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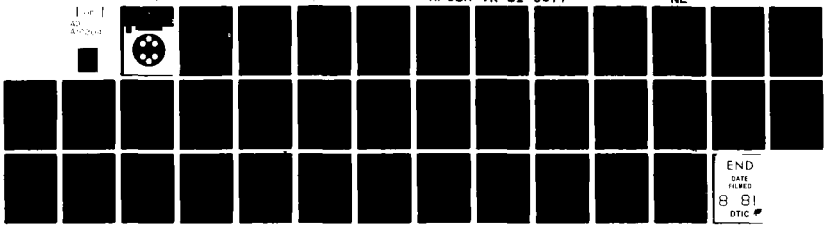

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(6) ALMOST SURE CONVERGENCE OF ADAPTIVE IDENTIFICATION PREDICTION AND CONTROL ALGORITHMS

by

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ALMOST SURE CONVERGENCE OF ADAPTIVE IDENTIFICATION PREDICTION AND CONTROL ALGORITHMS†

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March 1981

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ALMOST SURE CONVERGENCE OF ADAPTIVE IDENTIFICATION PREDICTION AND CONTROL ALGORITHMS

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Rajendra Kumar

ABSTRACT

The paper is concerned with the almost sure convergence of the adaptive parameter estimation, N-step ahead prediction and control algorithms based upon standard least square algorithm. With the usual stability and passivity assumptions for the prediction problem, it is demonstrated that the state estimation and the N-step ahead prediction errors converge to the optimum such errors achievable with known plant parameters, in the Cesaro sense. An additional regularity assumption on the signal model establishes the result that the state estimation and prediction errors also converge in the strong sense at an asymptotically arithmetic rate. Under an additional persistency of excitation condition it is shown that the parameter estimation error converges to zero at a rate specified by the degree of excitation. The persistency of excitation condition being of a trivial nature is also a necessary condition for parameter convergence. With the regularity condition holding, the convergence is also established for the adaptive control algorithms, e.g. self tuning regulators under the usual minimum phase restriction on the plant. In this case the tracking error equals the N-step ahead prediction error and thus converges to its optimum value at an asymptotically arithmetic rate.

1. INTRODUCTION

The paper is concerned with the almost sure convergence of the adaptive parameter estimation, N-step ahead prediction and control algorithms based upon standard least square algorithm. With the usual stability and passivity assumptions for the prediction problem, it is demonstrated that the state estimation and the N-step ahead prediction errors converge to the optimum such errors achievable with known plant parameters, in the Cesaro sense. Under an additional regularity assumption, the convergence of these errors and also that of the tracking error for the adaptive control, is shown to be in the strong sense and at an asymptotically arithmetic rate. An additional persistency of excitation condition of a trivial nature also establishes the parameter convergence.

In [1-6] the convergence analysis for the parameter estimation algorithm focuses mainly on the consistency and asymptotic properties of the parameter estimation errors. Of these the most recent work [2] establishes parameter convergence and also the convergence of the prediction error for the stochastic approximation algorithm using projections under relatively weak assumptions for such a convergence. Being concerned with establishing the parameter convergence, such analysis is bound to make assumptions which are not necessarily satisfied under situations where the parameter estimation is not the central issue, for example, in the case of adaptive prediction and control problems, see for instance [3,4].

Guided by this rationale, the work of [7-10] takes an alternative approach wherein the emphasis is placed on the convergence of the prediction error in the case of N-step ahead prediction problem or that of the tracking error for the case of adaptive control. In most of these references the issue of parameter convergence is completely ignored except in

[7] where simultaneously some results on the parameter convergence are also established. An important condition that emerges out from the analysis of [4-11] is that a system related to the signal generating system or frequently just the noise generating system be passive. In [11] unrealistic assumption on the adaptive predictor termed "persistency of excitation" condition are also made to derive convergence. The analysis of [4] essentially ignored the global stability problem and dealt with the local behavior only, when dealing with the adaptive control problem.

In [8] for the adaptive control problem, a scheme based upon stochastic gradient type of parameter estimation algorithm has been proposed along with its global convergence analysis, thus generalizing the earlier deterministic results of [14]. In [7,9,10] statistically more efficient algorithms based upon modified least square parameter estimation schemes have been proposed for the adaptive control and prediction problem. The modification in [9] is based upon the condition number check of a certain matrix so as to ensure convergence. In [7] the modification is based upon a stability measure and thus a weighted least square algorithm rather than the standard least square algorithm is used to ensure convergence. The N-step ahead prediction schemes of [10] use a bank of N least square parameter estimation algorithms to ensure global convergence. The convergence in all these references is established in the Cesaro sense.

The above schemes of [7-10] leave the question unanswered as to whether or not the schemes based upon the standard least square algorithm converge, without any modification, whenever the modified schemes do so. The question is of more than academic interest. Whereas the schemes of [10] involve extra complexity, which is not marginal, the ones of [7] have the problem that these afford more weight to the past measurements and

thus the adaptation of these algorithms to time-varying plants and the generalization of the convergence results may pose problems. In practical implementation, of course, this problem can be avoided by several possible schemes of ad hoc nature. Furthermore all the schemes to date with the exception of [2] establish convergence only in the Cesaro sense and thus leave open the possibility of the divergence to infinity of a subsequence of the prediction/tracking errors or even the plant input-output sequence in case of adaptive control. The schemes also ignore the parameter convergence except in [7] where only partial results on this issue appear.

The convergence analysis of this paper uses the standard least square algorithm for the parameter estimation and thus the results contain the convergence of the self-tuning regulators of [15,16], adaptive predictors of [17] and the adaptive Kalman prediction schemes of [11] as special cases. The distinctive feature of the analysis of this paper is that whereas in the absence of any regularity condition, convergence of the prediction error is established in the Cesaro sense, the regularity condition implies the convergence in the strong sense and asymptotically at an arithmetic rate. Furthermore, this strong convergence of prediction/tracking error is not a sufficient condition for the parameter convergence. However, an additional condition of a trivial nature also establishes the parameter convergence and its rate of convergence. The condition is trivial because in its absence one cannot expect parameter convergence with any algorithm, when noise is present in the measurements. It also emerges out from the theory that the convergence of parameter error need not always be at an arithmetic rate. In fact it could be slower depending upon the excitation, for example, at a logarithmic rate.

In Section 2, various algorithms, convergence conditions and the notations are introduced. The global convergence analysis is presented in Section 3 both for the adaptive N-step ahead prediction and the adaptive control problem. Some concluding remarks are made in Section 4.

2. DYNAMICS OF MODEL, ALGORITHM AND ESTIMATION ERRORS

Although the problem of one-step ahead prediction forms a special case of the N-step ahead prediction problem, for clarity of presentation, the two are treated separately in this section. This is so because the analysis of the N-step ahead prediction/control algorithm makes direct application of the results derived for the one-step ahead prediction/control problem. The development of this section follows closely that of [7].

Model Dynamics: For the adaptive control of the plants without a pure delay or for the one step ahead prediction problems, we consider the following multi-input multi-output model,

$$x_{k+1} = (F + G_1\theta')x_k + G_2\omega_k + f(u_k, z_k) \tag{2.1a}$$

$$z_k = \theta' x_k + \omega_k \tag{2.1b}$$

where u_k, z_k and x_k are the inputs, outputs and the states of the model. The matrices F, G_1, G_2 and the functions f are assumed known with the parameter matrix θ and the states x_k unknown. The noise ω_k is a zero mean white process or more precisely is a martingale difference sequence satisfying

$$E[\omega_k/F_{k-1}] = 0 \quad , \quad E[\|\omega_k\|^2/F_{k-1}] \leq \sigma^2 < \infty \tag{2.2a}$$

$$\limsup_{k\to\infty} \frac{1}{k} \sum_{i=0}^{k} \|\omega_i\|^2 < \infty \quad \text{a.s.} \tag{2.2b}$$

where F_k is the minimal σ-algebra generated by $\omega_1,\ldots,\omega_k$. As shown in [7,8], this model includes multi-input multi-output ARMAX model with exogeneous inputs and also the innovation models.

