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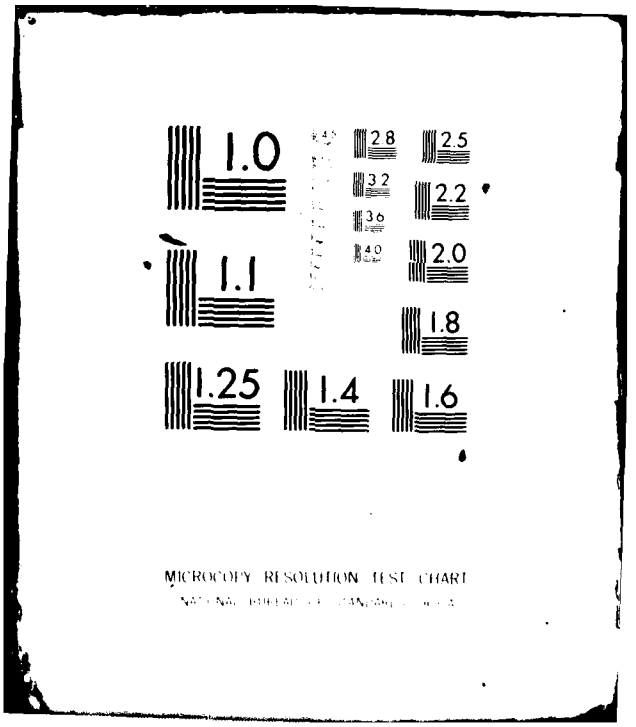

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ON THE RATES OF CONVERGENCE OF THE FINITE ELEMENT METHOD

I. Babuška
University of Maryland*

B. A. Szabo
Washington University

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ABSTRACT

The rate of convergence of the finite element method is a function of the strategy by which the number of degrees of freedom are increased. Alternative strategies are examined in the light of recent theoretical results and computational experience.

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1. Introduction

In finite element analysis convergence can be achieved in several different ways, nevertheless it is useful to distinguish among three basic modes of convergence: (1) The basis functions for each finite element can be fixed and the diameter of the largest element, h_{max}, allowed to approach zero. This mode is called h-convergence and its computer implementation the h-version of the finite element method. (2) The finite element mesh can be fixed and the minimal order of (polynomial) basis functions, p_{min}, allowed to approach infinity. This mode is called p-convergence and its computer implementation the p-version of the finite element method. (3) Mesh refinement can be combined with increments in the order of polynomial basis functions. There are many possible variations within each mode and there are other convergence processes as well. For example, we may concurrently decrease h_{max} and allow Poisson's ratio to approach the value of 1/2 to obtain the limiting case for incompressible solids.

The fact that convergence occurs has been the basis for justification of the finite element method but, as far as state of the art finite element analysis is concerned, convergence is not actually attempted in the computational process. Analysts are generally concerned with some specific finite element mesh and a corresponding fixed set of basis functions. The question of whether the choice of mesh and basis functions is adequate for the purposes of an analysis is not addressed directly. Rather, the analyst relies on his judgement and experience to ensure that the mesh is sufficiently fine or the polynomial orders are sufficiently high so that the error of analysis is small. In other words, finite element solutions are intended to be in the asymptotic range either with respect to $h_{max} \to 0$ or $p_{min} \to \infty$. Because the analysts' judgement is not always reliable, there is

a growing interest in adaptivity. Adaptivity is a procedure for efficient reduction of error on the basis of data already computed.

The most efficient error reduction technique is that for which the path on the error versus cost diagram is the steepest. If we simplify the problem and assume that cost is a simple ascending function of the number of degrees of freedom then the most efficient error reduction process is the convergence mode characterized by the highest rate of convergence. In reality the cost depends on other factors as well, some of which are difficult to quantify, nevertheless it is useful to establish asymptotic rates of convergence for the various modes.

It is possible to make statements about asymptotic rates of convergence a priori, i.e. without actually performing the computations, in terms of a property of the approximated function, called "smoothness". In this paper we present the definition of smoothness of functions; summarize the available theoretical knowledge concerning the asymptotic rates of the various modes of convergence and present example problems from two-dimensional elasticity. Finally we present some general conclusions concerning expected relative computational efficiencies and aspects of reliability of the various modes of convergence.

2. Basic Notation and Preliminaries

Throughout the paper, a point in the plane will be denoted by $x \equiv (x_1, x_2)$. Ω will represent a polygonal domain with vertices A_i and boundary Γ which is the union of (straight) line segments γ_i. The internal angles will be represented by θ_i. This notation is shown in Fig. 1.

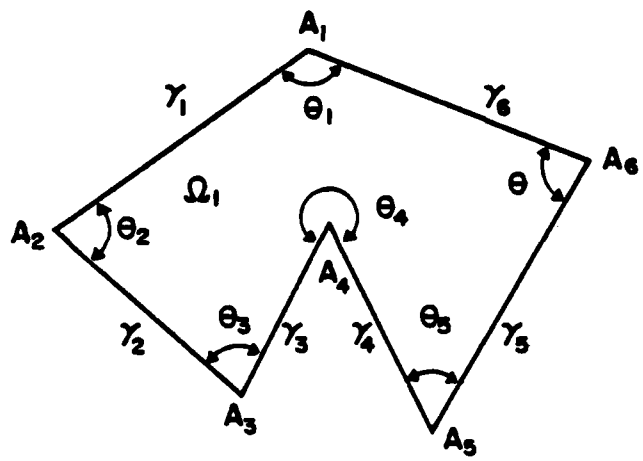

Fig. 1

Notation for the polygonal plane domain Ω

We shall be concerned with functions defined on Ω. It will be necessary to classify these functions with respect to their smoothness. In particular, we shall say that $u \in H^k(\Omega)$, $k \geq 0$ integer, when:

$$||u||^2_{H^k(\Omega)} \equiv \sum_{0 \leq \ell_1 + \ell_2 \leq k} \int_\Omega \left| \frac{\partial^{\ell_1+\ell_2}}{\partial x_1^{\ell_1} \partial x_2^{\ell_2}} u \right|^2 dx < \infty \qquad (1)$$

where $dx \equiv dx_1 dx_2$. The index k is a measure of the smoothness of u in that it indicates how many square integrable derivatives u has. We shall say that a function is smooth when k has a high value.

As an example, let us consider the case of linear elastic fracture mechanics, mode I: The displacement vector components in plane strain are [1]:

$$u_1 = \frac{K_I}{G}\left(\frac{r}{2\pi}\right)^{1/2} \cos\left(\frac{\theta}{2}\right)\left[1-2\nu+\sin^2\frac{\theta}{2}\right] + O(r^{3/2}) \tag{2a}$$

$$u_2 = \frac{K_I}{G}\left(\frac{r}{2\pi}\right)^{1/2} \sin\left(\frac{\theta}{2}\right)\left[2-2\nu-\cos^2\frac{\theta}{2}\right] + O(r^{3/2}) \tag{2b}$$

in which r and θ are polar coordinates, centered on the crack tip, G and ν are material constants and K_I is the stress intensity factor. In this case both u_1 and u_2 belong to $H^1(\Omega)$. Functions u_1 and u_2 are members of a class of functions which have great importance in finite element analysis. The general form of the class is:

$$v = \mathrm{Re}\,[r^{\gamma}(\log r)^{\delta} f(\theta)] \qquad \mathrm{Re}[\gamma] \equiv \alpha > 0,\ \delta \geq 0 \tag{3}$$

in which $f(\theta)$ is a smooth function of θ. The origin of the polar coordinates is typically located at a vertex of the domain Ω.

The definition of the space of functions $H^k(\Omega)$ in eq. (1) is for k integer only, nevertheless it is possible to generalize the notion of $H^k(\Omega)$ to $k \geq 0$ fractional. See, for example, [2].

The function v, defined by eq. (3), belongs to all spaces $H^s(\Omega)$, $s < \alpha+1$. In general, v does not however belong to $H^{\alpha+1}(\Omega)$. Thus, u_1 and u_2 defined in eq. (2) belongs to $H^{3/2-\varepsilon}(\Omega)$, $\varepsilon > 0$ arbitrarily.

The foregoing definitions provide for establishing two fundamental notions: If a function belongs to $H^k(\Omega)$ and the highest derivative in the strain energy expression is not greater than k, then the function has finite strain energy. Secondly, the fractional index k allows us to quantify the idea of "smoothness" of functions. For example if Ω contains the point $r = 0$ then $w_1 = r^{1/3}\cos\theta$ belongs to all spaces $H^s(\Omega)$, $s < 4/3$, and is smoother than $w_2 = r^{1/5}\cos\theta$ which belongs to all spaces $H^s(\Omega)$, $s < 6/5$.

Let $\Gamma^* = \bigcup_j \gamma_j$, $j = 1,2,\ldots,m$ be the union of some sides of the polygon Ω. Then if u belongs to $H^k(\Omega)$, $k \geq 1$, and $u = 0$ on Γ^*, then we write: $u \in H^k_{\Gamma^*}(\Omega)$. In the case of vector functions $u = (u_1,u_2)$, $u \in H^k(\Omega)$ *means that* $u_i \in H^k(\Omega)$, $i = 1,2$.

