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SOME REMARKS ON STRATEGY IN PLAYING TENNIS.(U)
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The paper shows how the probability of winning a game, a set and a match, depend on the probability of winning a ball (from one's own and from the opponent's serve). The conditions are given under which the usual serve strategy (strong-weak) is optimal. It is also shown that a game is "strategy-less" in the sense that if the player has k (= 1, 2 or 3) balls which he can play with increased probability of winning, then all strategies

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SOME REMARKS ON STRATEGY IN PLAYING TENNIS

Robert Bartoszyński and Madan L. Puri

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1. INTRODUCTION

The main purpose of this note is to study the problem to which degree the scoring system of tennis matches allows the difference in players' levels to be reflected in the result of the match. To put it more explicitly, assume that p_A and p_B are the probabilities of winning a single ball by A and by B from their serves. Then, if the balls are played independently, and no other factors intervene in the result, the probability of A winning the match is some function of p_A and p_B, say $M^A(p_A, p_B)$.

The conditions of fairness require that the function M^A equals ½ on the diagonal $p_A = p_B$ and satisfies $M^A(p_A, p_B) = 1 - M^A(p_B, p_A)$. If one wants the result to reflect the relative advantage of one player over another, one could require the function M^A to rise steeply from ½ as p_A increases from the value p_B. The latter requirement, while desirable from the point of view of "true" ranking of the players, may however cause lack of tension and drama of a tennis match, by decreasing the amount of randomness and uncertainty of the result.

In section 1, we consider the probability of winning a basic unit of tennis match, namely a game, as a function of probability of winning a ball. In section 2, we consider the latter probability as a dependent of the serving strategy, and analyse conditions under which the usual strategy (first serve strong, second weak) is indeed optimal. In section 3 we consider the simplest case of strategic analysis, when a player may increase the probability of

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winning just one ball during the game. The question then arises of optimal moment of the game at which such a special ball ought to be played. This situation is then generalized to the case of several "special" balls. Finally, in the last section, we provide some numerical results concerning the probability of winning the match by one player, given the probabilities of winning a ball from his own, and from his opponent's serve, thus providing some information relevant for the answer to the question formulated at the beginning of this section. We also give some numerical results concerning the strategy in the last two sets in the match.

2. ON THE ROLE OF TRAINING

Let us consider first the basic unit of a tennis match, namely a *game*. It is characterized by the condition that the serve belongs to one of the players throughout the whole game, and that for winning the game, one must win four balls, with the additional requirement that the number of balls won must exceed the number of balls lost by at least 2. By tradition, the first two balls won count 15 points each, the next two count 10 points each, and subsequently, the score is classified in terms of categories "deuce", "advantage A" and "advantage B". Thus, the game may be regarded as a walk over the graph represented on Fig. 1.

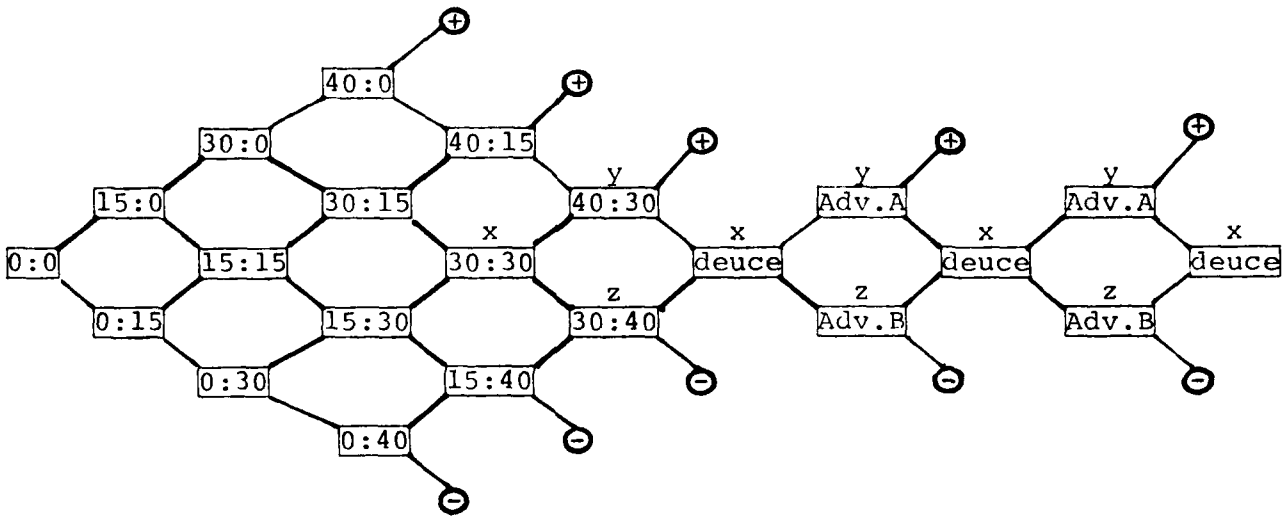

Fig 1.

