A GENERALIZATION OF THE KREISS MATRIX THEOREM

Shmuel Friedland

Mathematics Research Center
University of Wisconsin–Madison
610 Walnut Street
Madison, Wisconsin 53706

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ABSTRACT

Let \( A \) be a set of \( n \times n \) complex matrices \( A \) which satisfy the condition \( \| (I - zA)^{-1} \| < K/(1 - |z|)^{\alpha+1} \) for some \( \alpha > 0 \) and all \( |z| < 1 \).

Then it is shown here that there exists a constant \( p(\alpha,n) \) such that
\[ \| A^\nu \| < Kp(\alpha,n)^\nu, \nu = 0, 1, \ldots \] This forms a generalization of the Kreiss resolvent condition \( (\alpha = 0) \).

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SIGNIFICANCE AND EXPLANATION

Let $A$ be an $n \times n$ complex valued matrix. A standard and useful result in matrix theory claims that all powers of $A$ are bounded if and only if the spectral radius $\rho(A)$ is less or equal to one and for all eigenvalues $\lambda$ of $A$ such that $|\lambda| = 1$ the matrix $(I - zA)^{-1}$ has a simple pole at $z = \bar{\lambda}$. If we consider a more general problem, namely when the powers $A^v, v = 0, 1, \ldots,$ grow at most as $v^\alpha$, where $\alpha$ is a positive integer, then this condition holds if and only if $\rho(A) < 1$, and for all eigenvalues $\lambda$ of $A$ such that $|\lambda| = 1$ the matrix $(I - zA)^{-1}$ has at most a pole of order $\alpha + 1$ at $z = \bar{\lambda}$.

In the early sixties H. O. Kreiss, while studying stability of numerical schemes for partial differential equations, considered a generalization of the first problem described above. Namely, given a set $A$ of $n \times n$ complex valued matrices, when all powers of $A^v A$ are uniformly bounded. These sets—called the stable sets—were completely characterized by Kreiss by giving three equivalent conditions.

In this paper we consider $\alpha$-stable sets $A(\alpha \geq 0)$, such that for any $A \in A$ the powers $A^v$ are uniformly bounded by $|A^v| \leq C \rho(A^v)$. We generalize the Kreiss resolvent condition for $\alpha$-stable sets. It seems that $\alpha$-stable sets are related to the concept of weakly stable numerical schemes for partial differential equations.

The responsibility for the wording and views expressed in this descriptive summary lies with MPC, and not with the author of this report.
Introducing the iterative system of equations:
\[
x^{(i+1)} = A x^{(i)}, \quad i = 0, 1, 2, \ldots.
\]
where \(x^{(i)} \in \mathbb{C}^n\), \(A \in \mathbb{M}_n\), with \(\mathbb{C}^n\) being the set of \(n\) column complex vectors and \(\mathbb{M}_n\) the set of \(n \times n\) complex matrices. Clearly:
\[
x^{(1)} = A^0 x^{(0)}
\]
and to investigate the behaviour of \(x^{(1)}\) for large \(i\) one needs to study the powers \(A^i, i = 0, 1, \ldots\). Let \(\mathcal{A}\) be a set of \(n \times n\) matrices. \(\mathcal{A}\) is called an \(\alpha\)-stable set if:
\[
\|A^v\| < \alpha^0, \quad v = 0, 1, 2, \ldots.
\]
Here \(\alpha\) is a nonnegative number and \(\|\cdot\|\) is a norm on \(\mathbb{M}_n\). The concept of stability of the numerical schemes for solutions of partial differential equations is intimately connected with the notion of stable sets. Consult for example Kreiss (1962), Richtmyer and Morton (1967) and others. It seems that \(\alpha\)-stable sets are related to the concept of weakly stable numerical schemes for partial differential equations. See Kreiss (1962) and Forsyth and Wasow (1960). The stable sets were characterized completely by Kreiss (1962). In this paper we generalize the Kreiss result to \(\alpha\)-stable sets.