The model problems to be discussed have been taken from two-dimensional elasticity, assuming plane strain conditions. E and ν refer, respectively, to the modulus of elasticity and Poisson's ratio, $(0 \leq \nu < 1/2)$. Function $u = (u_1,u_2)$ is the displacement vector function. The strain energy function is defined by:

$$W(u) = \frac{E}{2(1-2\nu)(1+\nu)} \int_\Omega \left[(1-\nu)\left(\frac{\partial u_1}{\partial x_1}\right)^2 + 2\nu \frac{\partial u_1}{\partial x_1} \cdot \frac{\partial u_2}{\partial x_2} + (1-\nu)\left(\frac{\partial u_2}{\partial x_2}\right)^2 + \frac{1-2\nu}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)^2\right] dx \tag{4}$$

Let us now define on $\Gamma-\Gamma^*$ (the part of the boundary over which no displacements are prescribed) the function $T \equiv (t_1,t_2)$, representing prescribed tractions, and on Γ^* the function $\Phi \equiv (\varphi_1,\varphi_2)$, representing prescribed displacements. We assume that $\int_{\Gamma-\Gamma^*} t_i^2\, ds < \infty$ and there exists $\hat{\varphi}_i \in H^1(\Omega)$ and $\hat{\varphi}_i = \varphi_i$ on Γ^*, $i=1,2$. If $\Gamma^* \neq \emptyset$, then we have the well known uniqueness theorem:

<u>Theorem 1</u>. There exists exactly one function $u \equiv (u_1,u_2)$, $u_i \in H^1(\Omega)$ such that:

(a) $u_i = \varphi_i$ on Γ^*

(b) u minimizes the potential energy functional

$$\pi(u) = W(u) - \sum_{i=1}^{2} \int_{\Gamma-\Gamma^*} t_i u_i ds \tag{5}$$

among all vector functions $u = (u_1,u_2)$, $u_i \in H^1(\Omega)$ such that $u_i = \varphi_i$ on Γ^*. The proof of this theorem is given several textbooks, see for example, [3]. If $\Gamma^* = \emptyset$ and T satisfies the usual equilibrium conditions then u exists and is determined by theorem 1, except for arbitrary rigid body motion.

Assuming that T is smooth on every side $\gamma_i \in \Gamma-\Gamma^*$ and Φ is smooth on every side $\gamma_i \in \Gamma^*$, the solution u that minimizes $\pi(u)$ in theorem 1 is smooth on Ω except for possible singular behavior at the corners A_i of Ω. Such singular behavior is characterized by the coefficients α and δ in eq. (3) which depend on the angle θ_i and whether the two sides adjacent to A_i belong to Γ^* or $\Gamma-\Gamma^*$. More detailed discussion of this point is given in [4,5,6]. The important observation is that the

space $H^k(\Omega)$ to which the solution u belongs is determined primarily by the vertex angles of Ω and the boundary conditions imposed on Ω, provided that the loading and prescribed displacements are smooth.

The strain energy W(u) defined in eq. (4) has finite value for any $u \in H^1(\Omega)$. Furthermore, $\sqrt{W(u)}$ has all of the properties of a norm if $u \in H^1_{\Gamma^*}(\Omega)$ and $\Gamma^* \neq \emptyset$. In the following the norm $||u||_E = \sqrt{W(u)}$ plays essential role in measuring the error of finite element approximation. In fact $||u||_E$ is equivalent to the norm $||u||_{H^1(\Omega)}$ i.e. there exist C_1, C_2, independent of u, such that:

$$C_1||u||_{H^1(\Omega)} \leq ||u||_E \leq C_2||u||_{H^1(\Omega)}$$

3. The Finite Element Method

We shall consider two different families of meshes: The first family, denoted by γ, is the family of the usual triangularizations with bounded aspect ratio. A specific triangularization shall be denoted by τ. Thus: $\tau \in \gamma$. A specific triangle shall be denoted by τ_i. Thus: $\tau_i \in \tau \in \gamma$. The largest side of τ_i is denoted by h_i and the maximum of h_i: $h_{max}(\tau) = \max_i h_i$; the minimum of h_i: $h_{min}(\tau) = \min_i h_i$.

<u>Definition</u>: The family γ is quasiuniform if for any $\tau \in \gamma$ we have:

$$\frac{h_{max}(\tau)}{h_{min}(\tau)} \leq K < \infty$$

The other family to be considered is a family of square meshes G (with possible refinement). In this case we shall assume that all of the sides of Ω are parallel with the coordinate axes. Although this assumption may seem to be overly restrictive from the practical point of view, in reality it is not: Any plane domain can be subdivided into rectangular figures with curved (or straight) sides such that each figure can be mapped onto the unit square by smooth variable transformations.

An example of the square meshes is given in fig. 2. The nodal points of the mesh which are either lying on Γ or are common vertices of four squares are called regular, the others are called irregular. The term: "quasiuniform mesh" is applied to the square meshes in the same way as before, i.e. it means that the ratio of the largest element diagonal to the smallest is bounded as the size of the largest element approaches zero.

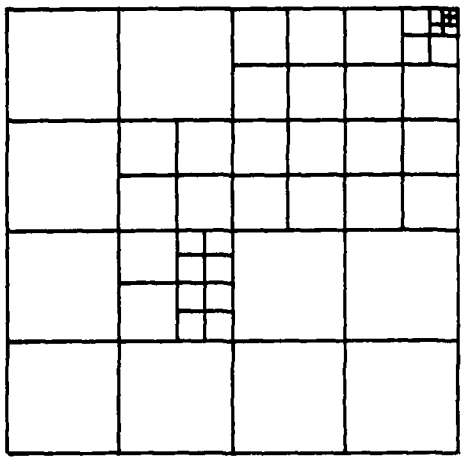

Fig. 2

Square mesh with refinement

Over each finite element we approximate $u = (u_1, u_2)$, $u \in H^1(\Omega)$, with polynomials of order p and we assume that $u \in H^1(\Omega)$. The polynomials assume the value of $\Phi = (\varphi_1, \varphi_2)$ on Γ^*. The polynomial order p may vary from element to element. In the following we shall assume also that the interelement continuity requirements are enforced exactly and minimally, i.e. overconformity and nonconformity are avoided. (The assumption that $u \in H^1(\Omega)$ guarantees conformity). The number of degrees of freedom, after enforcement of the principal boundary conditions, is denoted by N.

We shall denote the set of functions $u \in H^1(\Omega)$ which are polynomials of degree at most p_i on every $\tau_i \in \tau$ and satisfy the principal boundary conditions Φ on Γ^* by $M_{\Gamma*} = M_{\Gamma*}(\tau, p, \Phi)$. (The polynomial order may vary from element to element). Similarlv, we shall define the set of functions u, which are bilinear (i.e. products of linear polynomials in x_1 and x_2) on every element of a square mesh and $u = \Phi$ on Γ^* by $N_{\Gamma*}$.

The finite element solution $u_{FE} \in M_{\Gamma*}$ (respectively $u_{FE} \in N_{\Gamma*}$) is the function that minimizes the energy functional given by eq. (5) over the set $M_{\Gamma*}$ (respectively $N_{\Gamma*}$).

Let u be the exact solution by theorem 1. Then it is not difficult to prove that:

$$||u-u_{FE}||_E^2 = |\pi(u)-\pi(u_{FE})| = |W(u)-W(u_{FE})| \tag{6}$$

i.e. the energy of the error of the finite element solution equals the error of the energy [7].

4. The h and p Versions of the Finite Element Method

In the classical finite element method the polynomial degree is fixed, usually at some low p value (p=1,2, or 3) and the solution is constructed in $M_{\Gamma*}(\tau,p,\Phi)$ such that h_{max} is small. Increased accuracy is achieved by mesh refinement ($h_{max} \to 0$). This strategy for increasing the number of degrees of freedom is called *the h-version of the finite element method.*

We can also fix the mesh τ (with relatively small number of triangles) and increase p, either uniformly or selectively. This strategy for increasing the number of degrees of freedom is called *the p-version of the finite element method.*

The h and p versions can be viewed as special cases of the finite element method which allows increasing the number of degrees of freedom by concurrent refinement of the mesh and increases in p.

5. Model Problems

We now define the model problems used for illustrating the convergence properties of the various strategies used for increasing the number of degrees of freedom in the finite element method.

<u>Problem 1</u>: Square domain under plane strain conditions subjected to imposed shear displacement. Ω is the square, as shown in Fig. 3.

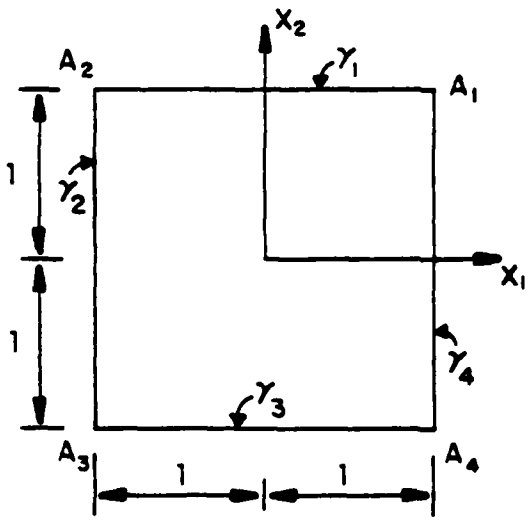

Fig. 3

Square domain, example problem 1.

$$\Gamma^* = \gamma_1 \cup \gamma_3$$

$$\varphi_1 = \begin{cases} +1 & \text{on } \gamma_1 \\ -1 & \text{on } \gamma_3 \end{cases}$$

$$\varphi_2 = 0 \quad \text{on } \gamma_1 \cup \gamma_3$$

$$T = (0,0) \text{ on } \gamma_2 \cup \gamma_4$$

We shall be concerned with the case of $E = 1$, $\nu = 0.3$ and $\nu = 0.4999$. Because x_1 and x_2 are axes of anti-symmetry, we shall compute the finite element solution u_{FE} for the right upper quarter of the domain only. Consequently the number of degrees of freedom shall be related to this quarter only. The estimated exact strain energies for the quarter domain are: $W = 0.130680$ for $\nu = 0.3$ and $W = 0.127035$ for $\nu = 0.4999$.

