N-WIDTH AND ENTROPY OF H(P)-CLASSES IN L(Q)(-1)(U)

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H. G. Burchard and K. Höllig

Mathematics Research Center
University of Wisconsin–Madison
610 Walnut Street
Madison, Wisconsin 53706

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ABSTRACT

The n-width $d_n$, approximation numbers $\delta_n$ and entropy $\epsilon_n$ of the Hardy spaces $H_p$ in $L_q(-1,1)$ are estimated. More precisely, denote by $F^r$ the space of continuous functions which satisfy a Lipschitz condition of order $r$ at $\pm 1$. It is shown that

$$\exp(-2\gamma n^{1/2}) \ll \delta_n (H_p \cap F^r, L_q), d_n (H_p \cap F^r, L_q) \ll \exp(-\alpha n^{1/2})$$

$$\exp(-2\beta n^{1/2}) \ll \delta_n (H_p, L_q), d_n (H_p, L_q) \ll \exp(-\beta n^{1/2}), \text{ for } p > q$$

$$\exp(-2\gamma n^{1/3}) \ll \epsilon_n (H_p \cap F^r, L_q) \ll \exp(-\gamma n^{1/3})$$

where "\ll" indicates that the inequalities hold except for polynomial factors in $n$. The constants $\alpha, \beta, \gamma$ depend on $p, q$ and $r$. For $p = q$, the factor 2 in the lower bound of the first inequality can be omitted.

AMS (MOS) Subject Classifications: 41A46, 30D55

Key Words: Hardy spaces, n-width, entropy, upper and lower bounds, Wittaker series

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SIGNIFICANCE AND EXPLANATION

For analytic functions many of the standard approximation processes converge at an exponential rate. Using more sophisticated methods, it is still possible to obtain exponential convergence even in the presence of singularities at the boundary. F. Stenger and A. A. Goncar, e.g., constructed rank n approximation methods $P_n$ such that

$$|f - P_n f|_{[-1,1]} < C \exp(-rn^{1/2})$$

for $f$ an analytic function, bounded in the unit disc, which satisfies a Lipschitz condition of order $r$ at $\pm 1$.

In this report it is shown that estimates of this type are optimal in the sense of $n$-width, i.e. the above rate of convergence is best possible for approximation by rank $n$ methods.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.
N-WIDTH AND ENTROPY OF $H_p^\infty$-CLASSES IN $L_q(-1,1)$

E. G. Burchard and K. Höllig

1. INTRODUCTION.

For analytic functions many of the standard approximation processes converge at an exponential rate. Using more sophisticated methods, it is still possible to obtain exponential convergence, even in the presence of singularities at the endpoints of an interval of approximation.

In this paper we obtain precise upper and lower bounds for optimal convergence rates of approximation processes for the natural imbeddings of Hardy spaces into $L^q_{\mathbb{R}}(-1,1)$ in the sense of n-width, approximation numbers (linear n-width) and also entropy. This makes it possible to assess the optimality of bounds previously obtained for special approximation operators.

As a model example, consider the class $H_\alpha$ of analytic functions $f$ bounded in the unit disc. To obtain convergence in $L_q(-1,1)$ of approximation methods, some mild additional assumptions must be imposed about the behaviour of $f$ at $\pm 1$. For this, let $F^r$ denote the class of functions in $L_q(-1,1)$ which satisfy a Lipschitz condition of order $r > 0$ at $\pm 1$ (c.f. (2.4)).

In [6] A. Goncar has constructed piecewise polynomial approximation operators $P_n$ of rank $n$ such that

$$
\|f - P_n f\|_{L_q(-1,1)} \ll \exp(-an^{1/2}) \left(\|f\|_{H_\alpha} + \|f\|_{F^r}\right)
$$

where $a = \log(1 + \sqrt{2})r^{1/2}$. Here, "\ll" indicates that the inequality holds except for a polynomial factor in $n$. R. DeVore and K. Scherer [4] showed that $\exp(-\sqrt{2}an^{1/2})$ is a lower bound for approximation by piecewise polynomial operators. In [9, 13-16] F. Stenger developed a theory for approximating analytic functions using Whittaker's cardinal series. In particular he obtained (1.1) with an improved value of the constant,

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\[ a = \frac{1}{2} r^{1/2}. \] For the approximation of functions \( f \) in \( H_n \cap F \) we obtain the sharp lower bound

\[ (1.2) \quad \text{If } - P^n f \|_{L_\infty(-1,1)} \gg \exp(-\frac{r}{2} (\text{rn})^{1/2})(\| f \|_{H_n} + \| f \|_{F}), \]

valid for an arbitrary rank \( n \) operator \( P_n \) (c.f. Theorem 1). This establishes that approximation by Whittaker's cardinal series is optimal and \( \exp(-\frac{r}{2} (\text{rn})^{1/2}) \) is the precise asymptotic order of the \( n \)-width of \( H_n \cap F \) in \( L_\infty(-1,1) \) up to a polynomial factor in \( n \). We obtain results analogous to (1.1) and (1.2) for the \( n \)-width and approximation numbers of \( H_p \cap F \) in \( L_\infty(-1,1) \) (c.f. Theorem 2) and of \( H_p \) in \( L_q(-1,1) \), \( p > q \), (c.f. Theorem 3).

For entropy, however, the asymptotic behavior is different. For our model example we obtain (c.f. Theorem 4)

\[ \exp(-2\gamma n^{1/3}) \ll \epsilon_n(H_n \cap F, L_\infty) \ll \exp(-\gamma n^{1/3}) \]

where \( \gamma = (\frac{r}{2} \log 2 \, r)^{1/3} \). Thus, our results show that \( n \)-width tends to zero more rapidly than entropy. These estimates are in remarkable contrast to the results for Sobolev spaces where \( \epsilon_n \ll \delta_n \) [7]. The slower decay of entropy seems to be typical for classes of analytic functions. E.g., the best known example appears to be the following. Set

\[ A = \{ w \in \mathbb{C} : |w| < 1 \}. \]

Then we have for \( \rho < 1 \)

\[ \delta_n(H_n \cap L_\infty(\rho A)) \ll \exp(-|\log \rho| n) \]

\[ \epsilon_n(H_n \cap L_\infty(\rho A)) \ll \exp(-(|\log \rho| n^{1/2})). \]

After stating our main results in section 2 we prove in section 3 auxiliary results regarding \( n \)-width and entropy. In section 4 we introduce equivalent approximation problems on the real line and obtain basic approximation properties of weighted cardinal series. The proofs of Theorems 1-4 are given in section 5.
2. MAIN RESULTS

Let $T : X + Y$ be a bounded linear operator between Banach spaces $X$ and $Y$. The $n$-width $d_n$, the approximation numbers (linear $n$-width) $\delta_n$, and the entropy $\varepsilon_n$ of $T$ are defined by

$$d_n(T) = \inf \sup \text{dist}_Y(Tx, Y)$$

$$\delta_n(T) = \inf \text{dim } \mathbb{P}(X,Y)$$

$$\varepsilon_n(T) = \inf \text{rank } \mathbb{P}(X,Y)$$

where $B(X)$ denotes the closed unit ball of the $B$-space $X$. If $T : X + Y$ is a continuous embedding we write $a_n(X,Y)$ in place of $\varepsilon_n(T)$. Here and in the sequel $\varepsilon_n$ stands for either one of the numbers $a_n, \delta_n$ or $\varepsilon_n$.

