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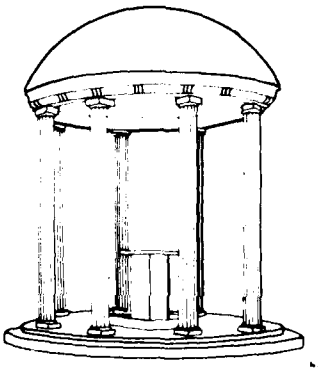

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6 (s,S) POLICIES FOR A DYNAMIC INVENTORY MODEL
WITH STOCHASTIC LEAD TIMES

9 Technical Report #17

10 Richard Ehrhardt

11 December 1980

12 17

Work Sponsored By

15 Office of Naval Research (N00014-78-C-0467)

Decision Control Models in Operations Research

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER Technical Report #17	2. GOVT ACCESSION NO. AD A094850	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) (s,S) POLICIES FOR A DYNAMIC INVENTORY MODEL WITH STOCHASTIC LEAD TIMES		5. TYPE OF REPORT & PERIOD COVERED Technical 6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Richard Ehrhardt		8. CONTRACT OR GRANT NUMBER(s) N00014-78-C-0467
9. PERFORMING ORGANIZATION NAME AND ADDRESS University of North Carolina at Chapel Hill Chapel Hill, North Carolina 27514		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Mathematical and Information Sciences Division Office of Naval Research, Code 434 Arlington, Virginia 22217		12. REPORT DATE December 1980 13. NUMBER OF PAGES 20
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified 15a. DECLASSIFICATION DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Inventory Control, (s,S) Policies

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A stochastic lead time inventory model is analyzed under the assumptions that (1) replenishment orders do not cross in time and (2) the lead time distribution for a given order is independent of the number and sizes of outstanding orders. This study corrects errors in the existing literature on the finite-horizon version of the model and yields an intuitively appealing dynamic program that is nearly identical to one that would apply in a transformed model with all lead times fixed at zero. Hence, many results that

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have been derived for fixed lead time models generalize easily. Conditions for the optimality of (s,S) policies are established for both finite and infinite planning horizons. The infinite-horizon model analysis is extended by adapting the fixed lead time results for the efficient computation of optimal and approximately optimal (s,S) policies.

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FOREWORD

As part of the on-going research program in "Decision Control Models in Operations Research," Mr. Richard Ehrhardt has investigated the structure of an inventory model with stochastic replenishment delivery lead times. The existing literature on this topic has been corrected and extended to encompass myopic optimal policies for finite planning horizons and optimal stationary (s,S) policies for infinite planning horizons. Efficient methods for computing infinite-horizon (s,S) policies are also given. An adaptation of the Power Approximation (Technical Report #7) is found to provide excellent performance relative to optimal policies. Several sections of this report parallel similar findings in earlier reports. Other related reports dealing with the research program are listed on the following pages.

Harvey M. Wagner
Principal Investigator

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Co-Principal Investigator

MacCormick, A. (1974), Statistical Problems in Inventory Control, ONR and ARO Technical Report 2, December 1974, School of Organization and Management, Yale University, 244 pp.

Estey, A. S. and R. L. Kaufman (1975), Multi-Item Inventory System Policies Using Statistical Estimates: Negative Binomial Demands (Variance/Mean = 9), ONR and ARO Technical Report 3, September 1975, School of Organization and Management, Yale University, 85 pp.

Ehrhardt, R. (1975), Variance Reduction Techniques for an Inventory Simulation, ONR and ARO Technical Report 4, September 1975, School of Organization and Management, Yale University, 24 pp.

Kaufman, R. (1976), Computer Programs for (s,S) Policies Under Independent or Filtered Demands, ONR and ARO Technical Report 5, School of Organization and Management, Yale University, 65 pp.

Kaufman, R. and J. Klincewicz (1976), Multi-Item Inventory System Policies Using Statistical Estimates: Sporadic Demands (Variance/Mean = 9), ONR and ARO Technical Report 6, School of Organization and Management, Yale University, 58 pp.