The solution u has singular behavior in the neighborhood of all vertices A_i of the form given in eq. (3) (with the origin of the polar coordinate system at vertex A_i). We observe that $u \in H^{k-\varepsilon}(\Omega)$, where k depends on Poisson's ratio, as shown in fig. 4. The data in fig. 4 were computed on the basis of reference 4.

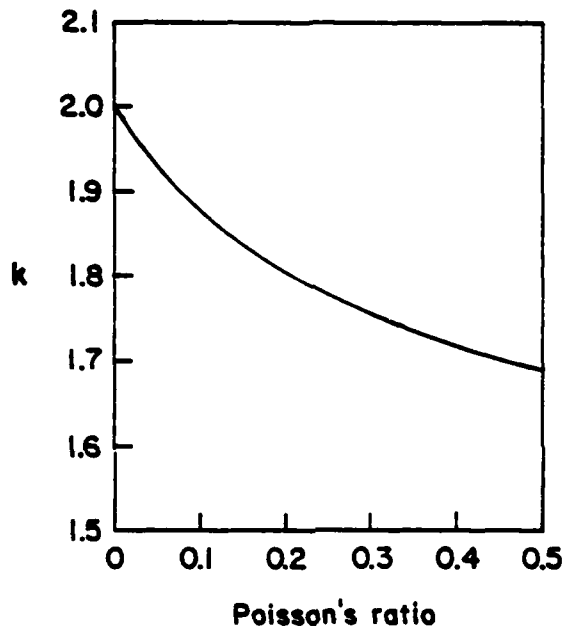

Fig. 4

The smoothness parameter $k \equiv 1+\alpha$ as a function of Poisson's ratio in example problem 1

<u>Problem 2</u>. The edge cracked square panel under plane strain conditions subjected to uniform tension as shown in Fig. 5. In this case $\Gamma^* = \emptyset$.

The boundary conditions are:

$T = (0,0)$ on $\Gamma - \gamma_1 \cup \gamma_6$

$T = (0,1)$ on γ_1

$T = (0,-1)$ on γ_6

We shall be concerned the case of $E = 1$, $\nu = 0.3$ and $\nu = 0.4999$. Because x_1 and x_2 are axes of symmetry, we shall compute the finite element solution u_{FE} for the right upper quarter of the domain only and the number of degrees of freedom shall be related to this domain only. The estimated exact strain energies for the quarter domain are: $W = 0.73422$ for $\nu = 0.3$ and $W = 0.60525$ for $\nu = 0.4999$.

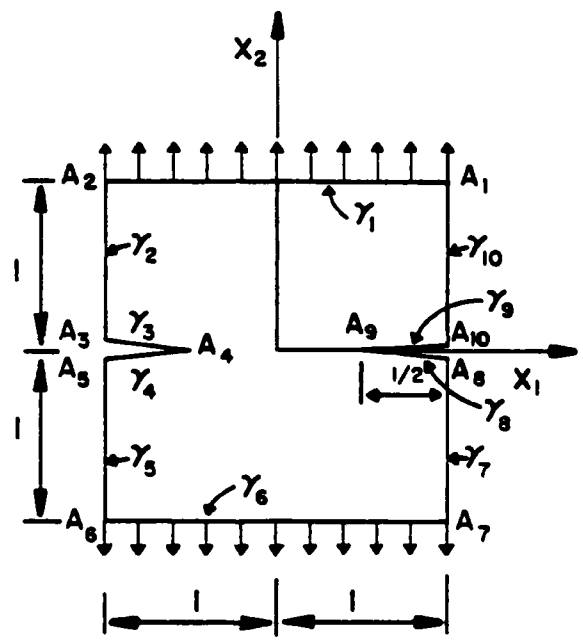

Fig. 5

Edge cracked square panel

The solution u has singular behavior at the crack tip (vertices A_4 and A_9) of the form given by eq's. (2a,b). Thus $u \in H^{3/2-\varepsilon}(\Omega)$. Unlike in example problem 1, the smoothness parameter k is independent of Poisson's ratio.

6. Asymptotic Error Analysis

When comparing alternative approaches in the finite element method, the error vs. cost relationship is of interest, with the error measured in terms of displacements, stresses, stress resultants at specific points, or in terms of energy. The error-cost relationship is often simplified by making the assumption that the cost is some simple function of the number of degrees of freedom. We shall follow the same course and use the error vs. number of degrees of freedom as our basis for comparison. Thus a given solution will be represented by a point on the error vs. N diagram. We shall be concerned with the rate of change of the error in energy with respect to N. N can be increased in various ways: uniform or quasiuniform mesh refinement; non-quasiuniform mesh refinement, uniform or non-uniform change in polynomial order on a fixed mesh, etc. These can be viewed as various "extensions" of the original solution. We shall say that an extension is asymptotic or pre-asymptotic depending on whether the error vs. N relationship is governed by one or more parameters. When N is sufficiently large then the error can be characterized well by a function $\psi(N)$. Function ψ depends in general on the smoothness of the approximated function. Thus the error can be written as:

$$||e||_E = \psi(N) + o[\psi(N)]$$

with:

$$\psi(N) = CN^{-\bar{\alpha}}$$

where $\bar{\alpha}$ $(\bar{\alpha}>0)$ is the governing parameter and $o[\psi(N)]$ represents terms that approach zero as $N \to \infty$ faster than $\psi(N)$. In order to ensure that the various extensions are compared on the same basis, we shall use $\bar{\alpha}$ as our basis for comparison. Thus we shall make the assumption that each extension is in the asymptotic range. This is important from the point of view of reliability: In the asymptotic range the error is not significantly influenced by optional input parameters (such as the kind of mesh divisions used) and the error is generally small.

7. The h-version

We shall now summarize theorems concerning the asymptotic rate of convergence in the h-version of the finite element method.

<u>Theorem 2</u>. Let the exact solution u belong in the space $H^k(\Omega)$. Then for a family of quasiuniform meshes γ and the space of polynomial approximating functions $M_{\Gamma*}(\tau,p,\Phi)$, in which p is fixed and uniform, we have:

$$||u-u_{FE}||_E \leq C(\Omega,k,p,\gamma)N^{-1/2\min(k-1,p)}||u||_{H^k(\Omega)} \tag{7}$$

In eq. (7) the constant C does not depend on u or N [2,8]. The absolute value of the exponent of N is called the asymptotic rate of convergence or simply rate of convergence.

An inverse theorem exists also, which can be summarized as follows: If we were able to observe the asymptotic rate of convergence for a given problem for <u>any quasiuniform triangular mesh</u> and fixed p (assuming complete polynomials) to be 1/2 α (i.e. $\bar{\alpha} = \frac{1}{2}\alpha$):

$$||u-u_{FE}||_E \leq CN^{-1/2\alpha}$$

then we would be able to make the following statements about the exact solution u:

1) If $1 < \alpha < p$ then $u \in H^{1+\alpha-\varepsilon}(\Omega)$, $\varepsilon > 0$ arbitrarily.
2) If $\alpha > p$ then u is a polynomial.
3) If $\alpha = p$ then $u \in H^{1+\alpha}(\Omega)$.

For analysis of the inverse theorem we refer to [9,10].

Theorem 2 and its inverse (summarized in the preceding statements) show that the asymptotic rate of convergence is completely characterized

by the smoothness of u. In most problems of practical importance k is between 1.5 and 2, and u is not a polynomial, hence the asymptotic rate of convergence is governed by k. Theorem 2 indicates that changing p when $p \geq k-1$ will not change the asymptotic rate of convergence, but will affect the accuracy of the solution because C depends on p.

We shall now illustrate theorem 2 on problem 1 with $\nu = 0.3$. The uniform mesh for the quarter domain (with h variable) is shown in Fig. 6.

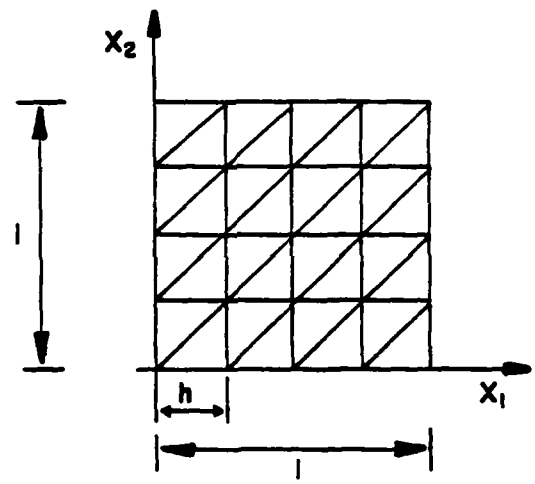

Fig. 6

Uniform triangulation (Example problems 1 and 2).

