REGULATION OF SPLIT LINEAR SYSTEMS OVER RINGS:
COEFFICIENT-ASSIGNMENT AND OBSERVERS

E. Emre and P.P. Khargonekar

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Regulation of Split Linear Systems Over Rings:
Coefficient-Assignment and Observers

E. EMRE AND P. P. KHARGONEKAR

Abstract — A theory of regulators is developed for finite-free split linear systems over a commutative ring $K$. This is achieved by developing a theory of coefficient-assignment and observers. It is shown that the problem of coefficient-assignment can be solved for reachable systems by using dynamic state feedback. For strongly observable systems, it is shown that observers with arbitrary dynamics can be constructed. Internal stability of observer and state feedback configuration is considered explicitly. Some examples are given to illustrate these techniques.
I. INTRODUCTION

Linear systems over commutative rings have been considered by several authors (see [21], [22], [5], [14], [23], and the references therein). For applications and a survey of results on systems over rings, the reader is referred to the survey papers by Kamen [15] and Sontag [23]. Among the classes of systems that can be formulated in this framework are, for example, delay differential systems, systems whose coefficients depend on parameters, whose coefficients are integers, multidimensional systems and systems described by discretized partial differential equations.

By a finite-free linear system over a commutative ring \( \mathbb{K} \), is meant a finitely generated free \( \mathbb{K} \)-module \( \mathbb{X} \) of rank \( n \), and \( \mathbb{K} \)-linear maps

\[
\begin{align*}
F : \mathbb{X} &\to \mathbb{X}, \\
G : \mathbb{K}^n &\to \mathbb{X} , \\
H : \mathbb{X} &\to \mathbb{K}^p .
\end{align*}
\]

Without loss of generality, one can assume that \( F, G, H \) are \( n \times n, n \times m, \) and \( p \times n \) matrices over \( \mathbb{K} \). We will denote such a system as \( \Sigma = (F, G, H) \) and will associate with it the abstract discrete-time system

\[
\begin{align*}
x(t+1) &= Fx(t) + Gu(t) \\
x(1) &= Hx(0)
\end{align*}
\]

This is only a conventional way of representing such systems. However, the constructions obtained based on such representations, in each case of interest, are valid for the actual interpretations of the system \( \Sigma \) through certain transforms. The following examples illustrate how this is done.

Example 1.1 — A Delay-Differential System: Consider the system

\[
\begin{align*}
\dot{x}_1(t) &= 5x_1(t-2) + x_2(t) + u(t) \\
\dot{x}_2(t) &= x_1(t) + x_2(t-2) + u(t-2) \\
y(t) &= x_1(t) - x_2(t-\sigma) \\
p_2(t) &= x_2(t).
\end{align*}
\]

Defining the delay operators \( \sigma_1, \sigma_2 \) by

\[
\begin{align*}
\sigma_1(x)(t) &:= x(t-2) \\
\sigma_2(x)(t) &:= x(t-\sigma),
\end{align*}
\]

we can represent the above system as

\[
\begin{bmatrix}
x_1(t) \\
x_2(t) \\
y(t) \\
p_2(t)
\end{bmatrix} =
\begin{bmatrix}
5 & 1 & 0 & 0 \\
1 & 0 & \sigma_2 & 0 \\
1 & -\sigma_2 & 0 & 1 \\
0 & 1 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x_1(t) \\
x_2(t) \\
y(t) \\
p_2(t)
\end{bmatrix} +
\begin{bmatrix}
0 \\
0 \\
1 & 0 & \sigma_2 & 0 \\
0 & 1 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
u(t) \\
u(t-2)
\end{bmatrix}
\]

Hence, a representation for this system is

\[
\Sigma := \left[ \begin{array}{cc}
5\sigma_1 & 1 \\
\sigma_1 & \sigma_2 + 1 \\
0 & \sigma_1 \\
0 & 1
\end{array} \right] \begin{bmatrix}
x_1(t) \\
x_2(t) \\
y(t) \\
p_2(t)
\end{bmatrix} + \begin{bmatrix}
0 \\
1 & 0 & \sigma_2 & 0 \\
0 & 1 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
u(t) \\
u(t-2)
\end{bmatrix}
\]

which is a system over the polynomial ring \( \mathbb{K} = \mathbb{R}[\sigma_1, \sigma_2] \) (where \( \mathbb{R} \) : real numbers). It carries all the information about the original system.

Example 1.2 — A System with Parameters: Suppose that we have a system \( \Sigma = (F, G, H) \) with parameter uncertainty. One could represent this uncertainty by assuming that the entries of \( F, G, H \) are functions of a finite number of parameters, say \( \sigma_1, \ldots, \sigma_n \). These functions, for example, could be polynomial functions in which case \( \Sigma \) would be a system over the polynomial ring \( \mathbb{K}[\sigma_1, \ldots, \sigma_n] \). An example is

\[
F = \begin{bmatrix}
1 & 1 \\
\sigma_1 & \sigma_2
\end{bmatrix},
\]

\[
G = \begin{bmatrix}
0 & \sigma_1 \\
1 & \sigma_2
\end{bmatrix},
\]

\[
H = \begin{bmatrix}
1 & \sigma_1^2 + \sigma_2^2 \\
\sigma_1 + 1 & \sigma_2
\end{bmatrix}.
\]

In such cases one would want to design a regulator which would work for every known value of \( \sigma_1 \) and \( \sigma_2 \) by adjustment of its parameters. This can be done by designing a regulator over the ring \( \mathbb{K}[\sigma_1, \sigma_2] \).

Further examples can be found in [23], [15].

A finite-free system \( \Sigma = (F, G, H) \) of dimension \( n \) over a commutative ring \( \mathbb{K} \) is reachable iff

\[
\mathbb{K}^* = \text{Im}(G) + \text{Im}(FG) + \cdots + \text{Im}(F^{n-1}G),
\]

i.e., the columns of the reachability matrix

\[
\Omega_0 = \begin{bmatrix}
H \\
HF \\
\vdots \\
HF^{n-1}
\end{bmatrix}
\]

are \( \mathbb{K} \)-linearly independent.

This is equivalent to the condition that \( x \in \mathbb{K}^* \), and \( \Omega_0 x = 0 \) implies \( x = 0 \).

\( \Sigma \) is strongly observable (respectively, weakly reachable) iff \((F', H')\) (transposes of \( F \) and \( H \)) (respectively, \( G', F' \)) is a reachable (respectively, observable) pair.