When the plant to be controlled has a delay of less than (N+1) units or when the N-step ahead prediction problem is considered, then a model of sufficient generality is the following,

$$x_{k+N} = Fx_{k+N-1} + G_1\theta' x_k + G_2 n_k + f(u_k, z_k) \tag{2.3a}$$

$$z_k = \theta' x_k + n_k \quad , \quad n_k = \omega_k + Q_0\omega_{k-1} + \cdots + Q_{N-2}\omega_{k+1-N} \tag{2.3b}$$

with ω_k satisfying (2.2). The actual values of the matrices Q_i are not of interest in the following and F, G_1, G_2 are again assumed known. For the ARMAX models these merely have as their elements, 0 and 1.

<u>State and Parameter Estimators</u> The state estimator for the model (2.1) is given, for some estimate $\hat{\theta}_k$ of the parameter matrix θ at time k, by

$$\hat{x}_{k+1} = (F + G_1\hat{\theta}_k')\hat{x}_k + G_2\tilde{z}_{k/k} + f(u_k, z_k) \tag{2.4a}$$

$$\tilde{z}_{k/k} = z_k - \hat{\theta}_k'\hat{x}_k \tag{2.4b}$$

and similarly for the model (2.3). The N-step ahead prediction of z_{k+N} is given by

$$\hat{z}_{k+N/k} = \hat{\theta}_k'\hat{x}_{k+N} \quad . \tag{2.4c}$$

The parameter estimation algorithm considered here is the standard least square algorithm for both the one-step and N-step ahead predictors/controllers,

$$\hat{\theta}_k = \hat{\theta}_{k-1} + \hat{P}_k \hat{x}_k \tilde{z}'_{k/k-1} \tag{2.5a}$$

$$\tilde{z}_{k/k-1} = z_k - \hat{\theta}'_{k-1} \hat{x}_k \tag{2.5b}$$

$$\hat{P}_k^{-1} = \hat{P}_{k-1}^{-1} + \hat{x}_k \hat{x}'_k \quad , \quad r_k = \text{trace } \hat{P}_k^{-1} \tag{2.5c}$$

or

$$\hat{P}_k = \hat{P}_{k-1} - \hat{P}_{k-1} \hat{x}_k (1 + \hat{x}'_k \hat{P}_{k-1} \hat{x}_k)^{-1} \hat{x}'_k \hat{P}_{k-1} \tag{2.5d}$$

<u>State and Parameter Estimation Error Equations</u> For the one-step ahead prediction/control situation, the equation for the state estimation error $\tilde{x}_k \triangleq x_k - \hat{x}_k$ is obtained from, (2.1), (2.4) and can be reorganized as

$$\tilde{x}_{k+1} = (F + G\theta')\tilde{x}_k + Gq_k \quad , \quad q_k \triangleq \tilde{\theta}'_k \hat{x}_k \quad , \tag{2.6a}$$

$$p_k = \theta' \tilde{x}_k + \frac{1}{2} \tilde{\theta}'_k \hat{x}_k \tag{2.6b}$$

The need to reorganize the $\tilde{x}_k$ generating system in this manner, will be apparent in the sequel. Similarly for the N-step ahead case, one obtains

$$\tilde{x}_{k+N} = F\tilde{x}_{k+N-1} + G(p_k + \frac{1}{2} q_k) \tag{2.7}$$

with p_k and q_k same as in (2.6). The parameter estimation error $\tilde{\theta}_k = \theta - \hat{\theta}_k$ obtained from (2.5) is as follows.

$$\tilde{\theta}_k = \tilde{\theta}_{k-1} - \hat{P}_k \hat{x}_k \tilde{z}'_{k/k-1} \quad , \quad \tilde{\theta}_k = \tilde{\theta}_{k-1} - \hat{P}_{k-1} \hat{x}_k \tilde{z}'_{k/k} \tag{2.8a}$$

$$\tilde{z}_{k/k-1} = \theta' \tilde{x}_k + \tilde{\theta}'_{k-1} \hat{x}_k + \omega_k \quad , \quad \tilde{z}_{k/k} = \theta' \tilde{x}_k + \tilde{\theta}'_k \hat{x}_k + \omega_k \tag{2.8b}$$

<u>Passivity Condition</u> As is common in literature [4-11], we make the assumption that the system (2.6a) or (2.7) with input q_k and output p_k is strictly input and strictly output passive. Thus for some $\kappa > 0$, $\varepsilon > 0$

and all integers m,

$$\sum_0^m p_k' q_k \geq -\frac{\kappa}{2} + \varepsilon \sum_0^m \|p_k\|^2 \quad , \quad \sum_0^m p_k' q_k \geq -\frac{\kappa}{2} + \varepsilon \sum_0^m \|q_k\|^2 \qquad (2.9)$$

We also require that the states $\tilde{x}_k$ of the system (2.6) or (2.7) are bounded in terms of $(p_k + \frac{1}{2} q_k)$. For the ARMAX models this is trivially satisfied since the matrix F has all of its eigenvalues at zero. For other models in addition to the input-output exponential stability implied by (2.9) we require that the matrix F in (2.6) or (2.7) has all the eigenvalues inside the unit circle. Thus

$$\sum_0^m \|\tilde{x}_k\|^2 \leq \kappa \left(\sum_0^m \|p_k + \frac{1}{2} q_k\|^2 \right) + \kappa \quad \text{some} \quad \kappa < \infty \qquad (2.10)$$

As shown in [18] the passivity condition (2.9) is equivalent to requiring that the transfer matrix $\{[I - z^{-(N-1)}\theta'(zI-F)^{-1}G]^{-1} - \frac{1}{2} I\}$ is strictly positive real. In single-input single-output ARMAX model case and for N = 1, the condition is well known and requires that $\{C^{-1}(z^{-1}) - \frac{1}{2}\}$ be strictly positive real. Here z denotes the Z-transform and $C(z^{-1})$ denotes in the Z-transform notation, the moving average polynomial associated with the noise ω_k.

<u>Stability/Minimum Phase Condition</u>: For the open loop prediction problem we require that for some κ and all $m > \bar{m}$, $\bar{m}$ being a finite integer,

$$\frac{1}{m} \sum_0^m \|x_k\|^2 \leq \kappa \qquad (2.11)$$

For the closed loop adaptive control we require that the trajectory z_k^* is bounded and the following minimum phase condition is satisfied. (For the N-step ahead case first sum in (2.13) has upper limit m-N.)

$$\frac{1}{m}\sum_{0}^{m}\|z_k^*\|^2 \leq \kappa \quad , \quad m \geq \bar{m} \quad \text{for some} \quad \kappa < \infty \tag{2.12}$$

$$\frac{\bar{\kappa}}{m}\sum_{0}^{m}\|u_k\|^2 \leq \frac{1}{m}\sum_{0}^{m}\|x_k\|^2 \leq \frac{\kappa}{m}\sum_{0}^{m}\|z_k\|^2 + \kappa, \text{ for some } \kappa,\bar{\kappa} < \infty \tag{2.13}$$

For the interpretation of (2.13) in terms of exponential stability and the complete observability/reachability of the inverse plant one may refer to [7].

For the convergence of the state estimation error and the N-step ahead prediction error in the Cesaro sense, in the case of N-step ahead prediction schemes, the assumptions so far made are sufficient. However, for the adaptive controller these assumptions turn out to be inadequate to establish the convergence of the tracking error to its minimum value. This problem has been addressed in [7] where a modification of the algorithm by an appropriate sequence of weighting coefficients, ensures convergence. In [9] an alternative scheme based upon the condition number of a certain matrix has also been proposed. However, both these schemes prove the convergence in the Cesaro sense and thus leave open the possibility of a subsequence of tracking errors diverging* to ∞. Secondly these schemes do not provide any analysis as to when the standard least square algorithm converges. Furthermore, whereas in [9] the problem of parameter convergence has not been addressed at all, the condition in [7] required for parameter convergence is of implicit nature. This is so because the condition in [7] is dependent upon the specific weighting sequence selected during a particular realization of the process.

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*This is a realistic problem in view of the fact that one is considering possibly unstable plants and in real life situations, nonlinearities due to saturation can lead to overall divergence. In particular, such a divergence can result if, even in the absence of nonlinearities, a subsequence of plant signals could have high magnitudes.

In this paper we make an assumption termed the regularity condition on the signal model as follows.

