_{1,n}(t)| = O((nh)^{-\frac{1}{2}}(\log n)^2) \qquad \text{a.s.}$$

using the differentiability and boundedness of ψ.

Lemma 3.3:

$$\| Y_{1,n} - Y_{2,n} \| = O_p(h^{\frac{1}{2}})$$

Proof: Note that the Jacobi of $T(x,y)$ is $f(x,y)$ hence

$$|Y_{1,n}(t) - Y_{2,n}(t)| = |(q(t)h)^{-\frac{1}{2}} \iint_{\Gamma_n} \psi(y-m(t)K((t-x)/h)f(x,y)dxdy| \cdot |W_n(1,1)|$$

It follows that

$$h^{-\frac{1}{2}} \| Y_{1,n}-Y_{2,n} \| \le |W_n(1,1)| \cdot \| g^{-\frac{1}{2}} \| \cdot \sup_{0\le t\le 1} h^{-1} \iint_{\Gamma_n} |\psi(y-m(t))K((t-x)/h)| f(x,y)dxdy$$

Since $\| g^{-\frac{1}{2}} \|$ is bounded by assumption and ψ is bounded we have

$$h^{-\frac{1}{2}} \| Y_{1,n}-Y_{2,n} \| \le |W_n(1,1)| \cdot C_4 \cdot h^{-1} \int (K((t-x)/h))dx = O_p(1).$$

Lemma 3.4:

$$\| Y_{2,n}-Y_{3,n} \| = O_p(h^{\frac{1}{2}})$$

Proof: The difference $|Y_{2,n}(t)-Y_{3,n}(t)|$ may be written as

$$|(q(t)h)^{-\frac{1}{2}} \iint_{\Gamma_n} [\psi(y-m(t))-\psi(y-m(x)]K((t-x)/h)dW_n(T(x,y))|$$

If we use the fact that ψ,m are uniformly continuous this is smaller than

$$h^{-\frac{1}{2}} |q(t)|^{-\frac{1}{2}} \cdot O_p(h)$$

and the lemma thus follows.

Lemma 3.5:

$$\| Y_{4,n}-Y_{5,n} \| = O_p(h^{\frac{1}{2}})$$

Proof:

$$|Y_{4,n}(t)-Y_{5,n}(t)| = h^{-\frac{1}{2}} |\int \{[\frac{q(x)}{g(t)}]^{\frac{1}{2}} -1\} K((t-x)/h)dW(x)| \le$$

$$\cdot\ h^{-\frac{1}{2}}\left|\int_{-A}^{A} W(t-hu)\ \frac{\partial}{\partial u}\{[\frac{g(t-hu)}{g(t)}]^{\frac{1}{2}} - 1\}\ K(u)du\right|$$

$$+\ h^{-\frac{1}{2}}|K(A)W(t-hA)\{\frac{g(t-Ah)}{g(t)}]^{\frac{1}{2}} - 1\}|$$

$$+\ h^{-\frac{1}{2}}|K(-A)W(t+hA)\{[\frac{g(t+h)A)}{g(t)}]^{\frac{1}{2}} - 1\}|$$

$$= S_{1,n}(t) + S_{2,n}(t) + S_{3,n}(t)\ ,\ \text{say.}$$

The second term can be estimated by

$$h^{-\frac{1}{2}}\|S_{2,n}\| \le K(A)\cdot \sup_{0\le t\le 1}|W(t-Ah)|\cdot \sup_{0\le t\le 1} h^{-1}|\{[\frac{g(t-Ah)}{g(t)}]^{\frac{1}{2}} - 1\}|$$

by the mean value theorem it follows that

$$h^{-\frac{1}{2}}\|S_{2,n}\| = O_p(1).$$

The first term $S_{1,n}$ is estimated as follows.