We should note that in general strong observability implies observability but not vice versa. However, in the case \( \mathbb{K} = \mathbb{R} \) these two notions coincide.

In general, a strictly proper rational transfer matrix over a commutative ring \( \mathbb{K} \) is defined as a formal power series

\[
Z = A_1z^{-1} + A_2z^{-2} + \cdots
\]

where \( A_i, i = 1, 2, \ldots, \) is a \( p \times m \) matrix over \( \mathbb{K} \), and the matrices \( A_i \) satisfy a recurrence relation

\[
A_{i+1} = \sum_{j=0}^{n-1} \sigma_j A_i A_j; \quad i = 1, 2, \ldots
\]

for some \( \sigma_j \in \mathbb{K} \) and some integer \( n \).

Such a transfer matrix always has a realization \( (F, G, H) \) such that

\[
Z = H(zI - F)^{-1}G.
\]

For general results on realization theory the reader is referred to [21]–[23], [15] and the references therein. For results of major importance to our paper the reader is referred to Section II.

The problem of finding an integer \( n \), and constructing feedback compensators for a system \((F, G, H)\) so that the closed-loop transfer matrix has a finite-free realization of dimension \( n \), \((F', G', H')\), where the characteristic polynomial of \( F' \) can be arbitrarily assigned is called the problem of coefficient-assignment. If we only require that the characteristic polynomial of \( F' \) has \( n \), roots \( z_1, \ldots, z_n \), with \( z \in \mathbb{K} \) which can be chosen arbitrarily, then this problem is called the problem of pole-placement. In general coefficient-assignment implies pole-placement but not vice versa.

In the case \( \mathbb{K} = \mathbb{R} \) the two imply each other, and are well known to be equivalent to the reachability of \((F, G)\) and have well known constructive solutions (see [25], [20], [1], [13], [6], [7], and the references therein).

However, the case where \( \mathbb{K} = \mathbb{R} \) is more difficult and only partial results have been obtained previously. The main approach has been to extend the results for the field case using constant state-feedback. Morse [19] has shown that if \( \mathbb{K} \) is a principal ideal domain and \((F, G)\) is reachable, then pole assignment is possible by nondynamic state-feedback. Sontag [23] has proved that for semilocal rings coefficient-assignment by nondynamic feedback is equivalent to the reachability of \((F, G)\).

Later Ching and Wyman [4] proved that Noetherian full quotient rings are semilocal rings and thus the results of Sontag [23] are applicable to this case. In general, Sontag [23] has proved that the reachability of \((F, G)\) is necessary for both pole-placement and coefficient-assignment by nondynamic feedback for any ring \( \mathbb{K} \).
Let $P$ be a multiplicatively closed subset of monic polynomials of the polynomial ring $K[x]$, called stable polynomials. Let $(F,G)$ be a given pair. We say that $(F,G)$ is regulable (or regulation is achieved) if there exists a finite-free system $\Sigma = (F,G,H,J)$ such that with dynamic feedback using $\Sigma$, the characteristic polynomial of the augmented closed-loop system is in $P$. (This latter property is usually referred to as internal stability as well.)

For principal ideal domains and semilocal rings and Noetherian full quantum rings, the results of [10], [23], [4], respectively, can be used to solve the regulation problem. Also, Byrnes [3] has given a stabilization result that can be used for regulation of systems over certain Frechet algebras.

However, for more general rings, there do not exist any coefficient-assignment and/or pole-placement results that can be used for regulation problem for a reachable pair $(F,G)$. In Section II of this paper we present for the first time a coefficient-assignment result by dynamic feedback which can be used to solve the regulation problem for finite-free reachable pairs $(F,G)$ defined over an arbitrary commutative ring $K$. Our result is based on an approach developed in [7], valid for an arbitrary commutative ring $K$, which also connects the problem to the reachability of $(F,G)$ in a natural way.

The general regulation problem is to achieve regulation as defined above when the plant is described as $\Sigma = (F,G,H)$. If a finite-free pair $(H,F)$ is strongly observable, one can build an observer in the form of a deterministic Kalman filter (for the rings mentioned above where a result of coefficient-assignment and/or pole-placement by nodynamic feedback is available). $\Sigma = (F,G,H)$ is called split iff it is reachable and strongly observable. For the motivation and details of the notion of split systems see [23], [16]. For a finite-free split system $\Sigma$ define over $K$ where $K$ is a principal ideal domain, the semilocal ring, one can solve the regulator problem by combining the results on observers for pairs $(H,F)$ and results on regulators for pairs $(F,G)$ or for $K = C$ a certain Frechet algebra, results of [3] can be used to solve regulation problem. Recently, Hautus and Sonntag [11] have obtained more general results on observers, also generalizing detectability ideas, for finitely generated algebras over fields. They have also considered the regulation problem for finite-free detectable and reachable systems over such rings, however their results cannot be used directly for regulation. In Section III of this paper, we give for the first time, a method to obtain an observer for a finite-free strongly observable system $\Sigma = (F,G,H)$ over $K$ an arbitrary commutative ring, based on the approach developed in [7]. This also connects this problem to the observability of $\Sigma$ in a natural way. Further, our observers can be built in such a way that the coefficient of the characteristic polynomial can be chosen arbitrarily. Also, our results allow the observers to be built by solving linear equations over $K$, and are guaranteed to have a realization over $K$.

Further, we show how one can combine our results on observers and coefficient-assignment to solve the general regulation problem for a finite-free split linear system $\Sigma = (F,G,H)$ over an arbitrary commutative ring $K$.

One can also use our coefficient-assignment result and the detectability result of [11] to solve the regulator problem for the case of finite-free detectable and reachable systems $\Sigma = (F,G,H)$ over finitely generated algebras over fields.

In Section IV, we give some examples to illustrate the applications of our results. We show that even if $\Sigma = (F,G,H)$ is not reachable and/or strongly observable, one can still approach and may be able to solve the problem using our method.

As for $\Sigma = (F,G,H)$ being split, over a commutative ring, every transfer matrix admits a free reachable or a free strongly observable realization. In general, reachability is known to be necessary for coefficient-assignment by state feedback [23]. For the existence of observers with arbitrary characteristic polynomial for a weakly reachable system, strong observability (as it will be seen in Section III) is also necessary. We show in Sections II and III that for free systems these are also sufficient. Furthermore, in the case of systems over $\mathbb{R}[x_1,\ldots,x_r]$, $\Sigma = (F,G,H)$ being reachable (strongly observable) is generic if and only if the number of inputs (outputs) exceeds $r$ (see [18]). Hence, in such cases, the assumption of spliteness is not very restrictive.