Let us consider the game from the point of view of Player A. For the moment, let us disregard the problems connected with the right to a double trial at serve, and assume that the balls are played independently, each of them being won by A with probability p, and lost by him with probability $q = 1 - p$.

Our first goal will be to determine the probability $G(p)$ of winning the game by A , i.e. the probability that the random walk originating at the vertex marked 0:0 will become absorbed by the upper boundary (marked +).

Denote by x, y and z the probabilities of winning the game by A given the deuce (or score 30:30) , advantage of A , and advantage of B respectively (see Fig. 1).

Observe first that the probability of an infite game is zero.

Indeed, an infinite game requires that the random walk passes through all vertices marked "deuce". Now, the probability of passing from one "deuce" to the next is $2pq < 1$, hence the probability of the game lasting for at least n deuces is $\sim(2pq)^n$, which tends to 0 as $n \to \infty$.

Next, we have

$$x = py + qz \ , \ y = p + qx \ , \ z = px \ , \tag{2.1}$$

which yields easily

$$x = p^2/Q \ , \ y = p(1 - pq)/Q \ , \ z = p^3/Q \ , \tag{2.2}$$

where $Q = 1 - 2pq$.

From Fig. 1 it is evident that the probabilities of winning the game by A, given the scores 40:15 and 15:40 are $p + qy$ and pz respectively. Proceeding in this way, we obtain after some calculations

Proposition 1. <u>The probability of winning the game by</u> A <u>is given by the formula</u>

$$G(p) = \frac{15p^4 - 34p^5 + 28p^6 - 8p^7}{2p^2 - 2p + 1} \ . \tag{2.3}$$

The graph of $G(p)$ is given on Figure 2. As may be seen, $G(p)$ is nearly linear on a fairly large central fragment of the interval $[0,1]$. One can find easily that $G'(\frac{1}{2}) = 5/2$. This means that for players of approximately equal strength, an increase of probability of winning a ball from p to $p + \Delta p$ yields an increase of the probability of winning by about $\frac{5}{2}\Delta p$. For example,

