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Iterated logarithm; Gaussian sequence; almost sure limit set.

SEE REVERSE
ITEM #20, CONTINUED:

Let \( \{x_n, n \geq 1\} \) be a stationary Gaussian sequence with \( \text{EX}_1 = 0, \text{EX}_1^2 = 1 \) and \( r_n = \text{EX} X_{n+1} \). Let \( Z^{(i)}_n \) denote the \( i \)th maximum of \( x_1, \ldots, x_n \) and \( a_n = (\ln \ln n) (2 \ln n)^{-1/2} \), \( b_n = (2 \ln n)^{1/2} - (\ln (4 \pi \ln n)) / (2 (2 \ln n)^{1/2}) \). Then assuming \( r_n (\ln n)^2 = o(1) \) the set of almost sure limit points of the vectors \( (Z^{(1)}_n - h_n) a_n^{-1}, (Z^{(2)}_n - h_n) a_n^{-1}, \ldots, (Z^{(T)}_n - h_n) a_n^{-1} \) is determined. The number of components \( T = T(n) \to \infty \) as \( n \to \infty \). This extends a result of Hebbar.
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Abstract

Let \( \{X_n, n \geq 1\} \) be a stationary Gaussian sequence with \( E X_1 = 0, \ E X_1^2 = 1 \) and \( r_n = E X_n X_{n+1}. \) Let \( Z_n^{(1)} \) denote the ith maximum of \( X_1, \ldots, X_n \) and \( a_n = (\ln n \ln n)(2 \ln n)^{-1/2}, \)
\[ b_n = (2 \ln n)^{1/2} - (\ln(4 \pi \ln n)) / (2 (2 \ln n)^{1/2}), \] Then assuming \( r_n (\ln n)^2 = o(1) \) the set of almost sure limit points of the vectors \( ((Z_n^{(1)} - b_n) a_n^{-1}, (Z_n^{(2)} - b_n) a_n^{-1}, \ldots (Z_n^{(\ell)} - b_n) a_n^{-1}) \) is determined. The number of components \( \ell = \ell(n) \to \infty \) as \( n \to \infty. \) This extends a result of Hebbar.

Keywords: Iterated logarithm, Gaussian sequence, almost sure limit set.

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1. Introduction

Let \( \{X_n, n \geq 1\} \) be a stationary Gaussian sequence with \( EX_1=0, EX_1^2=1 \) and \( r_n=EX_1X_{n+1} \). Let \( Z_n^{(i)} \) denote the ith maximum of \( X_1, ..., X_n \) that is \( Z_n^{(i)} = \frac{Z_n^{(i)}}{\sigma_n} \) equals the \( i \)-th order statistic. Set \( a_n = \sqrt{n} \log n / \sqrt{2 \pi n} \) and \( b_n = \sqrt{2 \pi n} - \log(4n\log n)/2\sqrt{2 \pi n} \). In [1]

Hebbar considers the set of almost sure limit points of the sequence of vectors

\[
\left\{ \left( \frac{Z_n^{(1)} - b_n}{a_n}, \frac{Z_n^{(2)} - b_n}{a_n}, ..., \frac{Z_n^{(\ell)} - b_n}{a_n} \right), n \geq 1 \right\}.
\]

Hebbar shows that under the assumption

\[
r_n(\log n)^{2+\epsilon} = o(1)
\]

for some \( \epsilon > 0 \) the above sequence has almost sure limit set equal to \( \left\{ (x_1, x_2, ..., x_{\ell}) : 0 \leq x_\ell \leq ... \leq x_1 \right\} \). In the present paper we strengthen this result in two directions. We relax the condition on \( r_n \) to \( r_n(\log n)^{2} = o(1) \) and further we allow the number \( \ell \) of extreme values considered to grow to infinity with \( n \). Let \( v_n^{(i)} = \frac{Z_n^{(i)} - b_n}{a_n} \). Then we consider the points in \( \mathbb{R}^\infty \) given by \( (v_n^{(1)}, ..., v_n^{(\ell)}), 0, 0, ... \) where \( \ell = \ell(n) \to \infty \) as \( n \to \infty \). In \( \mathbb{R}^\infty \) we consider two modes of convergence--pointwise convergence and 1-convergence. With \( \ell(n) \) suitably bounded we show that the almost sure limit set in \( \mathbb{R}^\infty \) is given by

\[
A = \left\{ (x_1, x_2, ...) : 0 \leq x_{i+1} \leq x_i, i = 1, 2, ..., \text{ and } \sum_{i=1}^{\infty} x_i \leq 1 \right\}
\]

2. Almost sure limit set

We consider two modes of convergence in \( \mathbb{R}^\infty \), pointwise which is metrized by

\[
d(x, y) = \sum_{n=1}^{\infty} \frac{\left| x_n - y_n \right|}{1 + \left| x_n - y_n \right|} 2^{-n} \text{ and } \ell_1.
\]

Let us observe that a point \( x \) is a limit point of a sequence \( x_n \) with respect to pointwise convergence if and only if for each fixed \( \ell \), \( (x_1, ..., x_\ell) \) is a limit point of \( (x_n^{(1)}, ..., x_n^{(\ell)}) \). Therefore with regard to pointwise convergence our extension of Hebbar's result is precisely to
weaken the mixing condition on \( r_n \) since finite dimensional results suffice to prove this case. Furthermore in this case we consider the almost sure limit points of the sequence \( \{(v_n^{(1)}, v_n^{(2)}, \ldots, v_n^{(n)}, 0, 0, \ldots), n \geq 1\} \) that is we take \( \ell(n) = n \).