The choice of stable polynomials $P$, of course, depends upon the application under consideration. For most of the applications, stability can be inferred from the characteristic polynomial. For example, for delay-differential systems, two-dimensional systems, systems with parameters, one can infer stability by examining the location of roots of the characteristic polynomial. Throughout this paper, we assume that $P$ is given and derive results for the given $P$. The only technical restriction on $P$, is that it be a multiplicative set.
then the sequence \([A_i]^T\), is the sequence of impulse response matrices of the \(i/o\) map associated with \(Z\). Let \(P, Q, K\) be \(p \times r, r \times r,\) and \(r \times m\) polynomial matrices with determinant of \(Q\) a monic polynomial such that

\[
Z = \sum_{i=1}^{\infty} A_i z^{-i}.
\]

The following lemma presents a natural realization of \(Z\) with \(K\) as the state-module. Since \([Q]\) is assumed to be monic, it is not difficult to see that \(K\) is a finitely generated \(K\)-module. The realization given in the following lemma is called the \(Q\)-realization of \(Z\). This result was proved in [10] for \(K\) a field. But since \(p\) is well defined, it is easily seen that the result holds for our case as well.

**Lemma 2.1 [10]:** Let \(\Sigma_Q = (F_Q, G_Q, H_Q)\) be defined as follows:

\[
\begin{align*}
G_Q & : K^{m} \to K^{n} : u \mapsto \sigma_Q(\tilde{R} u) \\
F_Q & : K^{n} \to K^{n} : x \mapsto \sigma_Q(x).
\end{align*}
\]

and

\[
H_Q : K^{n} \to K^{p} : x \mapsto (PQ)^{-1} x,
\]

where for \(a \in K_1(T), (a)_1\) denotes the coefficients of \(z^{-1}\). Then \(\Sigma_Q\) with \(K\) as the state-module is a realization of \(Z\).

For further details of the \(Q\)-realization, the reader is referred to [10], [8], and the references given there. Let \(\Sigma = (F, G, H)\) be a given finite-free split system over \(K\). Let \(Z\) be its transfer matrix, i.e.,

\[
Z = H(zI - F)^{-1} G.
\]

We will assume, without loss of generality, that \(Z\) can be written as

\[
Z = Q^1 R
\]

where \(Q, R\) are polynomial matrices. \(Q\) is row proper, and \(P, R\) satisfy the Bezout condition, i.e., there exist polynomial matrices \(X, Y\) such that

\[
QX + YR = I.
\]

If \(K\) a field such a representation is known to be possible for a reachable and observable system \(\Sigma\). But this is generally not possible for \(K\) a field. However, if \(\Sigma\) is split using a full state observer (as described in full detail in Section III) one can obtain a system whose transfer matrix is

\[
Z_L = (zI - F)^{-1} G
\]

which clearly is of the above form. Further, if this is done, then regulation of the composite system (original system \(\Sigma\) and the observer) is sufficient for regulation of \(\Sigma\), as it will be shown in Remark 3.9.

Now consider a dynamic feedback system \(\Sigma_L = (F_L, G_L, H_L, J_L)\) over \(K\), whose transfer matrix \(Z_L\) can be written as

\[
Z_L = P_L Q_L^{-1}
\]

where \(P, Q\) are polynomial matrices with \(Q\) column proper. (Later in this section we will show how to obtain such \(P, Q\) for regulation.) Then if we write the closed-loop state equations where the state is taken to be the external direct sum of the states of the observer and \(\Sigma\), it can be shown that [see Remark 3.9] the closed-loop system has the characteristic polynomial \(a_i\), where

\[
a_i = \text{det}(\Phi)|Q_0|
\]

Here

\[
Q_0 = [TG : R_I]
\]

will be the transfer matrix of the observer.

\[
\Phi = (zI - F)Q_i + GP_i
\]

and

\[
Q_0 = T(zI - F) + R_1 H.
\]

In this section, we will show how one can obtain polynomial matrices \(P, Q, K\) such that \(\Phi\) can be arbitrarily chosen. \(Q_0\) is column proper and \(P, Q_0^{-1}\) is a proper rational matrix. In Section III, we will show how one can obtain suitable polynomial matrices \(Q_0, T, R_1\).

We should note that dynamic state feedback corresponds to adding a number of integrators to the system \((F, G, I)\) and then applying state feedback to the resulting system. (See, for example, [11]) Hence, the results that we present in this section can also be interpreted as a method to determine a sufficient number of integrators, and the final state feedback matrix for regulation.

Our results in this section are based on the following theorem proved in [7, Theorem 3.1].

For each positive integer \(i\), define a \(K\)-submodule \(W_i\) of \(K\) as

\[
W_i = \text{Im} G_i := \text{Im} F_i G_i + \cdots + \text{Im} F_i G_i.
\]

In the following for a \(p \times p\) polynomial matrix \(\Phi\) and integers \((a, f), (b, e)\),

\[
\lim \{ \text{diag}(z^{-n}) \Phi \text{diag}(z^{-m}) \} = 0
\]

exists and is equal to \(\tilde{T}\), where \(\tilde{T}\) is a matrix over \(K\), means that

\[
Z_i = \text{diag}(z^{-n}) \Phi \text{diag}(z^{-m})
\]

is proper and its constant term is \(\tilde{T}\).

**Lemma 2.2 [7, Theorem 3.1]:** Let \(\Phi\) be a given \(p \times p\) polynomial matrix. Let \(Q\) be a \(p \times p\) nonsingular matrix with \(I_p\) as the lowest row degree coefficient matrix and with row degrees \(\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_p\). Let \(R\) be given \(p \times m\) polynomial matrix such that \(Q^{\gamma_p} R\) is strictly proper. Let \(P\) be a given \(m \times p\) polynomial matrix. Then there exists a column proper nonsingular polynomial matrix \(Q_1\) such that \(P_1 Q_1^{-1}\) is proper, and

\[
Q_1 + R P_1 = \Phi
\]

if and only if there exist integers \(\gamma_1, \gamma_2, \ldots, \gamma_p \geq 0\) such that

i) \[
\lim_{z \to \infty} \left[ \begin{array}{ccc} z^{-\gamma_1} & 0 & 0 \\
 & & \\
 & & \\
 & & \\
0 & z^{-\gamma_p} & 0 \end{array} \right] = 0
\]

exists and \(A\) is invertible over the ring \(K\), and

ii) \[
\gamma_1 > \delta_i(P_i) \geq \gamma_i - 1, \quad i = 1, 2, \ldots, p,
\]

where \(r_i\) is the least integer \(i\) for which the \(i\)th column of \(\sigma_i(\Phi)\). \(\gamma_i\), is in \(W_i\) and \(P_i\) is a polynomial matrix whose \(i\)th column is an input which drives \(S_{Q_0}\) from zero-state to \(\gamma_i\). Further if i) and ii) hold, then \(A\) is the highest degree column coefficient matrix of \(Q_1\) and \(\gamma_i\) is the \(i\)th column degree of \(Q_0\).

