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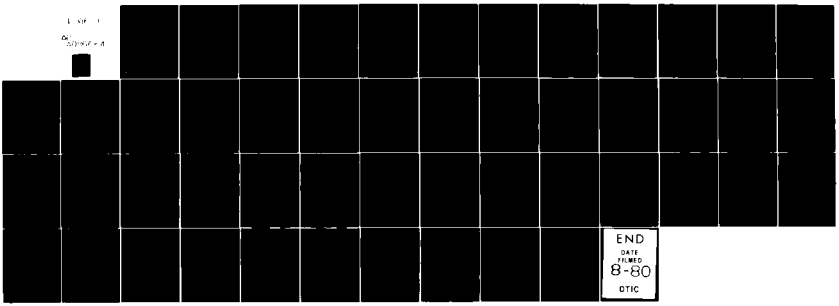

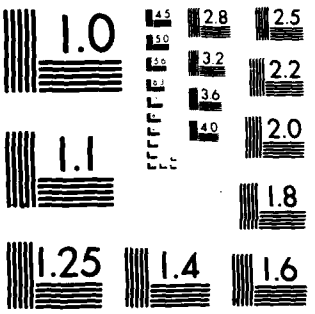

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THE STABILITY OF A COMPRESSIBLE FREE SHEAR LAYER

D. F. ROSCOE AND W. L. HANKEY

COMPUTATIONAL AERODYNAMICS GROUP
AERODYNAMICS AND AIRFRAME BRANCH
AIR FORCE FLIGHT DYNAMICS LABORATORY

OCTOBER 1979

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Inviscid flows with an inflection point in the velocity profile have been identified by Lord Rayleigh (1880) to be unstable. Michalke (1964) computed the amplification factors of the instability for a shear layer in incompressible flow. This report extends this work to include compressible flows. The results indicate large amplification rates occur through the transonic range but reduce drastically for supersonic flows.

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FOREWORD

This report is the result of work carried on in the Computational Aerodynamics Group, Aerodynamics and Airframe Branch, Aeromechanics Division, Flight Dynamics Laboratory, Air Force Wright Aeronautical Laboratories. It was performed by Dr. David Roscoe, visiting scientist, and directed by Dr. Wilbur L. Hankey under project 2307N418. Dr. Roscoe performed the work from January 1978 through September 1978, while employed by the University of Dayton Research Institute under Air Force Contract F33615-76-C-3145.

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1. Introduction

It is known that a two dimensional flow of inviscid fluid is unstable with respect to small disturbances if the velocity profile has an inflexion point in it. Rayleigh in 1880, [1] showed that the condition was necessary, and Tollmien [2] showed that under certain conditions, that it was sufficient.

The purpose of this report is to investigate the effects of compressibility on the stability of a free shear layer, whose profiles have points of inflexion on them. The hope is that the mechanisms involved in this type of instability will explain the dissimilarites observed between the stabilities of subsonic and supersonic flows.

As a starting point, we use Michalkes' work [3] that investigates the stability of the hyperbolic tangent velocity profile in an incompressible flow. We use this profile first to confirm (for our own comfort) Michalkes' results in the incompressible care, and then as a vehicle on which to complete the compressible study. The point of using this profile is that it can be analytically manipulated, allowing a relatively simple computational code to be used.

Before doing the compressible study proper, we first eliminated the possibilty of viscosity being a significant factor. This was done by demonstrating, that, for the incompressible case, viscosity has only a

slight attenuating effect on the type of instability being studied. This was the case for the range of Reynolds numbers $10^3 \leq Re < \infty$.

2. Governing Equations

The stability equation for a two dimensional inviscid compressible flow [4] is given by

$$\frac{d}{dy}\left[\frac{(v - c)\phi' - v'\phi}{g}\right] = \alpha^2(1 + 0.2M^2)(v - c)\phi \tag{2.1}$$

$$g = (1 + 0.2M^2)^{-1} - M_o^2(v - c)^2$$

$$M^2 = \frac{M_\infty^2 v^2}{1 + 0.2M_\infty^2(1 - v^2)}$$

$$M_o^2 = \frac{M_\infty^2}{1 + 0.2 M_\infty^2}$$

where $v(y)$ is the imposed velocity profile, M_∞ is the free stream Mach number, $\phi(y)$ is the complex amplitude of the disturbance and α is the complex wave number of the disturbance. A derivation is sketched in Appendix 1. Boundary conditions on (2.1) are derived by supposing that for an unbounded velocity profile, any finite disturbance will vanish at infinity. Thus, for boundary conditions on (2.1), we have

$$\phi(-\infty) = \phi(+\infty) = 0 \tag{2.2}$$

Equation (2.1) is derived by assuming a disturbance of the form

$$\Delta = \phi(y)\, e^{j\alpha(x - ct)}$$

Thus, the disturbance is fully described when $\phi(y)$, α and c are known.

Clearly, equation (2.1) with (2.2) is insufficient to determine all of these quantities, and so some further restriction is necessary. The classical viewpoint, [4] is that such disturbances were temporal by

nature, which is equivalent to saying that α is real (i.e. spatially bounded). Thus, for a specified wave number α (2.1) with (2.2) reduces essentially to a characteristic value problem, with c as the characteristic value and $\phi(y)$ as the characteristic function. In recent years, however, the view has formed that the disturbances are spatial in character, in which case αc is taken to be a real quantity. In this case if $\alpha = \alpha_R + j\alpha_I$ and $c = c_R + jc_I$, then αc = real implies $\alpha_R c_I + \alpha_I c_R = 0$ or

$$\alpha_I = -\alpha_R c_I / c_R. \tag{2.3}$$

Thus, for a specified α_R and α_I given by (2.3), and (2.1) with (2.2) again becomes a more or less conventional characteristic value/function problem in (c, ϕ) respectively.

Both approaches were considered in this study, and the conclusions concerning the behavior of the instabilities, with varying Mach number, were the same with both approaches. In the light of this coincidence we chose to consider the disturbance as temporal, mainly to facilitate comparison with published work (Michalke, [3]).

Thus, the mathematical problem as so far defined is as follows: for a specified range of real α's, solve the characteristic value problem defined at (2.1) and (2.2). Repeat the process for a range of M_∞ (free stream Mach number).

3. Organization of Equations into a More Convenient Form.

The characteristic value problem as defined at (2.1) and (2.2) is not easily solved, but a solution can be made easier by following the procedure

Michalke used for his incompressible study. Putting $M_\infty = 0$ into (2.1) gives, after rearrangement

$$(v - c)(\phi'' - \alpha^2\phi) - v''\phi = 0 \tag{3.1a}$$

with

$$\phi(-\infty) = \quad (+\infty) = 0 \tag{3.1b}$$

This is Rayleighs' stability equation, the subject of Michalkes' study. He reduced (3.1a) from a second order equation into a first order equation using the transformation

$$\phi(y) = \exp\,[\int\theta dy] \tag{3.2}$$

to obtain from (3.1a) the equation

$$\theta' + \theta^2 - \alpha^2 - \left(\frac{v''}{v - c}\right) = 0 \tag{3.3}$$

To obtain the boundary condition(s) on θ, we use the facts that $v''(-\infty) = v''(+\infty) = 0$ in (3.1a). Thus, at $y = \pm\infty$ (3.1a) gives

$$\phi'' - \alpha^2\phi = 0 \tag{3.4}$$

yielding at $y = \pm\infty$.

$$\phi(y) = \exp[\pm\alpha y] \tag{3.5}$$

Since we know that as $y \to -\infty$ then $\phi(y) \to 0$ then (3.5) gives as $y \to -\infty$, then $\phi(y) \to \exp[-\alpha y]$ (assuming $\alpha > 0$) and likewise, as $y \to +\infty$ then $(y) = \exp(+\alpha y)$. Comparison of these two solutions with (3.2) yield that as

$$y \to -\infty, \quad \text{then} \quad \theta \to +\alpha$$

and as (3.6)

$$y \to +\infty, \quad \text{then} \quad \theta \to -\alpha$$

Applying (6) onto the equation of the compressible case for $M_\infty > 0$ and following similar arguments for boundary conditions we obtain the system

$$(v - c)[g(\theta' + \theta^2) - g'\theta] = \alpha^2 g^2 (v - c)(1 + 0.2M^2) + v''\,g - v'g' \tag{3.7}$$

with the corresponding boundary conditions

$$\theta(-\infty) = \alpha \mathrm{Sign}(\alpha) \sqrt{g(-\infty)}$$
$$\theta(+\infty) = -\alpha \mathrm{Sign}(\alpha) \sqrt{1 + 0.2M_\infty^2} \quad \sqrt{g(+\infty)} \tag{3.8}$$

Here, the function Sign (α) is used to account for the (non-physical) mathematical possibility of α being negative. See Appendix 2 for details.

Superficially, the system (3.7) and (3.8) is overdetermined, in that (3.7) is first order, and yet (3.8) consists of two boundary conditions, to be satisfied by (3.2), but we must keep in mind that c is a parameter at our disposal, and may be adjusted to ensure the second (boundary) condition, given the first.