However when we consider the random element \( (v_n^{(1)}, \ldots, v_n^{(\ell)}, 0, 0, \ldots) \) as a point in \( \ell_1 \) then we must take into account the rate at which \( \ell(n) \) grows with \( n \). In this case we prove an iterated logarithm law result with \( \ell(n) = [\ln n] \). In the following we consider the \( \ell_1 \) case only since the pointwise convergence case immediately follows.

The proof closely follows the method in [1] although additional detail is required to accommodate the infinite dimensional setting. However Lemma 6 in [1] receives an entirely different proof here that depends on an extension of a result of Mittal [2].

**Remark:** Let \( x = (x_1, x_2, \ldots) \in A \) and assume \( x_1 > 0 \). Define the following sequences

\[
\lambda_k = [\ln(1 - \frac{1}{x_k})], \quad s_k = \sum_{i=1}^{\infty} x_i, \quad s = \sum_{i=1}^{\infty} x_i \quad (\text{assume } s < 1) \quad \text{and} \quad \alpha_k = \lceil \exp(k s_k) \rceil.
\]

Our program will be to show that the sequence \( \left( v_n^{(1)}, v_n^{(2)}, \ldots, v_n^{(\lambda_k)}, 0, 0, \ldots \right) \), \( k \geq 1 \) has \( x \) as a limit point almost surely. Then since \( \ell_{\alpha_k} \leq \lambda_k \) and \( \ell_{\alpha_k} \to \infty \text{ as } k \to \infty \) it follows easily that \( x \) is a limit point of \( \left( v_n^{(1)}, v_n^{(2)}, \ldots, v_n^{(\alpha_k)}, 0, 0, \ldots \right) \). In the lemmas which follow it will be assumed that \( r_n (\ell nn)^2 = o(1) \) and that \( s = \sum_{i=1}^{\infty} x_i < 1 \).

**Lemma 1.** For any \( \varepsilon > 0 \) we have

\[
(2.1) \quad P\{ \sum_{i=1}^{\lambda_k} (v_n^{(1)}) x_i > \varepsilon \} \quad \text{and} \quad v_n^{(1)} x_i, \quad i = 1, \ldots, \lambda_k, \quad \text{i.o.} = 0.
\]

**Proof:** To establish (2.1) it suffices to prove

\[
(2.2) \quad P\{ \max_{1 \leq i \leq \lambda_k} (v_n^{(1)}) x_i > \varepsilon / \lambda_k, \quad \min_{1 \leq i \leq \lambda_k} (v_n^{(1)}) x_i > 0, \quad \text{i.o.} = 0.
\]
Further by Borel Cantelli to establish (2.2) it suffices to show

\[(2.3) \sum_k [\lambda_k \max_{1 \leq j \leq \lambda_k} \mathbb{P}\{v^{(j)}_{\alpha_k} > x_j + \varepsilon/\lambda_k, v^{(1)}_{\alpha_k} > x_1, 1 \leq i \leq \lambda_k\}] < \infty.\]

Let \{v^{(1)}_{\alpha_k}, v^{(2)}_{\alpha_k}, \ldots, v^{(\lambda_k)}_{\alpha_k}\}, k \geq 1 be any triangular array with \(v^{(i)}_{\alpha_k} \geq x_i, 1 \leq i \leq \lambda_k\)
and \(\max_{1 \leq i \leq \lambda_k} (v^{(i)}_{\alpha_k} - x_i) > \varepsilon/\lambda_k\). Let \(\eta^{(1)}_{\alpha_k} = b_{\alpha_k} + y^{(1)}_{\alpha_k} / \alpha_k\). Then we establish (2.3) by showing that

\[(2.4) \sum_{k=1}^{\infty} \lambda_k \mathbb{P}\{Z^{(i)}_{\alpha_k} \geq \eta^{(i)}_{\alpha_k}, i=1, \ldots, \lambda_k\} < \infty.\]

Let \(Z_{n}^{(i)}\) be the ith maximum of a sample of size \(n\) of i.i.d. standard normal random variables. Then in order to show (2.4) it suffices to show

\[(2.5) \sum_{1}^{\infty} \lambda_k \mathbb{P}\{Z_{n}^{(i)} \geq \eta^{(i)}_{\alpha_k}, i=1, \ldots, \lambda_k\} < \infty \quad \text{and} \quad \sum_{1}^{\infty} \lambda_k \mathbb{P}\{Z_{n}^{(i)} \geq \eta^{(i)}_{\alpha_k}, i=1, \ldots, \lambda_k\} - \mathbb{P}\{Z^{(i)}_{\alpha_k} \geq \eta^{(i)}_{\alpha_k}, i=1, \ldots, \lambda_k\} < \infty.\]

