A note on "Geometric Transforms" of digital sets

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Abstract

A geometric transform on the digital plane is a function \( f \) that takes pairs \((P,S)\), where \( S \) is a set and \( P \) a point of \( S \), into nonnegative integers, and where \( f(S,P) \) depends only on the positions of the points of \( S \) relative to \( P \). Transforms of this type are useful for segmenting and describing \( S \). Two examples are distance transforms, for which \( f(S,P) \) is the distance from \( P \) to \( S \), and isovist transforms, where \( f(S,P) \) is (e.g.) the area of the part of \( S \) visible from \( P \). This note characterizes geometric transforms that have certain simple set-theoretic properties, e.g., such that \( f(S \cap T, P) = f(S, P) \land f(T, P) \) for all \( S, T, P \). It is shown that a geometric transform has this intersection property if and only if it is defined in a special way in terms of a "neighborhood base"; the class of such "neighborhood transforms" is a generalization of the class of distance transforms.

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1. Introduction

Given a subset $S$ of a digital picture, there are various useful ways of defining functions on $S$ that associate with each point $P$ of $S$ some geometric property of $S$ "relative to $P". An early example [1] is the distance transform, which associates with each $P \in S$ the distance (with respect to some given metric) from $P$ to $\bar{S}$ (the complement of $S$). This transform is a useful tool for describing or segmenting $S$; for example, the well-known "medial axis transformation" of $S$ is just the set of local maxima of its distance transform. A more recent example [2] is the class of "isovist transforms", which associate with each $P$ some property of the part of $S$ "visible" from $P$, e.g., its area; such transforms can be used, e.g., to find minimal sets of points from which all of $S$ can be seen. (A point $Q$ of $S$ is said to be visible from $P$ if the straight line segment $PQ$ lies entirely in $S$.)

In this note we give a general definition of such "geometric transforms" (for brevity: G-transforms). We also characterize G-transforms that have certain simple properties with respect to set-theoretic operations. In particular, we consider G-transforms having the "intersection property": for any two sets $S$ and $T$, the transform values for $S \cap T$ are (pointwise) the infs of the values for $S$ and for $T$. We show that a G-transform has this property iff it can be defined in a special way in terms of a "neighborhood basis"; the class of such transforms includes the class of distance transforms. Interestingly, the analogously defined "union property" implies that the transform must be trivial.
2. **G-transforms**

Let \( \Sigma \) be a bounded set of lattice points in the plane (e.g., a digital picture), let \( 2^\Sigma \) be the set of subsets of \( \Sigma \), and let \( f \) be a function defined on \( 2^\Sigma \times \Sigma \). For simplicity, we shall assume that \( f \) is integer-valued; that \( f(S,P) = 0 \) whenever \( P \notin S \); and that \( f(S,P) > 0 \) whenever \( P \in S \). We call \( f \) a **G-transform** if \( f(S,P) \) depends only on the positions of the (other) points of \( S \) relative to \( P \). This is a rather general definition; the following are a few examples of G-transforms:

a) The characteristic function, i.e., \( f(S,P) = 1 \) iff \( P \in S \)
b) The distance transform, i.e., \( f(S,P) = \text{the distance from } P \text{ to } S \)
c) The "area transform": \( f(S,P) = \text{the area of the connected component of } S \text{ that contains } P \)
d) The isovist transform: \( f(S,P) = \text{the area of the part of } S \text{ visible from } P \)

Since a G-transform is defined in terms of positions relative to \( P \), it is evidently shift-invariant -- in other words, shifting \( S \) cannot change the G-transform values of its points.* In particular, we have

**Proposition 1.** \( f(\{P\},P) \) has the same value for any \( P \).

For simplicity, we assume that this value is 1.

*We assume that when \( S \) shifts, it remains inside \( \Sigma \). Alternatively, we could allow cyclic shifts, and define \( f(S,P) \) in terms of the positions of the points of \( S \) relative to \( P \) "modulo \( \Sigma \)."
We say that $f$ has the **union property** if $f(S \cup T, P) = f(S, P) \lor f(T, P)$ for all $S, T, P$, and the **intersection property** if $f(S \cap T, P) = f(S, P) \land f(T, P)$ for all $S, T, P$. Evidently the characteristic function has both the union and the intersection property. In fact, it is the only G-transform that has the union property, as we see from

**Proposition 2.** A G-transform $f$ has the union property iff it is the characteristic function.

**Proof:** By Proposition 1, $f(\{P\}, P) = 1$ for all $P$. It follows from the union property that $f(\{P, Q\}, P) = f(\{P\}, P) \lor f(\{Q\}, P) = 1$ for all $\{P, Q\}$, i.e., for any two-element subset of $\Sigma$. By induction, the same is true for any finite subset of $\Sigma$.  

