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M.J. Shensa

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Reviewed by
P.M. Reeves, Head
Electronics Division

Under authority of
R.H. Hearn, Head
Fleet Engineering Department

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Doppler tracking is a common procedure which can be implemented by a multitude of techniques. Also, it is well known that in the absence of a velocity change in the receiver, the solution track is not unique. This paper examines the solution set of the Doppler tracking problem, presenting several new results and placing some of the lore in a more rigorous setting. In particular, it is shown that (in two dimensions) for an isovelocity receiver the solution is determined up to a rotation and reflection in the receiver's coordinate system. If a single velocity change is present, then there are exactly two solutions. Generalizations to three dimensions are also provided.

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SUMMARY

Doppler tracking is a common procedure which can be implemented by a multitude of techniques. Also, it is well known that in the absence of a velocity change in the receiver, the solution track is not unique. This paper examines the solution set of the Doppler tracking problem, presenting several new results and placing some of the lore in a more rigorous setting. In particular, it is shown that (in two dimensions) for an isovelocity receiver the solution is determined up to a rotation and reflection in the receiver's coordinate system. If a single velocity change is present, then there are exactly two solutions. Generalizations to three dimensions are also provided.

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CONTENTS

1. INTRODUCTION . . . page 1

2. ISOVELOCITY RECEIVER . . . 2

3. MANEUVERING RECEIVER . . . 5

4. EXTENSION TO THREE DIMENSIONS . . . 7

5. CONCLUSIONS . . . 7

APPENDIX A . . . 9

REFERENCE . . . 10

1. INTRODUCTION

This short paper is concerned with the following classical Doppler tracking problem: Given a source moving with constant velocity and emitting a tone of known frequency f_o, and a receiver following a known track with received frequencies f_i at times t_i, determine the track of the source (see Figure 1a). Assuming the data are exact, we wish to address the question, what is the nature of the solution set of the above problem? And, specifically, when is the solution unique? Although some of the results presented here are well known in the lore of Doppler tracking, to the author's knowledge a rigorous derivation has not appeared in the literature. Other results are new.

Before proceeding, it is both mathematically and descriptively convenient to transform to a coordinate system in which the receiver is at rest (Figure 1b). For this to be an advantage, we must assume that the Doppler shift depends only on the relative velocity $\bar{v} = \bar{v}_s - \bar{v}_r$ of the source and the receiver, an approximation which is valid as long as $|\bar{v}| \ll c$ where c is the propagation speed of the signal. In that case, the received frequency is given by

$$f_i = f_o \left(1 + \frac{|\bar{v}|}{c} \cos \theta_i\right) \tag{1}$$

with θ_i the angle between $\bar{v}$ and the position vector as illustrated in Figure 1b. We begin this study with the case in which the receiver's track is a single segment of constant velocity $\bar{v}_r$.

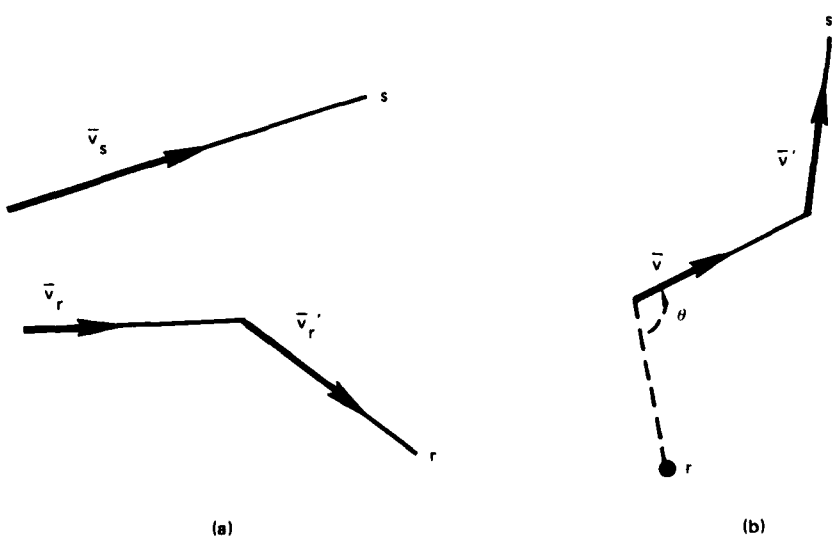

Figure 1. Example of source (s) and receiver (r) geometry: (a) fixed coordinate system, (b) coordinates relative to receiver.