Let $P_p$, $1 < p < \infty$, denote the Hardy space [5], i.e., $P_p$ is the class of analytic functions in the unit disc $\Delta$ for which

$$k_{P_p} = \sup \left( \frac{1}{2\pi} \int_0^{2\pi} |f(se^{i\theta})|^p d\theta \right)^{1/p}, \quad 1 < p < \infty,$$

$$k_{P_p} = \sup \left( \frac{1}{2\pi} \int_0^{2\pi} |f(se^{i\theta})|^p d\theta \right)^{1/p}, \quad 1 < p < \infty,$$

$$k_{P_p} = \sup |f(0)|.$$

is finite. Given a conformal homeomorphism $h$ of $\Delta$ onto a simply connected region $\Omega \subset \mathbb{C}$ one can define [6]

$$P_p(\Omega) = \{ f : \Omega \to \mathbb{C} : f \circ h \in P_p \}.$$

Different conformal maps result in equivalent norms.

We denote by $P$ the class of functions in $P_p(-1,1)$ which satisfy a Lipschitz condition of order $r > 0$ at $\pm 1$. Let $[x]$ $(\lceil x \rceil)$ denote the least (largest) integer not less (greater) than $x$. $P$ is the direct sum of $P_{[x]-1}$, the space of polynomials of degree at most $2[x] - 1$, and the space $P_{[x]-1}$ of functions with zeros of order $x$. 

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at \( \pm 1 \), \( \Psi(w) = 1 - w^2 \). Let \( p_r \) be the projection of \( F^* \) onto \( P_{2|2|} \) defined by the conditions

\[
(f - p_r f) \Psi^* \in L_\infty(-1,1).
\]

Then the norm on \( F^* \) can be defined by

\[
\| f \|_{F^*} = \| p_r f \|_{L_\infty(-1,1)} + \| (f - p_r f) \Psi \|_{L_\infty(-1,1)}.
\]

We study approximation of functions in \( H_p \cap F^* \). To state our results we use the following notions of asymptotic equivalence. Let \( a_n, b_n, n \in \mathbb{N} \), be two sequences of positive numbers. We write \( a_n < b_n \) if there exists a positive constant \( C \) such that

\[ a_n < C b_n \]

and \( a_n \ll b_n \) if there exists a positive constant \( j \) such that

\[ a_n \ll n^j b_n. \]

The symbols \( \gg \) and \( \ll \) are defined similarly.

**Theorem 1.** For \( r > 0 \) we have

\[
\delta_n(H_p \cap F^* L_\infty(-1,1)), \ d_n(H_p \cap F^* L_\infty(-1,1)) \ll \exp(-\frac{r}{2}(rn)^{1/2}).
\]

This result has already been mentioned in the introduction (c.f. (1.1), (1.2)). The upper estimate is due to F. Stenger [13] and our lower bound shows the optimality of the order \( \exp(-\frac{r}{2}(rn)^{1/2}) \).

**Theorem 2.** For \( r > 0 \) and \( 1 < p < \infty \) we have

\[
\exp(-2\alpha^{1/2}) \ll \delta_n(H_p \cap F^* L_\infty(-1,1)), \ d_n(H_p \cap F^* L_\infty(-1,1)) \ll \exp(-\alpha^{1/2})
\]

where \( \alpha = \frac{\sqrt{r}}{2(r + 1/p)^{1/2}} \).

It is interesting to compare these rates with the estimates of F. Stenger [15], who considered the classes \( \mathcal{K}^p = \mathcal{K}^p_{\infty} \), with \( \Psi(w) = 1 - w^2 \), and obtained

\[
\exp(-\sqrt{\alpha} + \varepsilon)n^{1/2}) \ll \delta_n(H^p \cap F^* L_\infty(-1,1)) \ll \exp(-\frac{\sqrt{r}}{2(p')^{1/2}} + \varepsilon)n^{1/2}), \ n > N(\varepsilon).
\]

---
Here $\delta_n$ is defined analogous to $\delta_n$ but restricting the class of rank $n$ approximations $P$ to methods based on point evaluation, i.e.

$$Pf = \sum_{j=1}^{n} f(x_j) a_j.$$ 

As we shall see in section 5, $H^s_p$ is similar to the (smaller) class $H^s_n \cap P^r$, $r = 1 - 1/p = 1/p'$. We obtain the bounds (valid also for $\delta_n$, $\Delta_n$)

$$(2.5) \quad \exp(-\pi \frac{1}{f_p^*} n^{1/2}) \ll \delta_n (H^s_p, L^q (-1, 1)) \ll \exp(-\frac{\pi}{2} \frac{1}{f_p^*} n^{1/2}).$$

In view of Theorem 1 we conjecture that the factor 2 in the lower bound of Theorem 2 can be omitted and $\exp(-cn^{1/2})$ is the precise asymptotic rate of the $n$-width.

**THEOREM 3.** For $1 < q < p < \infty$ we have

$$\exp(-2/\beta n^{1/2}) \ll \delta_n (H^s_p, L^q (-1, 1)), \quad \delta_n (H^s_p, L^q (-1, 1)) \ll \exp(-\beta n^{1/2})$$

where $\beta = \frac{\pi}{2} \left( \frac{1}{q} - \frac{1}{p} \right)^{1/2}$.

**REMARK.** The proofs of Theorems 1-3 will show that the results are true for any $s$-number in the sense of Pietsch [11].

As mentioned in the introduction, Vitushkin's [19] estimates for entropy of classes of analytic functions show exponential decay of $\epsilon_n (H^s_p, L^q (\rho A))$ as $\exp(-cm^{1/2})$ for $0 < s < 1$. Notice that in this case the functions approximated are analytic in a neighborhood of the domain $\rho A$ of approximation. In this paper we obtain estimates for entropy of imbeddings of analytic functions with singularities on the boundary $[-1, 1]$ of the interval of approximation, the rate being $\exp(-cm^{1/3})$. We attribute the curious exponent 1/3 to the fact that singularities are allowed here. A typical result is as follows.
THEOREM 4. For $r > 0$ and $1 < p < \infty$ we have

$$\exp(-2\gamma^n^{1/3}) \ll \epsilon_n(B_p \cap Y_{a_1,(-1,1)}) \ll \exp(-\gamma^{1/3})$$

where $\gamma = (\frac{x^2 \log x}{2(x + 1/p)})^{1/3}$. 

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3. GENERAL PROPERTIES OF $\delta_n$, $\alpha_n$ AND $\epsilon_n$

A. Pietsch has developed in [11] a general theory of "s-numbers" which includes $n$-width and approximation numbers as special cases. We list below some basic properties of $\delta_n$, $\alpha_n$ and $\epsilon_n$ which hold for entropy as well. Let $a_n$ denote either one of the numbers $\delta_n$, $\alpha_n$ or $\epsilon_n$ and let $T : X \rightarrow Y$ be a bounded linear operator, then we have

The numbers $a_n$ form a monotone decreasing sequence, i.e.

$$ (3.1) \quad a_0(T) > a_1(T) > \ldots $$

If $T$ admits the factorisation $T : Y \rightarrow X \rightarrow Z \rightarrow X$ we have

$$ (3.2) \quad a_n(T) \leq \max(a_n(T'), n+1) $$

The numbers $a_n$ are additive, i.e.

$$ (3.3) \quad a_n(T_0 + T_1) \leq a_n(T_0) + a_n(T_1) $$

Properties (3.1)-(3.3) are direct consequences of the definitions (c.f. [11]).

The following result is useful for obtaining lower bounds for $n$-width and approximation numbers.