$$h^{-1}S_{1,n}(t) = |h^{-1}\int_{-A}^{A} W(t-uh)\{K'(u)([\frac{g(t-uh)}{g(t)}]^{\frac{1}{2}} - 1\}\ du$$

$$-\tfrac{1}{2}\int_{-A}^{A} W(t-uh)K(u)[\frac{g(t-uh)}{g(t)}]^{-\frac{1}{2}}[\frac{g'(t-uh)}{g(t)}]du|$$

$$= |T_{1,n}(t) - T_{2,n}(t)|\ ,\ \text{say.}$$

$$\|T_{2,n}\| \le C_5\cdot\int_{-A}^{A}|W(t-hu)|du = O_p(1) \text{ by assumption on } g(t) = \sigma^2(t)\cdot f_X(t).$$

To estimate $T_{1,n}$ we again use the mean value theorem to conclude that

$$\sup_{0\le t\le 1} h^{-1}|[\frac{g(t-uh)}{g(t)}]^{\frac{1}{2}} - 1| < C_6\cdot|u|$$

hence

$$\|T_{1,n}\| \le C_6\cdot \sup_{0\le t\le 1}\int_{-A}^{A}|W(t-hu)K'(u)u|du = O_p(1).$$

Since $S_{3,n}(t)$ is estimated as $S_{2,n}(t)$ we finally obtain the desired result.

The next lemma shows that the truncation introduced through $\{a_n\}$ does not affect the limiting distribution.

Lemma 3.6:

$$\| Y_n - Y_{0,n} \| = O_p((\log n)^{-\frac{1}{2}}).$$

Proof: We shall only show that $g'(t)^{-\frac{1}{2}} h^{-\frac{1}{2}} \iint_{\mathbb{R}-\Gamma_n} \psi(y-m(t))K((t-x)/h)dZ_n(x,y)$ fulfills the lemma.

The replacement of $g'(t)$ by $g(t)$ may be proved as in Johnston (1982). The quantity above is less than $h^{-\frac{1}{2}} \| g^{-\frac{1}{2}} \| \cdot \| \iint_{\{|y|>a_n\}} \psi(y-m(\cdot))K((\cdot-x)/h)dZ(x,y) \|$.

It remains to show that the last factor tends to zero at a rate $O_p((\log n)^{-\frac{1}{2}})$. We show first that

$$V_n(t) = (\log n)^{\frac{1}{2}} h^{-\frac{1}{2}} \iint_{\{|y|>a_n\}} \psi(y-m(t))K((t-x)/h)dZ_n(x,y)$$

$$\xrightarrow{P} 0 \quad \text{for all } t$$

and then we show tightness of $V_n(t)$, the result then follows.

$$V_n(t) = (\log n)^{\frac{1}{2}}(nh)^{-\frac{1}{2}} \sum_{i=1}^{n} \{\psi(Y_i - m(t) I_{\{|y|>a_n\}}(Y_i)K((t-X_i)/h)$$

$$- E\psi(Y_i - m(t)) \cdot I_{\{|y|>a_n\}}(Y_i)K((t-X_i)/h)\}$$

$$= \sum_{i=1}^{n} X_{n,i}(t)$$

where $\{X_{n,i}(t)\}_{i=1}^{n}$ are iid for each n with $EX_{n,i}(t) = 0$ for all $t \in [0,1]$.

We have then

$$EX_{n,i}^2(t) \le (\log n)(nh)^{-1} E\psi^2(Y_i - m(t)) I_{\{|y|>a_n\}}(Y_i)K^2((t-X_i)/h)$$

$$\le \sup_{-A \le u \le A} K^2(u) \cdot (\log n)(nh)^{-1} E\psi^2(Y_i - m(t)) I_{\{|y|>a_n\}}(Y_i)$$

hence

$$\mathrm{var}\{V_n(t)\} = E(\sum_{i=1}^{n} X_{n,i}(t))^2 = n \cdot EX_{n,i}^2(t)$$

$$\le \sup_{-A \le u \le A} K^2(u) h^{-1} (\log n) \int_{\{|y|>a_n\}} f_y(y)dy \cdot M_\psi$$

where M_ψ denotes an upper bound for ψ^2.

