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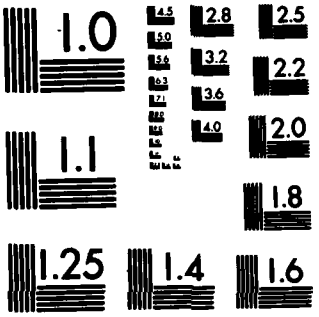

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EXISTENCE OF UNBIASED COVARIANCE COMPONENTS ESTIMATORS*

by

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SUMMARY

The condition of Pincus (1974) for the estimability of covariance components in normal models is extended to the case of singular covariance matrices.

KEY WORDS

Covariance component estimation, invariance.

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1. INTRODUCTION

In 1970 Seely derived a condition for the estimability of covariance components by a quadratic form in a general covariance component model. For normally distributed variables Pincus (1974) investigated the existence of arbitrary unbiased estimators and obtained the same characterization as Seely. In his paper Pincus assumed the parameter space has a nonempty interior consisting of regular covariance matrices. Later ~~(cf. Kleffe (1978))~~ there was a controversy whether Pincus' result remains valid for singular covariance matrices. As a matter of fact, one can dispense with the regularity but not with the nonempty interior. The latter condition, however, can be fairly weakened. As for invariant estimation one can even replace the assumption of normality by a weaker one. This is analogous to Theorem 2 of H. Bunke and O. Bunke (1974), which concerns estimability of the mean value.

The authors of this document ~~We~~ verify our result along the same lines as Pincus (1974) did in his original paper. But now a coordinate free presentation reduces the proof to its essential moments and in this way permits also singular covariance matrices. The crucial point turns out to be the completeness of the (locally) best linear unbiased estimator of the expectation.

2. RESULT

Let y be an n-dimensional random variable with $Ey=X\beta$, for some known $n\times k$ matrix X and unknown k-vector β of parameters. The variance-covariance matrix of y may have the linear structure

$$\operatorname{var} y = \sum_{i=1}^{m} \theta_i V_i = V_\theta \quad , \qquad \theta\in\Theta \subset R^m \quad ,$$

where V_i are given symmetric $n \times n$ matrices and Θ is a given subset of R^m. In this model the commonly used estimators for parametric functions $q'\theta$ are quadratic forms $y'Ay$ where A is a symmetric $n \times n$ matrix. This estimator is invariant with respect to the mean X if $AX = 0$. Denoting trace(AV) as (A,V) for symmetric matrices the expected value of a quadratic form $y'Ay$ reads

$$E(y'Ay) = \beta'X'AX\beta + (A,V_\theta).$$

Thus $y'Ay$ is unbiased for $q'\theta$ if

$$X'AX = 0 \quad \text{and} \quad (A,V_\theta) = q'\theta \quad \text{for } \theta\varepsilon\Theta.$$

An example of an estimable function that is not quadratically estimable is provided by $X=0$ and $V_\theta = I + \sigma^2 I$. Here $y'y-1$ is unbiased for σ^2, but no purely quadratic unbiased estimator exists for this parameter. To avoid such counterexamples we consider estimators of the form $c+y'Ay$. (Note that the class of estimable functions is not enlarged when linear terms $a'y$ are permitted too.) The unbiasedness condition for $c+y'Ay$ becomes

$$X'AX = 0 \quad , \quad (A,V_\eta) = q'\eta \quad \text{for } \eta \,\varepsilon\, \Theta-\Theta,$$
$$c = q'\theta - (A,V_\theta) \quad \text{for any } \theta\varepsilon\Theta.$$

Seely's (1970) condition of estimability is readily extended to the present case. We introduce some more notations to formulate his result properly. Let $P=XX^+$ be the projection onto im X, the column space of X, and let $M=I-P$. Further, let Q_0 and Q_1 be the $m \times m$ matrices with the entries

$$(Q_0)_{ij} = (V_i, MV_jM)$$

$$(Q_1)_{ij} = (V_i, V_j - PV_jP) \qquad i,j=1,\ldots,m .$$

Finally $\Theta^{\perp}$ stands for the subspace of all m-vectors orthogonal to Θ.

<u>Proposition</u> (Seely): (a) There exists an unbiased estimator $c+y'Ay$ of $q'\theta$ if an only if

$$q \in \operatorname{im} Q_1 + (\Theta-\Theta)^{\perp} .$$

(b) There exists an invariant unbiased estimator $c+y'Ay$ of $q'\theta$ if and only if

$$q \in \operatorname{im} Q_0 + (\Theta-\Theta)^{\perp} .$$

Our main result is now presented in the following proposition where span Θ denotes the subspace of R^m linearly generated by the elements of Θ.

<u>Proposition</u>: Let Θ satisfy the assumption

(1) $\qquad \operatorname{span}(\Theta-\Theta) = R \cdot (\Theta-\Theta)$.

(a) If y is normally distributed there exists an unbiased estimator of $q'\theta$ if and only if

(2) $\quad q \in \operatorname{im} Q_1 + (\Theta-\Theta)^{\perp}$.

(b) If the distribution of My depends only on $MV_{\theta}M$ $(\theta\epsilon\Theta)$ then there exists an invariant unbiased estimator of $q'\theta$ if and only if

(3) $\quad q \in \operatorname{im} Q_0 + (\Theta-\Theta)^{\perp}$.

<u>Remark</u>: Condition (1) is satisfied when Θ contains an inner point relative to the affine hull of Θ.