2. ISOVELOCITY RECEIVER

A solution track takes the form illustrated in Figure 2. It is immediately clear that given one solution, an arbitrary rotation of the coordinate axes will yield another; thus, the solution set is infinite. This leads rather naturally to the question of whether the solution is unique up to a rotation and reflection; i.e., are the speed $|\bar{v}|$, the distance to CPA (closest point of approach) R, and t_o the time of CPA, uniquely determined by the Doppler data? Since there are three unknowns, $|\bar{v}|$, R, and t_o, we suspect that three data points consisting of measurement times t_i and corresponding frequencies f_i, i = 1,2,3, will determine the solution(s). (It is clear that if the source is traveling directly away from the receiver the solution remains indeterminate, and we therefore exclude that case; i.e., we assume $f_i \neq f_j$ for $i \neq j$.)

Define the variables

$$\Delta t_i \triangleq t_i - t_o \tag{2}$$

and

$$q_i \triangleq \frac{|\bar{v}|\,\Delta t_i}{R} \tag{3}$$

so that

$$\cos\theta_i = \frac{q_i}{\sqrt{1+q_i^2}} \tag{4}$$

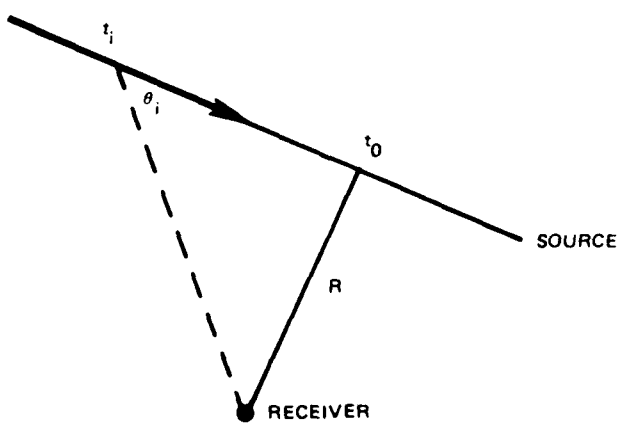

Figure 2. Geometry in receiver's coordinate system for isovelocity receiver.

Equation (1) becomes

$$\Delta f_i = f_i - f_o = \frac{f_o\,|v|}{c}\;\frac{q_i}{\sqrt{1+q_i^2}} \qquad i = 1,2,3 \tag{5}$$

and (3) implies

$$p = \frac{t_2 - t_1}{t_3 - t_1} = \frac{q_2 - q_1}{q_3 - q_1} \quad . \tag{6}$$

Although this formulation has increased the number of unknowns to four (q_i and $|\overline{v}|$), it shall shortly enable us to eliminate two.

We now introduce the variables

$$\begin{aligned} x &\triangleq \frac{q_1}{q_2} \\ y &\triangleq \frac{q_1}{q_3} \end{aligned} \tag{7}$$

and letting

$$\alpha \triangleq \frac{\Delta f_2}{\Delta f_1} \quad \text{and} \quad \beta \triangleq \frac{\Delta f_3}{\Delta f_1} \quad . \tag{8}$$

we have from (5)

$$\begin{aligned} \alpha^{-1} &= (\operatorname{sgn} x)\,\frac{\sqrt{x^2+q_1^2}}{\sqrt{1+q_1^2}} \\ \beta^{-1} &= (\operatorname{sgn} y)\,\frac{\sqrt{y^2+q_1^2}}{\sqrt{1+q_1^2}} \end{aligned} \tag{9}$$

or

$$1 + q_1^2 = \alpha^2 (x^2 + q_1^2)$$

$$1 + q_1^2 = \beta^2 (y^2 + q_1^2) \quad . \tag{10}$$

Eliminating q_1^2 yields

$$(1 - \alpha^2)\beta^2 y^2 - \alpha^2 (1 - \beta^2) x^2 = \beta^2 - \alpha^2 \quad . \tag{11}$$

Also, the substitution of (7) into (6) produces

$$p = \frac{y - xy}{x - xy} \tag{12}$$

or, after some rearranging,

$$\left(x - \frac{1}{1-p}\right)\left(y + \frac{p}{1-p}\right) = -\frac{p}{(1-p)^2} \quad . \tag{13}$$

Thus, the problem has been reduced to solving the two simultaneous quadratics (11) and (13) for x and y where α, β, and p are determined from the data t_1, Δf_i, $i = 1,2,3$.