This term tends to zero by assumption (A3). Thus by Markov's inequality we conclude that

$$V_n(t) \xrightarrow{p} 0 \qquad \text{for all } t \in [0,1].$$

To prove tightness of $\{V_n(t)\}$ we refer again to the following moment condition as stated in Lemma 3.1.

$$E\{|V_n(t) - V_n(t_1)|\cdot|V_n(t_2)-V_n(t)|\} \le C'\cdot(t_2-t_1)^2$$

C' denoting a constant, $t \in [t_1,t_2]$.

We again estimate the left hand side by Schwarz's inequality and estimate each factor separately.

$$E[V_n(t)-V_n(t_1)]^2 = (\log n)(nh)^{-1}E\{\sum_{i=1}^{n}\Psi_n(t,t_1,X_i,Y_i)\cdot I_{\{|y|>a_n\}}(Y_i)-$$
$$- E(\Psi_n(t,t_1,X_i,Y_i)\cdot I_{\{|y|>a_n\}}(Y_i))\}^2 ,$$

where $\Psi_n(t,t_1,X_i,Y_i) = \psi(Y_i-m(t))K((t-X_i)/h)-\psi(Y_i-m(t_1))K((t_1-X_1)/h)$

Since ψ,m,K are Lipschitz continuous it follows

$$\{E[V_n(t)-V_n(t_1)]^2\}^{\frac{1}{2}}$$
$$\le C_7\cdot (\log n)^{\frac{1}{2}}h^{-3/2}|t-t_1|\cdot\{\int_{\{|y|>a_n\}} f_y(y)dy\}^{\frac{1}{2}}$$

If we apply the same estimations to $V_n(t_2)-V_n(t_1)$ we finally have

$$E\{|V_n(t)-V_n(t_1)|\cdot|V_n(t_2)-V_n(t)|\} \le C_7^2(\log n)h^{-3}|t-t_1||t_2-t|$$
$$\cdot\int_{\{|y|>a_n\}} f_y(y)dy$$
$$\le C'\cdot|t_2-t_1|^2 \text{ since } t\in[t_1,t_2] .$$

by assumption (A3).

<u>Lemma 3.7</u>: Let $\lambda(K) = \int K^2(u)du$ and let $\{d_n\}$ as in the theorem. Then

$$(2\delta \log n)^{\frac{1}{2}}[\| Y_{3,n}\| /[\lambda(K)]^{\frac{1}{2}} - d_n]$$

has the same asymptotic distribution as

$$(2\delta \log n)^{\frac{1}{2}}[\| Y_{4,n}\| /[\lambda(K)]^{\frac{1}{2}} - d_n]$$

Proof: $Y_{3,n}(t)$ is a Gaussian process with

$$EY_{3,n}(t) = 0$$

and covariance function

$$r_3(t_1,t_2) = EY_{3,n}(t_1)Y_{3,n}(t_2)$$

$$= [g(t_1)g(t_2)]^{-\frac{1}{2}}h^{-1}\iint_{\Gamma_n}\psi^2(y-m(x))K((t_1-x)/h)K((t_2-x)/h)f(x,y)dxdy.$$

$$= h^{-1}[g(t_1)g(t_2)]^{-\frac{1}{2}}\iint_{\Gamma_n}\psi^2(y-m(x))f(y|x)dyK((t_1-x)/h)K((t_2-x)/h)f_X(x)dx$$

$$= h^{-1}[g(t_1)g(t_2)]^{-\frac{1}{2}}\int g(x)K((t_1-x)/h)K((t_2-x)/h)\ dx$$

$= r_4(t_1,t_2)$ the covariance function of the Gaussian process $Y_{4,n}(t)$, which proves the lemma.

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