In considering (2.5) observe that

\[\mathbb{P}\{Z_{n}^{(i)} \geq \eta^{(i)}_{\alpha_k}, i=1, \ldots, \lambda_k\} = \mathbb{P}\{Z_{n}^{(1)} \geq \eta^{(1)}_{\alpha_k}\}
- \sum_{i=2}^{\lambda_k} \mathbb{P}\{Z_{n}^{(1)} > \eta^{(1)}_{\alpha_k}, \ldots, Z_{n}^{(i-1)} > \eta^{(i-1)}_{\alpha_k}, Z_{n}^{(i)} \leq \eta^{(i)}_{\alpha_k}\}\]

Further it can be easily checked that

\[\mathbb{P}\{Z_{n}^{(1)} > \eta^{(1)}_{\alpha_k}\} = k \frac{1}{s_k} + o(\frac{1}{\alpha_k})\]
Thus by (2.8) and (2.9) we obtain that (2.7) equals
\[
\frac{\lambda_k}{1 - \frac{1}{s_k} \sum_{i=1}^{\infty} y_{\alpha_k}(t)} + \frac{1}{s_k} \sum_{i=1}^{\infty} y_{\alpha_k} - \frac{1}{s_k} \sum_{i=1}^{\infty} y_{\alpha_k} + O(\frac{1}{\alpha_k})
\]
Thus since \( \frac{1}{s_k} \sum_{i=1}^{\infty} y_{\alpha_k} > s_k + \epsilon \) and \( s_k \sum_{i=1}^{\infty} x_i \leq 1 \), (2.5) is established.

Next we consider (2.6). First observe that
\[
\begin{align*}
|p\{z_{\alpha_k}^*(1) > \eta_{\alpha_k}^{(1)}, i=1, \ldots, \lambda_k\} - p\{z_{\alpha_k}^{(1)} > \eta_{\alpha_k}^{(1)}, i=1, \ldots, \lambda_k\}| \\
\leq \left| p\{\eta_{\alpha_k}^{(1)} \leq z_{\alpha_k}^{*}(1) \leq z_{\alpha_k}\} - p\{\eta_{\alpha_k}^{(1)} \leq z_{\alpha_k}^{*}(1) \leq z_{\alpha_k}\} \right| \\
+ \sum_{i=2}^{\lambda_k} |p\{\eta_{\alpha_k}^{(1)} \leq z_{\alpha_k}^{*}(1) \leq z_{\alpha_k}, \ldots, \eta_{\alpha_k}^{(i-1)} \leq z_{\alpha_k}^{*}(i-1) \leq z_{\alpha_k}, z_{\alpha_k}^{*}(i) \leq \eta_{\alpha_k}\} \\
- p\{\eta_{\alpha_k}^{(1)} \leq z_{\alpha_k}, \ldots, \eta_{\alpha_k}^{(i-1)} \leq z_{\alpha_k}, z_{\alpha_k}^{(i)} \leq \eta_{\alpha_k}\}| \\
+ p\{z_{\alpha_k}^{*}(1) > z_{\alpha_k}\} + p\{z_{\alpha_k}^{*} > z_{\alpha_k}\}
\end{align*}
\]
\( (2.10) \)
where \( z_{\alpha_k} = 2\sqrt{\frac{\lambda_k}{\alpha_k}} \).

It can be checked that
\[
p\{z_{\alpha_k}^{*}(1) > z_{\alpha_k}\} = O\left(\frac{1}{\alpha_k}\right) \quad \text{and} \quad p\{z_{\alpha_k}^{*}(1) > z_{\alpha_k}\} = O\left(\frac{1}{\alpha_k}\right)
\]
\( (2.11) \)

\[
\left| p\{z_{\alpha_k}^{*}(1) > z_{\alpha_k}\} - p\{z_{\alpha_k}^{*}(1) > z_{\alpha_k}\} \right| \leq \alpha_k
\]
where \( \bar{r}_x = \sup_{i > x} |r_i| \). Thus by (2.11) \( \lambda_k (p\{z_{\alpha_k}^{*}(1) > z_{\alpha_k}\} + p\{z_{\alpha_k}^{*} > z_{\alpha_k}\}) \) is summable on \( k \).
Similarly it is easily checked that

\[
(2.12) \quad |p(\eta_{a_k}^{(1)} \leq \eta_{a_k}^{(1)}) - p(\eta_{a_k}^{(1)} \leq \eta_{a_k}^{(1)})| \leq (\text{CONST.})k \lambda_k.
\]

Since \( s_k < 1 \) and \( y_{a_k}^{(1)} > x_1 > 0 \) for all \( k \), the series in (2.12) is summable.