The complex system of (3.7) and (3.8) yields the pair of simultaneous equations.

$$\frac{d\theta_R}{dy} = -(\theta_R^2 - \theta_I^2) + \frac{(AC + BD)\,\theta_R + (AD - BC)\theta_I}{(A^2 + B^2)} + \frac{(AR_1 - BR_2)}{(A^2 + B^2)}$$
$$\frac{d\theta_I}{dy} = -2\theta_R\theta_I + \frac{(BC - AD)\,\theta_R + (BD + AC)\theta_I}{(A^2 + B^2)} + \frac{(BR_1 + AR_2)}{(A^2 + B^2)} \tag{3.9}$$

where

$$\theta \equiv \theta_R + j\theta_I$$
$$c \equiv c_R + jc_I$$
$$g \equiv g_R + jg_I$$
$$A = g_R(v - c_R) + g_i\, c_I$$
$$B = -g_I(v - c_R) + g_R c_I \tag{3.10}$$
$$C = g'_R(v - c_R) + g'_I c_I$$

$$D = -g'_I(v - c_R) + g'_R c_I$$

$$R_1 = \alpha^2(1 + 0.2M^2)[(g_R^2 - g_I^2)(v - c_R) + 2c_I g_R g_I] + v''g_R - v'g_R' \tag{3.10}$$

$$R_2 = \alpha^2(1 + 0.2M^2)[-c_I(g_R^2 - g_I^2) + 2g_R g_I(v - c_R)] + v''g_I - v'g_I'$$

with the boundary conditions as $y \to -\infty$

$$\theta_R = +\alpha \text{ Sign}(\alpha)(g_R^2 + g_I^2)^{\frac{1}{4}} \text{ Cos } \hat{\theta}/2$$

$$\theta_I = +\alpha \text{ Sign}(\alpha)(g_R^2 + g_I^2)^{\frac{1}{4}} \text{ Sin } \hat{\theta}/2$$

$$\hat{\theta} = \text{Principal } (\text{Tan}^{-1} g_I/g_R)$$

and

as $y \to +\infty$

$$\theta_R = -\alpha \text{ Sign}(\alpha)(1 + 0.2M_\infty^2)^{\frac{1}{2}} (g_R^2 + g_I^2)^{\frac{1}{4}} \text{ Cos } \hat{\theta}/2$$

$$\theta_I = -\alpha \text{ Sign}(\alpha)(1 + 0.2M_\infty^2)^{\frac{1}{2}} (g_R^2 + g_I^2)^{\frac{1}{4}} \text{ Sin } \hat{\theta}/2 \tag{3.10}$$

$$\hat{\theta} = \text{Principal } (\text{Tan}^{-1} g_I/g_R).$$

The derivations are given in Appendix 2. The system (3.9) and (3.10) is still not in a suitable form to be solved numerically, because the independent variable y is over the range $-\infty < y < +\infty$. To eliminate this problem, we introduce the independent variable transformation

$$z = 0.5(1 + \text{Tanh } (y)). \tag{3.11}$$

This transforms the range $-\infty < y < +\infty$ onto $0 \leq z \leq 1$. This transformation has a simple effect on (3.9), viz: if we label the two right hand sides

(3.9) as RHS_1 and RHS_2 respectively, then (3.9) becomes

$$\frac{d\theta_R}{dz} = \frac{1}{2z(1-z)} \quad (RHS_1)$$

and (3.12)

$$\frac{d\theta_I}{dz} = \frac{1}{2z(1-z)} \quad (RHS_2)$$

The boundary conditions remain the same, except that instead of being evaluated at $y = \pm\infty$, they are evaluated at $z = 0, 1$.

It is clear that any numerical treatment of (3.12) requires care at the two points $z = 0$ and $z = 1$. This treatment is described in Appendix 3.

There is a further point that needs to be clarified. It is indicated in Appendix 6 that when $c_I = 0$, i.e. at the neutral points, then α is necessarily real for $M_\infty \leq 2.5$. In turn, this leads to $g_I = 0$ so that in (3.12), the denominator $A^2 + B^2$ reduces to $g^2(v - c)^2$. Since $c \simeq 0.5$ and $0 \leq v \leq 1$, then clearly the region about $v = c$ needs to be carefully treated. This treatment is given in Appendix 4.

4. Strategy

Before describing in detail how to solve the system (3.9) and (3.10), it is instructive to consider the objective. We are primarily interested in the change in behaviour of the instabilities associated with a given velocity profile as Mach number increases in the range $0 \leq M_\infty \leq 2.5$, (the upper limit being decided by considerations to be discussed later). From [3] it can be seen that for temporal disturbances, for a given wave number α, the growth of the disturbance is determined by $e^{\alpha c_i}$. The intensity of a disturbance is experimentally observed to depend on its wave length $\lambda(=2\pi/\alpha)$.

It is instructive to plot $e^{\alpha c_I}$ against λ for a given, fixed M_∞. By forming a set of such curves for a discrete set of values $\{M_\infty\}$, a visual idea may be obtained of the effect of Mach number on instability.

Furthermore, from Appendix 1, we see that the argument of the derivative in equation (2.1) is directly related, in a fairly simple fashion to the complex static pressure field driving the disturbance. Since static pressure measurements may be (indirectly) obtained quite easily from experiment, it makes sense to plot the pressure fields for various Mach numbers.

5. Numerical Procedure

For a given α and M, the system (3.9) and (3.10) has the basic form

$$\begin{aligned} \dot{\theta}_R &= f_R(\theta_R, \theta_I, c_R, c_I, z) \\ \dot{\theta}_I &= f_I(\theta_R, \theta_I, c_R, c_I, z) \\ \theta_R(z = 0) &= \theta_{RO} \\ \theta_I(z = 0) &= \theta_{IO} \end{aligned} \tag{5.1}$$

where the pair (c_R, c_I) has to be such that the following conditions are true:

$$\begin{aligned} \theta_R(z = 1) &= \theta_{R_1} \quad \text{(specified)} \\ \theta_I(z = 1) &= \theta_{I_1} \quad \text{(specified)} \end{aligned} \tag{5.2}$$

The adopted procedure goes as follows: For a given (c_R, c_I), suppose that the solution of (5.1) at $z = 1$ is $[\theta_R(c_R, c_I, 1), \theta_I(c_R, c_I, 1)]$. Define the functional $F = [\theta_R(c_R, c_I, 1) - \theta_{R_1}]^2 + [\theta_I(c_R, c_I, 1) - \theta_{I_1}]^2$ (5.3)

The original characteristic value problem now reduces to the problem of minimizing F at (5.3) with respect to c_R and c_I. Clearly, $F_{min} = 0.0$, and this state occurs when (c_R, c_I) is such that

$$\begin{aligned} \theta_R(c_R, c_I, 1) &= \theta_{R_1} \\ \theta_I(c_R, c_I, 1) &= \theta_{R_1} \end{aligned} \tag{5.4}$$

i.e., F is minimized when the original characteristic value problem is solved. Thus, for a given fixed value of the pair (M_∞, α) we can generate the corresponding pair (c_R, c_I). To generate the trajectory (α, c_R, c_I) for a fixed M_∞, we advance α to $\alpha + \Delta\alpha$ and then calculate $(c_R + \Delta c_R, c_I + \Delta c_I)$. The process is repeated until a sufficient range of α is covered. As explained in the last section, we are interested in a plot of α against c_I, and so essentially, we form a projection of the trajectory (α, c_R, c_I) onto the (α, c_I) plane. In practice, it turns out that c_R varies only very slightly along such a trajectory (0.6% for $M_\infty = 1$), and so little if any information is lost by this particular representation.

5.1 Difficulties Involved

As with many apparently straight forward problems, hidden difficulties are often encountered. The present case is no exception. The whole method hinged upon our knowing for each value of M_∞, just one point on the trajectory (α, c_R, c_I). It transpired that it was very difficult to locate <u>any</u> point on it. The solution was to adopt the old African adage: "slowly, slowly catchee monkee...", and to proceed as follows. If the velocity profile is assumed to be the hyperbolic tangent profile, then at $M_\infty = 0$, it is known (Michalke) that

$$(\alpha, c_R, c_I) = (1, 0.5, 0.0) \tag{5.5}$$

is a point on the trajectory. It is the neutral point on the trajectory. We next assume that all trajectories corresponding to M_∞ in the range $0 \leq M_\infty \leq 2.5$ have such neutral points, $(\alpha, c_R, 0)$. Starting from $M_\infty = 0$ and by incrementing by some ΔM_∞, that is suitably small, a trajectory $(M_\infty, \alpha, c_R)^*$ may be generated for $c_I = 0$. From this trajectory, we can pick the neutral point $(\alpha, c_R, 0)$ for any of the trajectories M_∞ = constant. Knowledge of this point then allows the generation of the trajectories (α, c_R, c_I) for the given M_∞ = constant.

If the velocity profile is not the hyperbolic profile, it is just assumed that the neutral point for $M_\infty = 0$ lies in the region of $(\alpha, c_R, c_I) = (1, 0.5, 0)$ and a search is conducted. Having found that one point, everything proceeds as described above.

6. Results

The hyperbolic tangent profile is given by

$$v(y) = 0.5(1 + \text{Tanh}\ (y)). \tag{6.1}$$

In the range $-\infty < y < \infty$, then $0 \leq v(y) < +1$. At $y = 0$, there is a point of inflexion. The curve is sketched in Figure 1.

6.1 Effectiveness of Current Procedure

This profile was chosen because a study on the instabilities in an incompressible flow, using this profile, has been conducted by Michalke [3]. Thus, in the one case $M_\infty = 0$, his results form a ready basis for comparison with our results, giving us a measure of the effectiveness of the presently used procedure.

*Note: It transpires that for $c_I = 0$, then an algebraic relation $A(M_\infty, c_R) = 0$ exists. Details are explained in Appendix 4.

Table 1 displays the results of Michalke and those of the present method for $M_\infty = 0$. It is readily seen that (to within rounding error) perfect agreement exists. We were thus able to proceed with a certain degree of confidence in both the method and the code.