Ehrhardt, R. (1976), The Power Approximation: Inventory Policies Based on Limited Demand Information, ONR and ARO Technical Report 7, June 1976, School of Organization and Management, Yale University, 106 pp.

Klincewicz, J. G. (1976), Biased Variance Estimators for Statistical Inventory Policies, ONR and ARO Technical Report 8, August 1976, School of Organization and Management, Yale University, 24 pp.

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Klincewicz, J. G. (1976), The Power Approximation: Control of Multi-Item Inventory Systems with Constant Standard-Deviation-To-Mean Ratio for Demand, ONR and ARO Technical Report 10, November 1976, School of Business Administration and Curriculum in Operations Research and Systems Analysis, University of North Carolina at Chapel Hill, 47 pp.

Kaufman, R. L. (1977), (s,S) Inventory Policies in a Nonstationary Demand Environment, ONR and ARO Technical Report 11, April 1977, School of Business Administration and Curriculum in Operations Research and Systems Analysis, University of North Carolina at Chapel Hill, 155 pp.

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Schultz, C. R., R. Ehrhardt, and A. MacCormick (1977), Forecasting Operating Characteristics of (s,S) Inventory Systems, ONR and ARO Technical Report 13, December 1977, School of Business Administration and Curriculum in Operations Research and Systems Analysis, University of North Carolina at Chapel Hill, 47 pp.

Schultz, C. R. (1979), (s,S) Inventory Policies for a Wholesale Warehouse Inventory System, ONR Technical Report 14, April 1979, School of Business Administration and Curriculum in Operations Research and Systems Analysis, University of North Carolina at Chapel Hill, 75 pp.

Schultz, C. R. (1980), Wholesale Warehouse Inventory Control with Statistical Demand Information, ONR Technical Report 15, December 1980, School of Business Administration and Curriculum in Operations Research and Systems Analysis, University of North Carolina at Chapel Hill, 74 pp.

Ehrhardt, R. and G. Kastner (1980), An Empirical Comparison of Two Approximately Optimal (s,S) Inventory Policies, Technical Report 16, December 1980, School of Business Administration and Curriculum in Operations Research and Systems Analysis, University of North Carolina at Chapel Hill, 22 pp.

(s,S) POLICIES FOR A DYNAMIC INVENTORY MODEL WITH STOCHASTIC LEAD TIMES

Richard Ehrhardt

The University of North Carolina at Chapel Hill
(August 1980)

- Abstract -

A stochastic lead time inventory model is analyzed under the assumptions that (1) replenishment orders do not cross in time and (2) the lead time distribution for a given order is independent of the number and sizes of outstanding orders. This study corrects errors in the existing literature on the finite-horizon version of the model and yields an intuitively appealing dynamic program that is nearly identical to one that would apply in a transformed model with all lead times fixed at zero. Hence, many results that have been derived for fixed lead time models generalize easily. Conditions for the optimality of (s,S) policies are established for both finite and infinite planning horizons. The infinite-horizon model analysis is extended by adapting the fixed lead time results for the efficient computation of optimal and approximately optimal (s,S) policies.

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(s,S) POLICIES FOR A DYNAMIC INVENTORY MODEL WITH STOCHASTIC LEAD TIMES

We consider a periodic review, single-item inventory system where unfilled demand is backlogged, demands during review periods are independent, and the lead time between placement and delivery of an order may vary randomly. We require the joint distribution of lead times to have the properties that (1) replenishment orders do not cross in time and (2) the marginal lead time distribution of each order is independent of the number and size of outstanding orders. These assumptions could be appropriate in practice when, for example, only a single supplier is used and when the stocking organization places orders that are small and infrequent from the supplier's point of view. Replenishment costs are comprised of a setup cost and a cost that is linear in the amount ordered; holding and shortage costs are incurred in each period as a function of period-end inventory. Our criterion of optimality is minimization of the expected discounted cost incurred during a planning horizon which may be finite or infinite. Alternatively, when we consider an undiscounted, infinite-horizon model, our criterion is minimization of the expected cost per period.