Regularity Condition

$$\liminf_{k\to\infty} \frac{1}{\bar{r}_k} \sum_0^k x_i x_i' > \zeta I \quad \text{for some} \quad \zeta > 0 \quad , \quad \bar{r}_k = \sum_0^k x_i' x_i \tag{2.14a}$$

$$\limsup_{k\to\infty} \frac{\bar{r}_{k+1}}{\bar{r}_k} < \infty \tag{2.14b}$$

Remark 1. Condition (2.14a) is a restriction on the condition number of the "covariance" matrix* associated with the signal model. This condition is expected to be satisfied if the model has some type of exponential stability and is driven by persistent inputs. For the adaptive control problem, however, (2.14) needs to be interpreted in terms of z_k^* and ω_k. (See the remarks at the end of Section 3 in relation to 2.14.)

3. CONVERGENCE ANALYSIS

The convergence results are presented as three separate theorems. As the analysis of the N-step ahead predictor/controller makes direct use of the results for the 1-step ahead predictor/controller, the latter are presented first in the form of Theorem 3.1 and 3.2.

Theorem 3.1: Consider the plant and the least square estimation scheme of the previous section with noise condition (2.2a). Then, under

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*The matrix $\lim_{k\to\infty} \frac{1}{k} \sum_0^k x_i x_i'$ actually equals the covariance matrix of x_k provided that x_k is ergodic. Here, even though x_k is not ergodic we are referring to this matrix as "covariance" matrix.

the passivity condition of Section 2,

A.

(i) $$\sum_{k=0}^{\infty} r_k^{-1} \|\tilde{z}_{k/k} - \omega_k\|^2 < \infty \qquad \text{a.s.}$$

(ii) $$\sum_{k=0}^{\infty} r_k^{-1} \hat{x}_k' \hat{P}_{k-1} \hat{x}_k \|\tilde{z}_{k/k}\|^2 < \infty \qquad \text{a.s.}$$

(iii) $$\sum_{k=0}^{\infty} r_k^{-1} \|\tilde{x}_k\|^2 < \infty \qquad \text{a.s.}$$

Further, under the additional condition (2.14) one obtains (3.1a-d) below, wherein the condition (2.14b) is required only for the result (3.1d) and the result (3.1a) is also valid with r_k replaced by $\bar{r}_k$. For any arbitrary $\bar{\varepsilon} > 0$,

B.

(i) $$\limsup_{k\to\infty} \operatorname{tr}\{r_k^{-\bar{\varepsilon}} \tilde{\theta}_k' \hat{P}_k^{-1} \tilde{\theta}_k\} < \infty \qquad \text{a.s.}$$

$$\limsup_{k\to\infty} r_k^{(1-\bar{\varepsilon})} \|\tilde{\theta}_k\|^2 < \infty \qquad \text{a.s.} \qquad (3.1a)$$

$$\lim_{k\to\infty} r_k = \infty \implies \lim_{k\to\infty} \|\tilde{\theta}_k\| = 0 \qquad \text{a.s.}$$

(ii) $$\sum_{k=0}^{\infty} r_k^{-\bar{\varepsilon}} \|\tilde{z}_{k/k} - \omega_k\|^2 < \infty \qquad \text{a.s.}$$

$$\sum_{k=0}^{\infty} r_k^{-\bar{\varepsilon}} \|\tilde{x}_k\|^2 < \infty \qquad \text{a.s.} \qquad (3.1b)$$

(iii) $$\sum_{k=0}^{\infty} r_{k-1}^{(1-\bar{\varepsilon})} \|\hat{\theta}_k - \hat{\theta}_{k-1}\|^2 < \infty \qquad \text{a.s.} \qquad (3.1c)$$

(iv) $$\sum_{k=0}^{\infty} r_k^{-\bar{\varepsilon}} \|\tilde{z}_{k/k-1} - \omega_k\|^2 < \infty \qquad \text{a.s.} \qquad (3.1d)$$

<u>Proof</u>: Define V_k by

$$V_k = \mathrm{tr}\{\tilde{\theta}_k'\hat{P}_k^{-1}\tilde{\theta}_k\}\delta_k + \{\sum_0^k [2\delta_i p_i' q_i - \varepsilon\delta_i(\|q_i\|^2 + \|p_i\|^2)] + \kappa\} \tag{3.2}$$

where κ and ε are the constants appearing in (2.9), $\{\delta_k\}$ is some monotone nonincreasing sequence, $p_k = \theta'\tilde{x}_k + \frac{1}{2}\tilde{\theta}_k'\hat{x}_k$ and $q_k = \tilde{\theta}_k'\hat{x}_k$. Now $V_k \geq 0$ due to the passivity condition (2.9) and simple manipulations similar to those in [7] yield that

$$E[V_k|F_{k-1}] \leq V_{k-1} + E[\hat{\Delta}_k - 2\omega_k' q_k|F_{k-1}] - \varepsilon\delta_k E[\|q_k\|^2 + \|p_k\|^2|F_{k-1}] \tag{3.3}$$

where

$$\begin{aligned}\hat{\Delta}_k &= \mathrm{tr}\{\tilde{\theta}_k'\hat{P}_k^{-1}\tilde{\theta}_k - \tilde{\theta}_{k-1}'\hat{P}_{k-1}^{-1}\tilde{\theta}_{k-1} + 2\delta_k(p_k + \omega_k)'q_k\}\delta_k \\ &\leq -\eta_k\end{aligned}$$

with

$$\eta_k \triangleq \delta_k\hat{x}_k'\hat{P}_{k-1}\hat{x}_k\|\tilde{z}_{k/k}\|^2$$

The latter equality follows from a substitution of $\tilde{\theta}_{k-1}$ from the second part of (2.8a). Also from the definition of q_k and the first part of (2.8a),

$$E[2\delta_k\omega_k' q_k|F_{k-1}] = 2\delta_k E[\omega_k'\tilde{\theta}_k'\hat{x}_k|F_{k-1}] = -\beta_k$$

$$\beta_k \triangleq 2\delta_k\hat{x}_k'\hat{P}_k\hat{x}_k E[\|\omega_k\|^2|F_{k-1}] \tag{3.4}$$

The substitution of the above expressions in the second term on the right-hand side of (3.3) yields

$$E[V_k|F_{k-1}] \leq V_{k-1} - \varepsilon\delta_k E[\|q_k\|^2 + \|p_k\|^2|F_{k-1}] + \beta_k - \eta_k \tag{3.5}$$

Or with $\hat{V}_k = V_k + \varepsilon\delta_k(\|q_k\|^2 + \|p_k\|^2) + \eta_k$

$$E[\hat{V}_k|F_{k-1}] \leq \hat{V}_{k-1} - \varepsilon\delta_{k-1}(\|q_{k-1}\|^2 + \|p_{k-1}\|^2) + \beta_k - \eta_{k-1} \qquad (3.6)$$

Now to derive the results A of the theorem, make a specific selection for δ_k, namely $\delta_k = r_k^{-1}$. Result (i) of Lemma A2 of the Appendix then yields $\sum_0^\infty \beta_k < \infty$ and in view of the positivity of $\hat{V}_k$ and β_k, the application of the martingale convergence theorem [21] implies that $\hat{V}_k$ converges almost surely and $\sum_0^\infty \eta_k < \infty$, $\sum_0^\infty \|p_k\|^2 r_k^{-1} < \infty$, $\sum_0^\infty \|q_k\|^2 r_k^{-1} < \infty$ a.s. Thus with the application of second part of (2.8b) and the definition of η_k yields

$$\sum_0^\infty r_k^{-1} \|\tilde{z}_{k/k} - \omega_k\|^2 = \sum_0^\infty r_k^{-1}\|p_k + \frac{1}{2} q_k\|^2 \leq \frac{1}{2}\sum_0^\infty r_k^{-1}(\|q_k\|^2 + 4\|p_k\|^2) < \infty \quad \text{a.s.}$$

$$\sum_0^\infty r_k^{-1}\hat{x}_k\hat{P}_{k-1}\hat{x}_k\|\tilde{z}_{k/k}\|^2 < \infty \qquad \text{a.s.} \qquad (3.7)$$