The following theorem is an immediate consequence of Lemma 2.2

**Theorem 2.4:** Let \(Q\) be a \(p \times n\) polynomial matrix with \(I_p\) as the highest degree row coefficient matrix. Let \(R\) be a \(p \times m\) polynomial matrix such that the \(Q\)-realization \(\Sigma_Q\) of \(Q^{\gamma_p} R\) is reachable. Let \(r\) be the smallest integer such that

\[
K_0 = \text{Im} G_0 + \text{Im} F_0 G_0 + \cdots + \text{Im} F_0 G_0
\]

Let \(\gamma_i > \delta_i(P_i) \geq \gamma_i - 1, \quad i = 1, 2, \ldots, p,

where \(r_i\) is the least integer \(i\) for which the \(i\)th column of \(\sigma_i(\Phi)\). \(\gamma_i\), is in \(W_i\) and \(P_i\) is a polynomial matrix whose \(i\)th column is an input which drives \(S_{Q_0}\) from zero-state to \(\gamma_i\). Further if i) and ii) hold, then \(A\) is the highest degree column coefficient matrix of \(Q_1\) and \(\gamma_i\) is the \(i\)th column degree of \(Q_0\).

\[
\gamma_1 > \delta_i(P_i) \geq \gamma_i - 1, \quad i = 1, 2, \ldots, p,
\]

Let \(\beta\) be a given monic polynomial of degree

\[
\alpha = \beta(m, \gamma_i, \gamma)
\]
Then there exists a \( p \times p \) polynomial matrix \( Q_1 \) with column degrees
\( \rho_1, \rho_2, \ldots, \rho_p \) and \( I_p \) as the highest degree column coefficient
matrix, and there exists an \( m \times p \) polynomial \( P_i \) such that \( P_iQ_1^t \) is
a proper transfer matrix and \( B \) is the determinant of \( QQ_1 + RP_i \).

Proof. By hypothesis, we have
\[
\rho_1 \geq \rho_2 \geq \cdots \geq \rho_p.
\]

Following the procedure given in [24, Section 7.3], [12, Section 7.2] define
the polynomials \( \beta_1, \beta_2, \ldots, \beta_p \) by
\[
\beta = z^{\rho_1} + \beta_1 z^{\rho_1 - 1} + \cdots + \beta_p z^0,
\]
where \( \beta \) is a polynomial of degree less than \( \rho_1 \). Now if the \( p \times p \) polynomial
matrix \( \Phi \) is defined as
\[
\Phi = \begin{bmatrix}
z^{\rho_1} & \beta_1 & \beta_2 & \cdots & \beta_p \\
1 & z^{\rho_2} & \cdots & \cdots & \cdots \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 1 & \cdots & \cdots & \cdots 
\end{bmatrix},
\]
then \( \Phi \) is \( B \). Let \( \pi_i, \rho_i, \rho_i-1, \ldots, \rho_p \). Then
\[
\lim_{t \to \infty} \left( \begin{array}{ccc}
z^{\rho_1} & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & z^{\rho_p} 
\end{array} \right) \Phi = \left( \begin{array}{ccc}
z^{\rho_1} & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & z^{\rho_p} 
\end{array} \right) I_p.
\]

Now the conditions of Lemma 2.2 are satisfied, hence there exist polynomial
matrices \( Q_i \) and \( P_i \) with the required properties and
\[
QQ_1 + RP_i = \Phi.
\]

This completes the proof of the theorem. \( \Box \)

We also have the following.

Theorem 2.5. Let \( F \in K^{n \times n} \) and \( G \in K^{n \times m} \) be a reachable pair. Let \( v \)
be the smallest integer such that
\[
K^v = \text{Im} G + \text{Im} FG + \cdots + \text{Im} F^v G.
\]

Let \( p_1, p_2, \ldots, p_v \) be a given set of integers such that each \( p_i \geq v \), and let \( B \)
be a given monic polynomial of degree
\[
p = p_1 + p_2 + \cdots + p_v.
\]

Then there exists a non-scalar polynomial matrix \( Q \), with highest column
degree coefficient matrix \( I_1 \) with \( p_1 = 1 \) as its column degree, and an \( m \times n \) polynomial
matrix \( P \) such that \( P_iQ_1^t \) is proper and
\[
|z^{p(1-F)}Q_1 + GP_i| = B.
\]

Proof. If we choose
\[
Q = (z I - F), \quad R = G
\]
then the \( Q \)-realization of \( Q \) is \((I, F, G) \) with \( K^v \) as the state module.

Now we have
\[
\rho_1 \geq \rho_2 \geq \cdots \geq \rho_v = 1.
\]

The result follows immediately from Theorem 2.4. \( \Box \)

Remark 2.6. Let \( Q \) and \( R \) be as in Theorem 2.4 and let \( \Sigma_1 \) be reachable. It is well known (see, for example, [7]) that if \( Z_1 = -P_iQ_1^t \) is
chosen as a feedback compensator, then the closed-loop transfer matrix
\( Z \), can be written as
\[
Z = Q_1(QQ_1 + RP_i)^{-1}R
\]
which is the same as
\[
Z = Q_1(\Phi)^{-1}R.
\]

Note that \( \Phi \) is row proper. One can see by obtaining a \( \Phi \)-realization of
\( Z \), as in Lemma 2.1, and choosing \( S \) as a basis matrix for \( K_1 \), that \( F_0 \) has
a matrix representation which is in companion form with the characteristic
\[
|\Phi| = B.
\]

As this realization of \( Z \), is free, one obtains a realization of \( Z \), by matrix
transpositions. Hence, Theorem 2.5 not only provides us with a coefficient-
assignment result, but it also gives a realization of \( Z \), with a \( F_0 \) in
companion form.

