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610 Walnut Street
Madison, Wisconsin 53706

March 1980

(Received February 18, 1980)

Approved for public release
Distribution unlimited

Sponsored by
U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina  27709

National Science Foundation
Washington, D.C.  20550
We prove that if the second-order sufficient condition and constraint regularity hold at a local minimizer of a nonlinear programming problem, then for sufficiently smooth perturbations of the constraints and objective function the set of local stationary points is nonempty and continuous; further, if certain polyhedrality assumptions hold (as is usually the case in applications) then the local minimizers, the stationary points and the multipliers all obey a type of Lipschitz condition. Through the use of generalized equations, these results are obtained with a minimum of notational complexity.

AMS (MOS) Subject Classification: 90C30
Key Words: Nonlinear programming, Generalized equations
Work Unit No. 5 (Operations Research)

*Sponsored by the United States Army under Contract No. DAAG29-75-C-0024 and by the National Science Foundation under Grant No. MCS-7901066. Parts of the research for this paper were carried out at the Département de Mathématiques de la Décision, Université Paris-IX Dauphine, with financial support from the Centre National de la Recherche Scientifique, and at the Departamento de Matemáticas y Ciencias de la Computación, Universidad Simón Bolívar, Caracas, Venezuela, with support from the United Nations Educational, Scientific and Cultural Organization under Proyecto UNESCO VEN-77-002. The author greatly appreciates the hospitality and support extended by the institutions cited.
SIGNIFICANCE AND EXPLANATION

In practical problems from logistics, structural design, chemical engineering and other areas, it is often necessary to maximize or minimize a nonlinear function of several variables subject to nonlinear equation and/or inequality constraints. Since problem data or functional forms may not be known exactly, it is of interest to know whether a local solution to such a problem will persist under small changes in the data or in the problem functions. In this paper, we show that if two fairly well known "niceness" conditions are satisfied at a local solution of the unperturbed problem, then for small perturbations the perturbed problem will have one or more solutions near the original one. Moreover, these solutions will often display good continuity properties.
GENERALIZED EQUATIONS AND THEIR SOLUTIONS, PART II: APPLICATIONS TO NONLINEAR PROGRAMMING

Stephen M. Robinson*

Introduction. This paper deals with the stability of solutions and multipliers of nonlinear programming problems when the data of the problems are subjected to small perturbations. The problem with which we shall deal may be formulated by introducing functions \( f \) and \( g \) from an open convex set \( \mathcal{O} \subset \mathbb{R}^n \) to \( \mathbb{R} \) and \( \mathbb{R}^n \) respectively, a closed convex set \( C \subset \mathbb{R}^n \) and a closed convex cone \( Q \subset \mathbb{R}^n \). The problem of interest is then

\[
\begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad g(x) \in Q^* , \quad x \in C ,
\end{align*}
\]

where \( Q^* \) denotes the polar cone of \( Q \):

\[ Q^* := \{ y \in \mathbb{R}^n \mid (q,y) \leq 0 \text{ for each } q \in Q \} . \]

If we assume that \( f \) and \( g \) are Frechet differentiable and that certain regularity conditions are satisfied, it can be shown [17] that with each local minimizer \( x \) of (1.1) there are associated one or more multipliers \( u \in \mathbb{R}^n \) such that \((x,u)\) satisfy the necessary optimality conditions

\[
\begin{align*}
0 & \in \frac{\partial f}{\partial x}(x,u) + \partial \psi_C(x) \\
0 & \in \frac{\partial g}{\partial u}(x,u) + \partial \psi_Q(x) ,
\end{align*}
\]

where \( f(x,u) := f(x) + \langle u, g(x) \rangle \) is the standard Lagrangian, and where the subscript \( x \)

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or \( u \) denotes partial differentiation with respect to that variable. The notation \( \partial \Phi \)
denotes the normal cone at a point: thus,

\[
\partial \Phi C(x) := \begin{cases} 
\{ y \in \mathbb{R}^n | \langle y, c - x \rangle \leq 0 \text{ for each } c \in C \} & \text{if } x \in C \\
\emptyset & \text{if } x \notin C,
\end{cases}
\]

for a convex cone like \( Q \), the description of \( \partial \Phi \) is even simpler:

\[
\partial \Phi Q(u) = \begin{cases} 
\{ v \in Q^* | (v, u) = 0 \} & \text{if } u \in Q \\
\emptyset & \text{if } u \notin Q,
\end{cases}
\]

Any point \( x \) which, with some \( u \), satisfies (1.2) is called a stationary point of (1.1); there may well exist stationary points which are not local minimizers.

What we shall show here is that if \( x_0 \) is a local minimizer of (1.1) at which \( f \) and \( g \) are twice continuously Fréchet differentiable and at which certain regularity conditions hold (specifically, the second-order sufficient condition and constraint regularity), and if \( f \) and \( g \) are smoothly perturbed, then the set of stationary points near \( x_0 \), regarded as a multifunction (multivalued function) of the perturbations, is (nonempty and) continuous at \( x_0 \). Further, we show that the set of multipliers is upper semicontinuous, where these continuity properties are as defined by Berge [11]. These results are proved in Section 3, after a review in Section 2 of the necessary regularity conditions.

In Section 4, we show that if \( C \) and \( Q \) are assumed to be polyhedral (as will be the case in most applications), then even stronger results can be established: the distance from a stationary point of the perturbed problem to \( x_0 \), or from an associated multiplier to the set of multipliers associated with \( x_0 \), obeys a kind of Lipschitz condition. Examples are given, both in Section 3 and in Section 4, to show that certain stronger statements, although plausible, are not generally true.