A finite-horizon model of this system was analyzed by Kaplan [4] under the additional assumptions of stationarity of all model parameters and continuously distributed demand. The principal results were (1) that optimal policies can be computed using a dynamic program having only a scalar state variable, representing inventory on hand plus on order before ordering and (2) sufficient conditions can be found for

the optimality of base stock policies and (s,S) policies. Although the findings in [4] represent a breakthrough in the study of stochastic lead time systems, the results had two complicating features that are not present in fixed lead time models. First, the parameters of the dynamic program were not simply related to the marginal lead time distribution. Second, sufficient conditions were not found for the optimality of myopic ordering policies.

In this paper we correct two technical flaws existing in [4], allowing an intuitively appealing analogy with a zero-lead-time model. Then we establish conditions for the optimality of myopic base stock policies and present generalized conditions for the optimality of (s,S) policies. We also extend the model to encompass infinite planning horizons and show that optimal (s,S) policies exist under standard conditions on the cost functions. Finally, we present efficient algorithms for computing optimal and approximately optimal (s,S) policies in the infinite-horizon setting.

1. MODEL SPECIFICATION

We initially consider a finite planning horizon of N periods, numbered backwards from the end of the horizon; that is, the final period is given number 1, and the initial period is given number N. Demands in successive periods are independently, but not necessarily identically, distributed. Specifically, let the demand in period n be represented by the random variable D_n with mean μ_n and cumulative distribution function Φ_n. Also, let $\Phi_{i,j}$, $j > i$, be the be the convolution of $\Phi_i,\ldots,\Phi_j$. We assume complete backlogging of

unsatisfied demand, so negative inventory levels are permitted. Also, there are no losses from the system other than through demand satisfaction.

Costs in different periods are related by the single period discount factor α. Let $c_n(z)$ be the cost of ordering z units in period n, with

$$c_n(z) = K_n H(z) + c_n z ,$$

where

$$H(z) = \begin{cases} 0, & z = 0 \\ 1, & z > 0 . \end{cases}$$

We assume that both the setup cost K_n and the linear portion of the replenishment cost are paid upon delivery of the order. This assumption does not entail a loss of generality, since payment of either portion at the time of ordering can be described via scaling K_n or c_n by the expected value of α^L, where L is the random lead time.

Let $L_n(x)$ represent the holding and shortage costs in period n, where x is the ending inventory level in that period. Also we define the function

$$g_n(i,y) = EL_{n-i}[y - (D_n + \ldots + D_{n-i})]$$

$$= \int_0^\infty L_{n-i}(y-u)d\Phi_{n,n-i}(u) . \qquad (1)$$

We specify replenishment lead times as identically distributed random variables which can take on values from zero up to a fixed maximum m. Let a given lead time be represented by the random variable L, having the probability distribution

$$\ell_i = P\{L=i\},\ i=0,\ldots,m\ .$$

The joint distribution of lead times is characterized by our assumptions that (1) replenishment orders do not cross in time and (2) the lead time of an order is independent of the number and size of outstanding orders.

Following the development in [4], we focus on a single ordering decision, and let V be the number of outstanding orders immediately after the current order and before deliveries are received. Let U $(\cdot\ V)$ be the number of outstanding orders after the current delivery. Now if V is less than U in a given period, we know that only the oldest orders $(U-V$ of them$)$ must have been delivered. This is a consequence of our assumption that orders do not cross in time. Furthermore, our second lead time assumption (the lead time of an order is independent of the number and sizes of outstanding orders) implies that the dynamics of order deliveries are specified by a sequence of $m+1$ non-negative numbers $\{p_0,p_1,\ldots,p_m\}$. It is shown in [4] that

$$P\{U=j|V=i\} = \begin{cases} p_j, & j = 0 \text{ or } j < i \\ 1 - \sum_{k=0}^{i-1} p_k, & j = i\ , \end{cases}$$

where

$$\sum_{k=0}^{m} p_k = 1\ .$$

The analysis in [4] is conducted entirely in terms of the p_j's, as opposed to the ℓ_j's. An expression relating the p_j's to the