3. Proof: Let Q be the m×m matrix

$$(V_i, \Pi V_j) \qquad i,j = 1,\ldots,m$$

with $\Pi V = MVM$ or $\Pi V = V - PVP$ as to whether we are dealing with or without invariance. Π is an orthogonal projection in both cases. For m-vectors δ we have

$$\delta' Q \delta = \sum\sum \delta_i (V_i, \Pi V_j) \delta_j = (V_\delta, \Pi V_\delta) \geq 0.$$

Hence Q is nonnegative semidefinite and $Q\delta$ is 0 if and only if ΠV_δ vanishes.

Under the distributional assumptions of the proposition we are going to show that

$$(4) \qquad q \in [(\Theta - \Theta) \cap \ker Q]^{\perp}$$

if $q'\theta$ is (invariantly) estimable. Here ker Q stands for the subspace of all δ with $Q\delta = 0$. If in addition $\delta = \theta - \eta$ for some $\theta, \eta \in \Theta$ we have to verify that $q'\delta = 0$.

(a) In the non invariant case we consider the BLUE Gy of $X\beta$ in the normal model $y \sim N(X\beta, V_\theta)$. It is defined by

$$GX = X \qquad \text{and} \qquad GV_\theta M = 0$$

(see Drygas (1970), Theorem 3.9). Also $GV_\theta = GV_\theta G'$ holds true and therefore $GV_\theta(I-G)' = 0$. This implies the independence of $x = Gy$ and $z = y - Gy$. hence the distribution of (x,z) is

$$(x,z) \sim N(X\beta, GV_\theta) \times N(0,(I-G)V_\theta) = \mu_\beta \times \nu$$

Let $\phi(y)$ be any unbiased estimator of $q'\theta$. Defining $\psi(x,z) = \phi(x+z)$ we obtain

$$q'\theta = E\,\phi(y) = E\,\psi\,(Gy,\ y-Gy)$$
$$= \iint \psi(x,z)\ \nu(dz)\ \mu_\beta(dx)\ .$$

As recently shown by Drygas (1983) Gy is a complete statistic. Therefore the above equation leads to

$$f(x) := \int \psi(x,z)\ \nu(dz) = q'\theta \qquad \mu_\beta\text{-a.s.}$$

Let $N \subset \operatorname{im} X$ be the μ_β - null set of all x in $\operatorname{im} X$ such that $f(x) \neq q'\theta$. Since $\operatorname{im} GV_\theta$ is contained in $\operatorname{im} X$ we can write

$$\operatorname{im} X = \operatorname{im} GV_\theta \oplus \operatorname{im} X \cap (\operatorname{im} GV_\theta)^\perp$$

Let $X\beta = v_0 + u$ according to the above decomposition. Then μ_β is concentrated on $u + \operatorname{im} GV_\theta$ and is absolutely continuous with respect to the Lebesgue measure in this linear manifold. Thus

$$N_u := \{\ v \in \operatorname{im} GV_\theta\ :\ u+v \in N\ \}$$

is a null set with respect to the Lebesgue measure in $\operatorname{im} GV_\theta$. Since this holds true for arbitrary u, Fubini's Theorem yields: N is a null set with respect to the Lebesgue measure $\lambda^{\operatorname{im}X}$ in $\operatorname{im} X$. So we finally arrive at

$$f(x) = q'\theta \qquad \lambda^{\operatorname{im}X}\text{-a.s.}$$

Now we consider the model $N(X\beta, V_\eta)$. From $Q\delta = \theta$ we get $0 = \Pi V_\delta = V_\delta - PV_\delta P$. That means $V_\theta - PV_\theta P = V_\eta - PV_\eta P$ and therefore $V_\theta M = V_\eta M$. Thus the BLUE Gy of $X\beta$ in the model $y \sim N(X\beta, V_\theta)$ can as well

serve in the model $y \sim N(X\beta, V_\eta)$. Also it follows from $V_\theta - V_\eta = P(V_\theta - V_\eta)P$ that $G(V_\theta - V_\eta) = V_\theta - V_\eta$ and so $(I-G)V_\eta = (I-G)V_\theta$. Applying the same argument as above now leads to

$$f(x) = q'\eta \qquad \lambda^{\mathrm{im}X} - \text{a.s.}$$

from which $q'\delta = q'\theta - q'\eta = 0$ follows.

(b) To conclude the same for an invariant estimator $\phi(y)$ we write $\phi(y) = \psi(My)$. In the invariant case $Q\delta = 0$ means $0 = \Pi V_\delta = M V_\delta M$, i.e. $M V_\theta M = M V_\eta M$, and so

$$q'\theta = E_\theta \psi(My) = E_\eta \psi(My) = q'\eta .$$

because the distribution of My was assumed to depend on $MV_\theta M$ only.

It remains to be shown that (4) and (1) together imply $q \in \mathrm{im}\, Q + (\Theta-\Theta)^\perp$. From (1) it follows that

$$\begin{aligned} \mathrm{span}\,(\Theta-\Theta) \cap \ker Q &= R \cdot [(\Theta-\Theta) \cap \ker Q] \\ &= \mathrm{span}\,[(\Theta-\Theta) \cap \ker Q] \end{aligned}$$

Along with (4) this yields:

$$\begin{aligned} q \in [(\Theta-\Theta) \cap \ker Q]^\perp &= [\mathrm{span}(\Theta-\Theta) \cap \ker Q]^\perp \\ &= (\Theta-\Theta)^\perp + \mathrm{im}\, Q \end{aligned}$$

as was to be proved.

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