Remark 2.7. Theorem 2.5 is a coefficient-assignment result for reachable pairs \((F, G)\) by dynamic state-feedback which guarantees the internal
stability (see Remark 3.8 for details of this aspect) of the closed-loop
system, thus providing a solution to the regulator problem. All of the
previous work except for Hautus and Sonntag [11] has been concerned with
coefficient-assignment or stabilization by constant state feedback. In [11]
a special type of pole-placement result for reachable pairs \((F, G)\) by dynamic
state-feedback is given but the internal stability of the closed-loop system
is not considered.

Theorem 2.5 is the first coefficient-assignment result for reachable pairs
\((F, G)\) and further it also guarantees regulation. It will be seen in the
following section that one can build (at least) a full state observer for
finite-free strongly observable systems over arbitrary commutative rings.

These results together provide a solution to coefficient-assignment and
regulation problems (as shown in Remark 3.8 in detail) for finite-free split
linear systems defined over arbitrary commutative rings.

Remark 2.8. Concerning the construction of compensators, note that
\( Q_1 \) is row proper. Therefore, \( K_0 \) is finite and free. Hence, using Lemma
2.1, we can obtain matrix representations of the \( Q_1 \)-realization of \( Q_1^{-1}P_i \),
and then transpose them to obtain a realization of \( Z_1 = P_iQ_1^{-1} \).

III. OBSERVERS FOR LINEAR SYSTEMS OVER COMMUTATIVE RINGS

In this section, we will show how one can build observers for finite-free
strongly observable linear systems over a commutative ring \( K \). Throughout
this section, without loss of generality, we will assume that all the systems
under consideration have dynamic interpretations as discrete-time systems
over \( K \).

Recall that \( K[z] \) has a multiplicatively closed subset of monic
polynomials \( P_i \), called the set of stable polynomials. A rational function \( \phi \),
where \( p \neq q \in K[z] \) will be said to be stable if \( q \in P_i \).

Let \( \Sigma = (F, G, H) \) be a linear system over \( K \) with the dynamic interpreta-
\[
x(t + 1) = Fx(t) + Gu(t),
\]
\[
y(t) = Hx(t).
\]

We assume that \( F, G, H \) are \( n \times n, n \times m, m \times n \) matrices over \( K \) and \( K^\ast \)
the state module. \( K^\ast \) and \( K^\lambda \) are input and output value modules. We
define the transfer matrices \( Z \) and \( Z_i \) as
\[
Z = H(z I - F)^{-1}G
\]
\[
Z_i = (z I - F)^{-1}G.
\]

Let \( I \) be an \( l \times n \) matrix over \( K \). Let \( \Sigma_1 = (F, G, H), H_{10} = \{ 0 \} \), \( J \)
where \( F, G, H, J, \Sigma_1 \) are \( r \times r, r \times m, r \times r, \{ 0 \} \), and \( \Sigma \) is an \( p \times p 

matrices over \( K \) is a system with state module \( K^\ast \), input value module
\( K^\lambda \), output value module \( K^\lambda \). The dynamic interpretation of \( \Sigma_1 \) is
\[
\xi(t + 1) = F_0\xi(t) + G_0u(t) + G_2\psi(t)
\]
\[
\psi(t) = H_0\xi(t) + J_0\psi(t).
\]

We call \( \Sigma_1 \) an \( I \)-observer for \( \Sigma \) iff for every initial state \( x_0 \) of \( \Sigma \) and
every initial state \( \xi_0 \) of \( \Sigma_1 \) and for every input \( u \) each of the transfer
matrices from \( u, x_0, \xi_0 \) to
\[
e = \dot{x} - Lx
\]
is free.

We should note here that the degree of stability of the transfer matrices
in the above definition is also very important. Because this determines the rate of convergence (for the rings where convergence is well-defined), and as it will be shown in Remark 3.8 the characteristic polynomial of $F_k$ affects the regulation of the closed-loop system. Therefore, it is more desirable to be able to assign the coefficients of $|z| - F_k$ arbitrarily. It will be seen that this can be achieved with our approach.

We should note here that $x_\infty$, $i_\infty$ do not have to be in $K^n$ and $K^r$. In general, they may be in some other sets. But as we will see, our results are independent of the sets that $x_\infty$, $i_\infty$ may belong to. The reader is referred to Hautus and Sontag [11] for a discussion of this aspect of the problem.

To establish our main result, first we need the following two lemmas.

**Lemma 3.1.** Let $R_1$, $R_2$, $T$, $Q$, be $l \times m$, $l \times p$, $l \times n$, and $l \times 1$ polynomial matrices over $K$ such that

$$
(\dot{z} - F)T + H'R_2 = L'Q'
$$

Then

$$
Q_\nu^{-1}[R_1 : R_2][I_n] = 1Z_\nu.
$$

Conversely, if $(F, G)$ is weakly reachable and (3.2) is satisfied for some polynomial matrices $Q_\nu$, $R_1$, $R_2$, then there exists an $l \times n$ polynomial matrix $T$ such that (3.1a) and (3.1b) are satisfied.

**Proof:** (3.1a) and (3.1b) imply that

$$
T(\dot{z} - F) - Q_\nu L - R_2 H
$$

and

$$
TG = R_1
$$
or

$$
(Q_\nu L - R_2 H)(\dot{z} - F)^{-1}G = - R_1.
$$

which yields (3.2).

Conversely, if (3.2) holds, we can rewrite it as

$$
(\dot{z} - F)^{-1}G = - R_1.
$$

Let

$$
V(z) = R_1 H - Q_\nu L = \sum_{j=0}^{s} V_j z^j
$$

Then, as $R_1$ is polynomial,

$$
[V_0, \ldots, V_s] [1] [G : FG : \cdots] = 0.
$$

As $(F, G)$ is weakly reachable, we must have

$$
[V_0, \ldots, V_s] [1] [F^s] = 0.
$$

But this shows that $V(z)$ is right divisible by $(\dot{z} - F)$, i.e., there exists an $l \times n$ polynomial matrix $T$ such that

$$
R_2 H - Q_\nu L = - T(\dot{z} - F)
$$

and

$$
R_1 = TG.
$$

**Lemma 3.2.** Let $R_1$, $R_2$, $Q_\nu$ be as in Lemma 3.1 with $|Q_\nu| \in P_r$, and such that $Q_\nu^{-1}[R_1 : R_2]$ has a realization $\Sigma = (F_\nu, G_\nu, H_\nu, 0 : J_\nu)$ with the state module $K^r$ (for some integer $r \geq 0$), and $|z| - F_\nu = |Q_\nu|$. Then $\Sigma$ is an $L$-observer for $\Sigma_A$.