ℓ_j's is given on p. 495 of [4], but it is incorrect. We derive the correct relationship by noting that the probability ℓ_i that $\{L = i\}$ is given by the product of probabilities that it is not delivered in each of i delivery epochs $(0,1,2,\ldots,i-1$ periods after ordering) and the probability that a delivery is made i periods after ordering. We have

$$\ell_i = \begin{cases} p_0 & , i = 0 \\ \left(1-p_0\right)\left(1-p_0-p_1\right) \cdots \left(1 - \sum_{j=0}^{i-1} p_j\right) \sum_{j=0}^{i} p_j, & i=1,2,\ldots,m \,. \end{cases} \tag{2}$$

Although our derivation of (2) is original, we note that it also appears in Nahmias [5, p. 913]. Since the substantive results of [4] do not depend upon the relationship between the p_i's and ℓ_i's, they remain valid. We will show, however, that the interpretation of the dynamic program in [4] is simplified by (2).

The final aspect of model specification concerns the costs which must be included in computing optimal policies. We include all costs that are incurred during periods N through 1, plus those that occur in the following m periods due to orders placed during the planning horizon. A terminal reward (or salvage value) is also applied to the inventory level at the end of the horizon. This differs from [4], which only considers costs incurred in periods N through 1, and sets the terminal reward arbitrarily at zero for all terminal states. We regard this to be another flaw in [4]. Also, we will show that our change in cost accounting allows us to derive conditions for the optimality of myopic base stock policies.

2. FINITE PLANNING HORIZONS

The central finding in [4] is that all policy-dependent costs can be included in a dynamic program that has inventory on hand plus on order as its only state variable. Let $h_n(x)$ be the minimum expected discounted cost when x is the inventory on hand plus on order immediately before ordering in period n. We have

$$\left.\begin{aligned} h_n(x) &= \min_{y\geq x}\left\{\sum_{i=0}^{m} \alpha^i \ell_i [K_n H(y-x) + (y-x)c_n + g_n(i,y)] \right. \\ &\qquad\qquad \left. + \alpha E h_{n-1}(y-D_n)\right\} \\ h_0(x) &= -c_0 x \sum_{i=0}^{m} \alpha^i \ell_i \end{aligned}\right\} \quad (3)$$

Recursion (3) differs significantly from those given in [4] only in that our cost accounting includes additional terms at the end of the horizon, as described above at the end of Section 1. Now let

$$\beta \equiv E\alpha^L = \sum_{i=0}^{m} \alpha^i \ell_i \ ,$$

$$k_n(y) \equiv \sum_{i=0}^{m} \alpha^i \ell_i g_n(i,y)/\beta \ ,$$

and

$$F_n(x) \equiv h_n(x)/\beta \ .$$

Recursion (3) can be rewritten in terms of this notation as

$$\left.\begin{aligned} F_n(x) &= \min_{y\geq x}\left\{K_n H(y-x) + (y-x)c_n + k_n(y) + \alpha E F_{n-1}(y-D_n)\right\} \\ F_0(x) &= -c_0 x \end{aligned}\right\} \quad (4)$$

Notice that (4) is of the same form as a recursion for a zero-lead-time system with $k_n(\cdot)$ representing the single-period expected holding and shortage costs.