The computations for the uniform meshes were performed by means of COMET-X, a computer program developed at Washington University. COMET-X implements the p-version of the finite element method through the use of exactly and minimally conforming hierarchic finite elements. The range of polynomials permitted by COMET-X is from 1 to 8 [11,12].

Adaptively constructed non-quasiuniform meshes are shown in Fig. 7. These meshes were generated and the computations performed by FEARS (Finite Element Adaptive Research Solver), a computer program developed at the University of Maryland. FEARS gives a reliable estimate of error of the finite element solution [13].

The relative error is plotted against N on a log-log scale in Fig. 8 for various p values. The slope of these curves for large N is the asymptotic rate of convergence. We note that in Fig. 8 (and in the other figures illustrating the error vs. N relationship) we measure the error by the square of the norm $||\cdot||_E$. See eq. (6).

It is seen that for uniform mesh refinement the error depends on p but the slope is independent of p. With $k \equiv 1+\alpha = 1.76$ from Fig. 4, the slope of 0.76 is as predicted by theorem 2.

It has been proven that if the approximated function u is of the functional form of eq. (3) then there exists a sequence of non-quasiuniform meshes such that the rate of convergence depends only on p, not on γ and δ as would be the case if uniform or quasiuniform meshes were used. The proof is available in [8,10]. The sequence of meshes having this character can be constructed a priori or adaptively. The meshes constructed a priori cause higher approximation error than adaptively constructed ones, however.

As seen in Fig. 8, the error vs. N curve for adaptively constructed meshes approaches the slope of -1 for log-log scale, which is the maximum rate of convergence possible for p = 1.

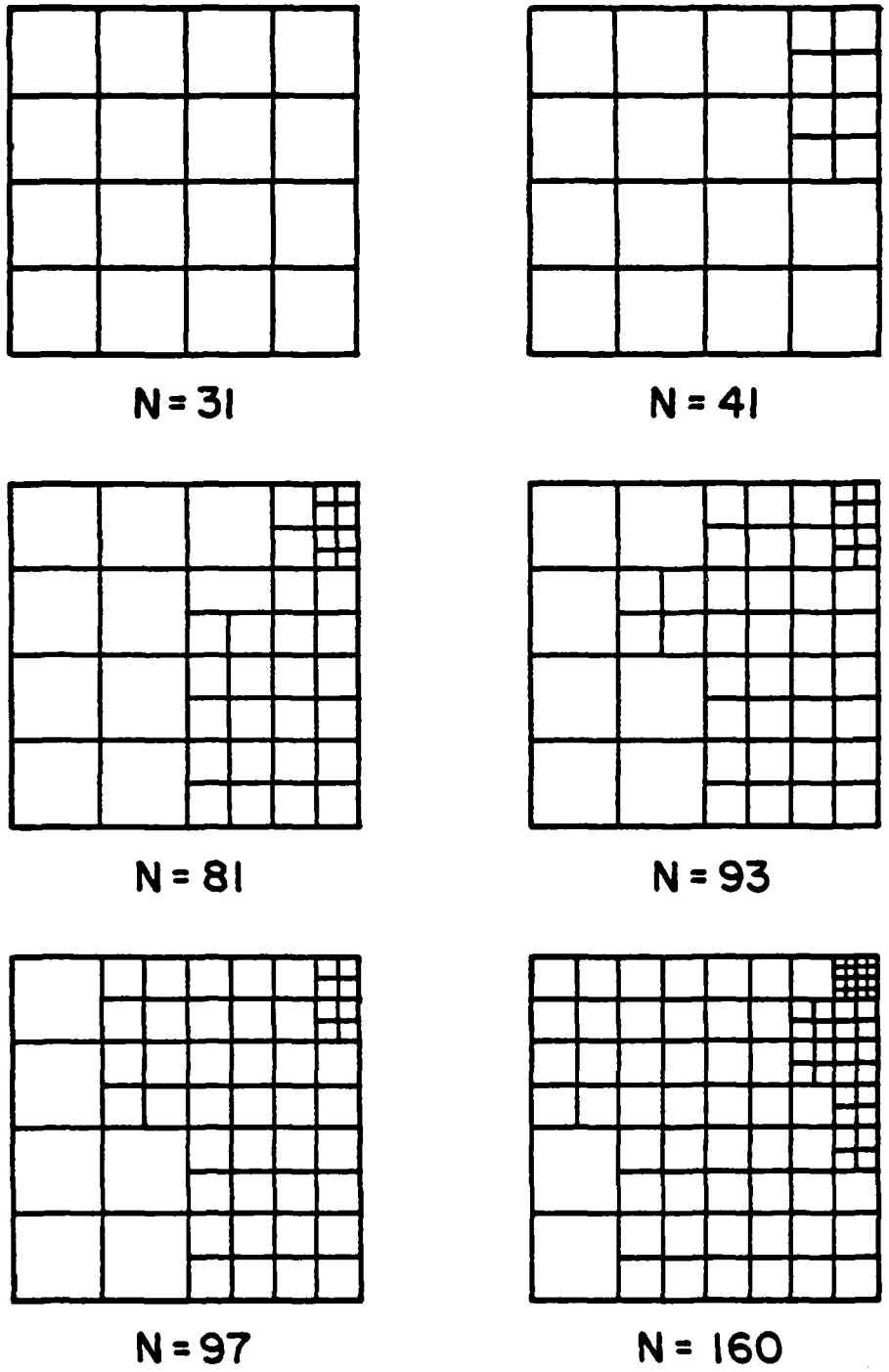

Fig. 7

Adaptively constructed non-quasiuniform meshes
for example problem 1

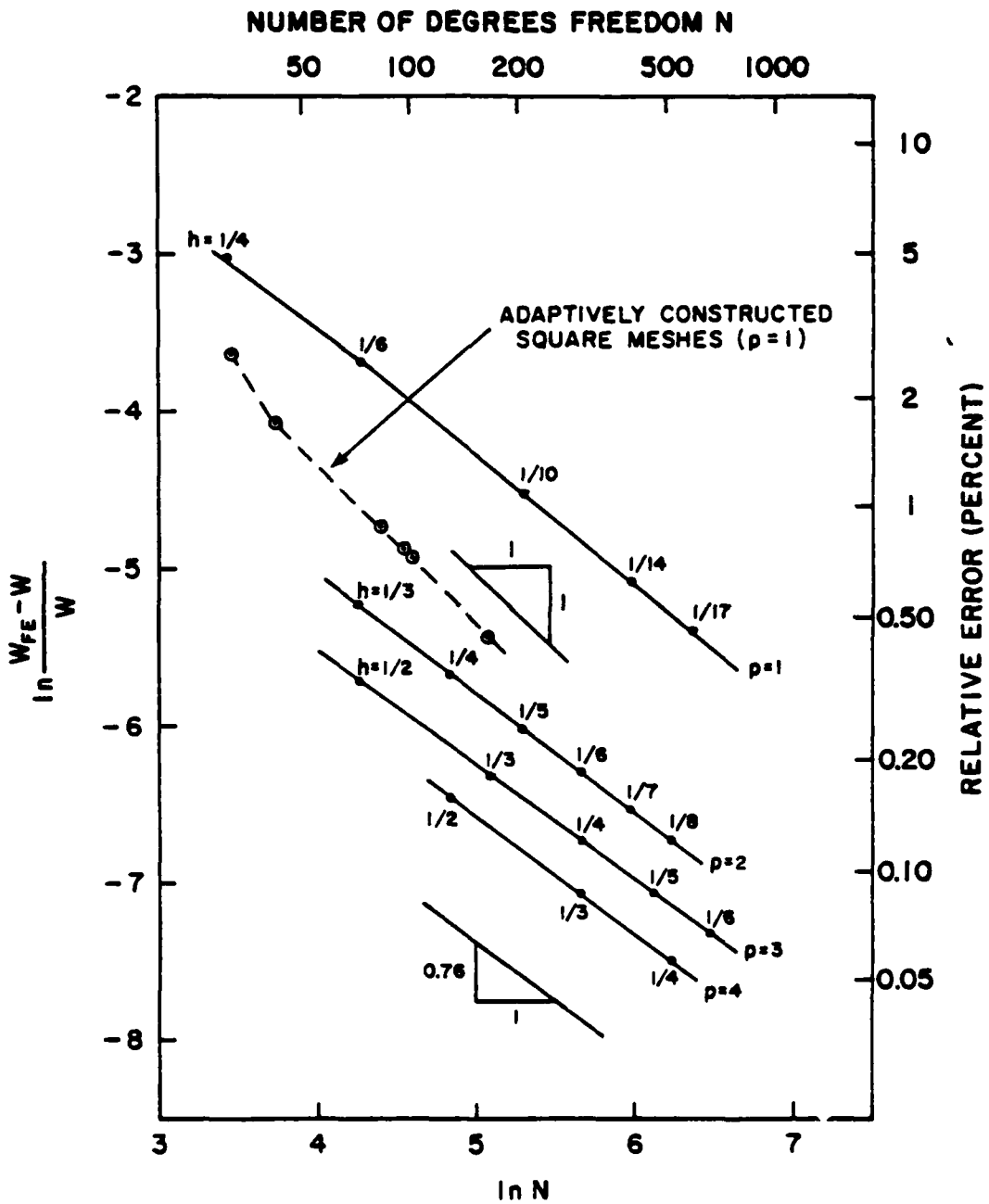

Fig. 8

Example problem 1: Relative error in energy vs. number of degrees of freedom. H-version, Poisson's ratio: 0.3