6.2 Effect of Viscosity

The next step was to determine the effect of Reynolds number on the instabilities in the flow [5]. The stability equation for the case of finite Reynolds numbers, and $M_\infty = 0$ is

$$(v - c)(\phi'' - \alpha^2\phi) + \frac{j}{\alpha Re}(\phi'''' - 2\alpha^2\phi'' + \alpha^4\phi) - v''\phi = 0 \qquad (6.2)$$

Application of the transformation procedure described in Section 3 yields the system

$$\Delta' + \Delta^2 - \alpha^2 = \theta$$

$$\frac{j}{\alpha Re}(\theta'' - \alpha^2\theta) + (v - c)\theta = v'' \qquad (6.3)$$

$$\phi = \exp(\int \Delta dy)$$

$$\theta(-\infty) = \theta(+\infty) = 0$$

The details of this derivation are given in Appendix 5. The system (23) was solved using a modified version of that procedure described in Section 5. See Appendix 5. Results are displayed only for the worst case, namely $Re = 10^3$, in Table 2. Column (a) gives the $Re = \infty$ case, and column (b) gives the $Re = 10^3$ case. It is plain that, even at this relatively low Reynolds number, the viscous attenuation of the instabilities is marginal.

Consequently, it was felt that viscosity could safely be ignored as a significant mechanism in the type of instabilities currently being considered.

6.3 Effect of Compressibility

Computations were performed for Mach numbers in the range $0 \leq M_\infty < 2.5$. The upper was forced upon us because at $M_\infty = 2.5$, the neutral point trajectory $(\alpha, c_R, 0)$ enters the complex plane, indicating the limit of the instability under consideration. See Appendix 6.

6.3.1 Behavior of the Neutral Point

Figure 2 displays the behavior of the neutral point $(\alpha, c_I) = (\alpha, 0)$ as M_∞ increases in the range $0 \leq M_\infty < 2.5$. Table 3 displays the same results numerically.

In essence, the neutral point appears to move monotonically from 1.0 to 0.0 as M_∞ moves from 0.0 to 2.5, indicating a progressive decrease in the range of unstable wavelengths $\lambda (= 2\pi/\alpha)$ for disturbances, as M_∞ increases. This last statement is qualified, because there seems to be an anomolous behavior present in α at about $M_\infty = 1.375$, not shown graphically, indicating the possible presence of a singularity in α. Unfortunately, the presently used code did not behave very well in the region concerned, and so we were unable to readily describe the events occurring.

From the gross behavior of the neutral point, two conclusions may be drawn - the results indicate

(a) Only large wavelengths are unstable $(\lambda > 2\pi\delta)$ or only low frequencies $(f < c_R/2\pi\delta)$ are unstable.

(b) At high Mach numbers, the range of unstable wavelengths is much less that at low Mach numbers.

These results appear to be consistent with the available experimental data [6, 7].

6.3.2 Behavior of Instabilities Over All Wavelengths

These results are displayed over the Mach number values $M_\infty = 0.0$, 0.5, 1.0 and 2.0. Tables 4a, 4b, 4c, and 4d give the numerical results for the four respective Mach numbers. Figures 3 gives the graphical displays. There are three conclusions to be drawn:

(a) There is apparently no high wavelength cut-off to the instability of disturbances.

(b) The magnitude of amplification decreases as Mach number increases

(c) The wavelength at which maximum amplication occurs increases as Mach number increases.

6.4 Determination of the Eigen Function

Solution of the characteristic equation produces eigen values for the complex velocity, $c(\alpha, M_\infty)$ and also the eigen function, $\phi(\alpha, M_\infty)$.

The function of most interest is the pressure $\hat{P}(y)$ since comparison with experimental pressure data may be possible. Referring to equation A.1.9 in Appendix 1

$$\frac{P}{\gamma \bar{P}} \equiv \Gamma = -\frac{j}{\alpha g}\left[(v - c)\phi' - v'\phi\right] \tag{6.4}$$

The real and imaginary values of Γ are shown in Figures 4 and 5 respectively. For a given M_∞, the ϕ plotted is the one whose associated c gives maximum amplification. For comparison with experiment the pressure may be obtained as follows:

$$P' = \text{Real } \hat{P}(y) \exp[j\alpha(x - ct)]$$

The two functions Real (Γ) and Imaginary (Γ), (referred to as $R(\Gamma)$ and $I(\Gamma)$ hereafter) are plotted in Figures 4 and 5 respectively.

They are plotted on the z-axis, $0 \leq z \leq 1$, corresponding to a y-axis plot $-\infty < y < +\infty$. Now, since as $y \to \pm\infty$, $\phi'(y) \to 0$ and $\phi(y) \to 0$, then from equation (6.4), $R(\Gamma) \to 0$ and $I(\Gamma) \to 0$ as $y \to \pm\infty$, i.e., as $z \to 0$ or $z \to 1$, then $I(\Gamma) \to 0$ and $R(\Gamma) \to 0$. From Figure 5, $I(\Gamma)$ is seen to behave correctly. From Figure 4, for $M_\infty = 0.0$ and 1.0, $R(\Gamma)$ also is seen to behave correctly, but for $M_\infty = 1.5$ and 2.0 $R(\Gamma)$ appears to be tending to some other value. However, at $z = 0$ and $z = 1$, then $\phi(z) = 0$ and by virture of (A.7.4), $\phi'(z) = 0$ at $z = 0, 1$. It follows that at $z = 0, 1$, then $R(\Gamma)$ must necessarily also be zero. This implies strange behavior for $R(\Gamma)$ when $M_\infty = 1.5$ and $M = 2.0$, which quite possibly is numerical in origin. It may also be connected, somehow, to the appearance of instabilities.

7. Summary

The stability of a compressible free shear layer has been investigated by utilizing the linearized equation resulting from a small perturbation analysis (Rayleigh equation). This eigen-value problem was solved numerically for various wave numbers (α) and Mach number for a mean velocity profile of the form of a hyperbolic tangent. This type of flow (with an inflection point) was shown to be unstable for low values of wave number and for Mach numbers below 2.5. These results are believed to be useful for analyzing aerodynamics instabilities [8] encountered in separated flows (buzz, cavity resonance, buffet, etc.).

APPENDIX A

DERIVATION OF THE GOVERNING EQUATIONS

An inviscid, compressible unsteady flow is assumed. Let u represent the streamwise cartesian velocity component, and v the cross flow component. p, ρ and T are the static pressure, density and temperature respectively.

The basic equations are: Continuity, two momentum and energy.

$$\frac{D\rho}{\rho Dt} + \nabla \cdot \underline{V} = 0$$

$$\rho \frac{Du}{Dt} = - \frac{\partial p}{\partial x}$$

$$\rho \frac{Dv}{Dt} = - \frac{\partial p}{\partial x} \qquad \text{(A.1.1)}$$

$$\frac{Dp}{pDt} = \frac{\gamma D\rho}{\rho Dt}$$

where $\frac{D}{Dt}$ = substantial derivative

Eliminating the density from the continuity and energy equation produces the following:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \frac{1}{\rho} \frac{\partial p}{\partial x} = 0$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + \frac{1}{\rho} \frac{\partial p}{\partial y} = 0 \qquad \text{(A.1.2)}$$

$$\frac{\partial p}{\partial t} + u \frac{\partial p}{\partial x} + v \frac{\partial p}{\partial y} + \gamma p\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

Now assume that the flow can be represented as a disturbance from a steady state shear flow [4, 5] in which

$$\bar{u}(y) = \text{streamwise velocity}$$

$$\bar{v} = 0 = \text{cross flow velocity} \qquad \text{(A.1.3)}$$

$$\bar{p} = \text{static pressure} = \text{constant}$$

It follows that $\frac{\partial \bar{u}}{\partial x} = \frac{\partial \bar{p}}{\partial x} = \frac{\partial \bar{p}}{\partial y} = 0$.

Let the time dependent disturbed flow be given by

$$\begin{aligned} u &= \bar{u} + u' \\ v &= v' \\ p &= \bar{p} + p' \end{aligned} \tag{A.1.4}$$

With this flow, the equations (A.1.2) give

$$\begin{aligned} &\frac{\partial u'}{\partial t} + \bar{u}\frac{\partial u'}{\partial x} + v'\frac{\partial \bar{u}}{\partial y} + \frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x} = 0 \\ &\frac{\partial v'}{\partial t} + \bar{u}\frac{\partial v'}{\partial x} + \frac{1}{\bar{\rho}}\frac{\partial p'}{\partial y} = 0 \\ &\frac{\partial p'}{\partial t} + \bar{u}\frac{\partial p'}{\partial x} + \gamma\bar{p}\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right) = 0 \end{aligned} \tag{A.1.5}$$

We now assume periodic disturbances of the form

$$\begin{aligned} u' &= \hat{u}(y)\exp[j\alpha(x - ct)] \\ v' &= \hat{v}(y)\exp[j\alpha(x - ct)] \\ p' &= \hat{p}(y)\exp[j\alpha(x - ct)] \end{aligned} \tag{A.1.6}$$

where $\hat{u}$, $\hat{v}$ and $\hat{p}$ are assumed to be complex. (A.1.6) substituted into (A.1.5) yields, in turn,

$$j\alpha(\bar{u} - c)\,\hat{u} + \bar{u}_y\hat{v} = -\frac{\hat{p}}{\rho}\,j\alpha \tag{A.1.7a}$$

$$j\alpha(\bar{u} - c)\hat{v} = -\frac{1}{\rho}\,\hat{p}_y \tag{A.1.7b}$$

$$j\alpha(\bar{u} - c)\hat{p} = -\gamma\bar{p}(j\alpha\hat{u} + \hat{v}_y) \tag{A.1.7c}$$

We seek to eliminate $\hat{p}$ and $\hat{u}$ from these three equations. From (A.1.7c)

$$j\alpha\hat{u} = -j\alpha(\bar{u} - c)\frac{\hat{p}}{\gamma\bar{p}} - \hat{v}_y \tag{A.1.8}$$