**LEMMA 1** [8]. Let $V$ be an $n+1$ dimensional subspace of $X$ and let $i : V \rightarrow X$ be the canonical injection. Then we have

$$ (3.4) \quad \delta_n(i) = \delta_n(i) = 1 $$

We shall need some estimates for $a_n$ in sequence spaces. By $c_0$, $c$ we denote $\ell^p$, $\ell^p$ with supremum norm. In addition, we define the weighted spaces $\ell_{\omega, p}$ by

$$ (3.5) \quad \ell_{\omega, p} = \{ f \in \ell_p : \| f \|_{\omega, p} = \sup \exp(p|w|) |f_{\omega}| < \infty \} $$

**LEMMA 2.** For $n > n$ we have

$$ (3.6) \quad \delta_n(\ell_{\omega, 0}) = \delta_n(\ell_{\omega, 0}) = 1 $$

and for $n = 2n - 1, 2n$

$$ (3.7) \quad \delta_n(\ell_{\omega, 0}, \ell_{\omega, 0}) = \delta_n(\ell_{\omega, 0}, \ell_{\omega, 0}) = \exp(-\pi n) $$

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Proof. The first part of the Lemma is a consequence of Lemma 1.

Let \( P_N \) be the projection of \( \mathbb{F}_\omega \) onto the span of the first \( N \) basis vectors \( (\delta_{\nu,N})_{|\nu| \leq N/2} \). Then

\[
1 - P_N : \mathbb{F}_\omega \to \mathbb{F}_\omega \text{ such that exp}(\rho |N/2|)
\]

is an upper bound for \( \delta_N (i), \ i : \mathbb{F}_\omega \to \mathbb{F}_\omega \) being the natural injection.

For the lower estimate consider the factorization of the identity

\[
\mathbb{F}_\omega^{N+1} \xrightarrow{\pi} \mathbb{F}_\omega \xrightarrow{\mathbb{I}} \mathbb{F}_\omega^{N+1}
\]

where \( \mathbb{I} \) is the canonical injection. Using (3.2) and Lemma 1 this yields

\[
1 = d_N (\mathbb{F}_\omega^{N+1}, \mathbb{F}_\omega^{N+1}) \leq \prod_{i} d_N (\delta_{\nu,i}, \delta_{\nu,i}) \leq \exp(\rho (N + 1)/2) d_N (\delta_{\nu,i}, \delta_{\nu,i})
\]

Lemma 3. For \( \rho > 0 \) we have

\[
e^{-2\rho \exp\left(-\left(\log 2 \rho\right)^{1/2}\right)} \leq \mathcal{A}(\mathbb{F}_\omega, \rho) \leq e^{\rho \exp\left(-\left(\log 2 \rho\right)^{1/2}\right)}.
\]

Proof. For \( \varepsilon > 0 \) the unit ball \( B \) of \( \mathbb{F}_\omega \) contains the finite subset

\[
\mathcal{A}(\varepsilon) = \{ a \in \mathbb{F}_\omega : \text{dist} (a, B) < \exp(-|\nu| \rho) \}.
\]

For the upper bound note that \( \mathcal{A}(2\varepsilon) \) is an \( \mathbb{F}_\omega \)-net for \( B \). It suffices therefore to show \( \text{card} \ \mathcal{A}(2\varepsilon) \leq 2^n \) when \( \varepsilon = 2e^{-\delta}, \ \delta = (\log 2 \rho)^{1/2} - \rho \). We estimate

\[
\text{card} \ \mathcal{A}(2\varepsilon) = \Pi_{\mathbb{F}_\omega} (2\exp(-|\nu| \rho + \delta)/4 + 1 < \Pi_{|\nu| < \delta/\rho} \exp(-\rho (\delta^2 \rho - 1)/\rho + \delta (2 \delta^2 \rho + 1)) = \exp(\delta^2 \rho + 2\delta) < 2^n
\]

as claimed.

For the lower bound fix \( \tilde{\varepsilon} = \frac{1}{2} e^{-\delta}, \ \delta = (\log 2 \rho)^{1/2} + 2\rho \). Then

\[
\text{card} \ \mathcal{A}(2\tilde{\varepsilon}) = \Pi_{|\nu| < \delta/\rho} \exp(-\rho (\delta^2 \rho + 1)/\rho + \delta (2 \delta^2 \rho + 1)) = \exp(\delta^2 \rho + 2\delta) > 2^n.
\]

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Given any $\mathfrak{N}$-net $N$ for $B$ with cardinality $2^n$, since $\text{card } A(2\tilde{\epsilon}) > 2^n$, at least one of the $\mathfrak{N}$-balls of radius $\epsilon$ with center in $N$ must contain two distinct points of $A(2\tilde{\epsilon})$, and this implies $\tilde{\epsilon} < \epsilon$. This establishes the lower bound.
4. APPROXIMATION PROCESSES ON THE LINE

For the proofs of our theorems in §3 it will be convenient to consider equivalent approximation problems on the line. The conformal mapping

\[ z = \log \frac{1 + w}{1 - w} \]

transforms the unit disk \( \Delta \) one-to-one onto the parallel strip \( \Omega = \{ z \in \mathbb{C} : |\Im z| < \frac{\pi}{2} \} \) and at the same time maps the interval \((-1,1)\) onto \( \mathbb{R} \). This substitution induces isometries from \( \mathcal{H}_p \) onto \( \mathcal{H}_p(\Omega) \) and from \( L_q(-1,1) \) onto the weighted space \( \phi^{1/q}L_q(\mathbb{R}) \), where

\[ \phi(z) = \frac{dz}{dw} = 1 + \cosh z \]

The norm on \( \phi^{1/q}L_q(\mathbb{R}) \) is given by

\[ \| f \|_{\phi^{1/q}L_q(\mathbb{R})} = \| f \phi^{1/q} \|_{L_q(\mathbb{R})} \]

We also need other weighted function spaces. Let

\[ \Omega_d = \frac{2\pi}{d} \Omega, \quad 0 < d < \pi \]

i.e., \( z \in \Omega_d \) iff \( |\Im z| < d \). Then, for real \( \lambda \), \( f \in \phi^\lambda \mathcal{H}_p(\Omega_d) \) iff \( \phi^{-\lambda} f \in \mathcal{H}_p(\Omega_d) \) and

\[ \| f \|_{\phi^\lambda \mathcal{H}_p(\Omega_d)} = \| \phi^{-\lambda} f \|_{\mathcal{H}_p(\Omega_d)} \]

Notice that \( \phi \) maps \( \Omega_y \) onto \( \mathbb{E}\{ x \in \mathbb{R} : x < 0 \} \) and so \( \phi^\lambda \) is holomorphic on \( \Omega_y \).

We note some simple properties of \( \phi(z) \). If \( z = x + iy \in \Omega_y \), then

\[ \phi(z) = \phi(x + iy) = c_y e^{x|y|} < |\phi(z)| = \cosh x + \cos y < 2 e^{x|y|}, \]

where \( c_y = \frac{1}{2} \) for \( |y| < \frac{\pi}{2} \) and \( c_y = (1 + \cos y)/2 \) for \( \frac{\pi}{2} < |y| < \pi \). If \( w \in \Delta \), then

\[ z = \log \frac{1 + w}{1 - w} \in \Omega_{y/2} \]

and

\[ 1 - |w|^2 = 2 \cos y/|\phi(z)| \]

We now establish equivalent approximation problems on the line. To do this, we first replace \( \mathbb{R}^n \) by the simpler space \( \phi^1L_\infty(-1,1) \).
Lemma 4. Let \( r > 0, \; k = 2|x|, \; 1 < p, q < \infty. \) Then for \( a_n = \delta_n d_n \)

\[
a_n^{nk}(H_p \cap F_p^r, L_q(-1,1)) \leq a_n(H_p \cap \tilde{F}_p^r, L_q(-1,1))
\]

\[
< a_n(H_p \cap F_q^r, L_q(-1,1)).
\]

For \( a_n = \varepsilon_n \), the second inequality is still valid but the left-hand inequality is replaced by

\[
\varepsilon_{mn}(H_p \cap F_p^r, L_q(-1,1)) - 2^{-m/k} \leq \varepsilon_n(H_p \cap \tilde{F}_p^r, L_q(-1,1)).
\]