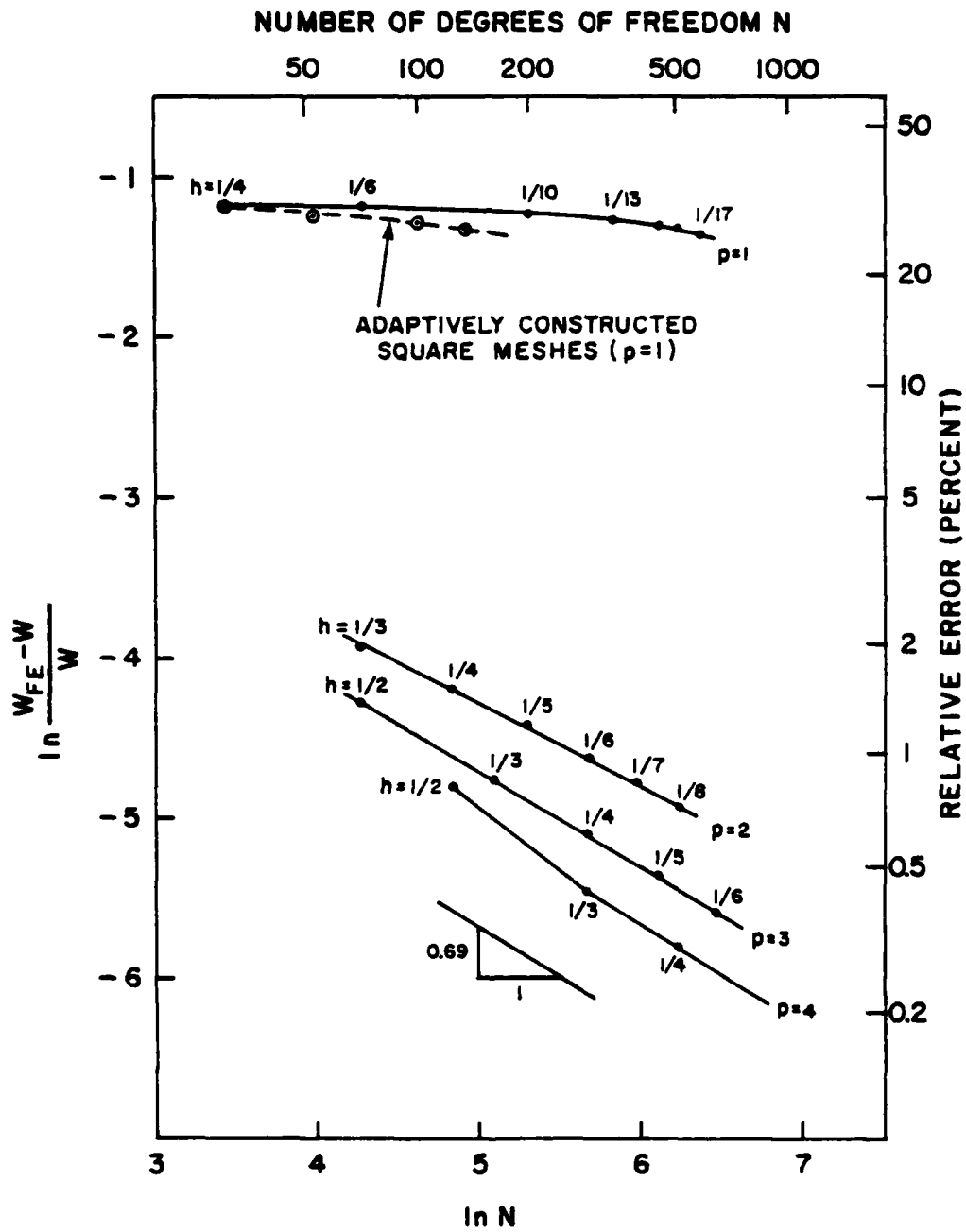

Fig. 9

Example problem 1: Relative error in energy vs. number of degrees of freedom. H-version, Poisson's ratio: 0.4999.

Eq. 7 is valid for all Poisson's ratios $0 \leq \nu < 1/2$, with the constant C possibly dependent on ν. Fig. 9 shows the behavior for $\nu = 0.4999$. We see that the rate of convergence approaches the value of k-1 (in this case 0.69) in accordance with theorem 2 but for $p = 1$ the asymptotic range is entered at high N values (in this case beyond the range of plotted values and probably beyond the round-off limitations of digital computers).

The nearly degenerate case of $\nu \approx 1/2$ deserves special consideration because two parameters are involved; $1/2 - \nu$ and h. The asymptotic theory is applicable only when h is small with respect to $1/2 - \nu$. It is well known that elements with $p = 1$ perform poorly when $\nu \approx 1/2$ and various special approaches, such as reduced integration, have been proposed. The results shown in Fig. 9 indicate that mesh refinement will not reduce the error when $p = 1$, on the other hand use of $p \geq 3$ essentially eliminates this difficulty. Theoretical analysis of this effect is not yet available.

8. The p-version

We shall now review the basic properties of the p-version.

<u>Definition</u>: Let p_{max} denote the largest polynomial order of the basis functions over all finite elements and let p_{min} denote the smallest. A sequence of p-distributions is quasiuniform if:

$$\frac{p_{max}}{p_{min}} \leq \kappa < \infty$$

<u>Theorem 3</u>. Let the exact solution u belong in the space $H^k(\Omega)$. Then for the space of exactly and minimally conforming polynomial approximating functions $M_{\Gamma*}(\tau,p,\Phi)$ in which the triangular mesh τ is fixed and the sequence of p-distributions is quasiuniform, for any $\varepsilon > 0$ we have:

$$||u-u_{FE}||_E \leq C(\Omega,k,\tau,\kappa,\varepsilon)N^{-1/2(k-1)+\varepsilon}||u||_{H^k(\Omega)} \tag{8}$$

Theorem 3 is similar to theorem 2, however here C is independent of p. The proof is given in [14]. Importantly, the inverse theorem, also given in [14], states that if $||u-u_{FE}|| \leq CN^{-1/2\alpha}$ under the conditions of theorem 3, then: (a) $u \in H^{1+\alpha-\varepsilon}(\Omega^*)$ where Ω^* is a subdomain of Ω not containing any of the finite element boundaries; (b) $u \in H^{1+\alpha/2-\varepsilon}(\Omega)$. This is significantly different from the inverse of theorem 2. In particular, if the singular behavior of u is confined to the boundaries of τ (i.e. the singularity is not in Ω^*) then the rate of convergence of the p version is <u>twice</u> the rate of convergence of the h-version, provided that $p \geq k-1$ in the h-version. In other words, the p-version can "absorb" singular behavior at the boundaries of finite elements.

A very important case is when the singularity is at the corner of the domain (and therefore at the vertex of one or more elements). Then we can prove the following theorem:

<u>Theorem 4</u>. Let $u = u_1+u_2$, u_1 having the functional form given in eq. (3) with the origin at the vertex of the domain and $u_2 \in H^k(\Omega)$. Then for triangular meshes:

$$||u-u_{FE}||_E \leq C(\tau,k,\alpha,\varepsilon)\left[N^{-\alpha+\varepsilon}||u_1||_{H^o(\Omega)} + N^{-1/2(k-1)+\varepsilon}||u_2||_{H^k(\Omega)}\right] \tag{9}$$

A conjectural statement concerning the existence of theorem 4 and a numerical demonstration of this theorem was given in [15]. The proof is given in [14].

In summary, the p-version cannot have lower rate of convergence than the h version based on quasiuniform meshes. The rate of convergence of the p-version is twice that of the h-version when the singularity is at element boundaries and quasiuniform meshes are used. On the other hand the h-version, with the use of optimally refined meshes (which are not quasiuniform) and sufficiently high p can have higher rate of convergence than the p-version.

Fig. 10 illustrates that the rate of convergence of the p-version is twice the rate of convergence of the h-version based on uniform mesh refinement in the case of example problem 2. The fact that the asymptotic range is entered at low p values is noted.