Now consider a term of the form

\[
|p(\eta_{a_k}^{(1)} \leq \eta_{a_k}^{(1)} \leq \eta_{a_k}^{(1)} \leq \eta_{a_k}^{(1)} \leq \eta_{a_k}^{(1)} \leq \eta_{a_k}^{(1)} \leq \eta_{a_k}^{(1)} \leq \eta_{a_k}^{(1)} \leq \eta_{a_k}^{(1)} | \leq \sum_{t_1, \ldots, t_{i-1}} |p(\eta_{a_k}^{(1)} \leq \eta_{a_k}^{(1)} \leq \eta_{a_k}^{(1)} \leq \eta_{a_k}^{(1)} \leq \eta_{a_k}^{(1)} \leq \eta_{a_k}^{(1)} \leq \eta_{a_k}^{(1)} \leq \eta_{a_k}^{(1)} | \leq S = \sum_{t_1, \ldots, t_{i-1}} |p(\eta_{a_k}^{(1)} \leq \eta_{a_k}^{(1)} \leq \eta_{a_k}^{(1)} \leq \eta_{a_k}^{(1)} \leq \eta_{a_k}^{(1)} \leq \eta_{a_k}^{(1)} \leq \eta_{a_k}^{(1)} \leq \eta_{a_k}^{(1)} | \leq S = S_0 + S_1 + \ldots + S_{i-2}
\]

where \( \{X^*_1, X^*_2, \ldots \} \) denotes an i.i.d. sequence of standard normal random variables and where the summation is over all \( 1 \leq t_1, \ldots, t_{i-1} \leq \alpha_k \) and \( t_u 
eq t_v \) of \( u \neq v \).

Let \( 0 < \theta < 1 \) be fixed and to be specified later. We write

\[
S = S_0 + S_1 + \ldots + S_{i-2}
\]

where \( S_u \) denotes the sum over all \( t_1, \ldots, t_{i-1} \) such that when the \( t \)'s are ordered \( t(1) \leq \ldots \leq t(i-1) \), there are exactly \( u \) indices \( h \) where \( t(h+1) - t(h) < \theta \alpha_k \).

Consider \( S_0 \). We have

\[
|p(\eta_{a_k}^{(1)} \leq X^*_{t_1} \leq \eta_{a_k}^{(1)} \leq X^*_{t_{i-1}} \leq \eta_{a_k}^{(1)} \leq X^*_t \leq \eta_{a_k}^{(1)} \leq \eta_{a_k}^{(1)} \leq \eta_{a_k}^{(1)} | \leq (\text{Const.})(T_0 + \sum_{0 \leq u < v \leq i-1} T_{u, v})
\]

where \( T_0 = \sum_{s, t, |r|} \phi(\eta_{a_k}^{(1)}, \eta_{a_k}^{(1)}, |r|)

\[
|p(\eta_{a_k}^{(1)} \leq X^*_{t_1} \leq \eta_{a_k}^{(1)} \leq X^*_{t_{i-1}} \leq \eta_{a_k}^{(1)} \leq X^*_t \leq \eta_{a_k}^{(1)} \leq \eta_{a_k}^{(1)} \leq \eta_{a_k}^{(1)} | \leq (\text{Const.})(T_0 + \sum_{0 \leq u < v \leq i-1} T_{u, v})
\]

and the summation is over all \( s, t, |r| \) such that \( s \neq t \) and \( t_1, \ldots, t_{i-1} \) such that when the \( t \)'s are ordered \( t(1) \leq \ldots \leq t(i-1) \), there are exactly \( u \) indices \( h \) where \( t(h+1) - t(h) < \theta \alpha_k \).
where \( r = r_{s,t} \) and \( \sum_{(0)} \) is summation over all \( s \neq t \) and \( s, t \neq t_1, \ldots, t_{i-1}, 1 \leq s, t \leq \alpha_k \) and where \( \phi(\cdot, \cdot, r) \) denotes the bivariate normal density with zero means, unit variances and correlation \( r \). Further for \( v > 0 \),

\[
T_{0,v} = \sum_{s} |r| \phi(\alpha_k, \alpha_k, |r|)
\]

\[
\cdot p(\alpha_k < x_{t_1} < z_{\alpha_k}, \ldots, \alpha_k < x_{t_{v-1}} < z_{\alpha_k}, \alpha_k < x_{t_{v+1}} < z_{\alpha_k}, \ldots, \alpha_k < x_{t_{i-1}} < z_{\alpha_k}) \]

where the sum is over \( s \neq t_1, \ldots, t_{i-1} \) and \( s \leq \alpha_k \) and \( r = r_{s,t} \).

\( T_{0,0} \) is defined in exactly the same way and finally for \( u, v > 0 \)

\[
T_{u,v} = |r| \phi(\alpha_k, \alpha_k, |r|)
\]

\[
p(\alpha_k < x_{t_1} < z_{\alpha_k}, \ldots, \alpha_k < x_{t_{u-1}} < z_{\alpha_k}, \alpha_k < x_{t_{u+1}} < z_{\alpha_k}, \ldots, \alpha_k < x_{t_{v-1}} < z_{\alpha_k}, \ldots, \alpha_k < x_{t_{i-1}} < z_{\alpha_k}) \]

We will give details only for the sum \( T_0 \) since the other sums are handled in the same way. For \( T_0 \) first consider the case when

(2.14) \( \min(|s-t_u|, |t-t_u|, u=1, \ldots, i-1) > 0 \)