Following the approach of Veinott [8], we establish sufficient conditions for the optimality of myopic base stock policies. Consider the zero-lead-time analogy for recursion (4), and let $F_n(x_n|Y)$ be the expected discounted cost in periods n through 0 when following a particular ordering policy Y and x_n is the starting inventory in period j. Also let $\{y_i,\ i=1,\ldots,N\}$ be the sequence of inventory on hand after ordering and before demand. Then one can show that

$$F_n(x_n|Y) = \sum_{i=1}^{n} \alpha^{n-i} E\left[K_i H(y_i - x_i) + G_i(y_i)\right] + \left[\sum_{i=0}^{n-1} \alpha^{n-i} c_i \mu_{i+1} - c_n x_n\right] \quad (5)$$

where

$$G_n(y) = (c_n - \alpha c_{n-1})y + k_n(y) \ . \quad (6)$$

The functions $G_n(\cdot)$ are composites of expected holding and shortage costs and the linear purchase costs. They can be interpreted (Veinott and Wagner [10]) as the conditional expected holding and shortage cost functions of an equivalent model with unit purchase costs c_n set equal

to zero. We shall use this interpretation, and hereafter refer to $G_n(\cdot)$ as a conditional expected holding and shortage cost function.

Consider the case of $K_n = 0$ for all n. It follows that if $-G_n(y)$ is unimodal with a minimum at $\bar{y}_n$, and if

$$\bar{y}_n - D_n \leq \bar{y}_{n-1}, \quad n=2,\ldots,N$$

with probability one, then it is optimal to order $\max(\bar{y}_n - x_n, 0)$ in period n. For alternative conditions that ensure the optimality of myopic base stock policies, see [8].

An interesting parallel to the fixed lead time problem arises in the solution for the base-stock values $\{\bar{y}_i, i=1,\ldots,N\}$ when demand has a density and the single-period holding and shortage costs are given by

$$L_n(x) = h \max(x,0) + p \max(-x,0), \quad n=-m+1,\ldots,N . \tag{7}$$

Then one can show that $\bar{y}_n$ is a solution to

$$\theta_n(\bar{y}_n) = (p-c_n+\alpha c_{n-1})/(h+p) ,$$

where

$$\theta_n(y) = \sum_{i=0}^{m} \alpha^i \ell_i \Phi_{n,n-i}(y)/\beta .$$

Notice that the functions $\theta_n(y)$ are linear combinations of convoluted demand distributions and are legitimate distribution functions in their own right.

When the model does not possess an optimal myopic base-stock policy, we consider the function

$$f_n(x_n|Y) = F_n(x_n|Y) - \left[\sum_{i=0}^{n-1} \alpha^{n-i} c_i \mu_{i+1} - c_n x_n\right]$$

$$= \sum_{i=1}^{n} \alpha^{n-i} E\left[K_i H(y_i - x_i) + G_i(y_i)\right] .$$

Notice that all policy-dependent costs are included in $f_n(x_n|Y)$. Therefore, an optimal policy can be found by computing

$$f_n(x) = \min_Y f_n(x|Y)$$

using the dynamic programming recursion

$$\left.\begin{aligned} f_n(x) &= \min_{y \geq x}\left\{K_n H(y-x) + G_n(y) + \alpha E f_{n-1}(y-D_n)\right\}, \quad n=1,\ldots,N \\ f_0(x) &= 0 . \end{aligned}\right\} \quad (8)$$

Expression (8) is easily recognized as a standard form in inventory theory. Therefore, conclusions about the structure of optimal policies are immediate. For example, if $K_n = 0$ for all n, one can show [8] that a base-stock policy is optimal when $G_n(y)$ is convex for all n. The base stock levels are given by the values $\{y_n^*, n=1,\ldots,N\}$ that minimize the expression in braces on the right-hand side of (8).

For models having $K_n > 0$ for at least one value of n, we cite two theorems from Denardo [1] which guarantee the optimality of (s,S) policies. Let

$$J_n(y) = G_n(y) + \alpha E f_{n-1}(y-D_n) .$$

and define S_n and s_n as solutions to

$$J_n(S_n) = \min_y \{J_n(y)\} \ , \tag{9}$$

and

$$s_n = \inf\{y \le S_n | J_n(y) = K_n + J_n(S_n)\} \ . \tag{10}$$

Theorem 1. *Suppose the following three conditions are satisfied:*

(a) $G_n(y)$ *is convex when* $n > 1$, *and* G_1 *is continuous and* K_1*-convex.*

(b) $K_{n+1} \ge \alpha K_n$ *for* $n=1,\ldots,N-1$.