In Fig. 10 we also show the performance of adaptively constructed square meshes for p = 1. Once again we see that the error vs. N curve approaches

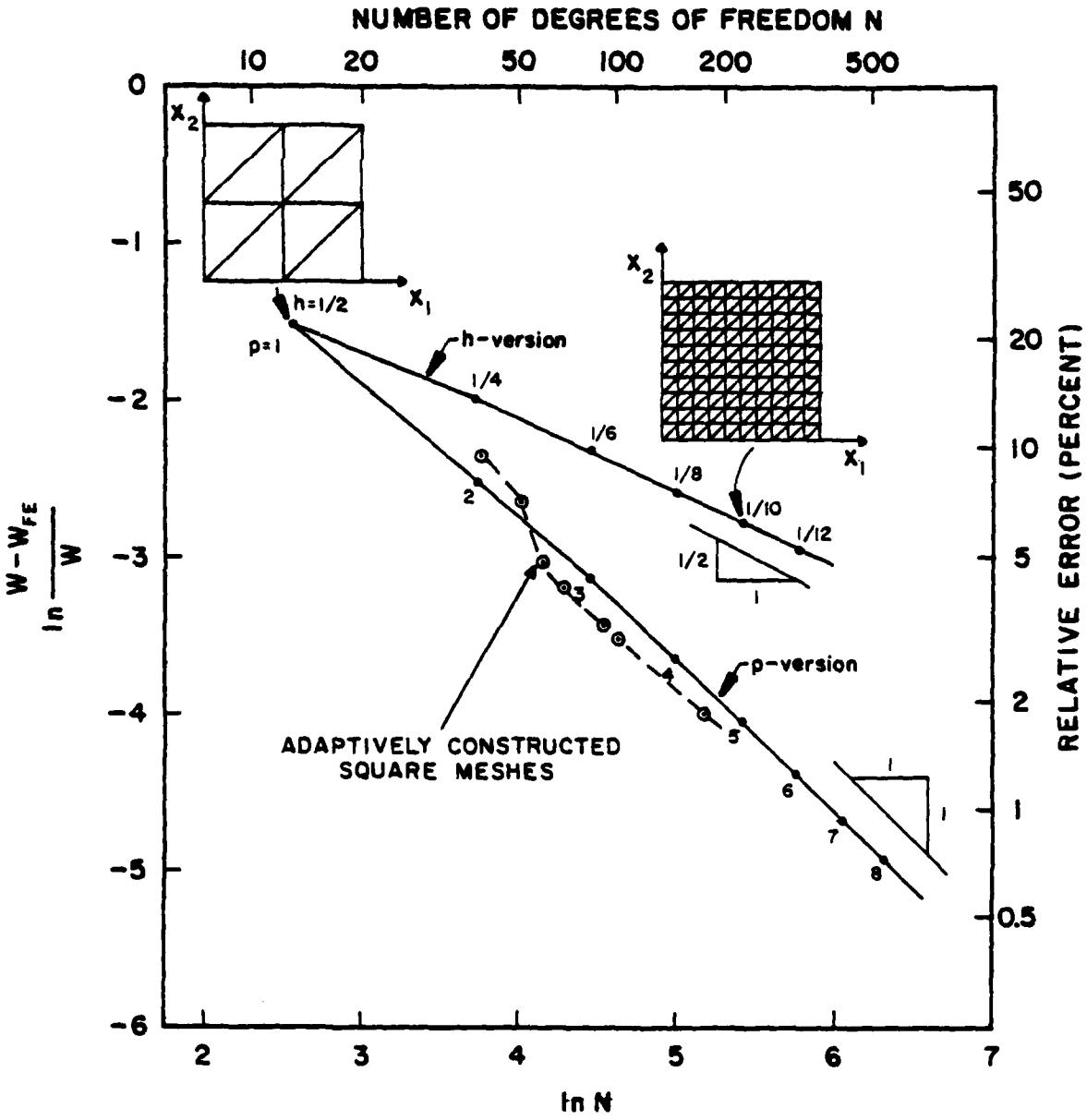

Fig. 10

Example problem 2: Relative error in strain energy vs. number of degrees of freedom. Comparison of the h and p versions. Poisson's ratio: 0.3

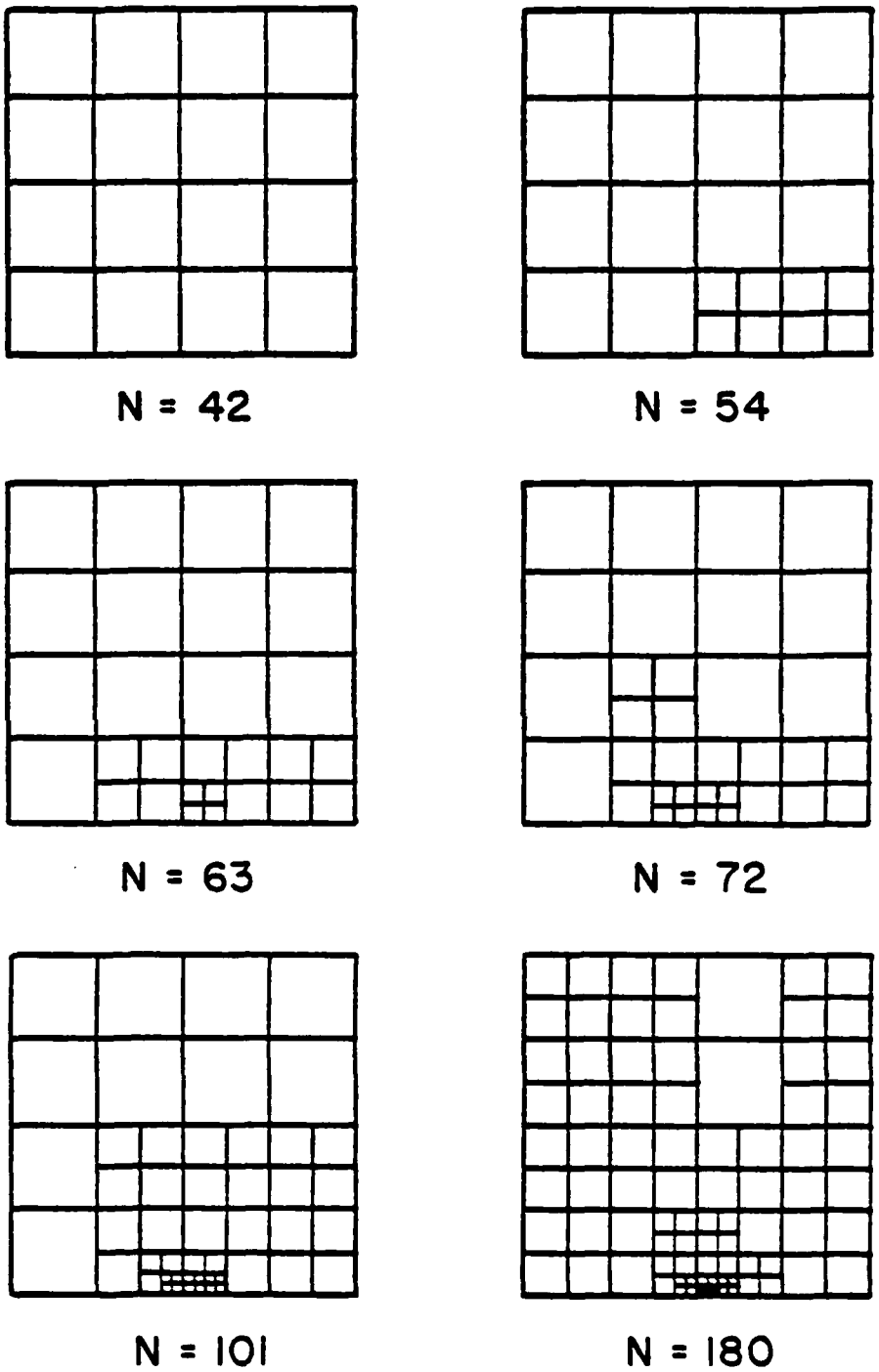

Fig. 11

Adaptively constructed non-quasiuniform meshes
for example problem 2.