**Theorem 1.** Let \(\alpha\) be a nonnegative number and \(\mathcal{A}\) be a set of \(n \times n\) complex valued matrices. Then the following two conditions are equivalent:

(A) There exists a constant \(K \geq 1\) such that for all \(A \in \mathcal{A}\) (1.3) holds.

(R) There exists a constant \(K \geq 1\) such that for all \(A \in \mathcal{A}\):
\[
|I - zA|^{-1} < K |1 - |z||^{-\alpha}, \quad |z| < 1.
\]

The implication (A) \(\Rightarrow\) (R) is obvious. The implication (R) \(\Rightarrow\) (A) is a consequence of Theorem 2 which estimates the Maclaurin coefficients of a certain family of rational functions.
functions in terms of the growth of their moduli. We were not able to give conditions analogous to the conditions (S) and (H) of Kreiss.

2. Coefficient Estimates for Certain Analytic Functions

Let \( D \) be a unit disc \( |z| < 1 \). Suppose that \( f(z) \) is an analytic function in \( D \).

Consider the Maclaurin expansion of \( f \)

\[
(2.1) \quad f(z) = \sum_{\nu=0}^{\infty} a_{\nu} z^{\nu}, \quad |z| < 1.
\]

Suppose that

\[
(2.2) \quad |a_{\nu}| < M_{\nu}, \quad \nu = 0, 1, 2, \ldots
\]

for \( \alpha > -1 \). It is a standard result in theory of special functions (e.g. Olver [1974, p. 119]) that

\[
(2.3) \quad a_{\nu} = (-1)^{\nu} \frac{(\alpha + 1)}{\nu} \frac{\Gamma(\alpha + \nu + 1)}{\Gamma(\nu + 1) \Gamma(\alpha + 1)}.
\]

Here two positive sequences \( \{u_{m}\} \) and \( \{v_{m}\} \) are called equivalent \( u_{m} = v_{m} \) if

\[
\lim_{m \to \infty} \frac{u_{m}}{v_{m}} = \beta, \quad 0 < \beta < \infty.
\]

Thus (2.3) implies

\[
(2.4) \quad |f(z)| < K |z|^{\alpha} (1 - |z|)^{-(\alpha + 1)}
\]

for some positive constant \( K \) with \( \alpha > -1 \). Conversely we have a weaker result.

Lemma 1. Let \( f(z) \) be analytic in \( D \). Assume that

\[
(2.5) \quad |f(z)| < K(1 - |z|)^{\alpha},
\]

for some \( \alpha > 0 \) and all \( |z| < 1 \). Then

\[
(2.6) \quad |z|^{\alpha} + \frac{|z|}{\alpha} + \frac{\alpha}{\alpha - 1} + \frac{\alpha}{\alpha + 1} < K \Gamma(\nu + 1)^{2}
\]

and this inequality is sharp.

Proof. As

\[
(2.7) \quad z = (z^{\alpha})^{-1} \quad f(z) z^{-(\alpha + 1)} Az
\]

for some positive constant \( A \),...
we get

\[(2.8) \quad |a_v| < \max_{|z|=r} |f(z)| r^{-\lambda} < K(1 - r)^{-\lambda} r^{-\nu} . \]

Note that

\[
\min_{0 < r < 1} (1 - r)^{1-\nu} = (1 - r)^{1-\nu} \left| \frac{r}{r+1} \right| = \frac{1}{1+1} (1+1)^{-\nu} \cdot 
\]

This establishes the first inequality in (2.6). To obtain the second inequality in (2.6) choose in (2.8) \( r = \frac{\nu}{\nu+1} \) and use the well known fact that \( (1 + \frac{1}{\nu})^\nu < e \). To see that (2.6) is sharp for each \( \nu \) consider the polynomial

\[(2.9) \quad p(z) = K(1 + \frac{\nu}{\nu+1})^\nu \left( \frac{\nu+1}{\nu} \right) z^\nu . \]