(c) $f_n \to \infty$ *as* $x \to \infty$ *for* $n=1,\ldots,N$, *and* $bf_n(y-D_{n+1})$ *is finite for* $n=1,\ldots,N-1$.

Then

$$f_n(x) = \begin{cases} K_n + J_n(S_n), & x < s_n \\ J_n(x) \quad , & x > s \ , \end{cases}$$

where S_n *and* s_n *are given by (9) and (10).*

Theorem 2. *The conclusions of* Theorem 1 *remain valid if its hypothesis is replaced by the following three conditions:*

(a) There exist $a_1 < a_2 < \ldots < a_N$ *such that* $J_n(y)$ *is non-increasing for* $y < a_n$ *and non-decreasing for* $y > a_n$. *Moreover,* $G_n(y)$ *is continuous, and* $K_n + G_n(S_n) = \lim_{y\to\infty}\{G_n(y)\}$.

(b) $f_{n+1} \geq \alpha K_n$ *for* $n=1,\ldots,N-1$.

(c) $EC_n(Q-D_{n+1})$ *is finite for* $n=1,\ldots,N-1$.

Other sets of conditions for the optimality of (s,S) policies exist. See, for example, Veinott [9] and Schäl [7].

3. INFINITE PLANNING HORIZONS

We consider an infinite horizon version of our model in which all data are stationary. Our notation is simplified in this setting by suppressing subscripts that denote period numbers whenever the quantity of interest does not vary with time. Hence, recursion (8) becomes

$$\left.\begin{aligned} f_n(x) &= \min_{y \geq x}\left\{KH(y-x) + G(y) + \alpha E f_{n-1}(y-D)\right\}, \quad n \geq 1 \\ f_0(x) &= 0 , \end{aligned}\right\} \quad (11)$$

where

$$G(y) = (1-\alpha)cy + k(y) , \quad (12)$$

and

$$k(y) = \sum_{i=0}^{m} \alpha^i \ell_i q(i,y)/B .$$

Recursion (11) is just like one for a fixed-lead-time model, with $G(y)$ representing the conditional expected holding and shortage costs. Hence, we know that if $G(y)$ is convex, a stationary (s,S) policy is optimal in (11) as n approaches infinity. The conclusion is supported by the argument in Iglehart [3], which also establishes the existence of

$$f(x) = \lim_{n\to\infty} f_n(x) .$$

Also from [3], we know that $f(x)$ satisfies the functional equation

$$f(x) = \min_{y\geq x}\left\{KH(y-x) + G(y) + \alpha Ef(y-D)\right\} . \tag{13}$$

The only difference between (13) and a fixed-lead-time model is in the function $G(\cdot)$. In fact, $G(\cdot)$ can be expressed in the same form as the conditional expected holding and shortage cost function of a fixed-lead-time model with a transformed demand distribution. We have

$$\begin{aligned} G(y) &= (1-\alpha)cy + k(y) \\ &= (1-\alpha)cy + \sum_{i=0}^{m} \alpha^i \ell_i g(i,y)/\beta \\ &= (1-\alpha)cy + \sum_{i=0}^{m} \alpha^i \ell_i \int_0^\infty L(y-u)d\Phi^{*(i+1)}(u)/\beta , \end{aligned}$$

where Φ^{*j} is the j-fold convolution of the demand distribution Φ. Hence, $G(\cdot)$ can be expressed in the form

$$G(y) = (1-\alpha)cy + \int_0^\infty L(y-u)d\Psi_\alpha(u) , \tag{14}$$

where

$$\Psi_\alpha(x) = \sum_{i=0}^{m} \alpha^i \ell_i \Phi^{*(i+1)}(x)/\beta . \tag{15}$$

Notice that the function $\Psi_\alpha(\cdot)$ has all the properties of a distribution function. Therefore, we call $\Psi_\alpha(\cdot)$ the discounted lead time demand distribution.