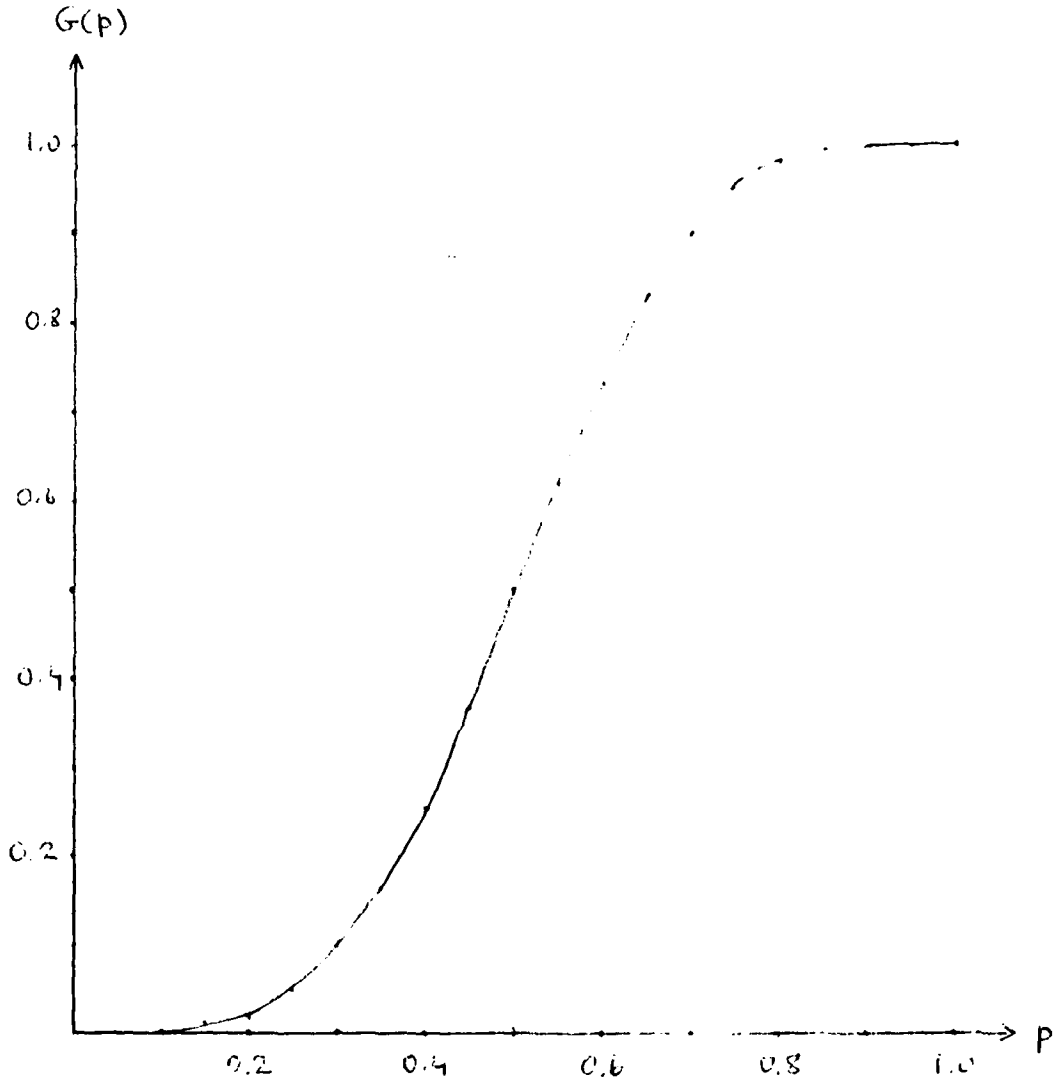

Fig. 2.

an increase of winning probability from $p = 0.5$ to $p = 0.55$ yields the increase of $G(p)$ by 0.123 from 0.5 to 0.623.

3. THE STRATEGY OF SERVING

Consider now in some more detail a node of the graph from Fig. 1, taking into account the fact that the players have the right to two trials at the serve. Let us assume that the player to serve is A .

A node of the graph on Fig. 1 will therefore take the form presented on Fig. 3.

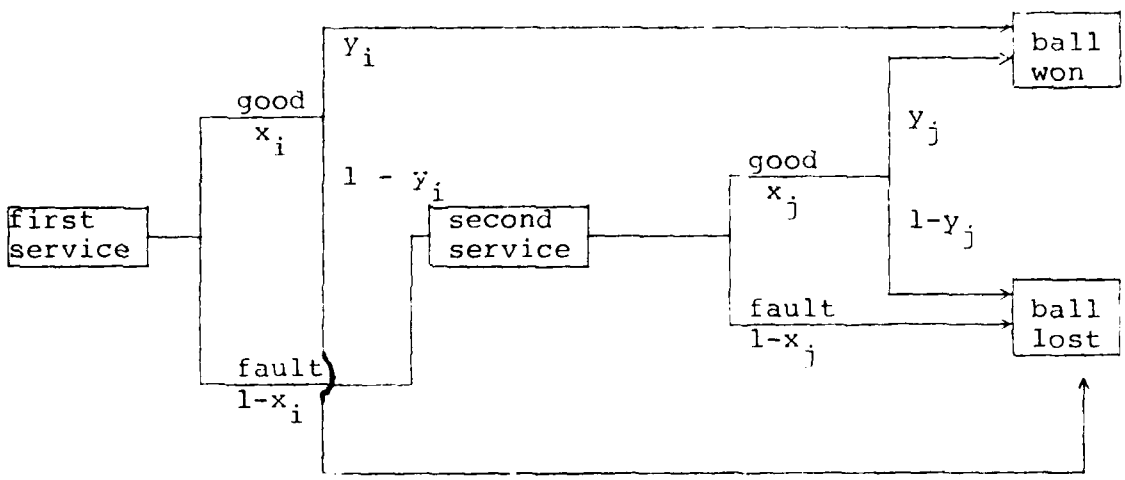

Fig. 3

Now, the players may use two kinds of serve, which will be referred to as <u>strong</u> (S) and <u>weak</u> (W). Let x_i denote the probability of a good (no fault) serve of type i , $i = S$ corresponding to a strong serve, and $i = W$ - to a weak serve. Similarly, let y_i be the probability of winning the ball from a serve of type i , given this serve is successful.

The four possible strategies of serving are (SS), (SW), (WS) and (WW). In practice of tennis matches, the players almost invariably use the strategy (SW). Let us therefore determine under which conditions this strategy is indeed the best among the four possible strategies.