Let \( B \) be a Banach space with a norm \( \| \cdot \| \). Assume that \( A : B \to B \) is a bounded linear operator. Suppose that the spectrum of \( A \) lies in the unit disc. Then expanding \( (I - zA)^{-1} \) in power series

\[(2.10) \quad (I - zA)^{-1} = \sum_{\nu=0}^{-\infty} z^\nu A^\nu \]

we get

\[(2.11) \quad A^\nu = (2\pi i)^{-\nu} \int_{|z|=r<1} (I - zA)^{-1} dz . \]

Thus if

\[(2.12) \quad \| (I - zA)^{-1} \| < K(1 - |z|)^{-\lambda}, \quad |z| < 1 \]

applying the results of Lemma 1 we obtain

\[(2.13) \quad \| A^\nu \| < Ke(\nu + 1)^{\lambda} . \]

It is an open problem whether the estimate (2.13) is sharp in some infinite dimensional Banach space. The following result enables one to improve the inequality (2.13) for matrices (i.e. \( B \) is finite dimensional).
Theorem 2. Consider all polynomials $p(z)$ and $q(z)$ of degrees $m$ and $n$ respectively such that the function $f(z) = p(z)/q(z)$ satisfies (2.5). Suppose that $x > 1$. Then there exists a positive constant $P(a,m,n)$ such that

$$|a_j| < P(a,m,n)^{(a-1)}.$$  

To prove this theorem we need the following lemma.

Lemma 2. Let $p(z)$ be a polynomial of degree $m$. Then there exists a constant $K(m)$ such that

$$\max_{|z|=r} |p(z)| \leq K(m) \max_{|z|=1} |p(\rho e^{i\theta})|.$$

Proof. It is enough to consider the case $r = 1$ with $p(z)$ of the form

$$p(z) = \prod_{\alpha=1}^{m} (z - \zeta_i), \quad |\zeta_1| < \cdots < |\zeta_m|.$$  

For $m = 1$ it suffices to choose $K(1) = 5$. Let $m > 1$. Define

$$K'(m) = \max_{0 < |\zeta| < \cdots < |\zeta_m|} \max_{|z|=1} \max_{|\rho|<\frac{1}{4}} |p(\rho e^{i\theta})|.$$  

In case that $|\zeta_m| > 1$ let $q(z) = \prod_{\alpha=1}^{m-1} (z - \zeta_i)$. Then

$$\max_{|z|=1} |p(z)| \leq (|\zeta_m| + 1) \max_{|z|=1} |q(z)|$$  

and

$$\leq 2(|\zeta_m| - 1)K(m - 1) \max_{|\rho|<\frac{1}{4}} |q(\rho e^{i\theta})| < 2K(m - 1) \max_{|\rho|<\frac{1}{4}} |p(\rho e^{i\theta})|.$$  

Put

$$K(m) = \max(K'(m), 2K(m - 1))$$  

and the lemma follows.

Proof of Theorem 1. Without restriction in generality we may assume that $p(z)$ and $q(z)$ do not have common zeros. Also it is enough to consider the case $K = 1$. The inequality (2.5) implies that we can choose $q$ and $p$ of the form

$$p(z) = \prod_{\alpha=1}^{m} (z - \zeta_i), \quad q(z) = \prod_{\alpha=1}^{n} (z - \zeta_i).$$  

Then
The inequality (2.5) yields \( |\zeta_i| < 1, i = 1, \ldots, n \). Put

\[
(2.16) \quad p(z) = z A \prod_{i=1}^{n-1} (1 - z \zeta_i), \quad q(z) = \prod_{i=1}^{n} (1 - z \zeta_i).
\]

Also

\[
(2.17) \quad g(z) = \prod_{i=1}^{n} (z - \zeta_i) / (z - \zeta_i),
\]

\[
(2.18) \quad |g(z)| < |z|^{m-n+q}/(|z| - 1)^2, \quad |z| > 1.
\]

Note that

\[
(2.19) \quad g(z) = \sum_{v=0}^{\infty} a_v z^{-v-n+m}, \quad |z| > 1.
\]