This establishes (i) and (ii) of result A. Now from the bounded input-bounded state property of the system, (2.10), it follows that

$$\sum_0^\infty r_k^{-1} \|\tilde{x}_k\|^2 < \infty \qquad \text{a.s.} \qquad (3.8)$$

To see this note that the system (2.6a) can be rewritten as

$$r_{k+1}^{-1/2} \tilde{x}_{k+1} = [r_{k+1}^{-1/2} F r_k^{1/2}] r_k^{-1/2} \tilde{x}_k + (G r_{k+1}^{-1/2} r_k^{1/2}) r_k^{-1/2}(p_k + \frac{1}{2} q_k)$$

has a bounded input-bounded state property. An intermediate step to achieve this is

$$\sum_0^k \|r_{k+1}^{-1/2} F^{k-i} r_i^{-1/2}\| \leq \sum_0^k \|F\|^{k-i} \leq \kappa < \infty$$

Now to derive the results B of the theorem, it is easily seen that,

$$x_k x_k' \leq 2\hat{x}_k \hat{x}_k' + 2\tilde{x}_k \tilde{x}_k'$$

or

$$\hat{x}_k \hat{x}_k' \geq \frac{1}{2} x_k x_k' - \tilde{x}_k \tilde{x}_k' \tag{3.9a}$$

and also

$$\bar{r}_k \geq \frac{r_k}{2} - \sum_{j=0}^{k} \|\tilde{x}_j\|^2 \quad , \quad \bar{r}_k \leq 2r_k + 2\sum_{j=0}^{k} \|\tilde{x}_j\|^2$$

Application of (3.8) and the Kronecker lemma then implies the existence of an integer m_1 for any given $\varepsilon_1 > 0$ such that

$$\frac{\bar{r}_k}{r_k} \geq \frac{1}{2} - \varepsilon_1 \quad , \quad \frac{\bar{r}_k}{r_k} < 2 + \varepsilon_1 \quad \text{for all} \quad k \geq m_1 \tag{3.9b}$$

Inequalities (3.9a,b) imply that

$$\frac{1}{r_k} \sum_{j=0}^{k} \hat{x}_j \hat{x}_j' \geq (\frac{1}{2} - \varepsilon_1) \frac{1}{\bar{r}_k} \sum_{j=0}^{k} x_j x_j' - \frac{1}{r_k} \sum_{j=0}^{k} \tilde{x}_j \tilde{x}_j' \quad \text{for} \quad k \geq m_1$$

The condition (2.14a) of Section 2 implies that there exist an $m_2 \geq m_1$ such that the first term on the right-hand side of the above inequality $> 2\varepsilon_2 I$ for some $\varepsilon_2 > 0$ and for all $k \geq m_2$. An application of (3.8) and the Kronecker lemma implies that

$$\frac{1}{r_k} \sum_{j=0}^{k} \hat{x}_j \hat{x}_j' \geq \varepsilon_2 I \ , \qquad k \geq m_3, \text{ for some } m_3 \geq m_2 \tag{3.10}$$

In other words $\qquad \lambda_{min} \hat{P}_k^{-1} \geq \varepsilon_2 r_k \quad$ for $k \geq m_3$

or $\qquad \lambda_{max} \hat{P}_k \leq \varepsilon_2^{-1} r_k^{-1} \quad$ for $k \geq m_3$ $\qquad$ (3.11)

Appealing again to (3.5), (3.6) with $\delta_k \equiv r_k^{-\bar{\varepsilon}}$ for some $\bar{\varepsilon} > 0$ and now defining

$$S_k = tr\{\tilde{\theta}_k' \hat{P}_k^{-1} \tilde{\theta}_k r_k^{-\bar{\varepsilon}}\} + \{\sum_{m_3}^{k} [2r_i^{-\bar{\varepsilon}} p_i' q_i - \varepsilon r_i^{-\bar{\varepsilon}} (\|q_i\|^2 + \|p_i\|^2)] + \kappa_1\}$$

for some $0 \leq \kappa_1 < \infty$, $\varepsilon > 0$

and

$$\hat{S}_k = S_k + \varepsilon r_k^{-\bar{\varepsilon}}(\|q_k\|^2 + \|p_k\|^2) + \bar{\eta}_k$$

$$\bar{\beta}_k \triangleq 2r_k^{-\bar{\varepsilon}}\hat{x}_k'\hat{P}_k\hat{x}_k E[\|\omega_k\|^2|F_{k-1}] \quad , \quad \bar{\eta}_k \triangleq r_k^{-\bar{\varepsilon}}\hat{x}_k'\hat{P}_{k-1}\hat{x}_k\|\tilde{z}_{k/k}\|^2$$

yields that for $k \geq m_3$

$$E[\hat{S}_k|F_{k-1}] \leq \hat{S}_{k-1} - \varepsilon r_k^{-\bar{\varepsilon}}(\|q_k\|^2 + \|p_k\|^2) + \bar{\beta}_k - \bar{\eta}_{k-1} \tag{3.12}$$

Application of (3.11) and the lemma A1 yields that $\sum_{k=m_3}^{\infty} \bar{\beta}_k < \infty$ and $\hat{S}_k \geq 0$ as before. The martingale theorem is once again applicable and thus $\hat{S}_k$ converges a.s. and $\sum_{m_3}^{\infty} r_k^{-\bar{\varepsilon}}\|p_k\|^2 < \infty$ and $\sum_{m_3}^{\infty} r_k^{-\bar{\varepsilon}}\|q_k\|^2 < \infty$. In a manner we obtained (3.6), (3.7), we also get the result

$$\sum_{m_3}^{\infty} \bar{\eta}_k < \infty \qquad \text{a.s.} \tag{3.13a}$$

$$\begin{aligned} &\sum_{k=m_3}^{\infty} r_k^{-\bar{\varepsilon}} \|\tilde{z}_{k/k} - \omega_k\|^2 < \infty \qquad \text{a.s.} \\ &\sum_{k=m_3}^{\infty} r_k^{-\bar{\varepsilon}} \|\tilde{x}_k\|^2 < \infty \qquad \text{a.s.} \end{aligned} \tag{3.13b}$$

Also, as all the terms in $\hat{S}_k$ are positive, we have that $\limsup_{k\to\infty} \hat{S}_k^{(i)} < \infty$ where $\hat{S}_k^{(i)}$ denotes *ith* of the terms in $\hat{S}_k$. Thus

$$\limsup_{k\to\infty} r_k^{-\bar{\varepsilon}} \tilde{\theta}_k'\hat{P}_k^{-1}\tilde{\theta}_k < \infty \qquad \text{a.s.} \tag{3.14}$$

which proves part a of result (i). With the application of (3.11), part a of result (i) implies (ib). Since all the signals are finite for finite k, the lower limits in (3.13) can be replaced by 0 thus yielding part (Bii) of the theorem. Similarly the application of (3.11), (2.8a), (3.13a) and

definition of $\bar{\eta}_k$ yields that

$$\sum_{k=m_3}^{\infty} r_{k-1}^{1-\bar{\varepsilon}} \|(\hat{\theta}_k - \hat{\theta}_{k-1})\|^2 < \infty \qquad \text{a.s.}$$

As commented before, the lower limit can once again be replaced by 0 thus implying (iii).

Now from (2.4b), (2.5b), notice that

$$\tilde{z}_{k/k-1} = \tilde{z}_{k/k} + (\hat{\theta}_k - \hat{\theta}_{k-1})'\hat{x}_k$$

Taking norm square on both sides and applying part (i) of the theorem's result B,

$$\sum_0^{\infty} r_k^{-2\bar{\varepsilon}} \|\tilde{z}_{k/k-1} - \omega_k\|^2 \leq 2\sum_0^{\infty} r_k^{-2\bar{\varepsilon}} \|\tilde{z}_{k/k} - \omega_k\|^2 + 2\kappa_1 \sum_{k=m_4}^{\infty} \frac{\hat{x}_k'\hat{x}_k}{r_{k-1}^{1+\bar{\varepsilon}}} + 2\sum_0^{m_4-1} \|(\hat{\theta}_k - \hat{\theta}_{k-1})'\hat{x}_k\|^2$$

for some $0 < \kappa_1 < \infty$ and some finite integer m_4. Application of (2.14b) and (3.9b) shows that for m_4 sufficiently large, r_{k-1} in the second sum on the right-hand side can be replaced by r_k. Application of Lemma A1, (3.1b) and the arbitrary nature of $\bar{\varepsilon}$ establishes (3.1d). Further the arbitrary nature of $\bar{\varepsilon}$, implies the strong convergence of various error terms to zero. Also in view of (3.9b) r_k can be replaced by $\bar{r}_k$ in (3.1a).