In evaluating \( T_0 \) we need to evaluate

(2.15) \( p(\alpha_k < x_{t_1} < z_{\alpha_k}, \ldots, \alpha_k < x_{t_{i-1}} < z_{\alpha_k}) \)
when

\[(2.16) \quad t_{(h+1)} - t_{(h)} > \alpha_k^\theta, \, h=1, \ldots, i-2\]

and \((2.14)\) hold. Now subject to \((2.14)\) we have that

\[(2.17) \quad E(X_t | X_s = \eta_k^{(i)}, X_t = \eta_k^{(i)}) = O((\ell n \alpha_k)^{-3/2}) \quad \text{and} \]

\[\text{CORR}(X_{t_u}, X_{t_v} | X_s, X_t) = r_{t_u, t_v}^* + O((\ell n \alpha_k)^{-4})\]

Therefore by \((2.17)\) the probability in \((2.15)\) is at most

\[(2.18) \quad P\{n^{(u)}_{\alpha_k} - c(\ell n \alpha_k)^{-3/2} \leq X_{t_u} \leq z_{\alpha_k} + c(\ell n \alpha_k)^{-3/2}, \, u=1, \ldots, i-1\} \]

for some constant \(c\) not depending on \(k\). Conditioning on \(X_t\) yields that \((2.18)\) is at most

\[(1-\Phi(\sqrt{\alpha_k}))P\{b^{(l-cr)} \leq X_{t_1} \leq z_{\alpha_k} \sqrt{\alpha_k} \}, \, u=2, \ldots, i-1\}

where \(\alpha_k = r_{\alpha_k}^*\).

Iterating the procedure yields that \((2.18)\) is at most

\[(2.19) \quad \prod_{u=1}^{i-1} [1-\Phi(b_{\alpha_k} (1-c\overline{r}^*)^u)]

Finally since \(\lambda_k = [\ell n \alpha_k]^{1/2}\) and \((1-c\overline{r}^*)^u \geq 1 - 2\lambda_k c\overline{r}^*, \, (2.19)\) is at most

\[(2.20) \quad [1-\Phi(b_{\alpha_k} - c \ell n \alpha_k (\ell n \alpha_k)^{3/2})](i-1) \quad , \quad c \text{ is some constant.}\]

In the same way it can be checked that if for some \(u_0, \, |s-t_{u_0}| < \alpha_k^\theta\) but \(|t-t_u| \geq \alpha_k^\theta, \, u=1, \ldots, i-1\) or the same case with \(s\) and \(t\) interchanged then \((2.15)\) is at most

\[(2.21) \quad (1-\Phi(\gamma b_{\alpha_k}))[1-\Phi(b_{\alpha_k} - c \ell n \alpha_k (\ell n \alpha_k)^{3/2})](i-2)\]
where \( y > 0 \).

And finally if both \(|s-t_u| < \alpha^i_k\) and \(|t-t_{u_0}| < \alpha^i_k\) for some \(u_0\) and \(v_0\) then (2.15) is at most

\[
(2.22) \quad (1-\Phi(\gamma b_{\alpha_k}))^2[1-\Phi(b_{\alpha_k} - c \frac{\ell_{nk}}{(\ell_{nk})^{3/2}})]^{i-3}
\]

Thus we have that provided (2.16) holds

\[
(2.23) \quad T_0 \leq \frac{(\text{CONST.})}{(\ell_{n\alpha_k})^{2y(i)+1}} \frac{[1-\Phi(b_{\alpha_k} - c \frac{\ell_{nk}}{(\ell_{nk})^{3/2}})]^{i-1}}{(\ell_{nk})^{2y(i)+1}} = \frac{\text{CONST.}}{(\ell_{nk})^{2y(i)+1}} \frac{1}{\alpha^i_k}
\]

Similarly if (2.16) holds we find

\[
(i) \quad T_{0,v} \leq \frac{(\text{CONST.})}{(y(i)+y(v)+1)} \frac{1}{\alpha^i_k}
\]

\[
(2.24) \quad (ii) \quad T_{u,0} \leq \frac{(\text{CONST.})}{(y(i)+y(u)+1)} \frac{1}{\alpha^i_k}
\]

\[
(iii) \quad T_{u,v} \leq \frac{(\text{CONST.})}{(y(i)+y(v)+1)} \frac{1}{\alpha^i_k}
\]

Thus by (2.23) and (2.24)

\[
(2.25) \quad S_0 \leq \frac{(\text{CONST.})}{k^{(i-1)}} \frac{(\ell_{nk})^2}{(1+2x)/s_k \alpha_k^{(i-1)}} \leq \frac{1}{k^{1+e}}
\]

for some \( e > 0 \).

Next consider \( S_h \). For simplicity let us consider a summand in (2.13) when \( t_1 < t_2 < \ldots < t_{i-1} \) and
(2.26) $0 < t_2 - t_1, t_3 - t_2, \ldots, t_{n+1} - t_n \leq \alpha_k^\theta$ and $t_{u+1} - t_u > \alpha_k^\theta$ \(u=1, \ldots, i-1\).