Let \( D_1, \ldots, D_p \) be \( p \)-mutually disjoint, open and bounded domains with the boundary \( \Gamma_1, \ldots, \Gamma_p \) respectively. Assume that \( \zeta_1 \in \bigcup_{j=1}^{p} D_j, i = 1, \ldots, n \). Then we obtain

\[
(2.20) \quad a_v = \sum_{j=1}^{P} (2\pi i)^{-1} \int_{\Gamma_j} g(z) z^{(v+n-m)} \, dz.
\]

To obtain the estimate (2.14) we are going to choose the domains \( D_1, \ldots, D_p \) according to the configuration of \( \zeta_1, \ldots, \zeta_n \) and the value of \( v \). First we group the points \( S_1, \ldots, S_q \) following Morton (1962). Let \( \zeta_{i_1} \) be one of the points with the largest modulus, \( |\zeta_{i_1}| = 1 - \delta_1 > |\zeta_i|, i = 1, \ldots, n \). Then we form \( S_1 \) from all those points which can be joined to \( \zeta_{i_1} \) by a chain of points, each link of which has length \( \zeta_{i_1} \).

In the same way \( S_2 \) is formed from the remaining points, and so on until all the points have been included in some \( S_q \). For each \( S_q \) we denote by \( 1 - \delta_q \) and \( 1 - \zeta_q \) the modulus of the largest and the smallest \( |\zeta_q|, \zeta_q \in S_q \). We rename \( \zeta_1, \ldots, \zeta_n \) so that
Consider any particular $S_j$ and let us denote its members by $\lambda_i, i = 1, 2, \ldots, k$, where

$1 - \zeta < |\lambda_i| < 1 - \zeta, i = 1, \ldots, k$. Let us also denote the points not in $S_g$ by $u_j, j = 1, 2, \ldots, n - k$. We claim

\begin{equation}
0 < \zeta < \cdots < \zeta_g.
\end{equation}

Indeed, the first two inequalities follow immediately from the assumption that there exists a chain of at most $k$ points between $\lambda_i$ and $\lambda_j$ such that the distance of any link $< \zeta$. The last inequality is a consequence of $u_j$ not being in $S_g$. Let

\begin{equation}
(2.23) \quad h(z) = \sum_{i=1}^{m-1} (z - u_i).
\end{equation}

For $\lambda_t \in S_g$ put

\begin{equation}
(2.24) \quad \eta = (1 + 2\zeta) |\lambda_t| / |\lambda_t|.
\end{equation}

Then

\begin{equation}
(2.25) \quad h(z) = \sum_{j=0}^{m-2} h_j(z - \eta)^3.
\end{equation}

We now estimate $h_j$. Let $T$ be a circle $|z - \eta| = \zeta$. Then

\[
|z - \zeta| < |z - \lambda_t| + |\lambda_t - \zeta| < |z - \eta| + |\eta - \lambda_t| + |\lambda_t - \zeta|.
\]

\[
= \zeta + 1 + 2\zeta - |\lambda_t| + |\lambda_t - \zeta| < (k + 3)\zeta + |\lambda_t - \zeta|.
\]

where the last inequality follows from (2.22). In particular

\[
|z - \lambda_j| < 2(k + 1)\zeta.
\]

in view of (2.22). Apply the Cauchy formula for $h_j$ and use (2.18) to get
\[ h_j = \frac{1}{(2\pi)^{-1}} \int h(z) dz (z-n)^{-(j+1)} \leq \delta^{-1} |z-n|^{-1} \]

(2.26)

\[ \leq \frac{1}{(2(k+1))^{n-k} \delta} \sum_{i=1}^{n-k} \frac{1}{\Gamma(k+3) \delta} + |\lambda_t - \nu_1| \]

We now consider the following three cases

(i) \( \delta \geq 1/(2n+2n) \),

(ii) \( \delta < 1/(4n) \),

(iii) neither (i) nor (ii) holds.