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Under the regularity condition (2.14) of the signal model, the results B of the theorem are end results in themselves. In view of the arbitrary nature of $\bar{\varepsilon}$, these imply that the one step ahead prediction error $\tilde{z}_{k/k-1}$, which is equal to the tracking error for the adaptive controller, converges to ω_k at a rate $1/k$. Also for the prediction problem the estimation error $\tilde{x}_k$ converges to zero at a rate $1/k$. Furthermore if $r_k \to \infty$ as k approaches ∞, then the parameter error $\|\tilde{\theta}_k\|$ converges

to zero, the rate at which this error converges is given by $1/r_k$ or with the application of (3.9b) by $1/\bar{r}_k$. Moreover, the parameter convergence does not require (2.14b). To obtain the result that the inputs and outputs remain bounded for the adaptive control situation, we exploit the minimum phase restriction on the plant.

In the absence of regularity condition (2.14) the results A of the theorem cannot be extended to conclude the convergence of $\tilde{z}_{k/k-1}$ or that of the tracking error for the case of adaptive control in any sense. For the prediction problem, however, these do denote some kind of convergence of the estimation error $\tilde{x}_k$. With the stability assumption (2.11) on the signal model, as shown subsequently, the result (Aiii) of the theorem also implies the convergence of state estimation error $\tilde{x}_k$ to zero and that of the prediction error $\tilde{z}_{k/k-1}$ to ω_k in the Cesaro sense.

Both the sets of results are given in the form of theorem 3.2.

<u>Theorem 3.2</u>. Consider the adaptive schemes of Section 2 under the passivity condition of the theorem 3.1, the noise condition (2.2a,b), and the stability assumption (2.11) for the adaptive prediction schemes, or under the bounded trajectory condition (2.12) and minimum phase condition (2.13) for the adaptive control schemes. Then the result A below holds for adaptive predictor in the absence of (2.14)

A.

(i) $$\limsup_{k\to\infty} \frac{1}{k} \sum_{0}^{k} \|\hat{x}_i\|^2 < \infty \qquad \text{a.s.} \qquad (3.15a)$$

(ii) $$\lim_{k\to\infty} \frac{1}{k} \sum_{0}^{k} \|\tilde{z}_{i/i} - \omega_i\|^2 = 0 \qquad \text{a.s.}$$

$$\lim_{k\to\infty} \frac{1}{k} \sum_{0}^{k} \|\tilde{x}_i\|^2 = 0 \qquad \text{a.s.} \qquad (3.15b)$$

(iii) $$\lim_{k\to\infty} \hat{x}_k' \hat{P}_{k-1} \hat{x}_k = 0 \qquad \text{a.s.}$$

$$\sum_0^\infty \|(\hat{\theta}_k - \hat{\theta}_{k-1})' \hat{x}_k\|^2 k^{-1} < \infty \qquad \text{a.s.} \tag{3.15c}$$

(iv) $$\lim_{k\to\infty} \frac{1}{k} \sum_0^k \|\tilde{z}_{i/i-1} - \omega_i\|^2 = 0 \qquad \text{a.s.} \tag{3.15d}$$

Further, under the regularity condition (2.14a) holding, then for any $\bar{\varepsilon} > 0$, the results (ii) and (iv) in A can be strengthened to (3.15e) (for either the adaptive prediction or control scheme), where the last part of (3.15e) also requires (2.14b).

B.

$$\left.\begin{aligned} &\sum_0^\infty k^{-\bar{\varepsilon}} \|\tilde{z}_{k/k} - \omega_k\|^2 < \infty && \text{a.s.} \\ &\sum_0^\infty k^{-\bar{\varepsilon}} \|\tilde{x}_k\|^2 < \infty && \text{a.s.} \\ &\sum_0^\infty k^{-\bar{\varepsilon}} \|\tilde{z}_{k/k-1} - \omega_k\|^2 < \infty && \text{a.s.} \end{aligned}\right\} \tag{3.15e}$$

Also for the case of adaptive controller with (2.14) holding, the states, inputs and outputs of the plant remain bounded.

C.

(i) $$\limsup_{k\to\infty} \frac{1}{k} \sum_0^k \|x_i\|^2 < \infty \qquad \text{a.s.} \tag{3.15f}$$

(ii) $$\limsup_{k\to\infty} \frac{1}{k} \sum_0^k \|u_i\|^2 < \infty \qquad \text{a.s.}$$

$$\limsup_{k\to\infty} \frac{1}{k} \sum_0^k \|z_i\|^2 < \infty \qquad \text{a.s.} \tag{3.15g}$$

<u>Proof</u>: Considering the case of adaptive prediction first, the application of the Kronecker lemma to result (Aiii) of Theorem 3.1 yields

$$\lim_{k\to\infty} \frac{1}{r_k} \sum_0^k \|\tilde{x}_i\|^2 = 0 \tag{3.16}$$

Now the application of $\|\hat{x}_i\|^2 \leq 2\|x_i\|^2 + 2\|\tilde{x}_i\|^2$, and (2.11) results in

$$\frac{1}{k} \sum_0^k \|\hat{x}_i\|^2 \leq \frac{\kappa_2}{k} \sum_0^k \|\tilde{x}_i\|^2 + \kappa_2 \quad , \quad 0 < \kappa_2 < \infty \tag{3.17}$$

Taking inverse on both sides of (3.17) and multiplying the resulting inequality by $\frac{1}{k} \sum_0^k \|\tilde{x}_i\|^2$ on both sides

$$\{\sum_0^k \|\hat{x}_i\|^2\}^{-1} \sum_0^k \|\tilde{x}_i\|^2 \geq \{\frac{\kappa_2}{k} \sum_0^k \|\tilde{x}_i\|^2 + \kappa_2\}^{-1} \frac{1}{k} \sum_0^k \|\tilde{x}_i\|^2$$

This inequality implies that $\limsup_{k\to\infty} \frac{1}{k} \sum_0^k \|\tilde{x}_i\|^2 < \infty$ a.s., for otherwise taking limits for a subsequence, there is a contradiction that $0 \geq 1$, thus establishing (3.15a). Substitution of (3.15a) in the results (Ai,iii) of Theorem 3.1 and the application of the Kronecker's lemma yields (3.15b). First part of (3.15c) follows from the result (ii) of the lemma A2. Now premultiplying second part of (2.8a) on both sides by $\hat{x}_k'$ and taking norm square yields,

$$\sum_0^\infty r_k^{-1} \|\hat{x}_k'(\hat{\theta}_k - \hat{\theta}_{k-1})\|^2 \leq \sum_0^\infty (\hat{x}_k'\hat{P}_{k-1}\hat{x}_k)\,[\hat{x}_k'\hat{P}_{k-1}\hat{x}_k r_k^{-1}\|\tilde{z}_{k/k}\|^2]$$

Now as $\|\tilde{z}_{k/k-1}\|^2 \leq 2\|\tilde{z}_{k/k}\|^2 + 2\|(\hat{\theta}_k - \hat{\theta}_{k-1})'\hat{x}_k\|^2$, application of the result (Aii) of Theorem 3.1 and the first part of (3.15c) then gives the desired result (3.15d) and the second part of (3.15c) in view of (3.15a). For the case of adaptive controller as shown in [7] for the weighted least square scheme, one obtains

$$\frac{1}{k} \sum_0^k \|\hat{x}_i\|^2 \leq \frac{\kappa_2}{k} \sum_0^k \|\tilde{z}_{i/i-1} - \nu_i\|^2 + \kappa_2 \quad , \quad 0 < \kappa_2 < \infty$$

Further the application of the Kronecker's lemma to (3.1d) yields

$\lim_{k\to\infty} \frac{1}{r_k} \sum_0^k \|\tilde{z}_{i/i-1} - \omega_i\|^2 = 0$ which combined with the above inequality, once again yields (3.15a). The application of (3.15a) in (3.1b,d) then, results in (3.15e). Applying $\|x_i\|^2 < 2\|\hat{x}_i\|^2 + 2\|\tilde{x}_i\|^2$, (3.15a) and the second part of (3.15b) establishes (3.15f). The first part of (3.15g) follows from (3.15f) and the minimum phase condition (2.13) whereas the second part is implied by (2.1b), the noise condition (2.2b) and (3.15f).