It is a plausible assumption that

$$x_S < x_W \quad \text{and} \quad y_S > y_W \,, \tag{3.1}$$

i.e. strong serve is more often faulty than a weak one (it is more difficult to hit the court with a strong serve); on the other hand, the probability of winning the ball from a strong serve is higher than from a weak one (a strong serve is more difficult to return).

The probability of winning the ball, if one applies the strategy (SW) equals

$$p_{SW} = x_S y_S + (1 - x_S) x_W y_W \tag{3.2}$$

and similar formulas hold for other strategies of serving.

We have therefore

$$\begin{aligned} p_{SW} - p_{WS} &= [x_S y_S + (1 - x_S) x_W y_W] - [x_W y_W + (1 - x_W) x_S y_S] \\ &= x_S x_W (y_S - y_W) > 0 \,, \end{aligned} \tag{3.3}$$

which means that under (3.1), if one decides to use both types of serve, it is better to start from the strong one.

Next, we have

$$p_{SW} - p_{SS} = [x_S y_S + (1 - x_S) x_W y_W] - [x_S y_S + (1 - x_S) x_S y_S]$$
$$= (1 - x_S)(x_W y_W - x_S y_S) \ . \tag{3.4}$$

We conclude therefore that if

$$x_W y_W > x_S y_S \ , \tag{3.5}$$

i.e., if the absolute probability of winning with the weak serve exceeds that of winning with the strong serve, then the strategy (SW) is better than the strategy (SS).

Observe that this conclusion did not require the use of assumption (3.1)

Finally,

$$p_{SW} - p_{WW} = [x_S y_S + (1 - x_S) x_W y_W] - [x_W y_W + (1 - x_W) x_W y_W]$$
$$= x_W y_W (x_W - x_S) - (x_W y_W - x_S y_S)$$
$$= x_S y_W \left\{ \frac{y_S}{y_W} - \frac{x_W}{x_S} [1 - (x_W - x_S)] \right\} \tag{3.6}$$

and we proved

Proposition 2. <u>Under condition</u> (3.1), <u>the strategy of serving (SW) is optimal if, and only if</u>

$$\frac{x_W}{x_S} [1 - (x_W - x_S)] < \frac{y_S}{y_W} < \frac{x_W}{x_S} \ . \tag{3.7}$$

4. SOME ELEMENTS OF STRATEGY IN A GAME

Consider again a game, in which A has probability p of winning a ball. Suppose that A may increase the probability

p to $p' > p$ just once during the game (e.g. if A has a way of distracting his opponent's attention in playing one ball, etc.). The problem arises when should the "special" ball be played so as to maximize the probability of winning the game.

Denote by $G_s(p,p')$ the probability of winning the game, if the "special" ball (with probability p' of winning it) is played according to the strategy s.

Here strategy s is any rule which determines when the special ball is to be played. If S stands for the class of all strategies, then S may be partitioned into two classes: S', say, of all strategies which will necessarily use the special ball, and the class S'' of those strategies for which this is not true (e.g. the strategy which tells to use the special ball when the score is 15:40 only, is in S'': the game might end without ever passing through the score 15:40).

We shall prove a somewhat unexpected

Proposition 3. <u>The game in tennis is "strategy-less", in the sense that</u>

$$G_s(p,p') = G_t(p,p') \quad \underline{\text{for all}} \quad s,t \in S' \tag{4.1}$$

$$G_s(p,p') > G_t(p,p') \quad \underline{\text{for all}} \quad s \in S',\ t \in S'' \ \underline{\text{and}}\ p' > p\ . \tag{4.2}$$

<u>Proof</u>. We shall proceed by finding the optimal strategy of placing the "special" ball, using the principle of backward induction.

For the score $u:v$ during the game, denote by $P_{u:v}$ the

probability of winning the game by A, *starting from the score* $u:v$, without ever playing the "special" ball. Next, let $P'_{u:v}$ denote the probability of winning by A, starting from the score $u:v$, if the first ball played is the "special" one. Similarly, $P''_{u:v}$ will denote the analogous probability given that the first ball played is a "normal" one (with probability p of success), and from then on, the game is played in an optimal way of placing the "special" ball. Finally,

$$P^*_{u:v} = \max\,(P'_{u:v}\,,\ P''_{u:v}) \tag{4.3}$$

is the probability of winning the game by A under the optimal strategy of placing the special ball, in a "partial game", starting from the score $u:v$.

We shall try to find the value $P^*_{0:0}$. The optimal strategy will then be determined by *finding those scores* $u:v$ *at which* $P^*_{u:v} = P'_{u:v} > P''_{u:v}$, being the scores at which the "special" ball must be played.

