ABSTRACT

This paper discusses autoregressive random field (ARF) models and derives a unilateral representation driven by uncorrelated noise.

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# A UNILATERAL REPRESENTATION FOR AUTOREGRESSIVE RANDOM FIELD MODELS

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### Unilateral Representation

This paper discusses autoregressive random field (ARF) models and derives a unilateral representation driven by uncorrelated noise.
1. Introduction

In this paper we shall deal with a subset of stationary (wide sense) processes with absolutely continuous spectral distributions which are rational functions of the two quantities $e^{i\theta_1}, e^{i\theta_2}$. More precisely we shall study the process $X_{[m,n]} \in \mathbb{R}^d, [m,n] \in \mathbb{Z} \times \mathbb{Z}$ on an infinite lattice, with covariance structure

$$E(X_{[m+s,n+t]}X_{[m,n]}^*) = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} e^{-is\theta_1 - it\theta_2} \rho(\theta_1, \theta_2) d\theta_1 d\theta_2,$$

(1.1)

and zero mean.

We assume $\rho(\theta_1, \theta_2)^{-1}$ exists and is finite at every $(\theta_1, \theta_2)$, and

$$\rho(\theta_1, \theta_2)^{-1} = (a[0,0]^+ \sum_{[m_1,m_2] \in \mathbb{N}^p} a[m_1,m_2]\cos(m_1\theta_1 + m_2\theta_2))$$

(1.2)

$a[0,0], a[m_1,m_2]$ are $p \times p$ matrices satisfying $a_{[s,t]} = a_{[-s,-t]}^*$. $V^*$ is the complex conjugate transpose of the vector $V$. $\mathbb{N}^p$ denotes the deleted $p \times p$ neighborhood of $[0,0]$, that is,

$$\{[m_1,m_2] : |m_1| \leq p, |m_2| \leq p, [m_1,m_2] \neq [0,0] \}.$$

Models of this type have been used as models of texture images [1,2]. In the case where $X[..]$ is a Gaussian process, it can be shown [3] that $X[..]$ is a Gauss-Markov process with respect to $\mathbb{N}^p$; that is,
\[ P \cdot \text{ob}(X[m,n] | X[s,t], [s,t] \notin [m,n]) = \]
\[ \text{Prob}(X[m,n] | X[s,t], [s,t] \in [m,n] + N^P) \]  
(1.3)

In fact, the process with spectral density \( f(\theta_1, \theta_2) \) satisfies the conditional model
\[ E(X[m,n] | X[s,t], [s,t] \notin [m,n]) = \]
\[ -a_{[0,0]}^{-1} \sum_{[m_1, m_2] \in N^P} a[m_1, m_2] x[m-m_1, n-m_2] \]  
(1.4)

Conditional models of this type have been found useful in the modeling of spatial patterns [7]. It is also known (see, for example, page 26 of [7]) that no finite one-sided representation for this model exists of the type (with \( S \) finite subset of \( \mathbb{Z} \times \mathbb{Z} \))
\[ b_{[0,0]} x[m,n] + \sum_{[m_1, m_2] \in S} b[m_1, m_2] x[m-m_1, n-m_2] = z[m,n] \]

where \( z[m,n] \) is a process of uncorrelated noise.

The purpose of this paper is to show that the collection of spectral representations of the process \( X[..] \) along one of the coordinates is representable as a one-sided finite order "time series" model along the other coordinate. Thus, in this sense it is seen that all ARF's have a "causal" representation.

This method of producing a one-sided representation can be contrasted with the so-called NSHP (non-symmetric half plane) representation of [3] and [6].
2. A Unilateral Representation

We consider the process $X_{[\ldots]}$ with spectral density

\[ f(\theta_1, \theta_2) = \left[a_{[0,0]} + \sum_{[m_1, m_2] \in \mathbb{N}} a_{[m_1, m_2]} \cos(m_1 \theta_1 + m_2 \theta_2) \right]^{-1}, \tag{2.1} \]

which is a $p$-th order autoregressive process.