Proof. Write the natural injection \( i : H_p \cap F_p^r \to L_q(-1,1) \) in the form

\[
i = (i - p_r) + p_r \]

where \( p_r \) is the projection of \( F_p^r \) onto \( F_{h-1} \), cf. (2.4). When \( a_n = \delta_n d_n \) it follows from (3.3) and rank \( p_r = k \) that

\[
a_n^{nk}(i) < a_n(i - p_r).
\]

This is the only step in the proof where entropy requires a different treatment, and we have

\[
(4.4) \quad \varepsilon_{mn}(i) \leq \varepsilon_n(i - p_r) + \varepsilon(p_r) \leq \varepsilon_n(i - p_r) + 2^{-m/k}.
\]

The second inequality comes from the factorization

\[
p_r : H_p \cap F_p^r \to F_{h-1} \hookrightarrow L_q(-1,1),
\]

where \( j \) is the identity on \( F_{h-1} \) and \( \varepsilon_n(j) = 2^{-m/k} \). We now proceed with \( a_n = \delta_n d_n \) or \( \varepsilon_n \). We may factor \( i - p_r \) as

\[
i - p_r : H_p \cap F_p^r \to H_p \cap \tilde{F}_p^r \to L_q(-1,1),
\]

with \( \tilde{F}_p^r = (i - p_r) \tilde{F}_p^r \), and \( \tilde{F}_p^r = 1 \) in view of (2.4) and the inequalities

\[
\|f - p_rf\|_{H_p} \leq \|f\|_{H_p} + \|p_rf\|_{H_p} \leq \|f\|_{H_p} + \|p_rf\|_{L_q(-1,1)} = \|f\|_{\tilde{F}_p^r}.
\]

The left-hand inequalities of the Lemma now follow from (4.3), (4.4) and (3.2). The right-hand inequality is obvious from (2.4).

The substitution \( s = \log \frac{1 + v}{1 - v} \) induces isometries \( H_p \to H_p(\Omega), \)

\( L_q(-1,1) + \psi_{1/q} L_q(\Omega) \) and \( \tilde{F}_p^r L_q(-1,1) + 2 \psi_{1/r} L_q(\Omega) \), noting that \( \psi(s) = \frac{2}{W(s)} \). As a consequence we obtain the following equivalence.
LEMA 5.

\[ a_n(\mathbb{H}_p \cap \mathbb{F}_{\mathbb{D}_{u}(-1,1)}, \mathbb{L}_{q}(-1,1)) = a_n(\mathbb{H}_p(\Omega) \cap \mathbb{F}_{\mathbb{D}_{u}(\mathbb{D})}, \mathbb{F}_{1/4q}(\mathbb{D})) \]

As one might expect when approximating functions on the line, the precise behavior of the functions near the boundary of \( \Omega \) is not important for the rate of \( a_n \). In fact, it turns out that what matters is merely the approximate rate of growth of \( |f(s)| \) for \( f \in \mathbb{H}_p(\Omega) \). The next lemma is what we need for the reduction from \( \mathbb{H}_p(\Omega) \) to \( \mathbb{H}_p(\mathbb{D}) \).

LEMA 6. For \( \varepsilon > 0 \) one has

\[ \varepsilon^{1/P} \| f \|_{\mathbb{H}_{1/2+\varepsilon}(\mathbb{D})} \leq \| f \|_{\mathbb{H}_p(\Omega)} \leq \varepsilon^{-1/P} \| f \|_{\mathbb{H}_{1/2-\varepsilon}(\mathbb{D})} \]

Proof. The lower bound follows from an inequality of Hardy and Littlewood [5]. For \( g \in \mathbb{H}_p = \mathbb{H}_p(\Delta) \)

\[ |g(w)| < 2^{1/P}(1 - |w|^2)^{-1/P} \| g \|_{\mathbb{H}_p} \]

For \( f \in \mathbb{H}_p(\Omega) \) let \( g(w) = f(\log \frac{1 + w}{1 - w}) \). Then \( \| g \|_{\mathbb{H}_p(\Omega)} = \| g \|_{\mathbb{H}_p} \) and hence by (4.2)

\[ |f(s)| < \cos^{-1/P} |g(s)|^{1/P} \| g \|_{\mathbb{H}_p(\Omega)} \]

For the upper bound we observe that

\[ \| g \|_{\mathbb{H}_p} \leq \| f \|_{\mathbb{H}_p} \leq \| f \|_{\mathbb{H}_p} \]

and

\[ \varepsilon^{-1/P} \| f \|_{\mathbb{H}_p} \leq \varepsilon^{-1/P} \| f \|_{\mathbb{H}_p} \]

and

\[ \| f \|_{\mathbb{H}_p} \leq \varepsilon^{-1/P} \| f \|_{\mathbb{H}_p} \]

Remark. In analogy to similar characterizations of Hardy spaces on the upper half-plane [5], one can show that \( \mathbb{H}_p(\Omega) = \mathbb{H}_p(\mathbb{D}) \), where \( f \in \mathbb{H}_p \) iff \( f \) is analytic in \( \Omega \) and

\[ \| f \|_{\mathbb{H}_p} = \sup_{|y| < \pi/2} \left\{ \int_{\mathbb{R}} |f(x + iy)|^p dx \right\}^{1/p} < \infty. \]
The proof of this non-elementary result makes use of the factorization theorem for the Nevanlinna class $N^+$ and the fact that $(1 - w^2)^{-1/p}$ is an outer function.

**Lemma 7.** Let $\varepsilon > 0$. Then

$$\|f\|_{L^q(R)} \leq \varepsilon^{-1/q} \|f\|_{L^q_n(R)}.$$ 

This inequality is useful when proving upper bounds, as it implies

$$a_n(x, \phi^{1/q}(x)) \leq \varepsilon^{-1/q} a_n(x, \phi^{1/q}(x)) L_w(R)).$$ 

The lower bounds require a different technique employing a regularization mapping $\phi^{1/q} L^q(R)$ into a weighted $L^q$-space, cf. Lemma 9 below.

Next, we describe the approximation processes on the line to be used in the proofs of our main theorems in §5. Let $\rho > 0$, $t > 0$, $\nu \in \mathbb{Z}$ and define the functions

$$s_\nu(z) = s_{\nu \phi}(z) = \frac{\sin(\nu t - \nu)}{\nu t - \nu}.$$

Notice that $s_\nu$ is holomorphic in $\Omega_\nu$ and $s_\nu(\nu t) = \delta_{\nu \nu}$. Let

$$S_{ntp} = \text{span}\{s_{\nu \phi}: |\nu| < n\}$$

and define the interpolatory projections

$$P_{ntp} : C(R) \to S_{ntp}$$

for

$$P_{ntp} f = \sum_{|\nu| < n} f(\nu t) s_{\nu \phi}.$$

In addition to these finite-rank approximations we also need the series

$$P_{t \phi} f = \sum_{\nu \in \mathbb{Z}} f(\nu t) s_{\nu \phi}.$$

For $\rho = 0$ this is the Whittaker cardinal series [18]. As mentioned in the introduction, the cardinal series was employed by Stenger [13] in obtaining his upper bounds. Lundin and Stenger [9] and Stenger [15] also used weighted cardinal series similar to ours. The next lemma implies that the series $P_{t \phi} f$ converges uniformly on $R$ if $(f(\nu t))_{\nu \in \mathbb{Z}}$ is a bounded sequence.

We now establish bounds on the condition number of the basis $(s_\nu)$. It is perhaps surprising that these crude estimates, where the coefficients grow as powers of $n$ (the
parameter \( t \) will turn out to be proportional to \( n^{1/2} \), suffice for determining the order of \( a_n \).