In case when $P'_{u:v} = P''_{u:v}$, the "special" ball may be played at $u:v$, or may be played later, according to optimal rules in partial games starting from the scores next after $u:v$, so that the optimal rule is not unique.

Let us begin with scores $40:30$ and $30:40$, and consider strategies $s \in S'$ (i.e. strategies which use the "special" ball with probability one).

We have, using (2.2)

$$P_{40:30} = y = \frac{p(1 - pq)}{1 - 2pq}\,, \quad P_{30:40} = z = \frac{p^3}{1 - 2pq}\,. \tag{4.4}$$

Clearly, if the special ball was not used until the score 40:30 or 30:40, it must be used at once, otherwise the strategy would not be in S'. Thus, using (2.2) again, we have

$$P^*_{40:30} = P'_{40:30} = p' + q'P_{deuce} \tag{4.5}$$
$$= p' + q' \frac{p^2}{1 - 2pq}$$

and

$$P^*_{30:40} = P'_{30:40} = p'P_{deuce} = p' \frac{p^2}{1 - 2pq} . \tag{4.6}$$

On the other hand,

$$P'_{30:30} = p'P_{40:30} + q'P_{30:40} \tag{4.7}$$
$$= p' \frac{p(1 - pq)}{1 - 2pq} + q' \frac{p^3}{1 - 2pq}$$

and

$$P''_{30:30} = pP^*_{40:30} + qP^*_{30:40} \tag{4.8}$$
$$= p(p' + q' \frac{p^2}{1 - 2pq}) + qp' \frac{p^2}{1 - 2pq} .$$

We check easily that $P'_{30:30} = P''_{30:30}$, which means that

$$P^*_{30:30} = P'_{30:30} . \tag{4.9}$$

Consequently, if the special ball was not played until the score 30:30, it may be played at this score, or it may be used at next score, i.e. at 40:30 or 30:40, whichever occurs.

Next, we have

$$P_{40:15} = p + qP_{40:30} = p + qy \tag{4.10}$$

$$= p + q\,\frac{p(1 - pq)}{1 - 2pq}\ .$$

Since at 40:15 the game may end the next ball, we have

$$P^{*}_{40:15} = P'_{40:15} = p' + q'\,\frac{p(1 - pq)}{1 - 2pq}\ . \tag{4.11}$$

Passing now to the score 30:15, we have

$$P'_{30:15} = p'P_{40:15} + q'P_{30:30} = p'(p + qy) + q'x \tag{4.12}$$

$$= p'(p + q\,\frac{p(1 - pq)}{1 - 2pq}) + q'\,\frac{p^2}{1 - 2pq}$$

and

$$P''_{30:15} = pP^{*}_{40:15} + qP^{*}_{30:30} = p(p' + q'y) + q(p'y + q'z) \tag{4.13}$$

$$= p(p' + q'\,\frac{p(1 - pq)}{1 - 2pq}) + q(p'\,\frac{p(1 - pq)}{1 - 2pq} + q'\,\frac{p^3}{1 - 2pq})\ .$$

Again we check easily that $P'_{30:15} = P''_{30:15}$, which means that at the score 30:15 one may either use the special ball, or use it later (at the score 40:15, and either at 30:30, or at 40:30 and 30:40).

Proceeding in the same way, we arrive finally at the values $P'_{0:0}$ and $P''_{0:0}$, and check that they are equal.

This proves the property (4.1). The property (4.2) follows at once from the observation that if $p' > p$, then each strategy in S'' may be modified to a strategy in S' which is superior to it. Thus, the proof of Proposition 3 is complete.

Some tedious but elementary calculations yield

$$P_{15:0} = p^3 + 3p^3q + 3p^2q^2y + 3pq^2x + pq^3z \tag{4.14}$$

and

$$P_{0:15} = p^4 + p^3qy + 3p^2qx + 3p^2q^2z \qquad (4.15)$$

with x,y,z given by (2.2)

Using the assertion of Proposition 3, we conclude that for all strategies $s \in S'$ we have $G_s(p,p') = P'_{0:0}$, hence

$$G_s(p,p') = p'P_{15:0} + q'P_{0:15} \qquad (4.16)$$

$$= G(p) + \Delta p(P_{15:0} - P_{0:15})$$

where $p' = p + \Delta p$.