Result B for the adaptive predictor follows by substituting (3.15a) in (3.1b,d).

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<u>N-Step Ahead Prediction Schemes</u> The convergence analysis of N-step ahead prediction/control schemes essentially involves the decomposition of the problem into N one-step ahead prediction/control problems. The direct application of the results of Theorem 3.1 and 3.2 then yields the desired convergence results.

<u>Theorem 3.3</u>: Consider the N-step ahead prediction/control schemes of Section 2 under the noise condition (2.2), passivity conditions (2.9), (2.10) and the minimum phase/bounded trajectory conditions (2.13), (2.12) for the closed loop adaptive control or the stability condition (2.11) for the open loop prediction schemes. Then in the absence of condition (2.14), the results A below hold for the N-step ahead predictor,

A.

(i) $$\limsup_{k\to\infty} \frac{1}{k} \sum_0^k \|\hat{x}_i\|^2 < \infty \qquad \text{a.s.} \qquad (3.18a)$$

(ii) $$\limsup_{k\to\infty} \frac{1}{k} \sum_{i=0}^k \|\tilde{z}_{i/i-N} - n_i\|^2 = 0 \qquad \text{a.s.} \qquad (3.18b)$$

Further, when the regularity condition (2.14) is satisfied, then for both

the prediction and adaptive control schemes, one also obtains for an arbitrary $\bar{\varepsilon} > 0$,

B.

(i) $$\limsup_{k\to\infty} r_k^{(1-\bar{\varepsilon})}\|\tilde{\theta}_k\|^2 < \infty \quad \text{a.s.}$$

$$\lim_{k\to\infty} r_k = \infty \Rightarrow \lim_{k\to\infty} \|\tilde{\theta}_k\| = 0 \quad \text{a.s.} \tag{3.18c}$$

(ii) $$\sum_0^\infty k^{-\bar{\varepsilon}} \|\tilde{x}_k\|^2 < \infty \quad \text{a.s.}$$

$$\sum_0^\infty k^{-\bar{\varepsilon}} \|\tilde{z}_{k/k-n_k}\|^2 < \infty \quad \text{a.s.} \tag{3.18d}$$

(iii) $$\sum_0^\infty k^{-\bar{\varepsilon}} \|\tilde{z}_{k/k-N-n_k}\|^2 < \infty \quad \text{a.s.} \tag{3.18e}$$

where part b of (2.14) is required only for (3.18e). Result (3.18c) is also valid with r_k replaced by $\bar{r}_k$.

For the N-step ahead adaptive controller, the plant states, inputs and outputs are also bounded, i.e.,

C.

$$\left.\begin{aligned} &\limsup_{k\to\infty} \frac{1}{k}\sum_0^k \|x_i\|^2 < \infty \quad \text{a.s.} \\ &\limsup_{k\to\infty} \frac{1}{k}\sum_0^k \|u_i\|^2 < \infty \quad \text{a.s.} \\ &\limsup_{k\to\infty} \frac{1}{k}\sum_0^k \|z_i\|^2 < \infty \quad \text{a.s.} \end{aligned}\right\} \tag{3.18f}$$

<u>Proof</u>: Following the procedure of [7,18], the error terms $\tilde{x}_k, \tilde{\theta}_k$ (2.7,2.8) are decomposed into N fictitious values as $\tilde{x}_k^{(1)},\ldots,\tilde{x}_k^{(N)}$, $\tilde{\theta}_k^{(1)},\ldots,\tilde{\theta}_k^{(N)}$ with the properties $\tilde{x}_k = \sum_{i=1}^N \tilde{x}_k^{(i)}$, $\tilde{\theta}_i = \sum_{i=1}^N \tilde{\theta}_k^{(i)}$ for all k, where

$$\tilde{\theta}_k^{(i)} = \tilde{\theta}_{k-1}^{(i)} - \hat{P}_k \hat{x}_k \tilde{z}_{k/k-1}^{(i)'} \tag{3.19a}$$

$$\tilde{\theta}_k^{(i)} = \tilde{\theta}_{k-1}^{(i)} - \hat{P}_{k-1} \hat{x}_k \tilde{z}_{k/k}^{(i)} \tag{3.19b}$$

$$\tilde{z}_{k/k-1}^{(i)} = \theta' \tilde{x}_k^{(i)} + \tilde{\theta}_{k-i}^{(i)} \hat{x}_k + Q_{i-2} \omega_{k+1-i}, Q_{-1} = I \tag{3.19c}$$

and

$$\tilde{x}_{k+N}^{(i)} = F \tilde{x}_{k+N-1}^{(i)} + G\left(\theta' \tilde{x}_k^{(i)} + \tilde{\theta}_k^{(i)'} \hat{x}_k\right) \tag{3.19d}$$

Defining function $\hat{V}_k^{(i)}$ by replacing $\tilde{\theta}_k, p_k, q_k$ by $\tilde{\theta}_k^{(i)}, p_k^{(i)}$ and $q_k^{(i)}$ respectively in the obvious notations, in (3.2) and in the equation for V_k, one has the property that $E[V_{k-1}^{(i)} | F_{k-i}] = V_{k-1}^{(i)}$, $E[\omega_{k+1-i} | F_{k-i}] = 0$, as $\hat{x}_k, \hat{P}_k, \tilde{x}_k \in F_{k-N} \subset F_{k-i}$ for $i = 1,2,\ldots,N$. Working with N supermartingales $\{\hat{V}_k^{(i)}, F_{k-i+1}\}$, then an analysis similar to that in Theorem 3.1 yields that

$$\sum_{k=0}^{\infty} r_k^{-1} \|\tilde{z}_{k/k}^{(i)} - Q_{i-2} \omega_{k+1-i}\|^2 < \infty \quad \text{a.s.}$$

$$\sum_{k=0}^{\infty} r_k^{-1} \|\tilde{x}_k^{(i)}\|^2 < \infty \quad \text{a.s.} \tag{3.20a}$$

$$\sum_{k=0}^{\infty} r_k^{-1} \hat{x}_k' \hat{P}_{k-1} \hat{x}_k \|\tilde{z}_{k/k}^{(i)}\|^2 < \infty \quad \text{a.s.} \tag{3.20b}$$

From (3.20a), by application of the inequality $\|\tilde{x}_k\|^2 \leq N \sum_{i=1}^{N} \|\tilde{x}_k^{(i)}\|^2$ and the Kronecker lemma, one obtains

$$\lim_{k\to\infty} \frac{1}{r_k} \sum_{0}^{k} \|\tilde{x}_i\|^2 = 0 \quad \text{a.s.} \tag{3.21}$$

The application of stability condition as in Theorem 3.2 yields that $\limsup_{k\to\infty} \frac{1}{k} \sum_{0}^{k} \|\hat{x}_i\|^2 < \infty$ a.s., thus establishing (3.18a). This result, its consequence that $\lim_{k\to\infty} \hat{x}_k' \hat{P}_{k-1} \hat{x}_k = 0$ a.s. (see lemma A2), (3.19b) and (3.20b) then yield,

$$\sum_{0}^{\infty} k^{-1} \|(\hat{\theta}_k^{(i)} - \hat{\theta}_{k-1}^{(i)})'\hat{x}_k\|^2 < \infty \qquad \text{a.s.}$$

or

$$\lim_{k\to\infty} \frac{1}{k} \sum_{j=0}^{k} \|\hat{x}_j'(\hat{\theta}_j^{(i)} - \hat{\theta}_{j-1}^{(i)})\|^2 = 0 \qquad \text{a.s.}$$