**Lemma 8.** For \((a_n)_{n \in \mathbb{N}} \) in \( L_1(\mathbb{R}) \), \( t > 1 \)

\[
1(a_n)_{n \in \mathbb{N}} < 1 \sum_{k \in \mathbb{N}} a_k \delta_{k/t}^* L_1(\mathbb{R}) < t \frac{t^{n/t}}{n} 1(a_n)_{n \in \mathbb{N}}
\]

**Proof.** For the upper bound we must estimate the Lebesgue function \( L(x) = \sum |s_{n \in \mathbb{N}}(x)| \).

By (4.1)

\[
\|L\|_{L_1(\mathbb{R})} < \sup \sum_{n \in \mathbb{N}} e^{-\rho|x-n/t|}
\]

The value of the last sum evidently depends only on the residue of \( xt \mod 1 \), hence let

\( 0 < xt < 1 \). Then \(|x - n/t| > (|x| - 1)/t \) and for \( t > 1 \)

\[
\|L\|_{L_1(\mathbb{R})} \leq e^{n/t} \sum_{n \in \mathbb{N}} e^{-\rho n/t} \leq t e^{n/t}/t.
\]

A similar estimate can also be proved for \( L_1(\mathbb{R}) \) replaced by \( L_q(\mathbb{R}) \).

**Lemma 9.** For a positive integer \( m, \rho > 0 \) and \( 1 < q < =

\[
(\rho + t)^{-1/q} n^{-1/q} \|a_n\|_{L_q(\mathbb{R})} \leq \|a_n\|_{L_q(\mathbb{R})} \leq \sum_{|v| < n} a_v \delta_{k/t}^* L_q(\mathbb{R})
\]

\[
< \frac{e^{-n}}{(\rho t)^{1/q}} \|a_n\|_{L_q(\mathbb{R})} \leq \|a_n\|_{L_q(\mathbb{R})} \leq \sum_{|v| < n} a_v \delta_{k/t}^* L_q(\mathbb{R})
\]

**Proof.** The right-hand inequality follows from (4.1). To show the lower bound, for each

\(|v| < n \) extend to \( L_q(\mathbb{R}) \) the linear functional on \( S_{nt\rho} \) given by \( \sum \frac{a_s \delta_{k/t}^* + a_v}{|x| < n} s \delta_{k/t}^* \). A convenient extension is \( L_q \), where
\[ L_{\nu} e = \frac{1}{2\pi} \frac{1}{|u|^\nu} \int_{u/t}^{u/t + \epsilon} s(x) dx, \quad s \in L_q(M), \]

and \( B = (b_{\nu}) \) is the inverse of \( A = (a_{\nu}) \),

\[ a_{\nu} = \frac{1}{2\pi} \frac{1}{u/t - \epsilon} \int_{u/t}^{u/t + \epsilon} s_k \rho \cdot d\theta, \quad \epsilon > 0, \quad |u|, |v| < \infty. \]

As \( s_k \rho (u/t) = \delta_{kp} \), we have the bound

\[ |a_{\nu} - \delta_{kp}| \leq \frac{1}{2\pi} \int_{u/t}^{u/t + \epsilon} |s_k \rho (u) - s_k \rho (u/t)| dx \]

\[ \leq \epsilon \sup_{x \in \mathbb{R}} \left| \frac{d}{dx} s_k \rho (u) \right| < M \epsilon (p + t) \]

where \( M \) is a constant. We choose \( \epsilon = \frac{1}{2n} (p + t)^{-1} n^{-1} \) and obtain

\[ |a_{\nu} - \delta_{kp}| < (2n + 1) M \epsilon (p + t) < \frac{1}{2}. \]

Therefore

\[ |B| < \frac{1}{1 - |A - \delta_{kp}|} < 2. \]

This gives for \( \frac{1}{q'} + \frac{1}{q} = 1 \)

\[ |L_{\nu} f|_{L_q(M)} \leq \frac{1}{\epsilon} \epsilon^{1/q'} = \epsilon^{-1/q} - (p + t)^{-1/q} \cdot n^{-1/q}. \]

Finally, in this section, we state a formula for the error of approximation

\( f - P_{nt} f \), which follows from the calculus of residues and was extensively used by P. Stenger [13].

**Lemma 10.** If \( f \in H_0(G_d), \quad 0 < d < n \) and \( x \in G_d \), then

\[ (f - P_{t, \rho} f)(x) = \frac{\sin \pi x}{2\pi} \int_{G_d} \frac{\delta^0 (x - z)f(z)}{(z - x) \sin (\pi z)} dz. \]

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Proof. Denote by \( R_n \) the rectangle

\[ \left\{ x + iy \in \Omega_d : |x| < \frac{n + 1/2}{t} \right\}. \]

If we replace \( P_{t_0, \rho} \) by \( P_{t, \rho} \) and \( \omega_d \) by \( \omega_n \), then (4.7) follows from the residue theorem when \( x \in R_n \). We now let \( \varepsilon \to 0 \) and \( n \to +\infty \) and apply the dominated convergence theorem. Here we are using the existence of nontangential limits in \( L^1(\Delta) \) for

\[ f \in H_n = H_\gamma(\Delta), \] as the lines \( |\text{Re } z| = \text{const.}, z \in \Omega_d \), transform conformally to nontangential curves in \( \Delta \) under the substitution \( z = \log \frac{1 + y}{1 - y} \). We also make use of (4.1) and of

(4.8) \[
|\sin(x + iy)| = \frac{1}{2} |2 \cosh 2y - 2 \cos 2x|^{1/2} > \frac{1}{2} (e |y| - 1).
\]

**Lemma 11.** Under the hypotheses of Lemma 10, when \( d = \frac{n}{2} \) and \( t > 1 \)

(4.9) \[
\| f - P_{t, \rho} f \|_{L^1(\Omega_d)} \leq \exp\left( - \frac{\rho^2}{4} t \right) H f_{\Omega_d}(\Omega_d).
\]

**Proof.** From (4.7) and (4.8)

\[
\| f - P_{t, \rho} f \|_{L^1(\Omega_d)} \leq (\exp\left( \frac{\rho^2}{4} t \right) - 1)^{-1} H f_{\Omega_d}(\Omega_d) \leq \exp\left( - \frac{\rho^2}{4} t \right) H f_{\Omega_d}(\Omega_d).
\]

For reference below we note

(4.10) \[
\sup_{y(tz - v) \leq \exp(\omega dt)} \left| \sin(y(tz - v)) \right| \leq \exp(\omega dt),
\]

a consequence of the maximum modulus theorem and (4.8).

We note that approximation in \( L^1(\Omega) \) can be replaced by approximation in \( C(\Omega) \), the subspace of continuous functions in \( L^1(\Omega) \). More precisely, for a compact linear operator \( T : X \to C(\Omega) \) we have

(3.5) \[
a_n(T) = a_n(jT)
\]

where \( j : C \to L^\infty \) is the injection. It is clear from (3.2) that \( a_n(jT) \leq a_n(T) \) as

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$|j| = 1$. To prove the reverse inequality note that the compactness of $T$ implies that there exists a modulus of continuity $w(\delta, t)$ such that

$$w(\delta, t) \to 0, \quad \delta \to 0,$$

$$w(\delta, t) \leq w(s, t'), \quad t < t',$$

and for all $f \in T(B(x))$

$$\sup_{|s|, |s'| \leq t \atop |s-s'| \leq \delta} |f(s') - f(s)| < w(\delta, t).$$