Table 1 gives probabilities $G(p)$ and differences $P_{15:0} - P_{0:15}$ for selected values of p , thus enabling calculation of $G_s(p,p')$ according to (4.16). For instance, at $p = 0.5$

TABLE 1

p	$G(p)$	$P_{15:0} - P_{0:15}$
0.00	0.0000	0.0000
0.05	0.0001	0.0012
0.10	0.0014	0.0089
0.15	0.0071	0.0278
0.20	0.0218	0.0602
0.25	0.0508	0.1055
0.30	0.0992	0.1597
0.35	0.1704	0.2160
0.40	0.2643	0.2658
0.45	0.3769	0.3002
0.50	0.5000	0.3125
0.55	0.6231	0.3002
0.60	0.7357	0.2658
0.65	0.8296	0.2160
0.70	0.9008	0.1597
0.75	0.9492	0.1055
0.80	0.9782	0.0602
0.85	0.9929	0.0278
0.90	0.9986	0.0089
0.95	0.9999	0.0012

we have $G(p) = 0.5$ and $P_{15:0} - P_{0:15} = 0.31$. This means that the increase from p to $p' = 0.55$ (hence with $\Delta p = 0.05$) in just one ball during the game, will increase the probability of winning the game by about $0.05 \times 0.31 = 0.0155$.

Proposition 3 may be generalized as follows. Consider a finite binary tree, the two branches leading out of a node marked by 1 and 0, Assume that to each terminal node, say $x, y, z, \ldots$ there is associated a number, say $W(x)$, $W(x), \ldots$ representing the payoff if the game described below terminates on the respective node.

For a given terminal node, say x , let $e_1, e_2, \ldots, e_n$ be the successive marks (1 or 0) assigned to the consecutive branches of the (unique) path leading to x . Let

$$e(x) = e_1/2 + e_2/2^2 + \ldots + e_n/2^n \ . \tag{4.17}$$

Assume that the payoffs W are monotone increasing with respect to the function e , i.e. for any two terminal nodes x and y , we have

$$W(x) < W(y) \quad \text{whenever} \quad e(x) < e(y) \ . \tag{4.18}$$

Consider now a random walk on the tree under consideration, in which consecutive steps are independent, and the branch marked 1 is chosen with probability p (and branched marked 0 is chosen with probability $q = 1 - p$).

The player may change the probability once during the whole random walk, from (p,q) into (p',q') with $p' > p$. He is to receive the reward equal to $W(x)$ if the random walk terminates

at node x .

Consider as before the strategies s of choosing the node at which the probabilities are to be modified, and let us partition the class S of all strategies into the set S' of all those strategies which modify the probabilities at some node with probability one, and the remainder class S'' .

Let V_s be the expected payoff associated with the strategy s . We have then

Proposition 4. <u>Under the assumption described above, we have</u>

$$V_{s_1} = V_{s_2} \quad \underline{\text{for all}} \quad s_1, s_2 \in S' \tag{4.19}$$

and

$$V_s > V_t \quad \underline{\text{for all}} \quad s \in S', \; t \in S'' \quad \underline{\text{and}} \quad p' > p. \tag{4.20}$$

The proof proceeds in the same way as for the tennis game, and will be omitted.

Proposition 3 may also be generalized by allowing more than one ball to be played in the "special" way (with higher probability of winning it). Naturally, the number of such "special" balls in the game cannot exceed 4, if one considers only the strategies from S' (which use all special balls with probability one). In case of 4 special balls, the only strategy in S' is to use them as the first four balls played. In the interesting cases of 2 and 3 special balls, one has the same "strategy-less" property of the tennis game as given in Proposition 3: it does not matter when the special balls are played, as long as one guarantees the use of all of them.

5. THE TENNIS MATCH

Let us at the end consider larger units of the game of tennis, namely the set and the match. The player who first wins 3 sets wins the match, so that the score in sets may be 3:0, 3:1 or 3:2.

Next, a set requires winning six games, with the difference of games won and lost being at least 2. During the set, the serve alternates from game to game. There is an additional provision that, if the score (in games) reaches 6:6 in any set except the fifth, the players play one tie-breaking game, and whoever wins it, wins the set (with the score recorded as 7:6). The rules of tie-breaker are such that after the first ball, the serve alternates at every two balls played, and one must win ~~six~~ seven balls, with the difference at least 2.

These rules are best illustrated on the graph of the set (see Fig. 4).