**Proof:** Let $u$ be a given input. Let $x_\infty$ be an initial state for $\Sigma$ and $i_\infty$ be an initial state for $\Sigma_A$. Then one can write the formal $z$-transform $\hat{z}$ of $\epsilon$ as

$$
\hat{z} = Z_\nu + Z_\infty + Z_\infty + Z_\infty + Z_\infty
$$

where

$$
Z_\nu = Q_\nu^{-1}[R_1 : R_2][I_n];
$$

$$
Z_\infty = Q_\nu^{-1}[R_1 : R_2][I_n] \frac{1}{z};
$$

$$
Z_\infty = H_\nu (\dot{z} - F)^{-1} z_i;
$$

$$
Z_\infty = L(\dot{z} - F)^{-1} z_i;
$$

$$
Z_\infty = T(\dot{z} - F)^{-1} z_i.
$$

But from Lemma 3.1 we have

$$
Z_\nu + Z_\infty + Z_\infty + Z_\infty + Z_\infty = 0.
$$

Hence,

$$
\hat{z} = Z_\nu + Z_\infty + Z_\infty
$$

But as $|z| - F_\nu = |Q_\nu|$ is in $P_r$, if $Z_\nu + Z_\infty$ depends on $x_\infty$ and $i_\infty$, through stable rational matrices, so does $\hat{z}$. Now we will prove that this is the case. Consider

$$
|z|^1 (Z_\nu + Z_\infty) = Q_\nu^{-1} (R_1 H - Q_\nu L)(\dot{z} - F)^{-1} z_i.
$$

From (3.1a) we have

$$
|z|^1 (Z_\nu + Z_\infty) = - Q_\nu^{-1} T x_i.
$$

As $|Q_\nu|$ is in $P_r$, $\hat{z}$ depends on $x_\infty$, $i_\infty$, and $u$ through stable rational matrices. Hence, by definition of an $L$-observer, $\Sigma$ is an $L$-observer for $\Sigma_A$.

Now we state our main theorem regarding $L$-observers.

**Theorem 3.4.** Let $(F, G)$ be reachable. Let $\mu$ be the first integer such that

$$
K^* = \lim_{n \to \infty} H^* + \cdots + \lim_{n \to \infty} F^* + \cdots.
$$

(i.e., $\mu$ is the reachability index of $(F, G)$). Let $y_1, \ldots, y_r$ be integers such that $y_r = \mu - 1$. Let $a$ be a given monic polynomial of degree

$$
y = \sum_{i=1}^{r} i \lambda_i.
$$

Then, there exist polynomial matrices $Q_\nu$, $R_1$, $R_2$, satisfying (3.1a) such that $Q_\nu$ is row proper with $r$ th row degree $y_r$, $(Q_\nu)^{-1} = a$, and such that $Q_\nu^{-1}[R_1 : R_2]$ has a finite realization

$$
\Sigma_A = (F_\nu, G_\nu, H_\nu, [G_\nu : G_\nu : [0 : J_\nu])
$$

with $|z| - F_\nu = a$. Further, if $a$ is in $P_r$, then $\Sigma_A$ is an $L$-observer for $\Sigma$.

For the proof of this theorem, we will need a result which can be easily inferred from a result of Emre [1979, Theorem 2.3] and Remarks 2.7 and 2.8 which is valid for $K$ an arbitrary commutative ring with identity.

**Lemma 3.5.** Let $(F, G)$ be reachable. Let $\Phi$ be a $P \times p$ nonsingular polynomial matrix such that $K^p$ is well defined and finite-free. Let $R$ be a $p \times m$ polynomial matrix such that $Q^{-1} R$ is strictly proper. Let $\Phi$ be a $p \times r$ polynomial matrix. Then there exist $p \times r$ and $m \times r$ polynomial matrices $Q$ and $P$, such that

$$
Q \Phi + R P = \Phi
$$

and if and only if $\text{Sp} \Phi(\Phi)$ is in the reachable subspace of the $Q$-realization $\Sigma_Q$ of $Q^{-1} R$. If this condition holds, $P_r$ can be chosen such that $\delta_0 (P_r) <$ reachability index of $\Sigma_Q$.

**Proof of the Theorem 3.4.** It is known (see the proof of Theorem 2.4)
how we can construct a column proper polynomial matrix \( Q_n \) with
column degrees \( \gamma_i \), such that \( |\gamma_i| = \alpha \).

Now with \( Q = (I - F)^{-1} R = H' \), Lemma 3.5 shows that we can find \( T' \), \( R_2 \), with \( R_2(\lambda I) \neq 0 \) such that

\[
(T(I - F))' + H'R_2 = L'Q_n.
\]

As \( Q_n \) is column proper with \( R_2(\lambda I) \neq 0 \), \( R_2(\lambda I)^{-1}Q_n \) is proper,

\[
T = -(I - F)' + (I - F)'L'Q_n
\]
or

\[
TQ_n^{-1} = -(I - F)' + (I - F)'L'Q_n^{-1} + (I - F)'I
\]
i.e., \( TQ_n^{-1} \) is strictly proper.

Then

\[
Q_n^{-1} \begin{bmatrix} R_1 & R_2 \end{bmatrix} = Q_n^{-1} \begin{bmatrix} TG & R_2 \end{bmatrix}
\]
is proper and \( Q_n^{-1} R_1 \) is strictly proper. If we choose \( \alpha \) to be stable, \( |\gamma_i| = \alpha \) will be stable If we choose \( \gamma_i \) to be the \( Q_n \)-realization of \( Q_n^{-1} \), \( Q_n^{-1} R_1 = 0 \) implies that \( Q_n \) is proper, the characteristic polynomial of \( F \) will be \( \alpha \). [See Remarks 2.6 and 2.7] Hence, we have an \( L \)-observer whose characteristic polynomial can be arbitrarily assigned.

Remark 3.8: From a practical point of view, choosing \( \gamma_i = \mu - 1 \) would give us reduced orders of observer of order \( \mu (\mu - 1) \). Note that we are free to choose \( \gamma_i \), as we want, as long as \( \gamma_i > \mu - 1 \). Note here that as \( \Sigma \) is the \( Q_n \)-realization of \( Q_n^{-1} \), \( Q_n^{-1} R_1 = 0 \), and \( Q_n \) is proper, the characteristic polynomial of \( F \) will be \( \alpha \). [See Remarks 2.6 and 2.7] Hence, we have an \( L \)-observer whose characteristic polynomial can be arbitrarily assigned.