Then

$$|p(\eta \leq x_1, \ldots, \eta \leq x_i) - p(\eta \leq x_1, \ldots, \eta \leq x_i, s = t_i) - p(\eta \leq x_1, \ldots, \eta \leq x_i, t = t_i)|$$

for all $t \neq t_1, \ldots, t_{i-1}$, \(1 \leq t \leq \alpha_k\)}

$$\leq T_0 + \sum_{u,v \leq t} T_{u,v}$$

where the $T_{u,v}$ have the same meaning as before except now condition (2.26) holds.

Consider $T_0$. We need to evaluate

$$p(\eta \leq x \leq z, u=1, \ldots, i-1)$$

subject to condition (2.26).

Let $K = K(\alpha_k) = [\exp(\sqrt{t\alpha_k})]$. Suppose that in addition to (2.26) we have that

$$t_2 - t_1, \ldots, t_m - t_{m+1} \leq K, \text{ and } K < t_{m+2} - t_{m+1} \leq \ldots, t_{i-1} - t_i.$$**

Then given (2.26) and (2.28), (2.27) equals at most

$$\prod_{u=0}^{m} (1-\Phi(b_u(1-\alpha_k^{\theta u}))) \prod_{u=m+1}^{h} (1-\Phi(b_u(1-\alpha_k^{\theta u}))) \prod_{u=h+1}^{i-2} (1-\Phi(b_u(1-\alpha_k^{\theta u}))) \leq (\text{CONST.})\alpha_k^{-Q}$$
where \( Q = 1 + \delta_1 - 6^m \delta_1 + (h-m)6^m \) and \( \delta_1 = 1 - c\bar{r}_1, \delta_k = 1 - c\bar{r}_k \)

where \( c \) is some constant and without loss of generality we can assume \( c\bar{r}_1 < 1 \) because if necessary we can work with the sequence \( \{x_{mn}, n \geq 1\} \) where \( m \) is some fixed integer.

Then as before we find that

\[
(i) \quad T_0 \leq \frac{(\text{CONST.})}{(2y\alpha_k^{(1)} + 1)} \frac{1}{Q} \frac{1}{\alpha_k}
\]

\[(2.29) \quad (ii) \quad T_{u,0} \leq \frac{(\text{CONST.})}{(y(\alpha_k) + y(1) + 1)} \frac{1}{Q} \frac{1}{\alpha_k}
\]

and

\[
(iii) \quad T_{u,v} \leq \frac{(\text{CONST.})}{(y(\alpha_k) + y(1) + 1)} \frac{1}{Q} \frac{1}{\alpha_k}
\]

Thus by the inequalities in (2.29) we have that

\[
S_h \leq \sum_{m=0}^{h} \frac{(\text{CONST.})}{Q} \frac{(\ell n_k)^2}{1 + 2x_i} \left( a_k[^{i-h-1} + \theta (h-m)] \right)^m_k
\]

from which it can be easily checked that

\[(2.30) \quad S_n \leq \alpha_k^{-f} \quad \text{for some } f > 0 \text{ not depending on } h.\]

Finally from (2.25) and (2.30) we have that

\[
S \leq \frac{(\text{CONST.})}{k^{1+e}} \quad k \geq 1 \text{ for some } e > 0 .
\]

Hence (2.6) holds completing the proof of lemma 1.
Lemma 2. Let \( c_n^{(i)} = b_n + x_1 a_n \). Then

\[
P\{Z_{\alpha_k}^{(i)} > c_n^{(i)}, i=1, \ldots, \lambda_k \text{ i.o.} \} = 1.
\]

Proof: Let \( \beta_k = [1/\alpha_k] \) and let \( Z_{\alpha_k}^{(i)} \) be the ith maximum of the random variables \( X_{\alpha_k} \cdot \beta_k + 1, \ldots, X_{\alpha_k} \). Let \( z \) be as in Lemma 1 and define \( I_k = \prod_{i=1}^{\lambda_k} I_{c_n^{(i)} < z_n^{(i)} < z_{\alpha_k}} \).

Let \( J_n = \sum_{k=1}^{\lfloor n \alpha \rfloor} I_k \) where \( 0 < a < 1 \) is a fixed real number. Then to show (2.31) it suffices to show

(i) \( EJ_n \to \infty \) as \( n \to \infty \) and

(ii) \( J_n/EJ_n \to 1 \) as \( n \to \infty \).

The proof of (2.32) follows the method of proof of Lemma 3 in [1] with changes similar to those in our Lemma 1. Therefore the details of this proof will be omitted.

Remark: The sequence \( (v_1^{(1)}, \ldots, v_{\alpha_k}^{(1)}, 0, 0, \ldots) \) has \( x \) as an almost sure limit point.