Also from (3.19b),

$$\frac{1}{N_1} \sum_{k=1}^{N_1} \|\hat{x}_k'(\tilde{\theta}_{k-j}^{(i)} - \tilde{\theta}_{k-j-1}^{(i)})\|^2 \le \frac{1}{N_1} \sum_{k=1}^{N_1} |\hat{x}_k' P_{k-j-1} \hat{x}_{k-j}|^2 \cdot \|\tilde{z}_{k-j/k-j}^{(i)}\|^2$$

for all i, j and $N_1 < \infty$

From Lemma A2, $|\hat{x}_{k+i}' \hat{P}_k \hat{x}_{k-j}| \to 0$ as $k \to \infty$ for all finite i and j. Thus $|\hat{x}_k' P_{k-j-1} \hat{x}_{k-j}| < \varepsilon_3$ for all $k \ge N_2$ and any given $\varepsilon_3 > 0$, N_2 being some finite integer. Thus for $N_1 > N_2$

$$\frac{1}{N_1} \sum_{k=1}^{N_1} \|\hat{x}_k'(\tilde{\theta}_{k-j}^{(i)} - \tilde{\theta}_{k-j-1}^{(i)})\|^2 \le \frac{1}{N_1} \sum_{k=1}^{N_2} |\hat{x}_k' P_{k-j-1} \hat{x}_{k-j}|^2 \cdot \|\tilde{z}_{k-j/k-j}^{(i)}\|^2$$

$$+ \frac{\varepsilon_3}{N_1} \sum_{N_2+1}^{N_1} \|\tilde{z}_{k-j/k-j}^{(i)}\|^2 \tag{3.22}$$

Now $\|\tilde{z}_{k-j/k-j}^{(i)}\|^2 \le 2\|\tilde{z}_{k-j/k-j}^{(i)} - Q_{i-2}\omega_{k-j+1-i}\|^2 + 2\|Q_{i-2}\|^2\|\omega_{k-j+1-i}\|^2$.

Application of (3.20a) and the noise condition (2.2b) in (3.22) yields, in view of arbitrary nature of ε_3

$$\limsup_{N_1\to\infty} \frac{1}{N_1} \sum_{k=1}^{N_1} \|\hat{x}_k'(\tilde{\theta}_{k-j}^{(i)} - \tilde{\theta}_{k-j-1}^{(i)})\|^2 = 0 \qquad \text{a.s.} \tag{3.23}$$

Now as

$$\tilde{z}_{k/k-N} = \tilde{z}_{k/k} + (\hat{\theta}_k - \hat{\theta}_{k-N})'\hat{x}_k \tag{3.24}$$

one obtains

$$\frac{1}{N_1}\sum_{k=1}^{N_1}\|\tilde{z}_{k/k-N}-n_k\|^2 \le (N+1)\left\{\frac{1}{N_1}\sum_{k=1}^{N_1}\|\tilde{z}_{k/k}-n_k\|^2 + \sum_{j=0}^{N-1}\frac{1}{N_1}\sum_{k=1}^{N_1}\|\hat{x}_k'(\tilde{\theta}_{k-j}-\tilde{\theta}_{k-j-1})\|^2\right\}$$

thus giving the desired result (3.18b) from (3.18a), (3.20a) and (3.23).

To establish result B, proceeding as in the proof of Theorem 3.1, under the regularity condition (2.14a), r_k^{-1} in (3.20) can be replaced by $r_k^{-\bar{\varepsilon}}$ to obtain

$$\begin{aligned} &\sum_{k=0}^{\infty} r_k^{-\bar{\varepsilon}}\|\tilde{z}_{k/k}^{(i)} - Q_{i-2}\omega_{k+1-i}\|^2 < \infty \qquad \text{a.s.} \\ &\sum_{k=0}^{\infty} r_k^{-\bar{\varepsilon}}\|\tilde{x}_k^{(i)}\|^2 < \infty \qquad \text{a.s.} \end{aligned} \tag{3.25a}$$

also implying that

$$\begin{aligned} &\sum_{k=0}^{\infty} r_k^{-\bar{\varepsilon}}\|\tilde{x}_k\|^2 < \infty \qquad \text{a.s.} \\ &\sum_{k=0}^{\infty} r_k^{-\bar{\varepsilon}}\|\tilde{z}_{k/k}^{(i)} - n_k\|^2 < \infty \qquad \text{a.s.} \end{aligned} \tag{3.25b}$$

Now the result that $\limsup_{k\to\infty} \operatorname{tr}\{\tilde{\theta}_k^{(i)\prime}\,\hat{P}_k^{-1}\,\tilde{\theta}_k^{(i)}\} < \infty$ a.s. for $i = 1, 2, \ldots, N$ and the inequality (3.11) following from (2.14a) and the first part of (3.25b) establish (3.18c).

Now the condition (2.14b) and (3.9b) which results due to the first part of (3.25b) as in the proof of theorem 3.1, imply that

$$\frac{r_k}{r_{k-1}} < \kappa_3 \quad \text{for all} \quad k > N_3\,, \quad 0 < \kappa_3 < \infty\,, \quad N_3 \text{ a finite integer} \tag{3.26}$$

Subtracting n_k from both sides of (3.24) and taking norm square on both sides, then application of (3.18c) yields

$$\|\tilde{z}_{k/k-N} - n_k\|^2 \leq (N+1)\left\{\|\tilde{z}_{k/k} - n_k\|^2 + \sum_{i=0}^{N-1} \|(\hat{\theta}_{k-i} - \hat{\theta}_{k-i-1})'\hat{x}_k\|^2\right\}$$

and

$$\sum_{k=0}^{\infty} r_k^{-2\bar{\varepsilon}}\|\tilde{z}_{k/k-N} - n_k\|^2 \leq (N+1)\left\{\sum_{0}^{\infty} r_k^{-2\bar{\varepsilon}}\|\tilde{z}_{k/k} - n_k\|^2 + \kappa_4 \sum_{k=N_4+1}^{\infty} \frac{\|\hat{x}_k\|^2}{r_k^{1+\bar{\varepsilon}}} \sum_{i=0}^{N-1} \sum_{k=0}^{N_4} \|(\hat{\theta}_{k-i} - \hat{\theta}_{k-i-1})'\hat{x}_k\|^2\right\}$$

where κ_4 is some finite constant and $N_4 \geq N_3$ is some integer. In view of (3.25b) and the lemma A1 the right-hand side of the above inequality is finite, thus

$$\sum_{0}^{\infty} r_k^{-\bar{\varepsilon}} \|\tilde{z}_{k/k-N} - n_k\|^2 < \infty \qquad \text{a.s.} \qquad (3.27)$$

The application of Kronecker lemma and the minimum phase condition (2.13) then implies $\limsup_{k\to\infty} \frac{r_k}{k} < \infty$ a.s., as for the case of adaptive controller based upon one-step ahead prediction, thus also establishing (3.18a) for the case of the N-step ahead adaptive control. Substitution of (3.18a) in (3.25b) and (3.27) yields (3.18d) and (3.18e). Due to (3.9b) which also remains valid for the N-step ahead case, r_k in (3.18c) can be replaced by $\bar{r}_k$. Results C of the theorem are established exactly as in Theorem 3.2, details being omitted here.

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Remarks

1. As can be inferred from the proof of Theorem 3.1, condition (2.14a) in the absence of the stability condition (2.11) is sufficient to guarantee that the state estimation error $\tilde{x}_k$ and a posteriori prediction error $\tilde{z}_{k/k}$ converges to zero in strong sense at a rate $1/k$.

2. For the convergence of $\tilde{z}_{k+N/k}$ however in the strong sense and at a rate $1/k$, additional condition (2.14b) is required. Note that a sufficient condition for (2.14b) is that $\limsup_{k\to\infty} \|x_k\|$ is finite. Other sufficient condition is where $\liminf_{k\to\infty} \bar{r}_k = \infty$ and $\|\hat{x}_k\|^2$ or $\|x_k\|^2$ does not increase faster than exponentially say.

3. The conditions (2.14) are also sufficient for the convergence of tracking error to zero at a rate $1/k$ in case of N-step ahead adaptive controllers.

4. For the convergence of prediction/tracking error to zero at a rate $1/k$ it is not necessary that $\bar{r}_k \to \infty$. In fact $\bar{r}_k$ may very well remain finite in the limit, thus the so-called persistency of excitation condition [11] is not satisfied.