When demand is discrete and $G(\cdot)$ is convex, expressions (14) and (15) allow a simple adaptation of the efficient Veinott-Wagner procedure [10] for computing optimal (s,S) policies. In [10] a fixed lead time λ is assumed, and the conditional holding and shortage cost function is computed using the convoluted lead time demand distribution $\Phi^{*(\lambda+1)}$. To adapt [10] for our stochastic lead time model, we merely substitute Ψ_α in place of $\Phi^{*(\lambda+1)}$ in any computation related to $G(\cdot)$. Specifically, the important expressions in [9] that require modification are (20), (21), (22), (23), (26), (27), and the two unnumbered expressions immediately preceding (21).

We have performed computations using the procedure described above. We summarize the results below, in Section 4.

We conclude this section with a discussion of approximately optimal (s,S) policies for the infinite horizon model. We have just shown how to compute optimal (s,S) policies by modifying a fixed-lead-time procedure. Basically the same kind of modification can be used to compute approximately optimal (s,S) policies as well.

For example, consider the common assumptions of $\alpha = 1$ and linear holding and shortage costs [as given by (7)]. This model was analyzed by Roberts [6], who used asymptotic renewal theory to characterize the limiting behavior of an optimal policy (s^*,S^*) as the parameters K and p grow large. He obtained the following expressions for optimal policy parameters s^* and $D^* = S^* - s^*$, as D^* grows large:

$$D^* = \sqrt{2K\mu/h} + o(D^*) ,$$

$$\int_{s^*}^{\infty}(u-s^*)d\Phi^{*(\lambda+1)}(u) = D^*/(1+p/h) + o(D^*) ,$$

where λ is the fixed lead time, μ is the single period demand mean, and $o(D^*)/D^*$ converges to zero as D^* becomes infinite. These expressions were used by Ehrhardt [2] to construct an approximately optimal policy (the Power Approximation) that is easy to compute and requires for demand information only the mean and variance of demand. Specifically, the Power Approximation requires the single-period demand mean and variance, μ and σ^2, as well as the mean and variance of $\phi^{*(\lambda+1)}$, μ_λ and σ_λ^2.

We suggest modifying the Power Approximation for our stochastic lead time model by replacing μ_λ and σ_λ^2 with the mean and variance of Ψ_α, μ_ℓ, and σ_ℓ^2. Specifically, expressions (13) - (16) in [2] require this change. In computing μ_ℓ and σ_ℓ^2, we note that $\alpha = 1$ implies that $\beta = 1$ also. Therefore

$$\mu_\ell = \int_0^\infty u d\Psi_1(u)$$

$$= \sum_{i=0}^{m} \ell_i \int_0^\infty u d\phi^{*(i+1)}(u)$$

$$\mu_\ell = \sum_{i=0}^{m} \ell_i (i+1)\mu = (EL+1)\mu \ , \tag{16}$$

and

$$\sigma_\ell^2 = \int_0^\infty u^2 d\Psi_1(u) - \mu_\ell^2$$

$$= (EL+1)\sigma^2 + \mu^2 Var(L) \ . \tag{17}$$

Notice that μ_ℓ and σ_ℓ^2 are merely the mean and variance of demand during (L+1) periods. The use of (16) and (17) in place of μ_λ and σ_λ^2 is a familiar heuristic approach for modifying a fixed lead time policy. Until now, however, this approach has not been theoretically justified for periodic review systems.

We assess the effectiveness of the modified Power Approximation in Section 4, below, where it is compared with optimal policies for a variety of parameter settings.

4. NUMERICAL RESULTS

We have performed computations using the procedures described above for infinite horizon problems. In this section we consider a set of twelve inventory items under a variety of assumptions about the distribution of lead time. First, we show how optimal expected costs vary with the variance of the lead time distribution. Then we compare the performance of optimal policies with that of the Power Approximation as modified by (16) and (17).