In (A.1.7a), (A.1.8) gives

$$(\bar{u} - c)\left[-j\alpha(\bar{u} - c)\frac{\hat{p}}{\gamma\bar{p}} - \hat{v}_y\right] + \bar{u}_y\hat{v} = -\frac{\hat{p}}{\bar{\rho}}\,j\alpha$$

$$\bar{u}_y\hat{v} - \hat{v}_y(\bar{u} - c) = -j\alpha\left[\frac{1}{\bar{\rho}} - \frac{(\bar{u} - c)^2}{\gamma\bar{p}}\right]\hat{p}$$

Use $\bar{\rho} = \bar{p}/R\bar{T}$, and we get

$$\frac{\bar{u}_y\hat{v} - \hat{v}_y(\bar{u} - c)}{\gamma R\bar{T} - (\bar{u} - c)^2} = - j\alpha \frac{\hat{p}}{\gamma\bar{p}} \tag{A.1.9}$$

Now eliminate $\hat{p}$ between (A.1.7b) and (A.1.9) and we get

$$\left[\frac{\bar{u}_y\hat{v} - (\bar{u} - c)\hat{v}_y}{\gamma R\bar{T} - (\bar{u} - c)^2}\right]_y = (j\alpha)\left(\frac{j\alpha}{\gamma R\bar{T}}\right)(\bar{u} - c)\hat{v}$$

or (A.1.10)

$$\left[\frac{\bar{u}_y\hat{v} - (\bar{u} - c)\hat{v}_y}{\gamma R\bar{T} - (\bar{u} - c)^2}\right]_y = - \frac{\alpha^2(\bar{u} - c)\hat{v}}{\gamma R\bar{T}}$$

Now use.

$$\gamma R\bar{T} = \frac{1 - 0.2\, M_o^2\, \bar{u}^2}{M_o^2} \tag{A.1.11}$$

where

$$M_o^2 = \frac{M_\infty^2}{1 + 0.2\, M_\infty^2} \tag{A.1.12}$$

and (A.1.10) becomes

$$\left[\frac{\bar{u}_y\hat{v} - (\bar{u} - c)\hat{v}_y}{(1 - 0.2\, M_o^2\, \bar{u}^2) - M_o^2(\bar{u}-c)^2}\right]_y = \frac{- \alpha^2(\bar{u} - c)\hat{v}}{(1 - 0.2\, M_o^2\, \bar{u}^2)} \tag{A.1.13}$$

Putting

$$M^2 = \frac{M_\infty^2\, \bar{u}^2}{1 + 0.2\, M_\infty^2\, (1 - \bar{u}^2)} \tag{A.1.14}$$

yields, after some algebra

$$\left[\frac{\bar{u}_y\hat{v} - (\bar{u} - c)\hat{v}_y}{(1 + 0.2\, M^2)^{-1} - M_o^2(\bar{u} - c)^2}\right]_y = - \alpha^2(1 + 0.2M^2)(\bar{u} - c)\hat{v} \tag{A.1.15}$$

A change of notation

$$\hat{v} \equiv \phi$$

$$\tilde{u} \equiv v$$

yields equations (2.1)

APPENDIX B

THE DERIVATION OF BOUNDARY CONDITIONS FOR THE TRANSFORMED STABILITY EQUATION

The untransformed stability equation for an inviscid, compressible flow is given by

$$\frac{d}{dy}\left(\frac{(v - c)\phi' - v'\phi}{g}\right) = \alpha^2(1 + 0.2M^2)(v - c)\phi \qquad (A.2.1)$$

We consider this equation for very large value of $|y| < \infty$.

As $y \to -\infty$, then $v \to 0$, $M \to 0$, $g \to 1 - M_o^2c^2$ and (A.2.1) may be approximated by

$$-c\phi'' = \alpha^2(1 - M_o^2c^2)(-c)\phi \qquad (A.2.2)$$

i.e. by

$$\phi'' - \alpha^2(1 - M_o^2c^2)\phi = 0 \qquad (A.2.3)$$

i.e. as $y \to -\infty$, then $\phi \to \exp(\pm\, \alpha(1 - M_o^2c^2)^{\frac{1}{2}}\, y)$ (A.2.4)

Since the boundary condition on ϕ at $-\infty$ is $\phi(-\infty) = 0$, then (A.2.4) must be

$$\phi = \exp(\alpha\hat{\alpha}(1 - M_o^2c^2)^{\frac{1}{2}}\, y) \qquad (A.2.5)$$

where

$$\hat{\alpha} = \mathrm{Sign}(\mathrm{Real}(\alpha(1 - M_o^2c^2)^{\frac{1}{2}})) \qquad (A.2.6)$$

Since by (3.2)

$$\phi = \exp(\int\theta dy) \qquad (A.2.7)$$

Then comparison of (A.2.7) with (A.2.5) yields that a $y \to -\infty$, then

$$\theta \to \alpha\,\hat{\alpha}(1 - M_o^2c^2)^{\frac{1}{2}}$$
$$\hat{\alpha} = \mathrm{Sign}(\mathrm{Real}(\alpha(1 - M_o^2c^2)^{\frac{1}{2}})) \qquad (A.2.8)$$

Put

$$g_o \equiv g_R + jg_I = (1 - M_o^2c^2), \qquad \text{then,}$$

$$g_o = (g_R^2 + g_I^2)^{\frac{1}{2}} \exp(j\mathrm{Tan}^{-1}(g_I/g_R)) \qquad (A.2.9)$$

Put $\hat{\theta} \equiv \mathrm{Tan}^{-1}(g_I/g_R)$, then (A.2.8) may be written

$$\begin{aligned} \theta &\to \alpha\hat{\alpha}(g_R^2 + g_I^2)^{\frac{1}{4}} \exp(j\hat{\theta}/2) \\ \hat{\alpha} &= \mathrm{Sign}(\alpha \ \mathrm{Cos}\ \hat{\theta}/2) \end{aligned} \tag{A.2.10}$$

Splitting (A.2.10) into its real and imaginary parts yields the boundary conditions at

$$\begin{aligned} y &= -\infty \\ \theta_R &= \alpha\hat{\alpha}(g_R^2 + g_I^2)^{\frac{1}{4}} \ \mathrm{Cos}(\hat{\theta}/2) \\ \theta_I &= \alpha\hat{\alpha}(g_R^2 + g_I^2)^{\frac{1}{4}} \ \mathrm{Sin}(\hat{\theta}/2) \\ \hat{\alpha} &= \mathrm{Sign}(\alpha \ \mathrm{Cos}(\hat{\theta}/2)) \\ \hat{\theta} &= \mathrm{Tan}^{-1}(g_I/g_R) \end{aligned} \tag{A.2.11}$$

As $y \to +\infty$, then $v \to 1$, $M \to M_\infty$ and $g \to g_\infty = (1 + 0.2\ M_\infty^2)^{-1} - M_o^2(1 - c)^2$, so that for large positive y, (A.2.1) may be approximated by

$$\phi'' - \alpha^2\ g_\infty(1 + 0.2\ M_\infty^2)\phi = 0 \tag{A.2.12}$$

Following similar arguments to those used to deduce (A.2.11), we arrive at the boundary condition at

$$\begin{aligned} y &= +\infty \\ \theta_R &= -\ \alpha\hat{\alpha}(g_R^2 + g_I^2)^{\frac{1}{4}} \ \mathrm{Cos}(\hat{\theta}/2) \\ \theta_I &= -\ \alpha\hat{\alpha}(g_R^2 + g_I^2)^{\frac{1}{4}} \ \mathrm{Sin}(\hat{\theta}/2) \\ \hat{\alpha} &= \mathrm{Sign}(\alpha \ \mathrm{Cos}(\hat{\theta}/2)) \\ \hat{\theta} &= \mathrm{Tan}^{-1}(g_I/g_r) \end{aligned} \tag{A.2.13}$$

A.B.

In evaluating $\hat{\theta} = \mathrm{Tan}^{-1}(g_I/g_R)$, care must be taken to ensure that the correct quadrant is used.

APPENDIX C

TREATMENT OF STABILITY EQUATION AT THE END POINTS, z = 0, 1

Equations (3.12) are

$$\dot{\theta}_R = \frac{1}{2z(1-z)} \quad (RHS_1)$$

$$\dot{\theta}_I = \frac{1}{2z(1-z)} \quad (RHS_2) \qquad (A.3.1)$$

where RHS_1 and RHS_2 are given at (3.9).