The constructive methods given in [7] in terms of linear equations over \( K \) can be applied to obtain observers (also, regulators) provided that one can solve the corresponding equations over \( K \).

Remark 3.7: For the regulator problem, one obvious choice for \( L \) would be \( L = L_0 \), in which case, we will have a full state observer. Then, as the transfer matrix of the \( \Sigma \) and \( \Sigma_L \) is \( L_0 \), one can apply dynamic state feedback as explained in Section II and construct a regulator to stabilize (coefficient-assignment) the closed-loop system. For a detailed explanation of the internal stability aspect of the regulators obtained this way, see Remark 3.8.

The advantage of choosing \( L \), other than \( L_0 \), would be that, first of all, we would need lower order observers. If there exists an \( L \) in \( K^{n \times n} \) such that \((I - F)^{-1}G_0\) can be expressed as \( Q^{-1} R \) with proper \( Q \), we can construct a lower order compensator for coefficient-assignment of the closed-loop system. Hence, we would reduce the orders of both the observer and the compensator for coefficient-assignment.

In particular, if \( F \) is cyclic with a generator \( h^k \), we can obtain \( h(1)(I - F)^{-1}G_0 \) using an observer of order \( \mu - 1 \). Then we can find a monic polynomial \( q \) and a polynomial row vector \( R \) such that

\[
q(1)^{-1} R = h(I - F)^{-1} G_0.
\]

Then using results of Section II, we can achieve coefficient-assignment with a compensator of order \( r - 1 \) where \( r \) is the reachability index of \( \Sigma \).

We should note here that if the transfer matrix of \( \Sigma = (F, G, H) \) can be expressed as \( Q^{-1} R \) with \( Q \) being polynomial matrices satisfying

\[
Q_4 + R_8 = I
\]
for some polynomial matrices \( A, B \) which is implied by \( \Sigma \) being split (see [16]) and if in addition such a \( Q \) can be chosen to be row proper one can directly use Theorem 2.4 bypassing the need for this latter condition is equivalent to the fact that the observability indices of \( (H, F) \) add up to the dimension of \( F \).

Remark 3.8.8: We will now show that our technique of regulator synthesis leads to an internally stable closed-loop system. Let \( \Sigma = (F, G, H) \) be a finite cyclic system. Let \( Q_n^{-1} \begin{bmatrix} R_1 & R_2 \end{bmatrix} \) be the transfer matrix of a full state observer as in Theorem 3.4. Then there exists a polynomial matrix \( T \) such that

\[
T(I - F) + R_2 H = Q_n.
\]

Furthermore, as \( Q_n \) is chosen to be row proper, \( K_{Q_n} \) is a free \( K \)-module (see Part II). Let \( \Sigma_0 = (F_0, G_0, H_0, [0 : J]) \) be a representation of the \( Q_n \)-realization of \( Q_n^{-1} \).

Let us define

\[
V_0 = -G_0 H_0 - G_0 H_0 (I - F) Q_n^{-1}.
\]

Multiplying the second row by \((I - F)^{-1} \), and further multiplying the second row by \( H_2 \), and adding to the first row, we get

\[
X = X_{F_0} \begin{bmatrix} I & 0 & -G_0 H_0 - G_0 H_0 (I - F) \end{bmatrix}^{-1} T
\]

where \( X_{F_0} = [I : F_0] \). Multiply the first row by \((I - F_0)^{-1} \), and further multiplying the first
row by \( GH_s \), and multiplying the second row by \( GJ/H_s \), and adding both to the third row, we get
\[
x = x_F X_r \left[ (z^2 - F) + GJ/H_s \right] GJ/H_s
\]
Noting that
\[
P_0 Q_0^{-1} \quad H_I(1 - F) \quad GJ + J_I,
\]
and combining terms, we have
\[
x = x_F X_r \left[ (z^2 - F) - GPQ_1^{-1} \right] X_r X_r \left[ (z^2 - F) - GPQ_1^{-1} \right]^{-1}
\]
As \( Q \) = \( x_r \), we have
\[
x = x_{F1} \Phi_1
\]
\[
\Phi_1 = x_F X_r.
\]
Since \( x_F = (Q_1) \) and \( \Phi_1 \) are chosen to be stable polynomials, \( x \) is a stable polynomial. This shows that our technique of regulator synthesis leads to an internally stable system, and hence provides a solution to the regulator problem for finite free split systems over an arbitrary commutative ring.

Now, \( H_I(1 - F) \quad GJ + J_I \) is observable for each \( \sigma \) if and only if \( G = \Phi_1 \) for arbitrary characteristic polynomial. Thus, we get
\[
\Phi = \begin{bmatrix} z^2 + 1 & 1 \\ -1 & z \end{bmatrix}
\]

Also,
\[
\Phi = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}
\]
If we now solve for \( P_0 \) and \( Q_0 \), in the polynomial equation
\[
(z^2 - F) Q_0 = G P_0 \Phi,
\]
we get
\[
P_0 = \begin{bmatrix} z^2 + 1 & 0 \\ 0 & z \end{bmatrix}, \quad Q_0 = \begin{bmatrix} z - \sigma & 0 \\ 0 & z \end{bmatrix}
\]
Therefore, the transfer matrix \( z_1 \) of the feedback compensator is given by
\[
Z_1 = P_0 Q_0^{-1} = \begin{bmatrix} \sigma (z - \sigma) & 1 \\ -z (z - \sigma) & 0 \end{bmatrix}
\]

With \( Z_1 \) as the feedback compensator and
\[
Z = \begin{bmatrix} z^2 - F \end{bmatrix}^{-1} G
\]
it can be easily checked that the overall closed-loop transfer matrix \( Z \) is given by
\[
Z = (1 + Z_1) \quad z = Z_1 \Phi_1, z \quad G
\]
\[
\begin{bmatrix} z - \sigma & 0 \\ 0 & z \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}
\]
The \( Q_1 \) realization of \( Q_1 \) is given by \( (F, G_1, H_1, J_1) \) where
\[
F_1 = \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix}, \quad G_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad H_1 = \begin{bmatrix} \sigma & 1 \\ 0 & 0 \end{bmatrix}
\]
Thus, \( (F_1, G_1, H_1, J_1) \) is a free realization of \( Z_1 \). The \( \Phi \)-realization \( (F, G, H, J) \) of the overall closed-loop system transfer matrix \( Z \), can be easily obtained as explained in Remark 2.6.