The sign of q_i is known from (5) which restricts the solution(s) to one quadrant (cf. (7)). The solution(s) will be the intersection(s) of a section of a hyperbola (eq. (13)) with a hyperbola or ellipse (eq. (11)). A detailed analysis is found in Appendix A. If the Δf_i are all of the same sign, the solution is unique. It is interesting to note that when this is not true there exist cases for which there are two (but never more than two) solutions. Observe that if there are five data points available, at least three must have the same sign and the solution of (11) and (13) is unique. Since we are assuming the data are exact, there is at least one solution, and we conclude:

THEOREM I

For an isovelocity receiver, the source track is uniquely determined up to rotations and reflections (with respect to the receiver's coordinate system) by five Doppler data points (satisfying $f_i \neq f_j$ for $i \neq j$). Three data points suffice for this uniqueness if the Doppler shifts are of the same sign.

3. MANEUVERING RECEIVER

Consider the case in which the receiver's track consists of two segments as in Figure 1; i.e., at some point in time there is a single change in velocity from $\bar{v}_r$ to $\bar{v}_r'$. Let $\Delta\bar{v} = \bar{v}_r - \bar{v}_r'$, then the relative velocities $\bar{v}$ and $\bar{v}'$ ($\bar{v} \triangleq \bar{v}_s - \bar{v}_r$) satisfy

$$\bar{v} - \bar{v}' = -\Delta\bar{v} \quad . \tag{14}$$

We may use the techniques of the previous section to solve for $|\bar{v}|$ and $|\bar{v}'|$ along their respective segments (assuming there are at least three data points for each). The triangle formed by $|\bar{v}|$, $|\bar{v}'|$, and $|\Delta\bar{v}|$ is determined by its three sides. Since $\Delta\bar{v}$ is a known vector, the orientation of this triangle is determined up to a reflection (see Figure 3a). Thus, given $|\bar{v}|$ and $|\bar{v}'|$, there are at most, two possibilities for $\bar{v}_s = \bar{v} + \bar{v}_r$. The velocity $\bar{v}_s$ completes the determination of the source track since it specifies the orientation in Figure 2.

Now, assume that the conditions of Theorem I have been met for each of the two segments. Then $|\bar{v}|$ and $|\bar{v}'|$ are uniquely determined, and there are at most two solutions (Figure 3a) for the source track. We shall demonstrate that there are at least two solutions.

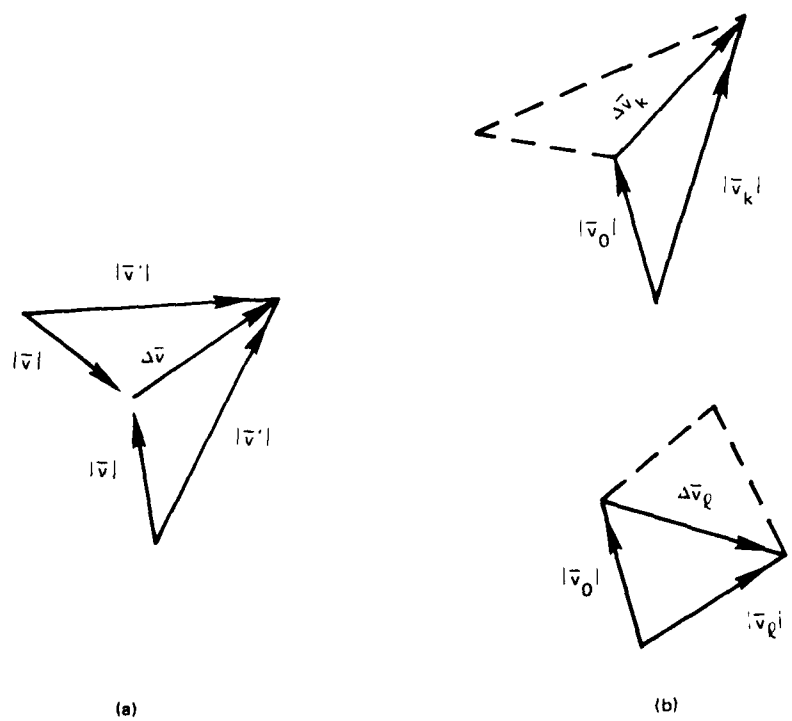

Figure 3. Illustration of possible solutions to $\Delta\bar{v} = \bar{v}' - \bar{v}$ where $\Delta\bar{v}$, $|\bar{v}|$, and $|\bar{v}'|$ are known; (a) single maneuver, (b) two maneuvers.