For $\varepsilon > 0$ we choose a function $h > 0$ such that $w(h(t), t) < \varepsilon$ and define a smoothing operator $R_\varepsilon : L_\infty \to C$ by

$$(R_\varepsilon f)(t) = \frac{1}{2h(t)} \int_{t-h(t)}^{t+h(t)} f(s) \, ds.$$ 

Then $|R_\varepsilon f| = 1$ and $s_n(T) > s_n(R_\varepsilon T) = s_n(T) - s_n(T) > s_n(T) - R_\varepsilon T$ by (3.1), (3.2) and (3.3). From the definition of $h$ we see that $s_n(T) - R_\varepsilon T < \varepsilon$ which, since $\varepsilon$ is arbitrary, finishes the proof.
Proof of Theorem 1. By Lemmas 4 and 5 it suffices to estimate \( d_n(1), \delta_n(1) \) with \( i : X \cap Y_{-1} + Y \). In view of the obvious inequality \( d_n \leq \delta_n \) we shall bound \( \delta_n \) from above and \( d_n \) from below. To prove the upper estimate we write \( i = (i - P_{tr}) + P_{tr} \) and by (3.1), (3.3) we have

\[
\delta_n(1) \leq \|i - P_{tr}\| + \delta_n(P_{tr}) .
\]

By Lemma 11 we have

\[
(i - P_{tr}) \leq \exp\left(-\frac{\pi^2}{2t}\right). \tag{5.1}
\]

To estimate the second term we factor \( P_{tr} \) as

\[
P_{tr} : Y_{-1} \overset{I}{\to} X_{-1} \overset{J}{\to} Y_{-1} \tag{5.2}
\]

where \((I_{tr}) = \hat{f}(\nu/t)\) and \(J_n = \frac{1}{\nu} \sum_{\nu \in N_{2tr}} \). Clearly \( II = 1 \) and since by Lemma 8

\[
\|I\| \leq t \text{ we obtain using (3.2) and Lemma 2}
\]

\[
\delta_n(P_{tr}) \leq t \exp\left(-\frac{\pi^2}{2t} n\right). \tag{5.3}
\]

Combining the estimate (5.1) with (5.2) and choosing \( t = \frac{1}{2} \) \((2\pi)\) gives the upper bound.

To prove the lower estimate we consider the following factorization of the identity on \( I_{2n} \),

\[
I_{2n} \overset{J_n}{\to} X \cap Y_{-1} \overset{I_n}{\to} Y, \tag{5.4}
\]

where \( I_n \) and \( J_n \) are defined analogous to \( I \) and \( J \). Using the estimates

\[
\|s_{\nu} I_n \| \leq \exp\left(\frac{\pi^2}{2t}\right) \]

\[
\|s_{\nu} J_n \| \leq \exp\left(\frac{\pi^2}{2t} n\right), \quad |\nu| \leq n
\]

for the norms of the basis functions \( s_{\nu} \) we obtain, choosing \( t = \frac{1}{2} \) \((2\pi)\) \(1/2\), \( m = \lfloor n/2 \rfloor \),
\[ |J_n| \leq n \left( \exp \left( \frac{c}{t} \right) + \exp \left( \frac{\lambda^2}{2} t \right) \right) \leq n \exp \left( \frac{c}{\sqrt{2}} (rn)^{1/2} \right). \]

The lower bound follows now from (3.2) and Lemma 1.

We now formulate a general result which allows a unified treatment of the proofs of Theorems 2-4 and is of independent interest.

**Theorem 5.** Let \( a_n \) denote either one of the numbers \( a_n \), \( b_n \) or \( c_n \). For \( \lambda > 0; \rho > 0 \) and \( t > 0 \), we have

\[ a_n (s_{\exp(t)}, a_n) \leq \exp \left( -\frac{\lambda^2}{2} \right) \left( \frac{\rho}{\lambda + \rho} n \right)^{1/2} \]

\[ \leq \exp \left( -\frac{\lambda^2}{2} \right) \left( \frac{\rho}{\lambda + \rho} n \right)^{1/2} \]

As a consequence, we obtain

\[ \exp \left( -\frac{\lambda^2}{2} \right) \left( \frac{\rho}{\lambda + \rho} n \right)^{1/2} \leq \delta_n (X \lambda \cap Y_{-\rho Y}) \]

\[ \leq \exp \left( -\frac{\lambda^2}{2} \right) \left( \frac{\rho}{\lambda + \rho} n \right)^{1/2} \]

and

\[ \exp \left( -\frac{\lambda^2}{2} \log 2 \right)^{1/2} \left( \frac{\rho}{\lambda + \rho} n \right)^{1/3} \leq \epsilon_n (X \lambda \cap Y_{-\rho Y}) \]

\[ \leq \exp \left( -\frac{\lambda^2}{2} \log 2 \right)^{1/2} \left( \frac{\rho}{\lambda + \rho} n \right)^{1/3}. \]

**Proof.** To prove the upper estimate in (5.4) we write the embedding \( i : X \lambda \cap Y_{-\rho} + Y \) in the form \( i = (i - P_{\Sigma}) + P_{\Sigma} \), where we choose \( \sigma > \lambda, \rho \). As in the proof of Theorem 1 we estimate...
and obtain for the second term (c.f. (5.2))

\[ a_n(P_{t_0}) = t \cdot a_n(\int_{-t_0}^{t_0} f(x) \, dx). \]

Therefore we have to show

\[ \|I_1 - P_{t_0}\| \ll \exp(-\frac{\beta^2}{2} \frac{1}{\lambda + \rho} t). \]

To estimate \( \sup \|f(x) - P_{t_0}f(x)\| \) we set \( \lambda = \frac{\beta^2}{2} \frac{1}{\lambda + \rho} t \) and consider two cases:

(i) \( |x| < \lambda \). Lemma 10 and the estimates (4.1), (4.8) imply that

\[ \|f(x) - P_{t_0}f(x)\| \leq \exp(-\frac{\beta^2}{2} t) \int_{-\lambda}^{\lambda} \exp(-\sigma |x - s| + \lambda |s|) \, ds \cdot \|f(x)\|_{L^1_{\lambda}}. \]

(ii) \( |x| > \lambda \). Since \( |f(x)| \leq \exp(-\rho |x|) \cdot \|f(x)\|_{L^1_{\rho}} \) it follows that

\[ \|P_{t_0}f(x)\| \leq \sum_{\nu} \exp(-\rho |\nu t| - \sigma |x - \nu t|) \cdot \|f(x)\|_{L^1_{\rho}} \leq t \exp(-\rho |x|) \cdot \|f(x)\|_{L^1_{\rho}}. \]

This implies that for \( |x| > \lambda \)

\[ \|f(x) - P_{t_0}f(x)\| \leq \exp(-\rho |x|) \cdot \|f(x)\|_{L^1_{\rho}}. \]

Combining the estimates (i) and (ii) completes the proof by our choice of \( \lambda \).

To prove the lower bound in (5.4) we consider for \( \varepsilon > 0 \) and \( \lambda = \left[ \frac{\beta^2}{2} \frac{1}{\lambda + \rho} t^2 \right] \)

the following factorization of the embedding \( j : \mathbb{R}^{\mathbb{N}} \rightarrow \mathbb{R}^{\mathbb{N}} \)

\[ \mathbb{R}^{\mathbb{N}} \xrightarrow{\mathbb{R}^{\mathbb{N}}} \xrightarrow{\mathbb{R}^{\mathbb{N}}} \mathbb{R}^{\mathbb{N}} \]

where \( (I_{\lambda} f) \nu = f((v + b(v))/t) \) and \( J_{\lambda} a = \sum_{\nu \in \mathbb{N}} a_{\nu} b_{\nu} (v) \cdot t, v, \sigma \)

Here \( b(v) \) is defined as \( b(v) = v + \lambda \cdot \text{sgn} \cdot \sigma \) and \( \sigma \) is chosen larger than \( \lambda \) and \( \rho \). Using the inequalities (4.1), (4.10) we obtain by a simple calculation, keeping in mind the choice of \( \sigma \),