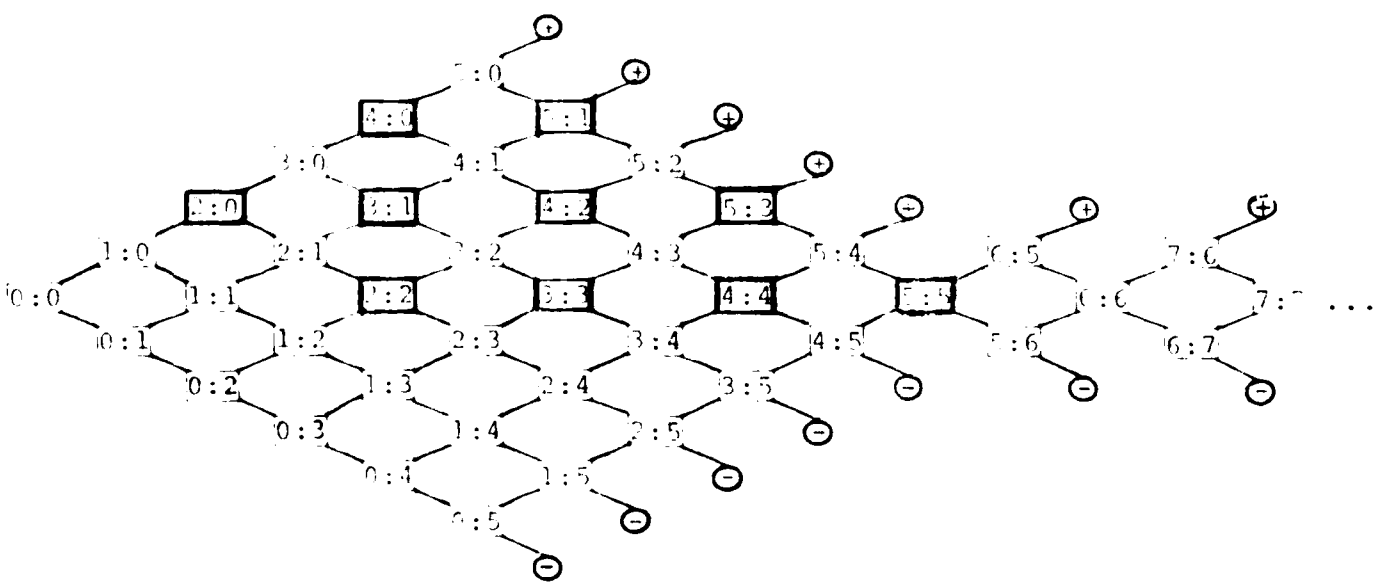

Fig. 4.

Suppose that A and B have probabilities of winning a single ball from one's own serve equal p_A and p_B, and suppose that A is to serve in the first game. The rules of the match are then as follows.

(a) The fifth set (if played) is a random walk on the infinite lattice from Fig. 4, with probabilities of going "up" equal alternatingly $G(p_A)$ and $1 - G(p_B)$, with G given by (2.3). The set is played until the random walk exits to a vertex marked + or − .

(b) In the sets other than the fifth, the situation is as above, with the additional provision that if the random walk reaches the node 6:6, the set is completed by a tiebreaker.

(c) The tiebreaker is a random walk on ~~the same~~ a similar lattice from Fig. 4, except that now the transitions occur not after a game, but after a single ball. The rules of changing the serve are such that (assuming A is to serve first), the successive probabilities of the random walk going "up" are

$$p_A\ ,\ 1 - p_B\ ,\ 1 - p_B\ ,\ p_A\ ,\ p_A\ ,\ 1 - p_B\ ,\ 1 - p_B\ ,\ p_A,\ p_A,\ \ldots$$

Let $\Pi_t = \Pi_t(p_A, p_B)$ and $\Pi_f = \Pi_f(p_A, p_B)$ denote the probabilities of winning a set with and without the tiebreaking rule. These probabilities do not depend on the choice of first server. Consequently, the probability of A winning the match equals

$$M^A(p_A, p_B) = \Pi_t^3 + 3\Pi_t^3(1 - \Pi_t) + 6\Pi_t^2(1 - \Pi_t)^2\Pi_f\ . \qquad (5.1)$$

Now, the probabilities Π_t and Π_f, as well as the probability of winning a tiebreaker, may in principle be written down explicitly, as function of p_A and p_B, in much the same way as

for the case of the game. Table 2 provides some interesting numerical values.