WRITE

$$RHS_1 = -(\theta_R^2 - \theta_I^2) + f_R$$

$$RHS_2 = -2\theta_R\theta_I + f_I \qquad (A.3.2)$$

then (A.3.1) becomes

$$\dot{\theta}_R = \frac{1}{2z(1-z)} (\theta_I^2 - \theta_R^2 + f_R)$$

$$\dot{\theta}_I = \frac{1}{2z(1-z)} (-2\theta_I\theta_R + f_I) \qquad (A.3.3)$$

Use L'Hopitals' rule as $z \to 0$ in (A.3.3):

$$\lim_{z \to 0} \dot{\theta}_R = 1/2(2\theta_I\dot{\theta}_I - 2\theta_R\dot{\theta}_R + \dot{f}_R)$$

$$\lim_{z \to 0} \dot{\theta}_I = 1/2(-2\theta_I\dot{\theta}_R - 2\dot{\theta}_I\theta_R + \dot{f}_I) \qquad (A.3.4)$$

Thus, evaluated at z = 0, (A.3.4) gives

$$(1 + \theta_R)\,\dot{\theta}_R - \theta_I\dot{\theta}_I = 1/2\,\dot{f}_R$$

$$\theta_I\dot{\theta}_R + (1 + \theta_R)\,\dot{\theta}_I = 1/2\,\dot{f}_I \qquad)A.3.5)$$

Solving (A.3.5) for $\dot{\theta}_R$ and $\dot{\theta}_I$ gives

$$\dot{\theta}_R\ (z = 0) = 1/2\left(\frac{(1 + \theta_R)\ \dot{f}_R + \theta_I\ \dot{f}_I}{(1 + \theta_R)^2 + \theta_I^2}\right)_{z = 0}$$

$$\dot{\theta}_I\ (z = 0) = 1/2\left(\frac{(1 + \theta_R)\ \dot{f}_I - \theta_I\ \dot{f}_R}{(1 + \theta_R)^2 + \theta_I^2}\right)_{z = 0} \qquad \text{(A.3.6)}$$

Following the same arguments at $z = 1$ gives

$$\dot{\theta}_R\ (z = 1) = -1/2\left(\frac{(1 - \theta_R)\ \dot{f}_I - \theta_I\ \dot{f}_R}{(1 - \theta_R)^2 + \theta_I^2}\right)_{z = 1}$$

$$\dot{\theta}_I\ (z = 1) = -1/2\left(\frac{(1 - \theta_R)\ \dot{f}_R - \theta_I\ \dot{f}_R}{(1 - \theta_R)^2 + \theta_I^2}\right)_{z = 1} \qquad \text{(A.3.7)}$$

APPENDIX D

TREATMENT OF SINGULARITY AT $v = c_R$ WHEN $c_I = 0$

In the case of $c_I = 0$, the transformed stability equations reduce to the single equation

$$\theta_R' = -\theta_R^2 + g' \frac{\theta_R}{g} + \alpha_R^2 g(1 + 0.2M^2) + \frac{v''g - v'g'}{g(v - c)} \qquad \text{(A.4b.1)}$$

since when c is real, then α is real for $M_\infty \leq 2.5$ (see final appendix).

It is clear from A.3b.1 that if it is not to contain a singularity, then as $v \to c$, then so must $v''g - v'g' \to 0$. From the requirement that

$$(v''g - v'g')\Big|_{v = c} = 0 \qquad \text{(A.4b.2)}$$

we can deduce a simple relationship between M_∞, and propagation velocity, c_R, of the neutral disturbance. We have

$$g = (1 + 0.2M^2)^{-1} - M_o^2(v - c)^2 \qquad \text{(A.4b.3)}$$

and

$$v = 0.5(1 + \text{Tanh}(y)) \qquad \text{(A.4b.4)}$$

(A.3b.3) gives

$$g' = -\frac{0.4MM'}{(1 + 0.2\,M^2)^2} - 2M_o^2(v - c)v' \qquad \text{(A.4b.5)}$$

as $v \to c$, then

$$g' \to -\frac{0.4MM'}{(1 + 0.2\,M^2)^2} \qquad \text{(A.4b.6)}$$

From (A.3b.4), with some manipulation

$$v' = -2v(v - 1) \qquad \text{(A.4b.7)}$$

and

$$v'' = 4v(v - 1)(2v - 1) \qquad \text{(A.4b.8)}$$

Thus, (A.3b.3) through(A.3b.8) in (A.3b.2) give the condition that at $v = c$, then,

$$4v(v - 1)(2v - 1)(1 + 0.2M^2)^{-1} + 2v(v - 1)\frac{(-0.4MM')}{(1 + 0.2M^2)^2} = 0 \qquad \text{(A.4b.9)}$$

$$2(2v - 1)(1 + 0.2M^2) - 0.4MM' = 0 \tag{A.4b.10}$$

Now use

$$M^2 = \frac{M_\infty^2 v^2}{1 + 0.2M_\infty^2(1 - v^2)} \tag{A.4b.11}$$

and we get, with $v = c$

$$\hat{M}^2 + 0.2 M_\infty^2 c^2 \hat{M} = \left(\frac{0.1 M_\infty^2}{2c - 1}\right)(-4c^2(c - 1)\hat{M} - 0.8c^4(c - 1)M_\infty^2) \tag{A.4b.12}$$

where

$$\hat{M} = 1 + 0.2 M_\infty^2 (1 - c^2) \tag{A.4b.13}$$

Now put $\sigma = 0.2 M_\infty^2$ and rearrange, these results

$$\sigma^2(-c^2 + 2c - 1) + \sigma(-c^2 + 4c - 2) + (2c - 1) = 0 \tag{A.4b.14}$$

This yields

$$\sigma = \frac{(2c - 1)}{(c - 1)^2}$$

$$M_\infty = \left\{\frac{5(2c - 1)}{(c - 1)^2}\right\}^{\frac{1}{2}} \tag{A.4b.15}$$

Alternatively, we can rearrange this to give

$$c = \frac{(1 + 0.2 M_\infty^2) - (1 + 0.2 M_\infty^2)^{\frac{1}{2}}}{0.2 M_\infty^2} \tag{A.4b.16}$$

The implication of this result is significant namely, that for a given profile $v(y)$, then the propagation velocity of the neutral distrubance is uniquely defined by the Mach number at infinity.

APPENDIX E

MATHEMATICAL AND NUMERICAL TREATMENT OF THE STABILITY EQUATION FOR INCOMPRESSIBLE, VISCOUS FLOWS

The equation is given by

$$(v - c)(\phi'' - \alpha^2\phi) + j/\alpha R_e(\phi'''' - 2\alpha^2\phi'' + \alpha^4\phi) - v''\phi = 0 \qquad (A.5.1)$$

Using the D-operator rotation this may be written as

$$(v - c)(D^2 - \alpha^2)\phi + j/\alpha R_e(D^2 - \alpha^2)(D^2 - \alpha^2)\phi - v''\phi = 0 \qquad (A.5.2)$$

Put $\phi = \exp(\int\Delta dy)$ in A.5.2, and we get

$$(v - c)(\Delta' + \Delta^2 - \alpha^2) + j/\alpha R_e(D^2 - \alpha^2)(\Delta' + \Delta^2 - \alpha^2) - v'' = 0 \qquad (A.5.3)$$

Put

$$\Delta' + \Delta^2 - \alpha^2 = \theta \qquad (A.5.4)$$

and the equations are now

$$\Delta^2 + \Delta^2 - \alpha^2 = \theta$$

$$j/\alpha R_e(\theta'' - \alpha^2\theta) + (v - c)\theta = v'' \qquad (A.5.5)$$

Using the arguments of Section 3, the boundary conditions of Δ are

$$\Delta(-\infty) = +\alpha, \qquad \Delta(+\infty) = -\alpha \qquad (A.5.6)$$

Using these in (A.5.4) gives

$$\theta(-\infty) = \theta(+\infty) = 0 \qquad (A.5.7)$$

The total system is

$$\begin{aligned} &\Delta' + \Delta^2 - \alpha^2 = \theta \\ &j/\alpha R_e(\theta'' - \alpha^2\theta) + (v - c)\theta = v'' \\ &\theta(-\infty) = \theta(+\infty) = 0 \\ &\Delta(-\infty) = +\alpha \end{aligned} \qquad (A.5.8)$$

where c is such that $\Delta(+\infty) = -\alpha$

The numerical procedure used was essentially that described in Section 5, except that for each value of c, the differential equation for θ must be solved. Note that as $R_e \to \infty$, the differential equation in θ becomes an algebraic relationship in θ, and the system reverts to the inviscid form.

APPENDIX F

THE UPPER LIMIT ON MACH NUMBER

In this appendix, it is shown that a necessary condition for the neutral point $(\alpha, c_I) = (\alpha, 0)$ to have a real αx is that $M_\infty \leq 2.5$. It is not shown that it is also sufficient, because the numerical calculations demonstrate that it is.

The stability equation is given by

$$\frac{d}{dy}\left(\frac{(v - c)\phi' - v'\phi}{g}\right) = \alpha^2(1 + 2M^2)(v - c)\phi$$

$$g = (1 + 0.2M^2)^{-1} - M_o^2(v - c)^2$$

$$M^2 = M_\infty^2 v^2/(1 + 0.2M_\infty^2\,(1 - v^2)) \qquad \text{(A.6.1)}$$

$$M_o^2 = M_\infty^2/(1 + 0.2M_\infty^2)$$

Suppose that for some y^* then g is such that $g(y) \leq 0$ for all $-\infty \leq y < y^*$. We will show that in this case that α^2 is necessarily complex.

It is known that as $y \to -\infty$, then $g \to$ constant, $-w^2$ say. Thus, for $y << y^*$ then $g \simeq -w^2 < 0$. Likewise, $v \simeq 0$ and $v' \simeq 0$. Hence, (A.6.1) may be written as

$$\phi'' = -w^2\alpha^2\phi$$

for $\quad -\infty < y << y^*$ $\qquad$ (A.6.2)

and $\quad \phi(-\infty) = 0$

If we insist on non-trivial solutions to (A.6.2), then α^2 must be complex, otherwise, the equation in (A.6.2) will only admit sinusoidal

solutions that cannot satisfy the boundary condition $\phi(-\infty) = 0$.

Thus, to obtain a real α^2, then g must remain positive <u>at least at its end points.</u> It is shown in Fig 5 that g is wholly positive for $0 \leq M_\infty < 2.5$, that $g(-\infty) = g(+\infty) = 0$ at $M_\infty = 2.5$ and that $g(-\infty) < 0$, $g(+\infty) < 0$ for $M_\infty > 2.5$.

Thus, $M_\infty = 2.5$ represents an upper bound on Mach numbers that allow real values of α at the neutral points.

APPENDIX G

USEFUL RELATIONSHIPS BETWEEN ϕ AND θ, FOR COMPUTATIONAL PURPOSES

Equation (3.2) relates $\theta(y)$ and $\phi(y)$ using $\phi(y) = \exp[\int \theta dy]$ (A.7.1)

where

$$-\infty < y < \infty .$$

For computational purposes it is necessary to "fix" a bottom limit to the integral in (A.7.1). Thus more precisely,

$$(y) = \exp[\int_o^y \phi(s)ds] \tag{A.7.2}$$

Fixing the bottom limit at $s = 0$ ensures that $\phi(0) = 1$ i.e., $\phi(y)$ is normalized to unity at the midpoint of the range of y.