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\[ \|s_{vB}(v)\|_{X^{\lambda}} \leq \exp\left(\frac{x^2}{2} t - \lambda \varphi/v\right) \]
\[ \|s_{wB}(v)\|_{X^{\rho}} \leq \exp(\rho(\Delta + |v|)/\epsilon) . \]

Therefore
\[ \|s_{vB}(v)\|_{X^{\lambda} \cap Y^{\rho}} \leq \sum |a_v| |s_{wB}(v)\|_{X^{\lambda} \cap Y^{\rho}} \]
\[ \leq \exp\left(\frac{x^2}{2} t - \lambda \varphi/v\right) \sum |a_v| + \exp(\lambda \varphi/v) \sum \exp(\rho |v|/\epsilon) |a_v| \]
\[ \leq (\exp\left(\frac{x^2}{2} t - \lambda \varphi/v\right) + \exp(\lambda \varphi/v)) \sum \exp(-\epsilon |v|) \|s_{\omega, \rho} \|_{X^{\lambda} \cap Y^{\rho}} \]

and by our choice of \( \lambda \) we get
\[ \|s_{vB}\| \leq \epsilon^{-1} \exp\left(\frac{x^2}{2} \frac{\rho}{\lambda + \rho} t\right) . \]

Since \( \|s_{vB}\| = 1 \) this implies
\[ a_n(j) \leq \epsilon^{-1} \exp\left(\frac{x^2}{2} \frac{\rho}{\lambda + \rho} t\right) a_n(1) . \]

Taking into account the asymptotic behaviour of \( a_n(j) = a_n(\delta, \rho/\epsilon, \epsilon, \delta) \) the lower estimate follows by the appropriate choice of \( \epsilon \).

The inequalities (5.5) and (5.6) follow from (5.4) by substituting the bounds for
\[ a_n(\delta, \rho/\epsilon, \epsilon, \delta) \]
\( a_n = \delta_n \) obtained in Lemmas 2 and 3 and choosing \( t \) appropriately. More precisely
for \( a_n = \delta_n \) we choose \( t = \frac{2}{\lambda + \rho} \) \((1 + \rho)n)^{1/2} \) and for \( a_n = \delta_n \) let
\[ t = \left(\frac{4 \log(1 + \rho)}{\varphi}\right)^{1/3} . \]

Using Theorem 5 we can now easily give the proofs of Theorems 2-4.

**Proof of Theorem 2.** In view of Lemmas 4 and 5 we have to estimate the \( n \)-width and approximation numbers of \( i : H_p(G) \cap Y^{\lambda} \to Y \).

For the upper estimate consider for \( \epsilon > 0 \) the following factorization of \( i \)
\[ H_p(G) \cap Y^{\lambda} \xrightarrow{l_{1/2}} X^{1/p} \cap Y^{\lambda} \xrightarrow{l_{2/3}} X^{1/2} \cap Y^{\lambda} \xrightarrow{l_2} X^{1/3} \cap Y^{\lambda} \xrightarrow{l_3} Y^{\lambda} \]
where $\lambda = \frac{\sqrt{2} - \varepsilon}{\sqrt{2}}$ and $T$ is defined by

$$(Ty)(s) = g(\lambda s).$$

Since by (4.1)

$$|g^\delta(s)| = |g^\delta(\lambda s)|, \text{ for } s, \lambda \in \mathbb{R}, \quad \lambda < 1,$$

we have that

$$PT : X_{p,d} + X_{\lambda, d, d/\lambda} \subseteq 1, \quad d/d \lambda < 1$$

$$PT : Y_{p} + Y_{-\lambda} \subseteq 1.$$

Using (3.2), Lemma 6 and (5.5) we obtain from the above factorization

$$\delta_n(i) \ll 1 + d_{n}^\delta(j_2) PT^{-1} \ll \varepsilon^{-1/p} \exp(-f(\varepsilon)n^{1/2})$$

where

$$f(\varepsilon) = \frac{\pi}{2} \left( \frac{\lambda^2}{\lambda^p + \lambda^p} \right)^{1/2}, \quad \lambda = \frac{\sqrt{2} - \varepsilon}{\sqrt{2}}.$$

Since $f$ is a smooth function of $\varepsilon$ we get by setting $\varepsilon = n^{-1/2}$

$$\delta_n(i) \ll \exp(-f(0)n^{1/2} - f'(\varepsilon)), \quad \varepsilon \in [0, n^{-1/2}].$$

This proves the upper bound in view of $f(0) = \frac{\pi}{2} \left( \frac{\varepsilon^2}{\varepsilon + \varepsilon} \right)^{1/2}.$

To prove the lower bound we consider for $\varepsilon > 0$ the embedding

$$j_1 : X_{1/p} \cap Y_{-\varepsilon} \to Y,$$

which may be factored as

$$X_{1/p} \cap Y_{-\varepsilon}_{j_2} \supseteq H_{p}(0) \cap Y \downarrow Y.$$

By Lemma 6 we have $1_{j_2} \ll \varepsilon^{-1/p}$. Applying (3.2) and substituting the estimate (5.5) for $d_n(j_1)$ we get

$$\exp(-\varepsilon \left( \frac{\varepsilon^2}{1/p - \varepsilon + \varepsilon} \right)^{1/2} n^{1/2}) \ll d_n(j_1) \ll \varepsilon^{-1/p} d_n(i).$$

As in the proof of the upper bound we write this inequality in the form

$$\varepsilon^{1/p} \exp(-g(\varepsilon)n^{1/2}) \ll d_n(i).$$

Writing $g(\varepsilon) = g(0) + \varepsilon g'(\xi)$ and setting $\varepsilon = n^{-1/2}$ finishes the proof.
The proof of Theorem 4 is completely analogous and therefore omitted. We simply use the estimate (5.6) for \( e_n \) instead of (5.5).

**Proof of Theorem 3.** Recall that by Lemmas 4 and 5 we may estimate \( d_n(i), \delta_n(i) \), where

\[ i : H_p(\Omega) \rightarrow \frac{1}{q} L_q(R) \]

To prove the upper estimate we consider for \( \varepsilon > 0 \), \( \lambda = \sqrt{\frac{2}{n}} - \varepsilon \) and \( \rho = 1/n - \frac{1}{2} (1/q + 1/p) \) the following factorization of \( i \)

\[ i : H_p(\Omega) \rightarrow \frac{1}{p} X_1/p, z/2 - \varepsilon \rightarrow X_1/p, j_2 \rightarrow Y_{1/q}, T^{-1} \rightarrow Y_{1/(q - p)/\lambda}, \frac{1}{q} L_q(R) \]

where \( T \) is defined by \( (Tg)(s) = g(\lambda s) \). As we already pointed out in the proof of Theorem 2, \( \|T^* T^{-1}\| \leq 1 \). By our choice of \( \lambda, \rho \) and since \( p > q \) we have \( \lambda/p < 1/q - \rho \) and \( (1/q - \rho)/\lambda < 1/q \). Therefore the embeddings \( j_1, j_2, j_3 \), are well defined. By Lemmas 6, 7 \( j_1, j_2, j_3 \) are well defined. By Lemmas 6, 7 we obtain

\[ \delta_n(i) \leq \varepsilon^{-1} \delta_n(j_2) \]

Since \( a_n(X, Y) = a_n(X, Y, \varepsilon, Y, \varepsilon) \), this and (5.5) imply

\[ \delta_n(i) \leq \varepsilon^{-1} \delta_n(X, Y, \varepsilon, Y, \varepsilon) \leq \varepsilon^{-1} \delta_n(X, Y, \varepsilon, Y, \varepsilon) \leq \varepsilon^{-1} \delta_n(X, Y, \varepsilon, Y, \varepsilon) \]

where \( h(\varepsilon) = \frac{\varepsilon}{2} (\lambda\rho - 1/q + \rho) \) with \( \lambda = \lambda(\varepsilon) \), \( \rho = \rho(\varepsilon) \) as defined above. Since \( \lambda(0) = 1, \rho(0) = 0 \) we may set \( \varepsilon = n^{-1/2} \) and complete the proof as in the previous cases.