The G-transforms that have the intersection property are less trivial; we shall characterize them in the next section.
3. **N-transforms**

Let \( n: \{O\}=N_0 \subset N_1 \subset \cdots \) be a nested set of finite subsets of \( \Sigma \) that contain the origin \( 0 \). For any point \( P \), let \( N_{pi} \) be the result of shifting \( N_i \) to bring \( 0 \) into the position of \( P \); thus \( n_p: \{P\}=N_{p0} \subset N_{p1} \subset \cdots \) is a nested set of sets that contain \( P \). We call \( n_p \) a **neighborhood basis** for \( P \).

Let \( l=n_0 \leq n_1 \leq n_2 \leq \cdots \) be any monotonic nondecreasing sequence of positive integers. For any \( S \subset \Sigma \) and any \( P \in S \), there is a largest \( i \), call it \( i(S,P) \), such that \( N_{pi} \subseteq S \). (Note that \( N_{p0}=\{P\} \subseteq S \), and that \( S \) is finite.) Let the G-transform \( f \) be defined by \( f(S,P)=n_i(S,P) \). We call such a G-transform an **N-transform**.

It is easily verified that a distance transform is a N-transform. In fact, let \( N_i \) be the "disk" of radius \( i \) centered at \( O \), i.e., the set of points whose distances from \( 0 \) are \( \leq i \), and let \( n_i=i+1 \); then the distance transform \( f(S,P) \) is just \( n_{pi} \) (\( i \) greater than the radius of the largest disk centered at \( P \) and contained in \( S \)). Note also that the characteristic function is an N-transform, if we simply take \( n_i=1 \) for all \( i \).

**Theorem 3.** A G-transform \( f \) has the intersection property iff it is an N-transform.

**Proof:** For any \( S \) and \( T \) we have \( i(S \cap T,P)=i(S,P) \land i(T,P) \), since the \( N_p \)'s are nested. Thus if \( f \) is an N-transform we have \( f(S \cap T,P)=n_i(S,P) \land n_i(T,P)=n_i(S,P) \land n_i(T,P) \) (since the n's are monotonic) = \( f(S,P) \land f(T,P) \), so that \( f \) has the intersection property.
Conversely, let $f$ be a G-transform and have the intersection property. For any $k$, if $f(S,P)=f(T,P)=k$, we have $f(S \cap T, P)=k$; thus if there are any sets $S$ such that $f(S,P)=k$, there is a smallest such set, call it $S_{P^k}$. By shift invariance, $f(S,P)>k$ implies $f(S',P')=k$, where $S'$ is $S$ shifted to make $P$ coincide with $P'$; thus $S_{P^k}$ exists iff $S_{P^k}$ does, and they are translates of one another. Let $l=k_0<k_1<\ldots$ be those $k$'s for which $S_{P^k}$ exists; then $n_p: \{P\}=N_{P_0}^C N_{P_1}^C \ldots$, where $N_{P_i}=S_{P_{k_i}}$, is a neighborhood basis for $P$. Moreover, for any $S$, let $i(S,P)$ be the largest $i$ such that $N_{P_i} \subseteq S$, and let $f(S,P)=m$. If we had $m=k_j>k_i$, $S$ would have to contain $S_{P_{k_j}}=N_{P_j}$, contradicting the definition of $i$. On the other hand, if $m=k_h<k_i$, by the intersection property $k_j=f(N_{P_i}, P)=f(S \cap N_{P_i}, P)=f(S, P) \wedge f(N_{P_i}, P)=k_h$, contradiction. Hence $f(S, P)=k_i$, so that $f$ is an N-transform.

Thus we see that the intersection property characterizes a class of G-transforms that constitute a natural generalization of the distance transforms.
4. **Concluding remarks**

The main result of this note has been a "set-theoretic" characterization of the "distance-like" G-transforms. It would be of interest to develop characterizations of other useful classes of G-transforms.

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**References**


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have certain simple set-theoretic properties, e.g., such that $f(S \cap T, P) = f(S, P) \land f(T, P)$ for all $S, T, P$. It is shown that a geometric transform has this intersection property if and only if it is defined in a special way in terms of a "neighborhood base"; the class of such "neighborhood transforms" is a generalization of the class of distance transforms.