A matrix representation of this realization is
\[
\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \end{bmatrix}
\]
\[
\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \end{bmatrix}
\]
It can be easily seen that the characteristic polynomial of \( F \) is \( (z^2 - F)^2 + 1) \).
Example 4.11: We will now illustrate our technique of observer and regulator synthesis for a delay-differential system. This example has been worked out in detail in [13] and [11, Example 5.6]. It will be seen that our observer turns out to be the same as that obtained in [11, Example 5.6]. We then complete the regulator synthesis by obtaining a feedback compensator to stabilize the system.

In this example, the system is not strongly observable. This example shows that observer synthesis can be approached with our results even if the system is not strongly observable. The reader is referred to [11] for a generalization of the concept of detectability for algebras over fields.

The delay-differential system is given by

\[
\begin{align*}
\dot{x}_1 &= x_1(t-1) + u(t), \\
\dot{x}_2 &= x_1(t-1) + x_2(t) + u(t), \\
y(t) &= x_2(t).
\end{align*}
\]

(4.12)

Let \( a \) denote the shift operator \( a(x(t)) = x(t-1) \). Then the delay-differential system (4.12) can be modeled as a linear system over \( R[a] \).

For details, see [14]. The matrices \((F, G, H)\) are given by

\[
F = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 1 \end{bmatrix}.
\]

It is shown in [11] that this system is not strongly observable and is unstable. It is also shown that the polynomial \( z + \mu a \) is stable for \( 0 < \mu < 1/2 \). We will construct a full state observer whose characteristic polynomial is \( z + \mu a \). With the notation of Section III, we need to solve for \( T \) and \( R \) in the equation

\[
(\mathbf{I} - F)T + H'R = Q_n.
\]

(4.14)

Let us choose

\[
Q_n = \begin{bmatrix} z + \mu a & 0 \\ 0 & 1 \end{bmatrix}.
\]

(4.15)

Now a solution to (4.14) is given by

\[
R_n = \begin{bmatrix} \mu^2 + \sigma - \mu \\ 1 \end{bmatrix}, \quad T = \begin{bmatrix} 1 & 0 \end{bmatrix}.
\]

(4.16)

Furthermore,

\[
R_n = TG \begin{bmatrix} 1 - \mu \sigma - \mu \\ 0 \end{bmatrix}.
\]

Therefore, the observer transfer matrix is given by

\[
Q_n[R_n : R_n] = \begin{bmatrix} (1 - \mu \sigma - \mu)(z + \mu a)^{-1} & (\sigma - \mu \sigma)(z + \mu a)^{-1} + \mu \\ 0 & 1 \end{bmatrix}.
\]

(4.17)

The following observer equations can be immediately obtained from (4.17) as explained in Section III:

\[
\begin{align*}
\dot{x}_1 &= \mu \dot{x}_1(t-1) + (1 - \mu)u(t) - \mu u(t-1) \\
&+ (1 - \mu^2)y(t-1) - \mu y(t), \\
\dot{y}_1(t) &= \dot{x}_1(t) + \mu y(t), \\
\dot{y}_2(t) &= y(t).
\end{align*}
\]

(4.18)

If \( \mu \) is chosen such that \( 0 < \mu < 1/2 \), then the system (4.18) is a full state observer for the system (4.12). It is thus seen that observer synthesis is possible even if the original system is not strongly observable. The key point is to choose \( Q_n \) such that \( |Q_n| \) is stable, and such that solutions to (4.14) exist.

We will now construct a feedback compensator to stabilize the overall system. Using the notations of Section II, we have \( r = 2 \). Let us choose \( p_1 = p_2 = 2 \). Let

\[
\begin{align*}
\beta &= z^2 + 2z + 1 = (z^2 + 1)^2, \\
\Phi &= (z^2 + 2z + 1)(z^2 + 1) \\
&\quad - 1 \\
&\quad z^2.
\end{align*}
\]

(4.19)

It can be easily checked that \( \beta \) is a stable polynomial for continuous-time systems. The polynomial matrix \( \Phi \) turns out to be

\[
\Phi = \begin{bmatrix} z^2 + 2z + 1 \\ -1 \end{bmatrix}.
\]

(4.20)

We need to solve for \( P_1 \) and \( Q_1 \) in the polynomial equation

\[
(\mathbf{I} - F)Q_1 + GP_1 = \Phi.
\]

(4.21)

It is easy to check that

\[
\eta_{II} f_s(\Phi) = \begin{bmatrix} \sigma^2 - 1 & 0 + 1 \\ 3\sigma - 1 & 5\sigma^2 + 2\sigma + 1 \end{bmatrix}.
\]

A solution for \( P_1 \) and \( Q_1 \) is given by

\[
P_1 = \begin{bmatrix} (\sigma - \sigma^2 - a) - 4 \end{bmatrix} + (\sigma - 2\sigma^2 + 4a + 3) \\
\begin{bmatrix} \sigma^2 - 1 & 0 + 1 \\ 3\sigma - 1 & 5\sigma^2 + 2\sigma + 1 \end{bmatrix}.
\]

(4.22)

Now the transfer matrix of the feedback compensator is given by \( P_1Q_1^{-1} \), with \( P_1, Q_1 \) as in (4.22). The \( Q_1 \)-realization for the compensator can be obtained as explained in Remark 2.6.

The overall transfer matrix with the observer and feedback compensator is given by

\[
Z_n = HQ_1^{-1}T.
\]

(4.23)

The \( T \)-realization for (4.23) is given by the following equations:

\[
\begin{align*}
\dot{x}_1(t) &= \dot{x}_1(t), \\
\dot{x}_2(t) &= \dot{x}_2(t) + u(t), \\
\dot{y}_1(t) &= \dot{y}_1(t), \\
\dot{y}_2(t) &= \dot{y}_2(t) - 2\dot{y}_2(t) - 3\dot{y}_2(t) - 5\dot{y}_2(t) + u(t), \\
y(t) &= \dot{y}_2(t) - \dot{y}_2(t-1) - \dot{y}_2(t-2) + \dot{y}_2(t) + 4\dot{y}_2(t) - 2\dot{y}_2(t-1) + \dot{y}_2(t-2).
\end{align*}
\]

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REFERENCES

[3] [14]


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