5. In the absence of persistency of excitation, $\bar{r}_k \not\to \infty$, $\hat{\theta}_k$ does not converge to θ, however, the norm square of the parameter estimation error $\|\tilde{\theta}_k\|^2$ is of the order of $1/\bar{r}_k$. Moreover if $\bar{r}_k \to \infty$ then $\hat{\theta}_k \to \theta$ a.s. Further, for the convergence of parameter, condition (2.14b) is not required.

6. For the parameter convergence it is not important that $\bar{r}_k$ approaches ∞ at a certain rate say as k. In fact it may do so at a very slow rate say, $\log \log k$. In the particular case when $\bar{r}_k$ increases at a rate k, then $\|\tilde{\theta}_k\|^2$ decreases as $1/k$.

7. For the closed loop adaptive control, the regularity conditions need interpretation. To develop some intuition for these, consider the case of single-input single-output ARMAX model. Then $x_k' = [z_{k-1} \cdots z_{k-n}, u_{k-1} \cdots u_{k-n}, \omega_{k-1} \cdots \omega_{k-n}]$ which may be written as $x_k' = [(z^*_{k-1} + \tilde{z}_{k/k-1}), \cdots (z^*_{k-n} + \tilde{z}_{k-n,k-n-1}), u_{k-1} \cdots u_{k-n}, \omega_{k-1} \cdots \omega_{k-n}]$. In view of the minimum phase condition, u_k is obtained as the output of a stable system with inputs ω_k and $(z^*_k + \tilde{z}_{k/k-1})$. If $\tilde{z}_{k/k-1}$ may be shown to converge to zero in some sense independent of the condition (2.14), then (2.14) will be implied by a certain regularity condition on a vector $\phi_k = [z^*_{k-1} \cdots z^*_{k-n}, \omega_{k-1} \cdots, \omega_{k-n}]'$. However the theory of this paper shows only the convergence of $\tilde{z}_{k/k}$ independent of (2.14). This problem is solved fully in a subsequent paper where somewhat different convergence analysis using a priori prediction error method and the truncation of various estimates is used to establish the complete results for the adaptive control.

4. CONCLUSIONS

The paper has presented the analysis demonstrating under the usual conditions of the literature, the convergence of the adaptive N-step ahead prediction schemes based upon the standard least square algorithm for parameter estimation, the convergence of the prediction error being understood in the Cesaro sense. Under a regularity condition on the signal model, it has been further proved that the convergence of the prediction error and also that of the tracking error in the case of adaptive control, is also in the strong sense, with the rate of convergence being asymptotically arithmetic. Under an additional persistency of excitation condition of a trivial nature, the consistency of parameter estimates is also established. The dependence of the rate of parameter convergence on the "degree" of persistency of excitation, a scalar parameter of the signal model, is also established. This parameter is just the trace of certain matrix in the signal model and has the intuitive content of signal power.

For the prediction problem, the regularity condition is a very natural condition on the signal model which is already assumed to be stable. For the adaptive control problem, this condition needs an interpretation in terms of the noise and the trajectory z_k^*, the external inputs to the closed loop controller. In a subsequent paper using a priori prediction error method, the convergence analysis for the case of adaptive control is established with the regularity condition expressed directly in terms of the noise ω_k and the trajectory z_k^*.

The convergence results of this paper have applicability to the adaptive control of the general non-minimum phase plants of [22].

APPENDIX

Lemma A1. For an arbitrary $\bar{\varepsilon} > 0$, with r_k defined as in (2.5c), one obtains

$$\sum_{0}^{\infty} \hat{x}_k' \hat{x}_k \; r_k^{-(1+\bar{\varepsilon})} < \infty$$

Proof: Select an integer m such that $\bar{\varepsilon} \geq \frac{1}{2^m}$ and define $N = 2^m$, then the following straightforward manipulations yield the desired result,

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{\hat{x}_k' \hat{x}_k}{r_k^{1+\bar{\varepsilon}}} &\leq \sum_{k=0}^{\infty} \frac{\hat{x}_k' \hat{x}_k}{r_k^{(1+1/N)}} \\
&= \sum_{k=0}^{\infty} \frac{(r_k - r_{k-1})}{r_k^{(1+1/N)}} \\
&\leq \sum_{k=0}^{\infty} \frac{2^m r_k^{(1-1/N)} (r_k^{1/N} - r_{k-1}^{1/N})}{r_k^{(1+1/N)}} \\
&\leq 2^m \sum_{k=0}^{\infty} \left(\frac{1}{r_{k-1}^{1/N}} - \frac{1}{r_k^{1/N}} \right) \\
&\leq 2^m \, r_{-1}^{-1/N} < \infty
\end{aligned}$$

Note that to obtain the second inequality above, we repeatedly use the identity $(a-b) = (\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b})$; $a,b \geq 0$. $\nabla \nabla \nabla$

Lemma A2. For the update of $\hat{P}_k$ given by (2.5c,d),

$$\sum_{k=0}^{\infty} \frac{\hat{x}_k' \hat{P}_k \hat{x}_k}{r_k} < \infty$$

Furthermore, when $\limsup_{k \to \infty} \frac{r_k}{k} < \infty$ a.s., then one obtains

(ii) $\lim_{k\to\infty} \hat{x}_k'\hat{P}_k\hat{x}_k = 0$ a.s., $\lim_{k\to\infty} \hat{x}_k'\hat{P}_{k-1}\hat{x}_k = 0$ a.s.

(iii) $\lim_{k\to\infty} \hat{x}_{k+i}'\hat{P}_k\hat{x}_{k-j} = 0$ a.s. for all finite i,j

Proof: Results (i), (ii) follow from Reference [10]. To obtain (iii) use the following identity (matrix inversion lemma) repeatedly.

$$\hat{P}_k = \hat{P}_{k+1} + \hat{P}_{k+1}\hat{x}_{k+1}(1 - \hat{x}_{k+1}'\hat{P}_{k+1}\hat{x}_{k+1})^{-1}\,\hat{x}_{k+1}'\hat{P}_{k+1}$$

For example, for $i = 2$, $j = 0$

$$\hat{x}_{k+2}'\hat{P}_k\hat{x}_k = \hat{x}_{k+2}'\hat{P}_{k+1}\hat{x}_k + (\hat{x}_{k+2}'\hat{P}_{k+1}\hat{x}_{k+1})(\hat{x}_{k+1}'\hat{P}_{k+1}\hat{x}_k)\{1 - \hat{x}_{k+1}'\hat{P}_{k+1}\hat{x}_{k+1}\}^{-1}$$

Now

$$|\hat{x}_{k+2}'\hat{P}_{k+1}\hat{x}_k| \le \frac{1}{2}\,[\hat{x}_{k+2}'\hat{P}_{k+1}\hat{x}_{k+2} + \hat{x}_k'\hat{P}_{k+1}\hat{x}_k]$$

and

$$|\hat{x}_{k+2}'\hat{P}_{k+1}\hat{x}_{k+1}| \le \frac{1}{2}\,[\hat{x}_{k+2}'P_{k+1}\hat{x}_{k+2} + \hat{x}_{k+1}'P_{k+1}\hat{x}_{k+1}]$$

Application of these inequalities and result (ii) yields (iii) for this case. The result for general i and j is similarly obtained.

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The paper is concerned with the almost sure convergence of the adaptive parameter estimation, N-step ahead prediction and control algorithms based upon standard least square algorithm. With the usual stability and passivity assumptions for the prediction problem, it is demonstrated that the state estimation and the N-step ahead prediction errors converge to the optimum such errors achievable with known plant parameters, in the Cesaro sense. An additional regularity assumption on the signal model establishes the result that
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the state estimation and prediction errors also converge in the strong sense at an asymptotically arithmetic rate. Under an additional persistency of excitation condition it is shown that the parameter estimation error converges to zero at a rate specified by the degree of excitation. The persistency of excitation condition being of a trivial nature is also a necessary condition for parameter convergence. With the regularity condition holding, the convergence is also established for the adaptive control algorithms, e.g., self tuning regulators under the usual minimum phase restriction on the plant. In this case the tracking error equals the N-step ahead prediction error and thus converges to its optimum value at an asymptotically arithmetic rate.

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