To see this let \( N_n = \{ \omega: \sum_{i=1}^{\lambda_k} (v_{\alpha_k}^{(i)} - x_i) > 1/n, v_{\alpha_k}^{(i)} > x_i, i=1, \ldots, \lambda_k \text{ i.o.} \} \) and \( N = \lim_{n \to \infty} N_n \). Let \( A = \{ \omega: v_{\alpha_k}^{(i)} > x_i, i=1, \ldots, \lambda_k \text{ i.o.} \} \) and \( \Lambda = \Lambda \cap N \). It is easy to check that if \( \omega \in \Lambda \), then \( (v_{\alpha_k}^{(1)}(\omega), \ldots, v_{\alpha_k}^{(\lambda_k)}(\omega), 0, 0, \ldots) \) has \( x \) as a limit point and by Lemmas 1 and 2 \( P(\Lambda) = 1 \). Therefore by the Remark preceding Lemma 1, we have that \( x \) is an almost sure limit point of \( (v_{\alpha_k}^{(1)}, \ldots, v_{\alpha_k}^{(\lambda_k)}, 0, 0, \ldots) \).

Lemma 3. Let \( x = (x_1, x_2, \ldots) \) be any point in \( \mathbb{R}^\infty \) with \( 0 < x_{i+1} \leq x_i, i=1, 2, \ldots \) and \( \sum_{i=1}^{\infty} x_i > 1 \). Then \( x \) cannot be an a.s. limit point of the sequence \( (v_{\alpha_k}^{(1)}, \ldots, v_{\alpha_k}^{(\ell_n)}, 0, 0, \ldots) \)
Proof: Let \( m \) be such that \( s_m = \frac{m}{m} x_i > 1, \) and \( x_i > 0. \) Let \( z_i = \frac{s_m+1}{2s_m} x_i, \) \( i=1, \ldots, m. \) Then since \( \frac{m}{m} z_i > 1, \) it follows as in \([1]\) that

\[
P\{v_n(i) > z_i, i=1, \ldots, m \ \text{i.o.}\} = 0.
\]

Let \( N = \{\omega: v_n(1) > z_1, \ldots, v_n(m) > z_m, \ \text{i.o.}\}. \) Then if \( \omega \in N^c, x \) cannot be a limit point of \((v_n(1), \ldots, v_n(m), 0,0,\ldots)\) because for all \( n \) sufficiently large

\[
\frac{1}{n} \sum_{i=1}^{n} |v_n(i) - x_i| \geq \min_{1 \leq i \leq m} (x_i - z_i) = \frac{s_m + 1}{2s_m} x_m.
\]

A useful uniform bound on the tail probabilities of the normalized maxima for a Gaussian sequence is provided by Lemma 1 in \([2]\). We state a version of this result which is suited to our problem.

Lemma 4. Let \( c_n = n \). Let \( \{X_k, n\}, k=1, \ldots, n, n=1,2,\ldots \) be a triangular array of standard normal random variables. Then setting \( r_n(i,j) = E X_i \cdot X_j, n \)

\[M_n = \max_{1 \leq k \leq n} X_k, n \text{ and } \delta_n(x) = \sup_{|i-j| \geq n} |r_n(i,j)| \text{ we have}
\]

\[e^{tA^2} P\{c_n(M_n - b_n) < -A\} = o(1) \text{ as } A \to \infty
\]

uniformly in \( n \) for all \( t \) in a neighborhood of zero provided

(i) \( \overline{\lim}_{n \to \infty} \delta_n(1) < 1 \)

(ii) \( \delta_n(0) \overline{\lim}_{n \to \infty} = 0(1) \) for some fixed \( 0 < \alpha < 1. \)

Lemma 5. For any fixed positive integer \( \ell \) and \( \epsilon > 0, P\{z_n(\ell) < b_n - \epsilon a_n, \ \text{i.o.}\} = 0. \)

It is easily checked that it is sufficient to show \( P\{z_n(\ell) < b_n - \epsilon a_n, \ \text{i.o.}\} = 0. \)

Also since for \( k \) sufficiently large \( b_{n_k} - \epsilon a_{n_k} < b_{n_k+1} - \epsilon a_{n_k+1} \)

it is enough to show

\[(2.33) \ P\{z_n(\ell) < b_{n_k} - \epsilon a_{n_k}, \ \text{i.o.}\} = 0. \]
Observe that
\[
P\{Z_n^{(\ell)} < b_n - \varepsilon a_n \leq Z_n^{(1)} > b_n + 2a_n \}
\]
\[
(2.34)
+ \sum_{i=2}^{\ell} P\{Z_n^{(i)} < b_n - \varepsilon a_n < Z_n^{(i-1)} < Z_n^{(1)} < b_n + 2a_n \}
\]

Now
\[
P\{Z_n^{(1)} > b_n + 2a_n \} \leq n(1 - \Phi(b_n + 2a_n)) = \frac{1}{(\ell n)^2}
\]

Further we have that
\[
P\{Z_n^{(i)} < b_n - \varepsilon a_n < Z_n^{(i-1)} < Z_n^{(1)} < b_n + 2a_n \}
\]
\[
(2.36)
= \sum_{t_1, \ldots, t_{i-1}} P\{X_j \leq b_n - \varepsilon a_n, j \neq t_1, \ldots, t_{i-1}, 1 \leq j \leq n \}
\]
and
\[
b_n - \varepsilon a_n < X_{t_1} < b_n + 2a_n, u = 1, \ldots, i-1
\]