Consider a system of 12 inventory items, each having a negative binomial demand distribution with a variance-to-mean ratio of 3. Mean demand μ has three values, 2, 4, and 8. Each item has linear holding and shortage costs as given by (7). Since the total cost function is linear in the parameters h, p, and K, the value of the unit holding cost is a redundant parameter which is set at unity. The unit shortage costs are 4 and 9, and the setup cost values are 32 and 64. All combinations of these parameter settings are included, yielding 12 items.

We consider the four lead time distributions displayed in Table I. Each is a symmetrical triangular distribution over the range [0,4], with a mean value of 2. The variance of lead time ranges from a minimum of 0 for the deterministic case to a maximum of 2 for the uniform distribution. We also list the coefficient of variation γ of each lead time distribution, which is defined as the ratio of the standard deviation to the mean.

TABLE I

Lead Time Distributions

Probability Mass ℓ_i, $i=0,\ldots,4$							
i: 0	1	2	3	4	EL	Var L	γ
0	0	1	0	0	2	0	0
0	.250	.500	.250	0	2	.50	.354
.0667	.2333	.4000	.2333	.0667	2	1.00	.500
.2	.2	.2	.2	.2	2	2.00	.707

Table II shows optimal total cost per period as a function of parameter values for each of the four lead time distributions given in Table I. Notice that the total aggregate cost of the 12 items, increases monotonically with lead time variance. The largest lead time variance yields an optimal total cost of 327 for the 12 items, 17% higher than the deterministic lead time cost of 280. When costs are aggregated by parameter values, we see that the larger lead time variances produce slightly larger cost increases for items with a

penalty cost p of 9 than for those with p equal to 4. Slightly larger cost increases are also displayed for items with a setup cost K of 32 as opposed to those with K equal to 64. The bulk of the cost increase, however, can be attributed to items with the largest value of mean demand. Notice that items with μ equal to 8 shows a 24% increase in total cost from 126 for the deterministic lead time system to 156 for the high lead time variance system. The corresponding percentage increase for items with μ equal to 2 is merely 8%. This fact is not surprising, since we have held the demand variance-to-mean ratio constant. Therefore, items with the largest mean demand also have the largest variance of demand, yielding especially large values of σ_ℓ^2 in (17).

TABLE II

Optimal Total Cost per Period of 12 Items for 4 Different Lead Time Distributions

Lead Time		Total Aggregate Cost	Costs Aggregated by Parameter Value						
			Penalty Cost		Setup Cost		Mean Demand		
Mean	Variance		4	9	32	64	2	4	8
2	0	280	129	150	124	156	64	90	126
2	1/2	293	135	159	131	162	65	93	135
2	1	306	140	166	137	168	66	96	143
2	2	327	149	178	149	179	69	102	156

Table III lists percentage increases in expected total cost per period when Power Approximation policies are compared with optimal policies. The Power Approximation yields costs within a few tenths of a percent of optimal for all parameter settings. This level of performance is comparable with the data in [2], where only deterministic lead times were considered.

TABLE III

Percentages Above Optimal Total Cost per Period for 12-Item Systems Under Approximately Optimal Control

Lead Time		Total Aggregate Cost	Costs Aggregated by Parameter Value						
			Penalty Cost		Setup Cost		Mean Demand		
Mean	Variance		4	9	32	64	2	4	8
2	0	0.1	0.2	0.1	0.1	0.3	0.2	0.2	0.1
2	1/2	0.2	0.2	0.1	0.1	0.2	0.3	0.1	0.2
2	1	0.2	0.3	0.1	0.2	0.1	0.2	0.2	0.1
2	2	0.3	0.3	0.2	0.4	0.2	0.0	0.3	0.4

ACKNOWLEDGEMENTS

The author is pleased to acknowledge the helpful comments and suggestions of his colleagues, Professors Alastair MacCormick and Harvey M. Wagner. He also expresses his appreciation to James Lent who performed the computer calculations. This research was supported by the Office of Naval Research under Contract N00014-78-C-0467.

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