The lower estimate, however, cannot be obtained by this technique since there is no embedding \( L_p(\Omega) \rightarrow Y_\lambda \). But we may work with \( L_q \) directly and proceed similarly as in the proof of Theorem 1. By Lemma 6 and isometries we obtain

\[ d_n(H_p(\Omega), \frac{1}{q} L_q(R)) \geq \varepsilon^{-1} d_n(X, Y, \varepsilon, Y, \varepsilon) \]

Let \( \rho = 1/q - 1/p + \varepsilon \). Then \( \rho > 0 \) and we can factor the identity

\[ I : L_\varepsilon \rightarrow \frac{1}{p} L_q(R) \rightarrow \frac{1}{q} L_q(R) \]

\[ X_\varepsilon \rightarrow L_\varepsilon \rightarrow L_\varepsilon \]

Here, \( I_\varepsilon((a_v)) = \sum_{\|v\| \leq n} a_v \varepsilon \in \rho \). From Lemma 9 and the Hahn-Banach Theorem there exist linear functionals \( L_v \) on \( L_q(R) \), \( |v| \leq n \), such that
(actually, an explicit construction of $L_y$ is provided in the proof of the lemma). Let $I_1$ be defined by $I_1^2 = (L_y f | |v| < n^2$. Then

$$I_1^{1/2} < (p + t)^{1/2} n^{1/2}.$$ 

To estimate $II_2^{1/2}$ we have from (4.1), (4.9)

$$I_{n+t}^{1/2} \leq \sup_{|x| < n^2} \exp(-p|x|/t - \varepsilon^{1/2}) \exp(-\varepsilon^{1/2} |x|/t - t/2).$$

Therefore

$$II_2^{1/2} \leq n \exp(\varepsilon n/2) + \varepsilon^{1/2} t).$$

From Lemma 1 and (3.2) there follows now

$$1 = d_n (I) \leq II_1 d_n (j) II_2^{1/2},$$

hence

$$d_n (j) \leq (p + t)^{-1/2} n^{-1/2} \varepsilon^{1/2} \exp(-\varepsilon^{1/2} n(t) - t/2).$$

The proof is completed by the choice of $t = \sqrt{n}/2$ and then $\varepsilon = n^{-1/2}$.

**Remark.** Combining the ideas of the preceding proofs, we obtain the estimates (2.5).

Indeed we have, cf. Lemma 5,

$$\delta_n (H_p^{1/2} L_u (-1, 1)) = 2 \delta_n (\ell_p^2 (O), L_u (M)).$$

Let $p = 1 - 1/p = 1/p'$. From the obvious injection $X_{-\rho} \to X \cap Y_{-\rho}$, by Lemma 6, (3.5), (5.5) and using $T, \lambda$ as defined above

$$\delta_n (\ell_p^2 (O), L_u (M)) \leq \varepsilon^{1/p} \delta_n (X_{-\rho}, \varepsilon^{1/2} Y)$$

$$\leq \varepsilon^{1/p} \delta_n (X_{-\rho}, \varepsilon^{1/2} Y) \leq \varepsilon^{1/p} \delta_n (X \cap Y_{-\rho}, \varepsilon^{1/2} Y) \ll \exp(-\frac{n}{2} \varepsilon^{1/2} (\varepsilon n)^{1/2}),$$

$\varepsilon = n^{-1/2}$, as in the proof of Theorem 2.
For the lower bound

\[ \delta_n \left( \phi^{-1} \mathcal{H}_p(\Omega) \cap L_{\infty}(\mathbb{H}) \right) > \varepsilon^{1/p} \delta_n \left( Y_p - \varepsilon^2 \right) \gg \exp(-\kappa n^{1/2}), \]

were we proceed as in the proof of Theorem 3, but using Lemma 8 instead of Lemma 9.

The following slight extension of our results is now easily obtained. It concerns the question about the dependence of \( a_n(\mathcal{H}(D) \cap F^p, L^q(-1,1)) \) on the domain \( D \). A domain much smaller than \( \Delta \) might be useful to consider when it is a matter of approximating functions with singularities very near the interval of approximation \((-1,1)\). Or, conversely, one might be interested in functions known to have singularities far from \((-1,1)\). Suitable domains generalizing \( \Delta \) are given by

\[ \Delta_d = \{ w \in \mathbb{C} : |\text{arg} \left( \frac{1 + w}{1 - w} \right) | < d \}, \quad 0 < d < \pi, \quad \Delta = \Delta_{\pi/2}. \]

These were considered by Stenger [13-16], who obtained upper bounds. The substitution

\[ z = \log \frac{1 + w}{1 - w} \]

maps \( \Delta_d \) conformally and 1-to-1 onto \( \mathcal{G}_d \). Using the isometry

\[ (Tg)(z) = g(\lambda z), \quad \lambda = \frac{2d}{\pi}, \]

we find we can reduce the problem of bounding

\[ a_n(\mathcal{H}_p(\Delta_d) \cap F^p, L^q(-1,1)) \]

to that of bounding

\[ a_n(\mathcal{H}_p(\Omega) \cap Y_{-\lambda^d}, L^q(-1,1)) \]

which we have already solved. We state only two of the resulting estimates for illustration:

\[ a_n(\mathcal{H}_p(\Delta_d) \cap F^p, L^q(-1,1)) \ll \exp\left( -\kappa n^{1/2} \right) \]

where \( r > 0, \lambda = \frac{2\pi}{d} \), and for \( p > q/\lambda \)

\[ \exp(-2\kappa n^{1/2}) \ll a_n(\mathcal{H}_p(\Delta_d) \cap L^q(-1,1)) \ll \exp(-\kappa n^{1/2}) \]

when \( \beta = \frac{\pi}{2} \left( \frac{\lambda}{d} - \frac{1}{p} \right)^{1/2} \) and similarly in the remaining cases.
REFERENCES


**Title:**
H-Width and Entropy of \( H_p \)-Classes in \( L_q (-1,1) \)

**Authors:**
H. G. Burchard and K. Höllig

**Performing Organization Name and Address:**
Mathematics Research Center, University of Wisconsin
610 Walnut Street
Madison, Wisconsin 53706

**Prepared for:**
University of Wisconsin

**Prepared by:**
University of Wisconsin

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November 1982

**Monitoring Agency Name and Address:**
U.S. Army Research Office
P.O. Box 12211
Research Triangle Park
North Carolina 27709

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**Abstract:**
The n-width \( d_n \), approximation numbers \( \delta_n \) and entropy \( \varepsilon_n \) of the Hardy spaces \( H_p \) in \( L_q (-1,1) \) are estimated. More precisely, denote by \( f^r \) the space of continuous functions which satisfy a Lipschitz condition of order \( r \) at \( \pm 1 \). It is shown that

\[ (\text{cont.}) \]
20. ABSTRACT (cont.)

\[ \exp(-2m^{1/2}) \ll \delta_n(H_p \cap F^r, L_o) \ll \exp(-m^{1/2}) \]
\[ \exp(-2m^{1/2}) \ll \delta_n(H_p, L_q) \ll \exp(-m^{1/2}), \text{ for } p > q \]
\[ \exp(-2m^{1/3}) \ll \varepsilon_n(H_p \cap F^r, L_o) \ll \exp(-m^{1/3}) \]

where "\ll" indicates that the inequalities hold except for polynomial factors in \( n \). The constants \( \alpha, \beta, \gamma \) depend on \( p, q \) and \( r \). For \( p = \infty \), the factor 2 in the lower bound of the first inequality can be omitted.