From (A.7.2) there follows

$$\begin{aligned} \phi_R(y) &= \exp[\int_o^y \theta_R \, ds] \cos[\int_o^y \theta_I \, ds] \\ \phi_I(y) &= \exp[\int_o^y \theta_R \, ds] \sin[\int_o^y \theta_I \, ds] \end{aligned} \tag{A.7.3}$$

$$\begin{aligned} \frac{d\phi_R}{dy} &= \theta_R \, \phi_R - \theta_I \, \phi_I \\ \frac{d\phi_R}{dy} &= \theta_R \, \phi_I - \theta_I \, \phi_R \end{aligned} \tag{A.7.4}$$

In the z-plane, (A.7.2) becomes

$$\phi(z) = \exp[\int_{0.5}^{z} \frac{\phi(s)}{2s(1 - s)} \, ds] \tag{A.7.5}$$

Relationships corresponding to (A.7.3) follow directly, and hence, values for ϕ'_R and ϕ'_I can be found on the z-plane using (A.7.4).

TABLE 1

Comparison of Michalkes Results With Those of the Present Method, for Incompressible Inviscid Flow

C_I	α(MICHALKE)	α(PRESENT METHOD)
0.5000	0.0	0.0
0.3487	0.2000	0.2002
0.2133	0.4446	0.4443
0.1442	0.6000	0.5995
0.0674	0.8000	0.7995
0.0000	1.0000	0.9997

TABLE 2

Effect of Reynolds Number on Amplification
Of Disturbances for Incompressible Flow

	(a)	(b)
C_I	α, Re = ∞	α, Re = 10^3
0.5000	0.0	0
0.3487	0.2002	0.2016
0.2133	0.4443	0.4404
0.1442	0.5995	0.5939
0.0674	0.7995	0.7920
0.0000	0.9997	Not computed

TABLE 3

Behavior of the Neutral Point

$(\alpha, C_R, C_I) = (\alpha, C_R, 0)$ as Mach Number Increases

M_∞	α	C_R
0.0	0.9973	0.500
0.2	0.9934	0.501
0.4	0.9747	0.504
0.6	0.9440	0.509
0.8	0.9022	0.515
1.0	0.8502	0.523
1.2	0.7888	0.532
1.4	0.7187	0.541
1.6	0.6396	0.552
1.8	0.5509	0.562
2.0	0.4483	0.573
2.2	0.3194	0.584
2.4	0.0796	0.505
2.5	Not Computed	0.600

TABLE 4a

Amplication Against Wavelength for Mach Number 0.0

$M_\infty = 0.0$	
$\exp(\alpha C_I)$	$\lambda = 2\pi/\alpha$
1.0000	6.28
1.0605	8.21
1.0701	9.10
1.0926	10.88
1.0947	11.33
1.0965	11.82
1.0978	12.35
1.0987	12.90
1.0992	13.50
1.0994	14.14
1.0992	14.83
1.0987	15.57
1.0978	16.38
1.0967	17.25
1.0952	18.19
1.0935	19.22
1.0914	20.34
1.0892	21.58
1.0866	22.94

TABLE 4b

Amplification Against Wavelength for Mach Number 0.5

$M_\infty = 0.5$	
$\exp(\alpha C_I)$	$\lambda = 2\pi/\alpha$
1.0000	6.54
1.0164	6.93
1.0304	7.34
1.0482	8.01
1.0653	8.94
1.0779	10.05
1.0845	11.00
1.0891	12.15
1.0909	13.01
1.0916	13.63
1.0919*	14.65
1.0916	15.41
1.0911	16.22
1.0901	17.11
1.0889	18.08
1.0873	19.13
1.0855	20.30
1.0833	21.59
1.0809	23.01
1.0783	24.60
1.0753	26.38
1.0722	28.39

TABLE 4c

Amplification Against Wavelength for Mach Number 1.0

$M_\infty = 1.0$	
$\exp(\alpha C_I)$	$\lambda = 2\pi/\alpha$
1.0000	7.39
1.0129	7.81
1.0267	8.36
1.0361	8.85
1.0445	9.38
1.0517	9.97
1.0597	10.84
1.0657	11.85
1.0700	13.03
1.0723	14.40
1.0731*	16.03
1.0729	17.96
1.0724	17.98
1.0715	19.12
1.0702	20.37
1.0686	21.77
1.0667	23.34
1.0645	25.12
1.0620	27.14
1.0593	29.46
1.0562	32.15

TABLE 4d

Amplification Against Wavelength for Mach Number 2.0

$M_\infty = 2.0$	
$\exp(\alpha C_I)$	$\lambda = 2\pi/\alpha$
1.0000	13.82
1.0044	14.42
1.0083	15.16
1.0119	15.93
1.0151	16.74
1.0180	17.62
1.0205	18.58
1.0227	19.63
1.0245	20.78
1.0271	23.49
1.0279	25.10
1.0284	26.93
1.0285	29.03

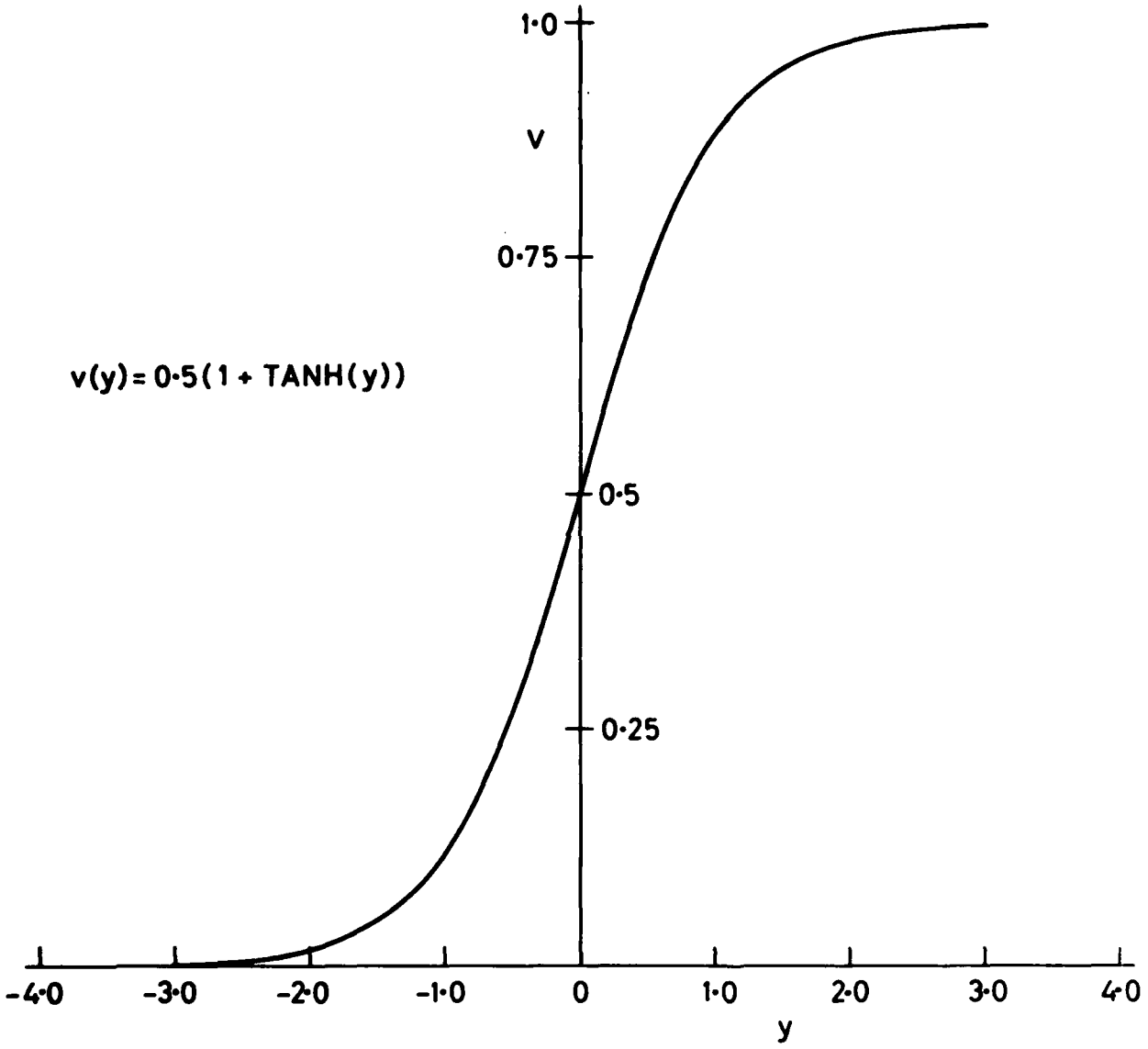

FIGURE 1. PLOT OF v(y) = 0·5(1 + TANH(y)).