Case (i). Let \( C_k \) be a disc \(|z - \zeta_j| < \delta \), for \( \zeta_j \in S_k \). Then

(2.27)

\[ D = \bigcup_{i=1}^{n} C_i = \bigcup_{j=1}^{p} D_j \]

where each \( D_j \) contains a subset of some \( S_k \) and \( D_j \cap D_k = \emptyset \) for \( j \neq k \). Let \( \Gamma_j \) be the boundary of \( D_j \). Then \( \ell(\Gamma_j) \) - the length of \( \Gamma_j \) - satisfies the inequality

(2.28)

\[ \ell(\Gamma_j) \leq 2n(n) \delta \]

where \( n(D_j) \) is the number of points \( \zeta_1, \ldots, \zeta_n \) in \( D_j \). Let \( z \in \Gamma_j \). Clearly

\[ z = \lambda_t + \omega, \quad |\omega| = \delta \leq 2 \delta \]

Thus

(2.29)

\[ \sum_{i=1}^{n} |z - \zeta_i|^{-1} \leq \sum_{j=1}^{n-k} \frac{1}{\Gamma(k+3) \delta} + |\lambda_t - \nu_1|^{-1} \]

Also for \( n \) of the form (2.24) we have
\[ |z - n| < |z - \lambda_t^+| + |\lambda_t^- - n| < 2 \delta_\beta + 1 + 2 \delta_\beta - |\lambda_t^-| < (k + 3) \delta_\beta. \]

Combine (2.25)-(2.26) with the above equality to deduce

\[ \text{(4.30)} \quad |h(z)| \leq [2(k + 3)]^{n+t} |z|^{2} \sum_{i=1}^{n} |\lambda_i - \mu_i|. \]

Finally we deduce

\[ \text{(2.31)} \quad |g(z)| \leq [16(n + 2)]^{n+t} |\delta_\beta - \alpha|. \]

Using the equality (2.20) and the inequalities (2.28), (2.31) for \( v \geq m - n \) we get

\[ |g(z)| \leq \frac{1}{16(n + 2)} \max_{n} |g(z)|^{(v+n-1)/2} dz | \leq n/16(n + 2)^{n+t} |\delta_\beta - \alpha|^{v+1} \]

\[ \leq n^{3}[16(n + 2)]^{n+t} |\alpha + 1|, \]

as \( \alpha > 1 \). Thus we have shown (2.14) \( (K = 1) \).

**Case (iii).** Let \( C_i \) be an open disc with center at \( \zeta_i/|\zeta_i| \) and radius \( 1/2v \). Form \( D \) by (2.27). Assume that \( z \in \Gamma_j \). So

\[ z = \zeta_i/|\zeta_i| + \sigma, \quad |\sigma| = \frac{1}{2v}. \]

We now estimate

\[ K(\Gamma) = \max_{z \in \Gamma} |h(z)|, \quad \Gamma = \{z, z = \frac{\zeta_i}{|\zeta_i|} (1 + \frac{1}{2v} e^{i \theta}), \quad |\theta| < \frac{\pi}{4} \}. \]

According to (2.18)

\[ K(\Gamma) \leq e(4v)^{3}[\max_{z \in \Gamma} \frac{n}{t=1} |z - \zeta_t|], \]

for \( v \geq m - n + a \). Let \( t_i = (1 + \frac{1}{2v}) |\zeta_i|/|\zeta_i| \). Clearly \( t_i \in \Gamma_j \). We claim that for \( z \in \Gamma \) or \( z \) of the form (2.32) which is in \( \Gamma_j \) we have

\[ \frac{|\zeta_i|}{3} \leq |z - \zeta_t| \leq 3|\zeta_i - \zeta_t|. \]

Indeed it is easy to see that for such \( z \) the following inequalities hold:
\[ |z - z_t| > \frac{1}{40}, \quad |\eta_n - z_t| > \frac{1}{20}, \quad |z - \eta_n| < \frac{1}{7}. \]