Table 2

Probability of Winning by Player A

p_A	p_B	A game from own serve $G(p_A)$	A game from opponent's serve $1-G(p_B)$	Tiebreak	Set with tiebreak rule T_t	Fifth set T_f	Match $M^A(p_A,p_B)$
0.51	0.50	0.5250	0.5000	0.5148	0.5356	0.5369	0.5670
0.52		0.5499		0.5295	0.5709	0.5734	0.6321
0.53		0.5746		0.5442	0.6056	0.6092	0.6934
0.54		0.5990		0.5589	0.6394	0.6441	0.7497
0.55		0.6231		0.5736	0.6721	0.6778	0.7997
0.60		0.7357		0.6455	0.8117	0.8202	0.9519

0.61	0.60	0.7562	0.2643	0.5152	0.5340	0.5361	0.5644
0.62		0.7759		0.5304	0.5675	0.5719	0.6267
0.63		0.7947		0.5458	0.6003	0.6069	0.6853
0.64		0.8126		0.5611	0.6320	0.6410	0.7391
0.65		0.8296		0.5765	0.6625	0.6739	0.7874
0.70		0.9008		0.6532	0.7930	0.8143	0.9400

0.71	0.70	0.9122	0.0992	0.5166	0.5304	0.5372	0.5593
0.72		0.9228		0.5334	0.5601	0.5745	0.6169
0.73		0.9324		0.5504	0.5891	0.6116	0.6715
0.74		0.9412		0.5676	0.6173	0.6484	0.7224
0.75		0.9492		0.5849	0.6444	0.6843	0.7687
0.80		0.9782		0.6731	0.7627	0.8418	0.9520

0.76	0.75	0.9564	0.0508	0.5180	0.5282	0.5414	0.5578
0.77		0.9629		0.5362	0.5560	0.5836	0.6142
0.78		0.9686		0.5548	0.5830	0.6261	0.6681
0.79		0.9737		0.5737	0.6094	0.6684	0.7187
0.80		0.8782		0.5928	0.6349	0.7098	0.7651
0.85		0.9929		0.6922	0.7499	0.8839	0.9426

As may be seen, the tiebreaker favours the weaker player in the sense that he has higher chances of winning a tiebreaker (and consequently, a set with tiebreaking rule) than the set without the tiebreaker. The probability of winning a match appears quite sensitive to player's advantage in winning a ball, i.e. sensitive to the difference $p_A - p_B$.

It may be shown easily that Proposition 3 applies as well to a single set. Imagine namely that the player may increase his chances of winning a number of games during the set (say, only in games from his own serve). Then the probability of winning the set does not depend on the choice of games which are played with the increased probability of winning, as long as the maximal allowed number of games is played in this way.

This property does not, however, carry over to the case of a match. One may namely consider the situation when A can increase the probability of winning a game (from his own serve, say) from $G(p_A)$ to some larger value p', in total of k games during the match. Then it is no longer irrelevant where the games with probability p' of winning are placed.

The determination of the optimal strategy is not simple; below, we give some numerical values for a special case.

For simplicity, assume that the increase in probability may apply only to the games with player's own serve, while the probability of winning a game with the opponent's serve is $1 - G(p_B)$. Assume also that the increase does not apply to tiebreak.

If $p_A = 0.63$ and $p_B = 0.60$, then $G(p_A) = 0.795$ and $G(p_B) = 0.736$, while the probability of winning the tiebreak by

A equals 0.546.

Suppose that 3 sets were already played, and A is to serve in the first game of the fourth set. Assume that he can change the probability of winning a game from $G(p_A) = 0.795$ to $p' = 0.9$ in the total of $k = 4$ games in the rest of the match.

If the score in sets is 2:1 , the optimal strategy (obtained by applying the principle of backward induction, in much the same way as the determination of optimal strategy in the proof of Proposition 3) is to play the games with probability of winning p' in any game in the fourth set in which A is either tied, or has the advantage, except the scores 0:0 and 1:1 . Thus A is to use the increased probability of winning at the scores 2:0, 4:0, 3:1, 2:2, 5:1, 4:2, 3:3, 5:3, 4:4 and 5:5 , that is, the scores marked on Fig. 4 (A is to serve when the total of games played is even). If he loses the fourth set, he is left with a certain number x $(0 \leq x \leq 4)$ games in which he can increase the probability of winning. These he may play in the fifth set in any way he chooses, provided he uses all available games.

Clearly, the optimal strategy here is aimed at winning the match in four sets. The probability of winning the match, given the score 2:1 in sets, in "normal situation" equals $\pi_t + (1 - \pi_t)\pi_f = 0.8429$; with optimal usage of 4 "special" games it is 0.9036.

On the other hand, if the score in sets is 1:2 , then A should use his options at all scores in the fourth set when he is either tied, or his opponent has an advantage, except for the scores 0:0 and 1:1 . The situation is therefore symmetric, and the optimal strategy is aimed primarily at not losing in the fourth

set. Here the probability of winning the match in a "normal situation equals $\pi_t \pi_f = 0.3643$, while with optimal use of four "special" games it is 0.4689.

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