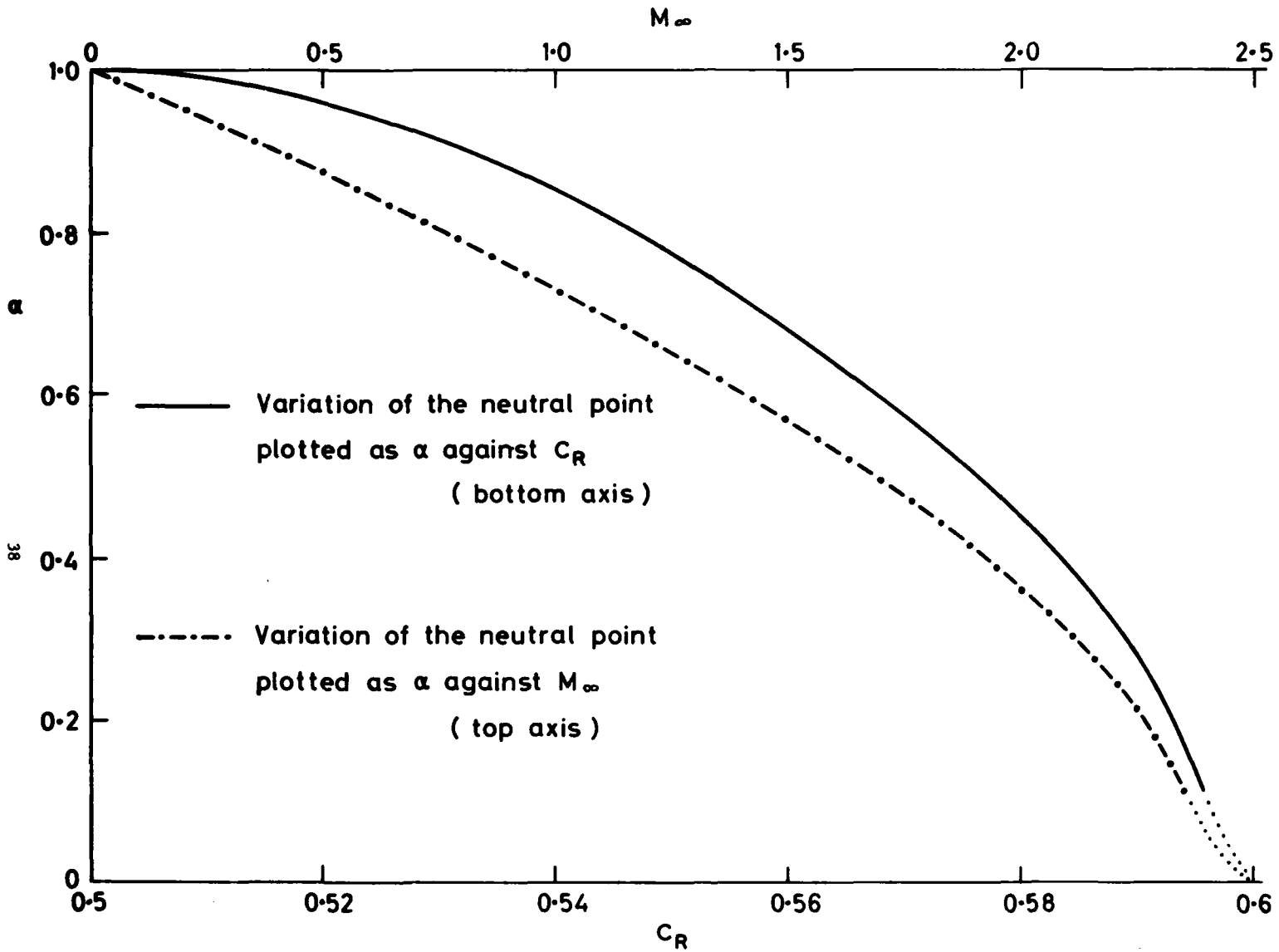

FIGURE 2. THE VARIATION OF THE NEUTRAL POINT PLOTTED AS α AGAINST (1) C_R AND (2) M_∞.

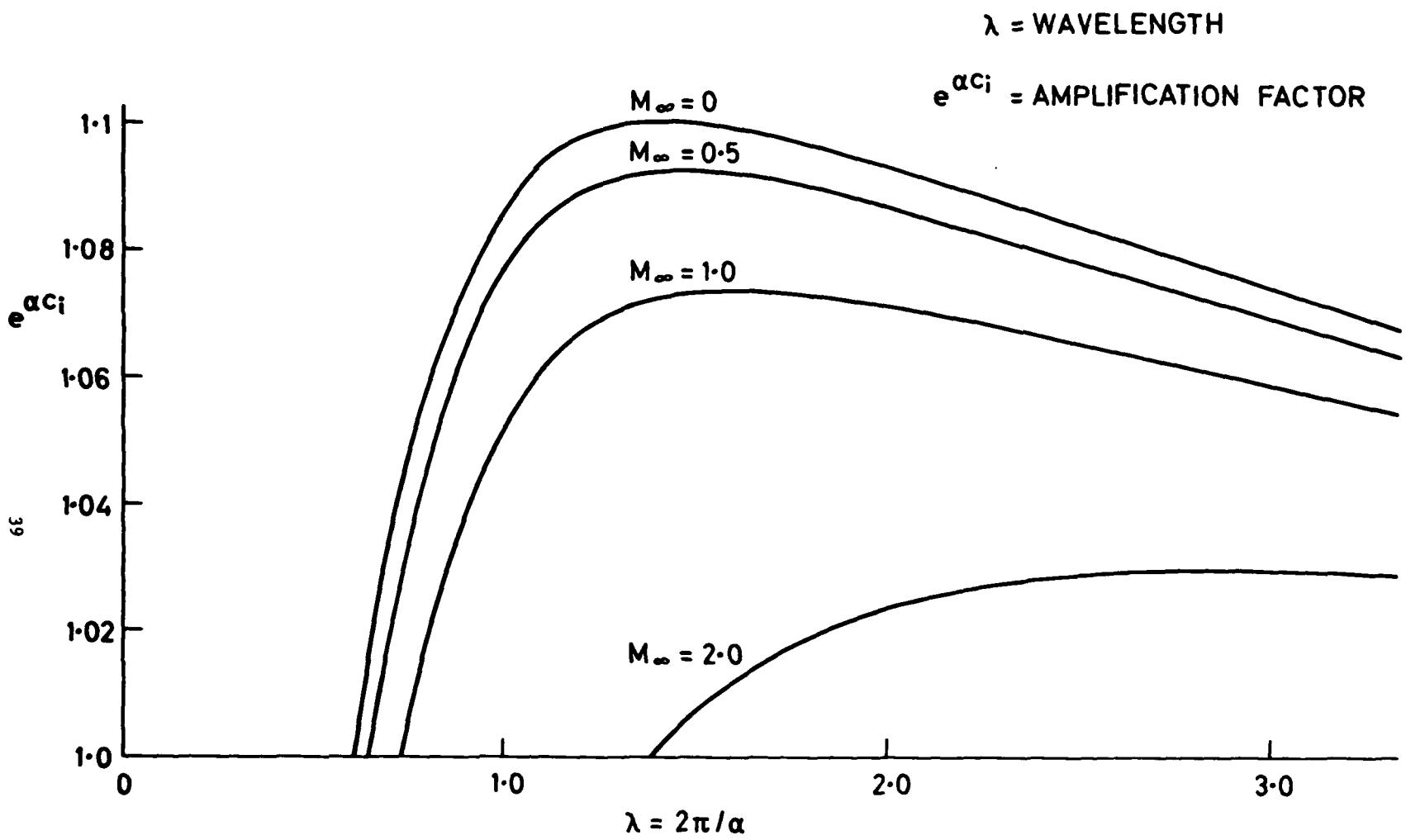

FIGURE 3. AMPLIFICATION AS A FUNCTION OF WAVELENGTH FOR MACH NUMBERS 0, 0·5, 1·0 AND 2·0.

REAL PART OF THE FUNCTION $\frac{-j}{\alpha}\left(\frac{(v-c)\phi' - v'\phi}{g}\right) \equiv \Gamma = \hat{P}$