So

\[ |\eta_n - z_t| < |z - \eta_n| + |z - z_t| < 4|z - z_t| + |z - \eta_n| = 5|z - z_t|, \]
\[ |z - z_t| < |z - \eta_n| + |\eta_n - z_t| < 2|\eta_n - z_t| + |\eta_n - \eta_n| = 3|\eta_n - \eta_n|. \]

Therefore

\[ K(7) < e^{3(4\nu)^2} \sum_{t=1}^{n} |z - z_t|. \]

Let \( z = \frac{\zeta}{I} > 0 \), \( \lambda \) = \( \frac{1}{2\nu} \). Then by Lemma 2 and the above inequalities

\[ |g(z)| < K(7)K(m - \xi) < K(m - \xi)(15)\nu e^{(4\nu)^2} \]
\[ \sum_{t=1}^{n} |z - z_t|. \]

and

\[ |g(z)|^{v+m-n-1} < K(m - \xi)(15)^2 \nu e^{(1 + \frac{1}{2\nu})^{v+m-n-1}} < K(m - \xi)(15)^2 \nu e^{2} \]

for \( v + m - n + 1 < 3 \). As the length of the boundary of \( D \) does not exceed \( \alpha n/\nu \) from (2.20) we get

\[ \lambda < K(m - \xi)(15)^2 \nu e^{2} \alpha^{-1}. \]

Case (iii). In this case we claim that there exists \( 1 < \gamma < s \) such that

\[ \delta_{s+1} = \frac{1}{2^n+2n+\nu} \max_{i \leq s} \delta_i , \quad \xi = 0, \ldots, \gamma - 1, \quad \zeta_{\gamma} \neq \xi_\gamma. \]

and

\[ \delta_{\gamma+1} > \frac{1}{2^n+2n+\nu} \max_{i \leq \gamma} \zeta_i , \]

otherwise either (i) or (ii) hold. (Note the inequality 2.21). Put

\[ t = \left( \max_{t} \xi_t \right) + \frac{1}{2^n+2n+\nu}. \]
It is not difficult to show that $r < \frac{1}{2\nu}$. Let $C_1 \subseteq \mathbb{R}$. For $\theta < \gamma$ denote by $C_1$ a disc with center at $z_1/|z_1|$ and radius $r$. For $\theta > \gamma$ let $C_1$ be a disc with center at $z_1$ and radius $\frac{1}{2}$. As before define $D$ by (2.27). Now estimate $a_\nu$ from the equality (2.20) using the arguments of the Cases (i) and (ii) in accordance with $\theta > \gamma$ or $\theta < \gamma$ to induce (2.14). This concludes the proof of Theorem 2.

Remark 1. A special case of Theorem 2, namely $\alpha = 1$ and $m = n - 1$ was established in Morton [1964].

Proof of Theorem 1.

(A) $\Rightarrow$ (P). Follows immediately from (2.3).

(P) $\Rightarrow$ (A). Let $(I - zA)^{-1} = (f_{ij}(z))_{ij}$.

Then $f_{ij}(z) = p_{ij}(z)/q_{ij}(z)$, where the degrees of $p_{ij}$ and $q_{ij}$ are $n - 1$ and $n$ respectively. Now (1.3) follows from Theorem 2.

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Shmuel Friedland

Mathematics Research Center, University of Wisconsin
610 Walnut Street
Madison, Wisconsin 53706

U.S. Army Research Office
P.O. Box 12211
Research Triangle Park, North Carolina 27709

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)
Let A be a set of n x n complex matrices A which satisfy the condition

\[ \left\| \left( I - zA \right)^{-1} \right\| \leq \frac{K}{\left( 1 - \left| z \right| \right)^{n+1}} \]

for some \( \alpha \geq 0 \) and all \( \left| z \right| < 1 \). Then it is shown here that there exists a constant \( B(n) \) such that

\[ \left\| A^k \right\| \leq K \left( B(n) \right)^{\frac{k}{\alpha}}, \]

\( k = 0, 1, \ldots \). This forms a generalization of the Kreiss resolvent condition

\( \alpha = 0 \).