THEOREM II

Let the receiver's track consist of at most two isovelocity segments. Then, given one solution to the Doppler tracking problem, there exists a distinct second solution.

The proof relies on the following lemma:

LEMMA 1

Given two vectors $\bar{v}_1$ and $\bar{v}_2$, there exists a (non-unique) vector $\bar{v}_o$ such that $\bar{v}_1 - \bar{v}_o$ and $\bar{v}_2 - \bar{v}_o$ are parallel.

PROOF: The construction is illustrated in Figure 4a. A rigorous proof may be supplied by the reader.

PROOF OF THEOREM II: We now transform to a coordinate system moving with velocity $\bar{v}_o$ such that $\bar{v}_r - \bar{v}_o$ and $\bar{v}_r' - \bar{v}_o$ are parallel. It is clear from Figure 4b that given one solution (solid line), there exists a second solution (dotted line) obtained by a reflection in the receiver's track.

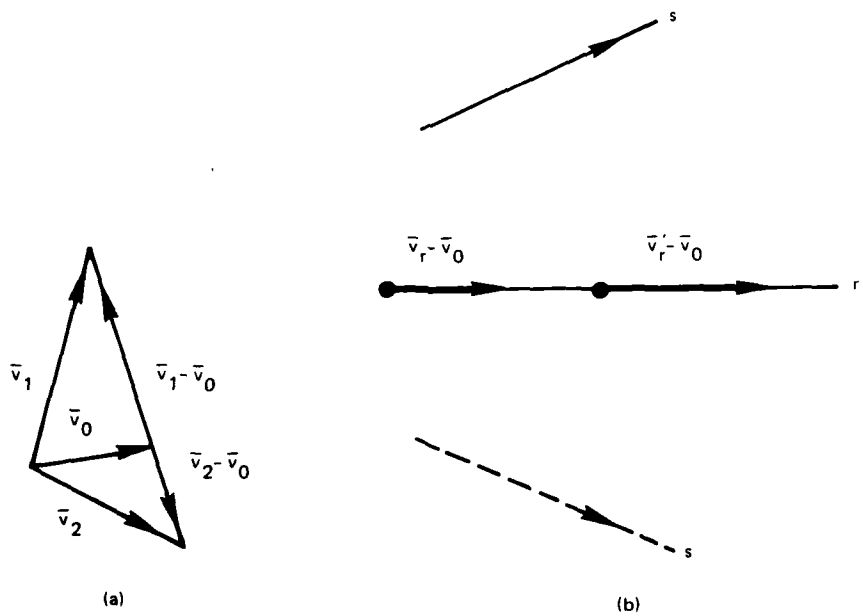

Figure 4. (a) Example of Lemma 1; (b) Illustration of Theorem II.

Combining these results with those of the previous section, we have:

THEOREM III

Let the receiver's track consist of two segments, each containing at least five Doppler points (with $f_i \neq f_j$ for $i \neq j$). Then there exist exactly two solutions to the Doppler tracking problem.

Suppose now that there are further maneuvers. Label the corresponding velocity differences with a subscript i so that $\Delta\overline{v}_i = \overline{v}_{ri} - \overline{v}_{ro}$ where $\overline{v}_{ro}$ is the velocity of the initial segment. It is easily seen from Figure 3b that if there exist at least two differences $\Delta\overline{v}_k$ and $\Delta\overline{v}_\ell$ which are not parallel, the Doppler-tracking solution must be unique (because both $\overline{v}_o$ and its reflection (dotted line in Figure 3b) in the triangle containing $\Delta\overline{v}_k$ cannot match their counterparts in the triangle containing $\Delta\overline{v}_\ell$).