For a fixed 0 < \theta < 1 and n sufficiently large
\[
P\{X_j \leq b_n - \varepsilon a_n, j \neq t_1, \ldots, t_{i-1}, 1 \leq j \leq n \}
\]
and
\[
b_n - \varepsilon a_n < X_{t_u} < b_n + 2a_n, u = 1, \ldots, i-1
\]
\[
\leq \int_{b_n - \varepsilon a_n}^{b_n + 2a_n} \cdot \int_{b_n - \varepsilon a_n}^{b_n + 2a_n} P\{X_j \leq b_n - \varepsilon a_n, |j - t_u| > n^\theta \}
\]
\[
(2.37)
= \frac{1}{(\ell n)^2}
\]
\[
dP\{X_{t_1} \leq x_{t_1}, \ldots, X_{t_{i-1}} \leq x_{t_{i-1}} \}
\]
\[
\leq P\{X_j \leq b_n - \varepsilon/2 a_n, |j - t_u| > n^\theta, u = 1, \ldots, i-1, 1 \leq j \leq n \}
\]
\[
P\{b_n - \varepsilon a_n \leq X_{t_u} \leq b_n + 2a_n, u = 1, \ldots, i-1 \}
\]
where CORR(\(\tilde{X}_j, \tilde{X}_k\)) = \(r_{jk} + o(r_{jk}^2)\).

Let \(1 \leq t_1, t_2, \ldots, t_{i-1} \leq n\) be chosen to maximize

\[
P(\tilde{X}_j < b_n - \varepsilon/2 - a_n, |j-t_u| > \theta, u=1, \ldots, i-1, 1 \leq j \leq n)
\]

Let \(Y_{1,m}, Y_{2,m}, \ldots, Y_{m,m}\) represent the \(X_j\), \(|j-t_u| > \theta, u=1, \ldots, i-1, 1 \leq j \leq n\) in their natural order and let \(M_m = \max Y_k\). Note \(n - 2(i-1)n^\theta \leq m \leq n\). Then the sum in (2.36) is at most

\[
(2.38) \quad P(M_m \leq b_m - \varepsilon/4 - a_m) \cdot \sum_{t_1, \ldots, t_{i-1}} P\{b_n - \varepsilon - a_n \leq \tilde{X}_j \leq b_n + 2a_n, u=1, \ldots, i-1\}
\]

It is easily checked that the \(Y_{k,m}\) satisfy the hypothesis of Lemma 4. Hence

\[
(2.39) \quad P\{M_m \leq b_m - \varepsilon/4 - a_m\} \leq e^{-c(\ell \ln \ln n)^2}
\]

for some constant \(c > 0\) not depending on \(n\).

Further if \(\min\{|t_u - t_v|: 1 \leq u < v \leq i-1\} \geq \theta\), then following the approach in Lemma 1, one can check that

\[
P\{b_n - \varepsilon - a_n \leq \tilde{X}_j \leq b_n + 2a_n, u=1, \ldots, i-1\} \leq (1-\phi(b_n - \varepsilon a_n))^{i-1}
\]

\[
= (1/n \cdot e^{-\ell \ln \ln n})^{i-1}
\]

While if there are exactly \(h\) indices say \(u_1, \ldots, u_h\) such that when the \(t\)'s are ordered \(t_{(u_1+1)} - t_{(u_1)} < \theta, \ldots, t_{(u_h+1)} - t_{(u_h)} < \theta\) then

\[
P\{b_n - \varepsilon - a_n \leq \tilde{X}_j \leq b_n + 2a_n, u=1, \ldots, i-1\} \leq (1-\phi(b_n) - \delta h(1-\phi(b_n - \varepsilon a_n)))^{i-h-1}
\]

where \(0 < \delta < 1\) is some constant not depending on \(n\)

\[
(2.41) \quad \leq (1/n)^{h \delta^2 + i-h-1} e^{-i^2(\ell \ln \ln n)}
\]

Therefore by choosing \(\theta < \delta^2\) we have by (2.39), (2.40) and (2.41) that (2.38) is at most \(e^{-c(\ell \ln \ln n)^2}\) for some \(c > 0\). Therefore by (2.34), (2.35) and the above
we see that
\[ p\{z_n^{(\ell)} < b_n - c a_n \} \leq \frac{1}{(\ell n)^2}. \]
Since this series evaluated at the \( n_k \) is summable on \( k \), (2.33) holds completing the proof of Lemma 5.

**Theorem 1.** Under the assumptions of Lemma 1 the almost sure limit points of the sequence \( \left( v_n^{(1)}, \ldots, v_n^{(\ell n)}, 0, 0, \ldots \right) \) in \( \ell_1 \) coincide with the set

\[ A = \{(x_1, x_2, \ldots) : 0 \leq x_{i+1} \leq x_i, \ i=1,2,\ldots, \sum_{i=1}^{\infty} x_i \leq 1\} \]

**Proof:** Lemmas 1 and 2 establish that each point of \( A \) is an almost sure limit point while Lemmas 3 and 5 establish that no point in \( A^c \) can be an almost sure limit point.

**REFERENCES**