Let $z = e^{-i\theta_1}, w = e^{-i\theta_2}$, and rewrite the above equality as

\[ f(\theta_1, \theta_2)^{-1} = a_0(w) + a_1(w)z + \ldots + a_p(w)z^p \]

\[ + a_1^*(w)z^{-1} + \ldots + a_p^*(w)z^{-p}. \]

For each fixed $w$ we can consider $f(\theta_1, \theta_2)$ as a spectral density in $\theta_1$. We next produce a causal factorization of $f(\theta_1, \theta_2)$ in the form

\[ f(\theta_1, \theta_2)^{-1} = \left[ c_0^*(w) + c_1^*(w)z^{-1} + \ldots + c_p^*(w)z^{-p} \right] \left[ c_0(w) + c_1(w)z + \ldots + c_p(w)z^p \right], \tag{2.2} \]

where, for each $w = e^{-i\theta_2}$, $c_0(w) + c_1(w)\xi + \ldots + c_p(w)\xi^p$ has no roots inside and on the complex unit circle $|\xi| = 1$. ([4], page 65).

We next consider the spectral representation of the process $X_{[\ldots]}$ along the second coordinate:

\[ X_{[n,m]} = \int_0^{2\pi} \int_0^{2\pi} dY_n(\theta), \tag{2.3} \]

where $Y_n(\theta)$ is the spectral representation of the process $X_{[n,\ldots]}$. ([5], page 481).

Next expand each of $c_0(w), \ldots, c_p(w)$ in a Fourier expansion.
\[ c_j(w) = \sum_{k=-\infty}^{\infty} e^{ik \theta} c_{[j,k]} \]

Then \((\text{[4]}, \text{page 61})\) the process satisfies the autoregression

\[ \sum_{j=0}^{\infty} \sum_{k=-\infty}^{\infty} c_{[j,k]} X_{n-j,m+k} = Z_{[n,m]} \]

where \(Z_{[\ldots]}\) is an uncorrelated white noise process. Let \(W_n(\theta)\) be the spectral representation of the process \(Z_{[n,\ldots]}\):

\[ z_{[n,m]} = \int_{\pi}^{\pi} e^{-im \theta} dW_n(\theta) \]

We conclude with the following

**Theorem:** Let \(\{Y_n(\theta)\}, \{W_n(\theta)\}\) be the spectral representations of the processes defined above. They satisfy the stochastic differential equation

\[ \sum_{k=0}^{\infty} c_k (e^{-i \theta}) dY_{n-k}(\theta) = dW_n(\theta) \]

**Proof:** In the above autoregressive representation we substitute the spectral integrals and get (after combining terms)

\[ \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \sum_{j=0}^{\infty} \sum_{k=-\infty}^{\infty} c_{[j,k]} e^{i(m+k) \theta} dY_{n-j}(\theta) - e^{-i \theta} dW_n(\theta) = 0 \]

Factoring out \(e^{im \theta}\) we have

\[ \int_{-\pi}^{\pi} e^{im \theta} \sum_{j=0}^{\infty} \sum_{k=-\infty}^{\infty} c_{[j,k]} (e^{-i \theta}) dY_{n-j}(\theta) - dW_n(\theta) = 0 \]

As any continuous function \(f(\theta), \theta \in [-\pi, \pi]\) can be approximated in mean square by linear combinations of \(e^{im \theta}\), the result follows.
3. The finite version.

The above calculation can be carried out in the case where we have a finite number of values

\[ X[n,0], \ldots, X[n,M-1] \]

in the vertical direction.

Let \( \psi_M = e^{2\pi i/M} \). The finite versions of the above spectral representations are as follows:

\[
X_{[1,m]} = \sum_{k=0}^{M-1} \psi_M^{km} Y(n,k) \quad (3.1)
\]

or

\[
\Delta Y(n,k) = \sum_{j=0}^{1} \psi_M^{-jk} X[n,j]. \quad (3.2)
\]

That is, \( \Delta Y(n, \cdot) \) is the finite Fourier transform of the data \( X[n, \cdot] \). Similarly

\[
\Delta W(n,k) = \sum_{j=0}^{1} \psi_M^{-jk} Z[n,j]. \quad (3.3)
\]

The finite analogue of the above Theorem is

\[
\sum_{k=0}^{M-1} \sum_{j=0}^{M} \psi_M^{-k} \{ \sum_{n-j} \psi_M^{-k} \Delta Y(n-j,k) - \Delta W(n,k) \} = 0
\]

concluding with

\[
\sum_{n-j} \psi_M^{-k} \Delta Y(n-j,k) = \Delta W(n,k). \quad (3.4)
\]
References

1. R. Chellappa, "Fitting Random Field Models to Images", TR-928, Computer Vision Laboratory, University of Maryland, College Park, Maryland, August, 1980.