4. EXTENSION TO THREE DIMENSIONS

The previous results generalize directly to the three-dimensional case. Once again we transfer to the receiver's coordinates. For an isovelocity receiver, the source track and its perpendicular to the receiver at CPA determine a plane. In that plane the situation is exactly that of Figure 2, only we now have spherical symmetry. Thus, the solution is indeterminate up to two angles (the direction of R) rather than one angle plus a reflection. Note that the wording in Theorem I is such that it remains valid.

When the track has two segments, Figure 3a is still valid (since $\overline{v}_1$ and $\overline{v}_2$ determine a plane); however, it may be rotated about the vector $\Delta\overline{v}$ (Figure 5a). Thus, we find that the set of solutions is unique up to a one parameter set of rotations.

In the case of three segments there are at most two solutions, provided the conditions of the previous section are satisfied; i.e., provided there exist k and ℓ such that $\Delta\overline{v}_k$ and $\Delta\overline{v}_\ell$ are not parallel. This situation is illustrated in Figure 5b. $\Delta\overline{v}_k$ and $\Delta\overline{v}_\ell$ determine the plane of the figure. The point Q, representing one solution, does not necessarily lie in that plane. The complete solution set is the intersection of two non-identical circles, which can be at most two points. If $\overline{v}$ is coplanar with $\Delta\overline{v}_k$ and $\Delta\overline{v}_\ell$ (i.e., if the receiver's track lies in a plane) there is only one solution. If the receiver's track does not lie in a plane there is a second solution, namely, the reflection of Q in the plane of the paper. Finally, for four or more segments the solution is unique.

5. CONCLUSIONS

We have examined the solution set of the Doppler tracking problem for a fixed velocity source and possibly maneuvering receiver. The problem as stated involved four unknown parameters. For a sufficient number of data points three of these parameters are determined by one isovelocity receiver track segment. Note that one of these parameters

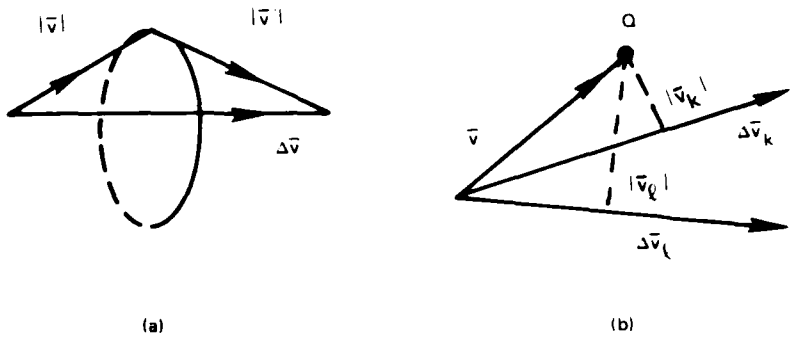

Figure 5. (a) Three dimensional case. The vector $\bar{v}$ may lie anywhere on the circle.
(b) Example with two maneuvers. The circular loci do not lie in the same plane and meet at the single point Q.

is relative source speed. A second segment, (i.e., a maneuver), determines the fourth parameter up to a reflection. In three dimensions, three segments are required to achieve uniqueness up to a single reflection.

In addition to the geometric insight provided, the analysis may be used to find a starting point for the solution of the nonlinear equations typical of least-square fits to Doppler data. In the case of a single maneuver, there are two solutions, a piece of information which can be critical in solving nonlinear equations. Finally, we note that the construction in Section 2 reduces the problem for exact data to the simultaneous solution of two quadratics which is much simpler than the original problem in four unknowns.

APPENDIX A

Given three Doppler points, we order them so that $t_3 < t_2 < t_1$. Thus, $0 < p < 1$. We shall refer to the situation in which all three Δf_i are non-negative as Case 1. Note that our conclusions concerning the solution set in this case will remain valid for $\Delta f_i \leq 0$, $i = 1,2,3$ by symmetry. The remaining possibility, where one Δf_i differs in sign from the other two, shall be termed Case 2. Also, from symmetry, we may assume without loss of generality in Case 2 that $t_2 \leq t_0 < t_1$.

CASE 1

Our assumptions imply (see (3) and (7)) that

$$0 < y < x < 1 \tag{A-1}$$

and (eq. (8))

$$1 < \alpha < \beta \quad . \tag{A-2}$$

In addition $(q_1)^2$ must be positive so that equations (10) imply

$$x < \frac{1}{\alpha} \quad , y < \frac{1}{\beta} \quad . \tag{A-3}$$

Equation (11) is therefore a hyperbola with part of a single branch in the first quadrant and similarly for equation (13) (since $0 < p < 1$). In fact it is easily shown that the situation is as illustrated in Figure A-1. There are thus at most two solutions. Note, however, that the point $x = 1$, $y = 1$ is a solution of both equations. But this point does not satisfy (A-1). Thus, there is a maximum of one solution.