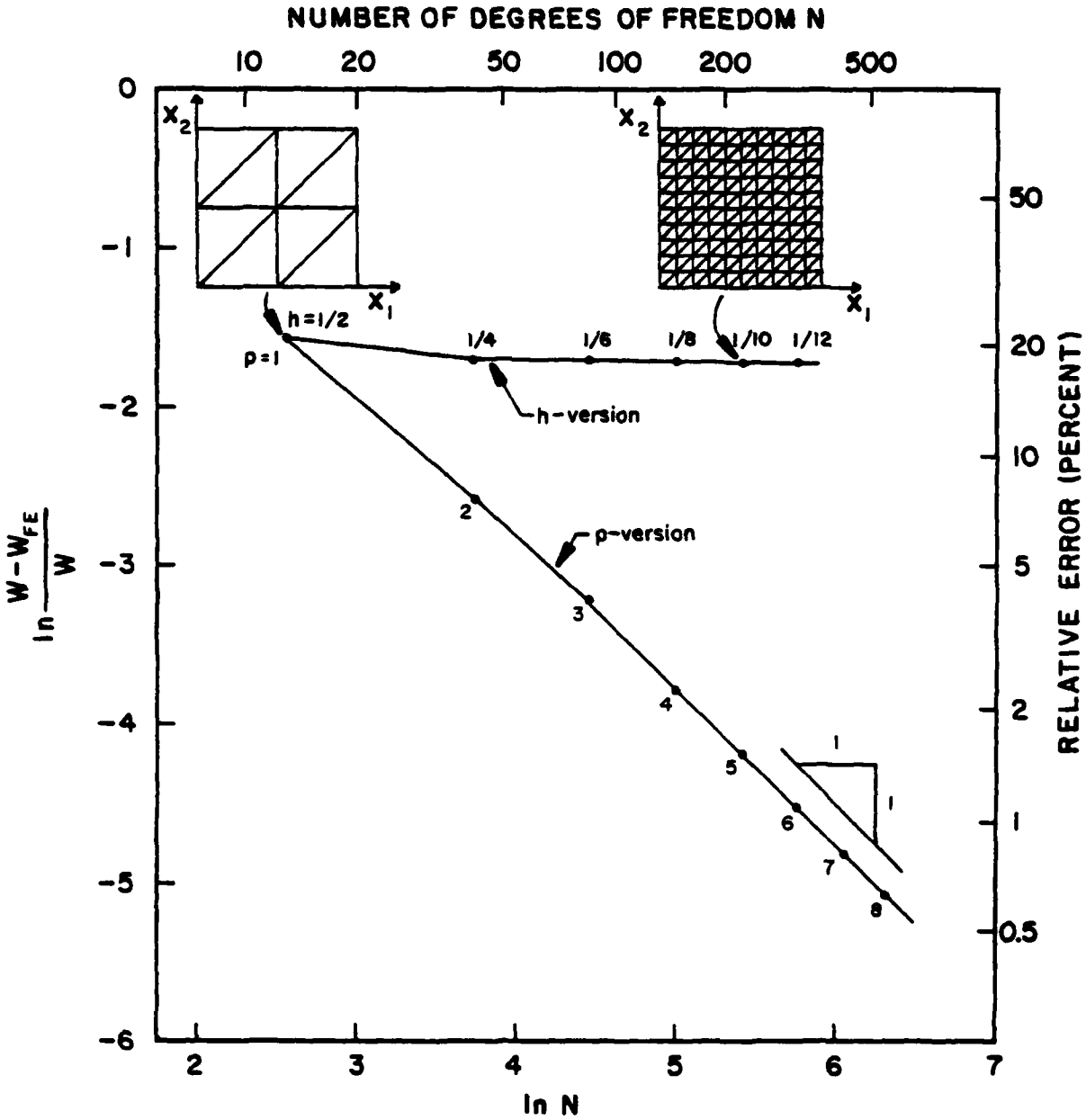

Fig. 12

Example problem 2: Relative error in strain energy vs. number of degrees of freedom. Comparison of the h- and p-versions. Poisson's ratio: 0.4999.

the slope of -1 on log-log scale for large N, which is the largest asymptotic rate possible for p = 1. The larger pre-asymptotic rate is due to the fact that at the beginning a very small number of additional elements at the crack tip increase the accuracy significantly. Subsequently the domain must be refined away from the singularity, as shown in Fig. 11, which causes the rate of convergence to become slower.

Fig. 12 illustrates that the point of entry into the asymptotic range is not affected in the p-version to a significant degree when Poisson's ratio is close to 0.5, but is strongly affected in the h-version when p = 1. As in example 1, mesh refinement is not effective for error reduction.

9. Rate of Convergence when p_{max} is Increased with Concurrent Non-quasiuniform Mesh Refinement.

In this section we analyze some convergence properties of the finite element method under the condition that mesh refinement is accompanied by increases in p. First we quote the following basic theorem:

<u>Theorem 5</u>: Let $u \in H^k(\Omega)$; let γ be a family of quasiuniform triangular meshes and let u_{FE} be the finite element solution based on the space of piecewise polynomial functions $M_{\Gamma*}(\tau,p,\Phi)$ with $\tau \in \gamma$; p quasiuniformly distributed, and $p_{min} \geq k-1$. Then, for any $\varepsilon > 0$

$$||u-u_{FE}||_E \leq C(\Omega,k,\varepsilon)N^{-\frac{k-1}{2}+\varepsilon}||u||_{H^k(\Omega)} \tag{10}$$

The proof of this theorem is given in [16]. The significance of this theorem is that it joins theorems 3 and 4, making the constant C independent of both p and τ. An inverse theorem exists but is not known as yet.

In Sections 7 and 8 we demonstrated that it is possible to generate proper sequences of mesh such that the rate of convergence is independent of the singularity but depends on the p-distribution. In the case of corner singularities (of the form given by eq. 3) it can be shown (see [16]) that properly refined sequences of mesh combined with suitable sequences of p-distribution result in the error bound:

$$||u-u_{FE}||_E \leq C(\beta)N^{-\beta} \tag{11}$$

with β arbitrarily large. It is also known that under certain conditions the bound is exponential [16].

We shall demonstrate on example problem 1 that β can be arbitrarily large. The sequence of meshes and corresponding p-distributions are shown in fig. 13. Each refinement reduces the size of the triangles at the singularity by the factor $(1-\rho)$. Thus the corresponding sides of the triangles at the singularity are in geometric progression with common ratio $(1-\rho)$. We note that this refinement is not quasiuniform. The p-distribution is also not quasiuniform: $p = 1$ for the two elements at the singularity; $p = 2$ for the next group of four elements then p progressively increases by increments of one for each additional group of four elements away from the singularity.

The relative error in strain energy vs. N is plotted on log-log scale in Figures 14 and 15 for two refinements characterized by $\rho \cong 0.62$ (the "golden rule" refinement which is shown in Fig. 13) and $\rho = 0.90$, a much stronger refinement. It is seen that the slope of the relative error vs. N curve progressively increases with N, indicating that β in eq. (11) is an ascending function of N and can be arbitrarily large.

The results obtained with the p-version for two elements are also shown in Figures 14 and 15. The indications are that within the range of accuracy of practical interest the p-version is as effective in reducing the error as the strategy just outlined.

Optimal combination of non-quasiuniform mesh refinement and p-distribution is a delicate matter. For very high accuracy one can expect the polynomial degrees for elements at the singularity to be smaller than for the larger triangles away from the singularity. Nevertheless, in the pre-asymptotic range the optimal p-distribution can be

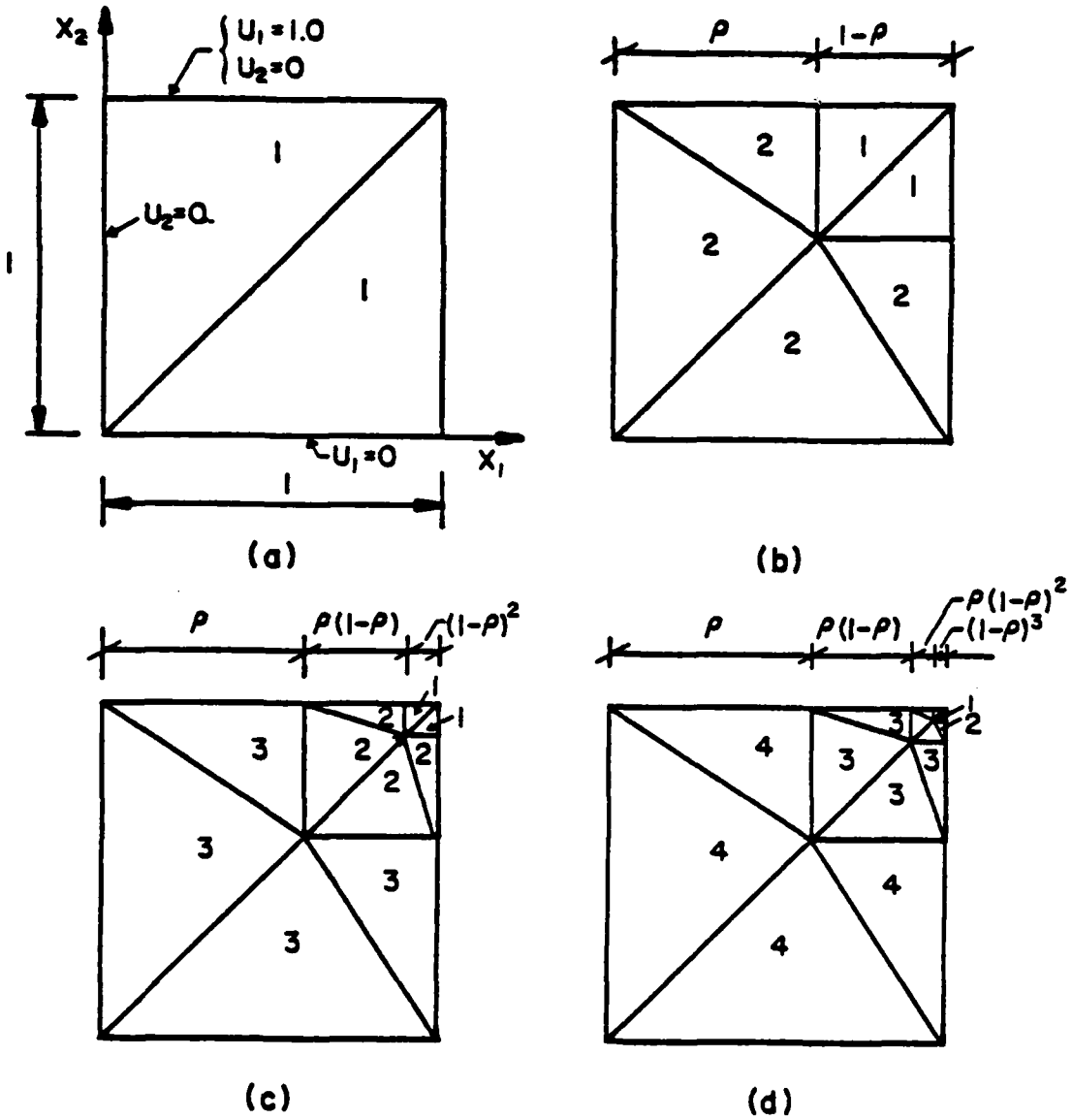

Fig. 13

Example problem 1: Non-quasiuniform mesh refinement and p-distribution. (The p-values are shown for each element). The 18-element mesh with $p_{max} = 5$ is not shown.