PLOTTED FOR MACH NUMBERS 0·0, 1·0, 1·5 AND 2·0

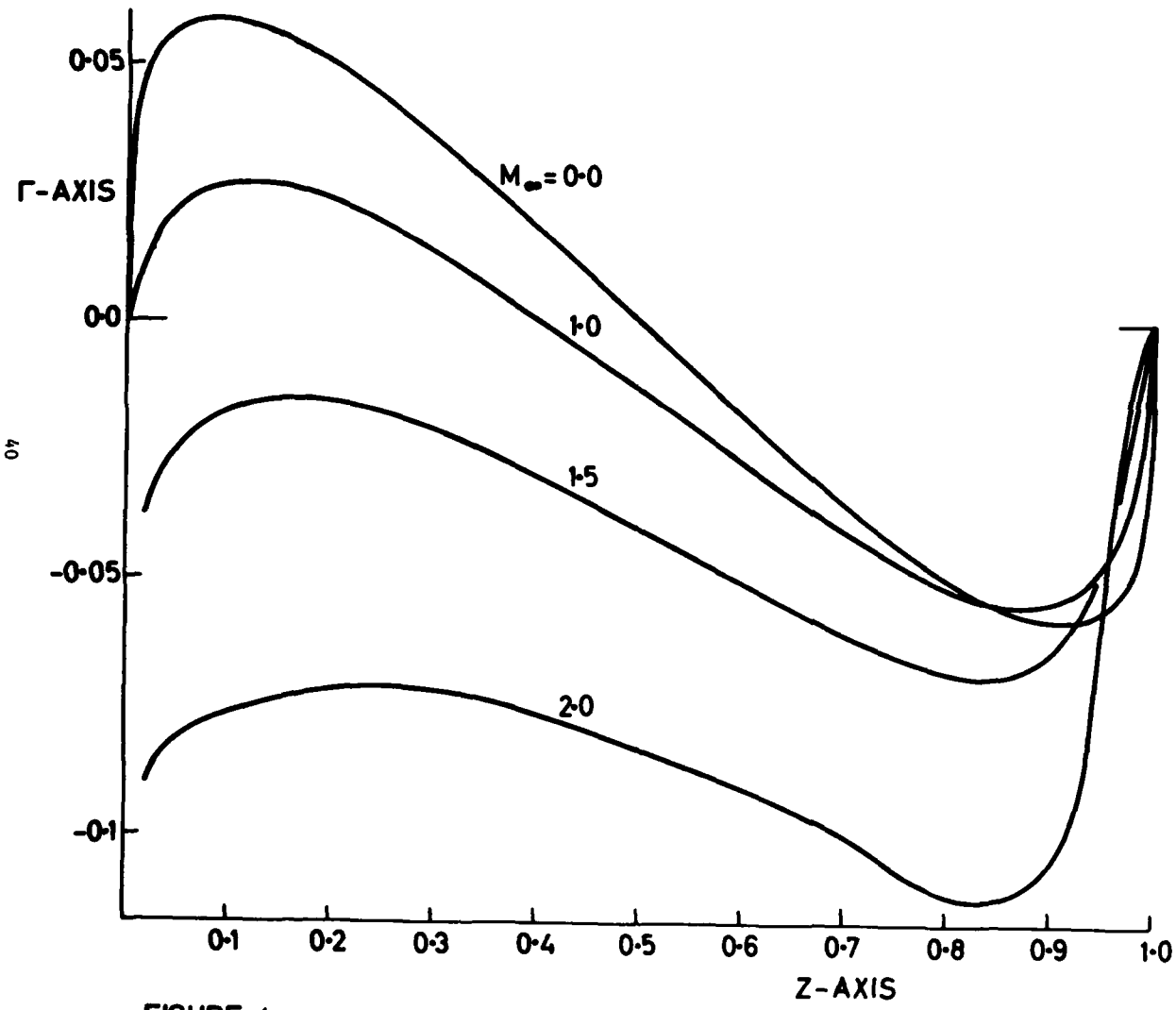

FIGURE 4

IMAGINARY PART OF THE FUNCTION $\frac{-j}{\alpha}\left(\frac{(v-c)\phi' - v'\phi}{g}\right) \equiv \Gamma$

PLOTTED FOR MACH NUMBERS 0·0, 1·0, 1·5 AND 2·0

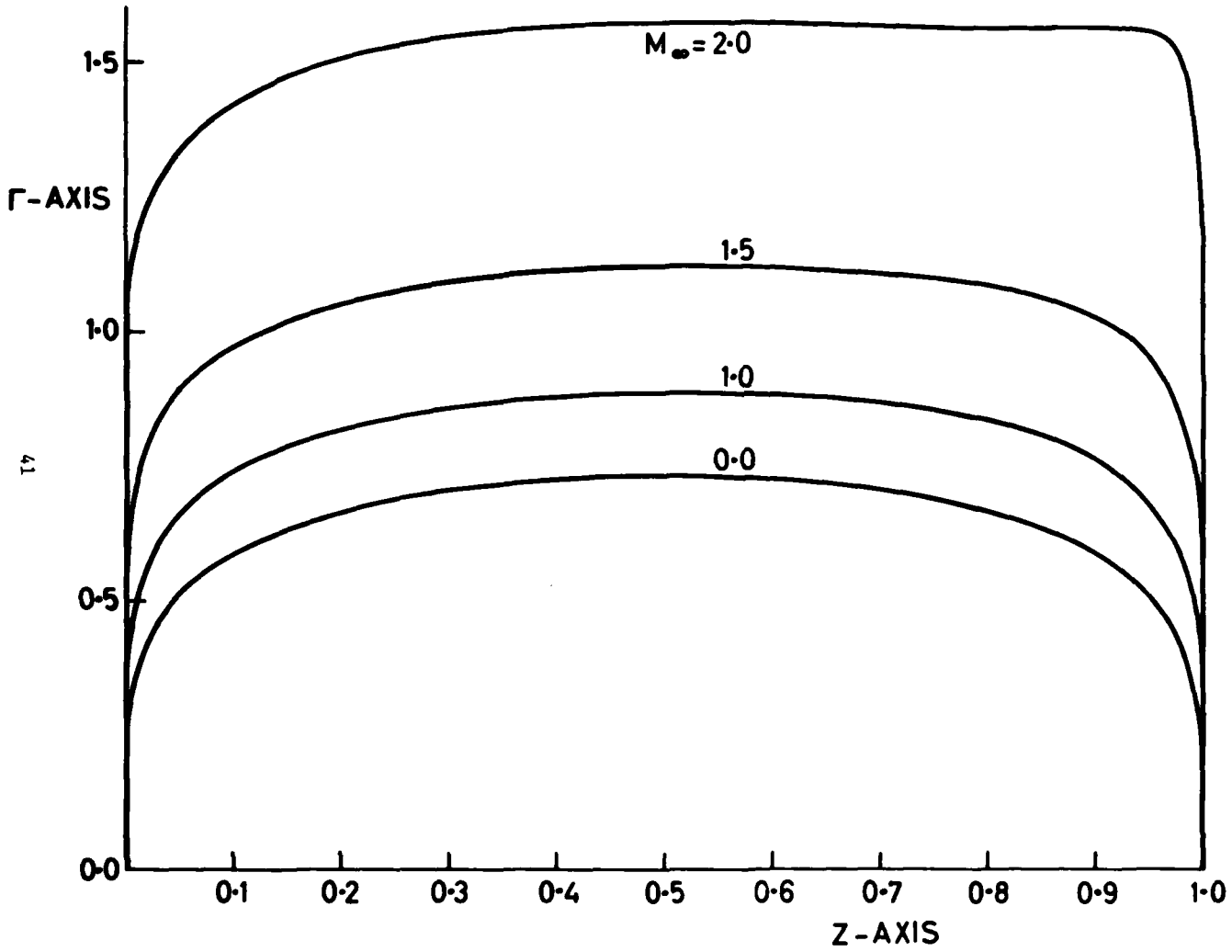

FIGURE 5

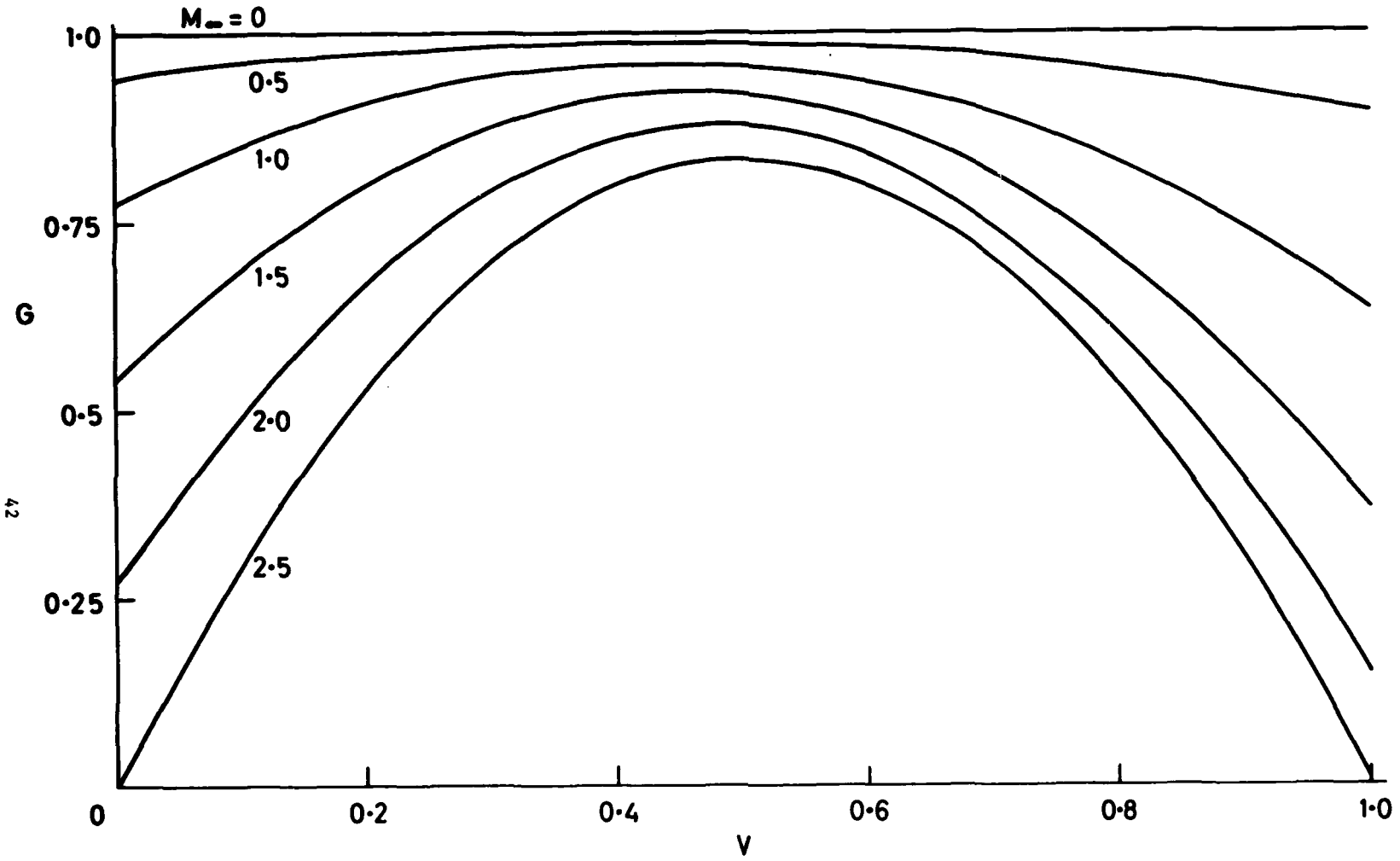

FIGURE 6. THE FUNCTION G PLOTTED AGAINST V FOR MACH NUMBERS 0, 0·5, 1·0, 1·5, 2·0 AND 2·5.

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