CASE 2

In this case

$$x < y < 0 \tag{A-4}$$

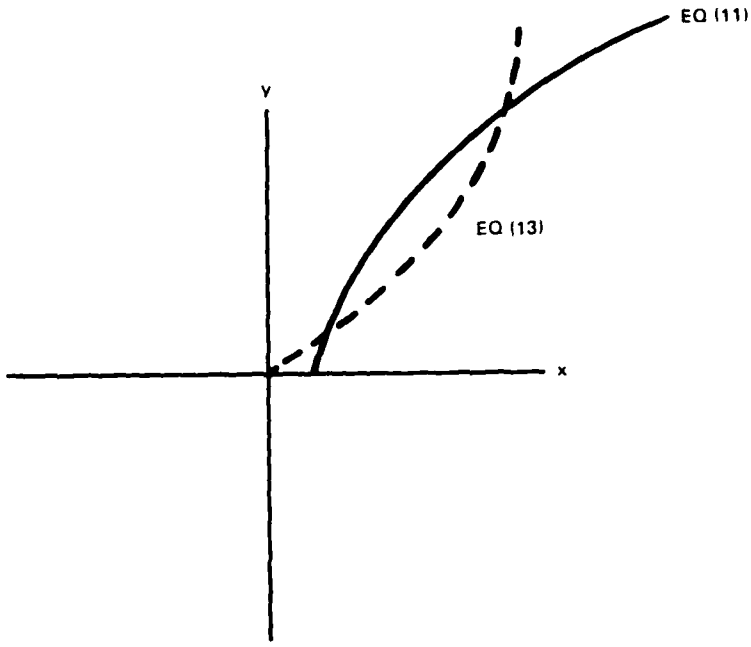

Figure A-1. Plot of hyperbolae (11) and (13) for Case 1.

and

$$\beta < \alpha < 0 \quad . \tag{A-5}$$

This gives rise to three subcases: (a) $|\alpha| > 1$, $|\beta| > 1$; (b) $|\alpha| < 1$, $|\beta| > 1$; and (c) $|\alpha| < 1$, $|\beta| < 1$. Note that in (b), equation (11) represents one quadrant of an ellipse. Also, as in Case 1, equations (10) put bounds on x and y since $(q_1)^2$ must be positive. These three subcases are pictured in Figure A-2. Note that there are at most two solutions. Examples of zero, one, and two solutions may be constructed [1].

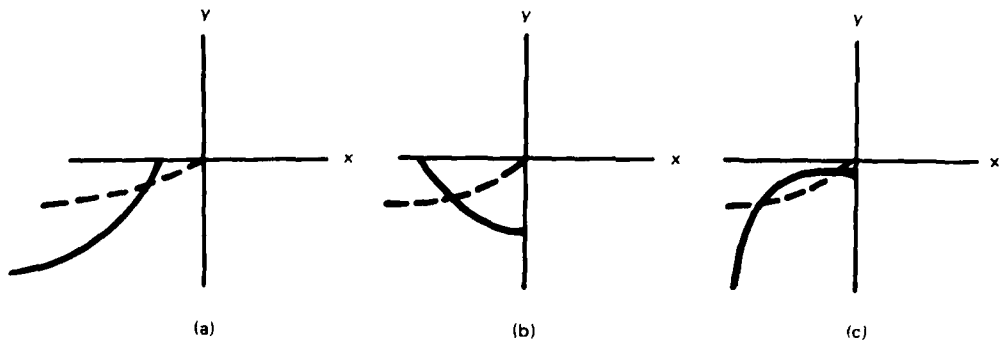

Figure A-2. Plot of equations (11) (solid line) and (13) (dashed line) for Case 2. (a) $|\alpha| > 1$, $|\beta| > 1$; (b) $|\alpha| < 1$, $|\beta| > 1$; (c) $|\alpha| < 1$, $|\beta| < 1$.

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1. M. Shensa, unpublished results, Naval Ocean Systems Center, San Diego, CA.

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