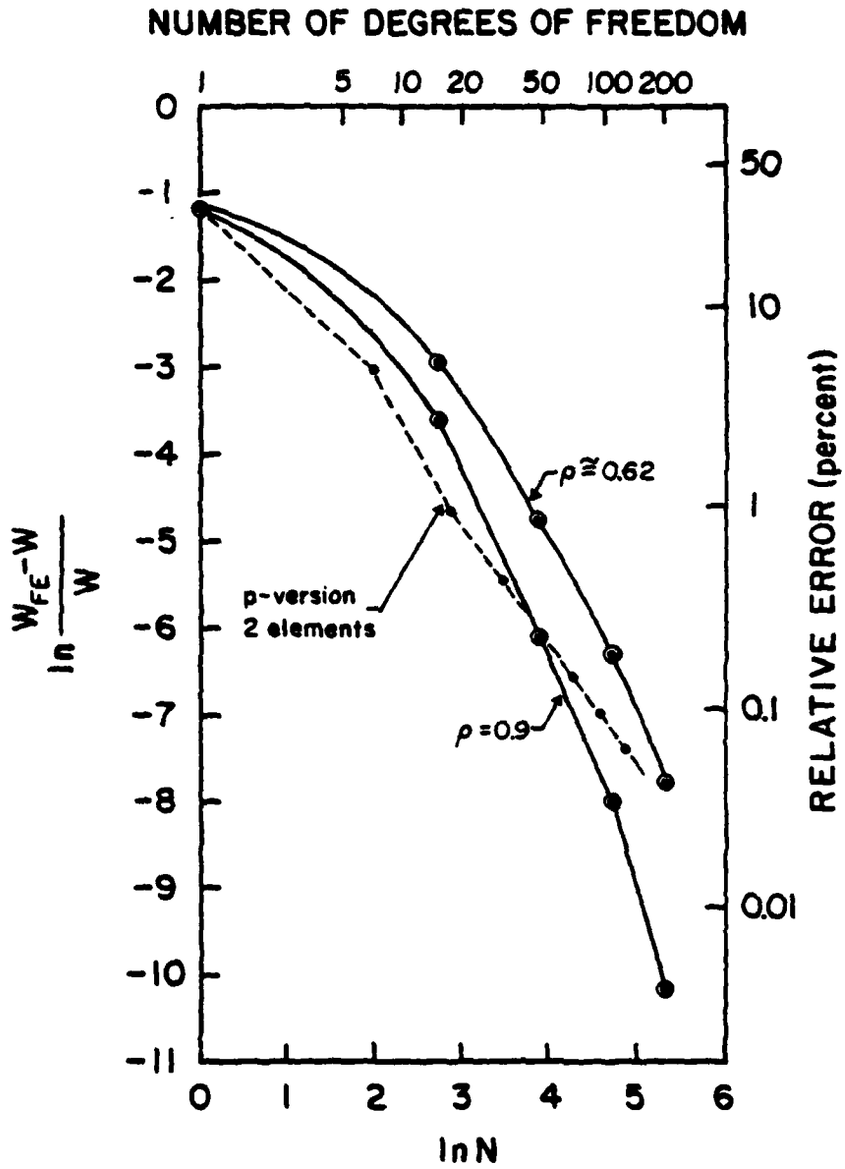

Fig. 14

Example problem 1: Relative error in strain energy vs. number of degrees of freedom for non-quasiuniform mesh refinement and p-distribution. Poisson's ratio: 0.3

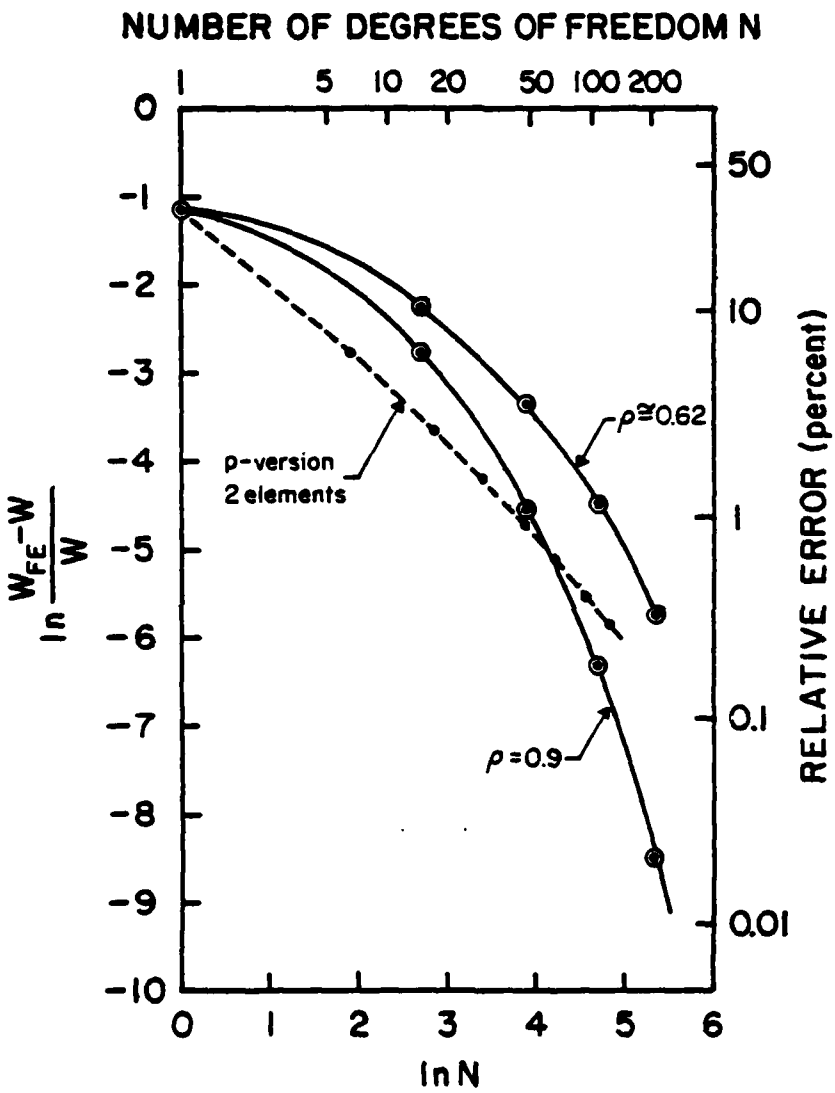

Fig. 15

Example problem 1: Relative error in strain energy vs. number of degrees of freedom for non-quasiuniform mesh refinement and p-distribution. *Poisson's ratio:* 0.4999

quite different. For general theoretical results and some numerical results in one dimension the reader is referred to [16].

10. Adaptivity

We have seen that if the number of degrees of freedom is increased through proper combinations of mesh refinement and p-distribution, the error will decrease very rapidly with increasing N. The choice of proper mesh refinement and p-distribution depends on the smoothness of the solution, however, and cannot in general be determined a priori.

It is possible to compute local measures of error, which indicate the contribution of each element to the total error of approximation. The local error measures provide a basis for establishing proper distributions of the degrees of freedom.

A more detailed discussion of adaptivity will be presented in a future paper.

11. Conclusions

We can summarize the main results concerning asymptotic rates of convergence in the finite element method as follows:

1. The asymptotic rate of convergence depends on the smoothness of the function to be approximated and the order of the polynomial basis functions (p). Smoothness is measured by the number of square integrable derivatives (k) over the domain of interest, with k generalized to fractional values.
2. In the h-version of the finite element method the rate of convergence is the smaller of p and k-1 if uniform or quasiuniform mesh refinement is used. When the singular behavior is caused by corners, there is a sequence of not quasiuniform meshes (called proper mesh refinement) for which the rate of convergence is dependent only on p.
3. In the p version of the finite element method the rate of convergence cannot be slower than in the h-version, provided that quasiuniform mesh refinement is used in the h-version. When the singularity is at the boundaries of finite elements, the rate of convergence of the p-version is twice that of the h-version provided that quasiuniform mesh refinement is used and $p \geq k-1$ in the h-version.
4. It is possible to design optimal sequences of meshes and p-distributions for which the rate of convergence in the presence of singularities is arbitrarily large; in fact the convergence can be exponential. Such sequences are not quasiuniform and depend on the function to be approximated. For this reason, the sequences can be determined in practice only by an adaptive approach.

5. For obvious practical reasons, finite element analyses should be both efficient and reliable. This means that the point of entry into the asymptotic range should occur at small values of N and should not be sensitive to the input parameters. The p-version meets this requirement in general better than the h-version. This was demonstrated through the example of nearly incompressible solids, in which the point of entry into the asymptotic range was not affected in the p-version to an important degree but was significantly shifted in the h-version.

6. Our discussion and comparisons were based on error vs. number of degrees of freedom relationships. This is a simplified treatment of the more important error vs. cost relationship. Detailed analysis of marginal cost vs. error reduction is beyond the scope of this paper, but we note that reasonably designed fixed meshes, combined with uniform or selective increases in p provide the most promising approach to efficient quality control in finite element analysis.

12. Acknowledgements

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