In several parts of this paper we use as a device for simplifying and motivating results the concept of a generalized equation. These objects, introduced in Part I of this paper [19], are relations of the form...
\[ 0 \in F(z) + T(z), \]

where \( F : \mathbb{R}^k \times \mathbb{R}^k \) and \( T \) is a closed multifunction from \( \mathbb{R}^k \) to itself (often a normal-cone operator). For example, if we rewrite (1.2) in terms of \( f \) and \( g \), and use the fact that \( \partial \psi_{CQ}(x,u) = \partial \psi_C(x) \times \partial \psi_Q(u) \), we obtain the generalized equation

\[
0 \in \begin{bmatrix} f'(x) + g'(x)u \\ -g(x) \end{bmatrix} + \partial \psi_{CQ}(x,u),
\]

(1.3)

and this relation will be used in several ways in our analysis.

Many papers have already been written about various stability questions connected with (1.1), often in the special case in which \( C = \mathbb{R}^k \) and \( Q = \mathbb{R}_+^k \times \mathbb{R}^l \) (where \( \mathbb{R}_+^k \) is the non-negative orthant in \( \mathbb{R}^k \)). This special case formulates the standard nonlinear programming problem

\[
\begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad g_i(x) \leq 0 \quad (i = 1, \ldots, k) \\
& \quad g_i(x) = 0 \quad (i = k + 1, \ldots, k + l).
\end{align*}
\]

(1.4)

We shall not attempt to review all of these papers here; rather, we mention only a few which illustrate the different types of assumptions that have been imposed.

Fiacco and McCormick [3, §§2.4, 5.2] formulated a basic technique for analyzing (1.4) under the assumptions of strict complementary slackness, linear independence of the gradients of the binding constraints, and the second-order sufficient condition. This technique was refined, to deal with general perturbations, in [15] and in [4]; the basic tool, in all three cases, was the standard implicit-function theorem.

If the assumption of strict complementary slackness is dropped, then the standard implicit-function theorem can no longer be used. However, if the linear independence assumption and a somewhat strengthened form of the second-order sufficient condition are retained, then one can still show that the stationary point and associated multiplier are (locally) single-valued functions of the perturbations, and that they are locally Lipschitzian if appropriate continuity assumptions are made on the problem functions. This is shown for the problem (1.4) in [20], as a by-product of a general implicit-function theorem.
theorem for generalized equations. A similar result, without the Lipschitz continuity, was established by Kojima [10, Th. 6.4] as part of his investigation of "strongly stable" solutions of (1.4).

If one weakens the hypotheses still further by dropping the assumption of linear independence, then the appropriate condition to assume in its place is regularity of the constraints [18] (see Section 2; in the case of (1.4) this is the constraint qualification of Mangasarian and Fromovitz [12]). With this assumption, together with the strengthened form of the second-order sufficient condition previously mentioned, Kojima [10, Th. 7.2] has shown the existence of a locally unique stationary point which is continuous under small perturbations. Under similar hypotheses, Levitin [11, Th. 4] stated that if the minimizer exists for small perturbations then various properties, including Lipschitz continuity, followed. However, in Section 4 below we give an example which appears to satisfy Levitin's hypotheses but for which local Lipschitz continuity does not hold. Finally, Sargent [23] has studied the existence and continuity of local minimizers under constraint regularity; however, his methods are quite different from those of the other papers mentioned here, and it appears that some of the results in [23] may not be completely correct.

We begin our analysis, in the next section, by reviewing a generalized form of the well known second-order sufficient condition, and exploring some of its properties.

2. A review of the second-order sufficient condition. The second-order sufficient condition is a very well known regularity condition in nonlinear programming. It is discussed for the standard nonlinear programming problem (1.4) in [3, §2.3]; versions adapted to problems in more general spaces are given by, e.g., Guignard [6], Maurer and Zowe [13] and Maurer [14].

In this section we exhibit a form of this condition suitable for the problem (1.1). Although we have not seen this particular form in the literature, it is not likely to be very surprising to anyone familiar with the field. What may be less familiar, however, is
its motivation in terms of generalized equations and its use to prove isolation results like Theorem 2.3.

To develop a second-order sufficient condition for (1.1), we return to the generalized equation (1.3) which formulates the necessary optimality conditions. We have shown in [19, 20] that important aspects of the behavior of a generalized equation are captured in its linearization about a given point. To make use of this linearization in the present case, we assume that \((x_0, u_0)\) is a point satisfying (1.2); the linearized form of (1.2) (or equivalently, of (1.3)) at \((x_0, u_0)\) is then

\[
0 \in \begin{bmatrix}
L_x(x_0, u_0) & L_{xx}(x_0, u_0) & L_{xu}(x_0, u_0)
\end{bmatrix}
\begin{bmatrix}
x - x_0 \\
u - u_0
\end{bmatrix}
+ \delta_{C^L}(x, u).
\]

Simplifying this expression, we obtain

\[
0 \in \begin{bmatrix}
L_x(x_0, u_0) & 0
\end{bmatrix}
\begin{bmatrix}
x - x_0 \\
u - u_0
\end{bmatrix}
+ \delta_{C^L}(x_0, u_0).
\]

Examination of the form of (2.1) leads us to the realization that it formulates the necessary optimality conditions for a certain quadratic programming problem, namely

\[
\begin{align*}
\text{minimize} & \quad f^*(x_0)(x - x_0) + \frac{1}{2}(x - x_0)^T L_{xx}(x_0, u_0)(x - x_0) \\
\text{subject to} & \quad g(x_0) + g^*(x_0)(x - x_0) \in Q^*, \quad x \in C.
\end{align*}
\]

The quadratic programming problem (2.2) is not derived by any straightforward approximation of the functions in (1.1) (note that the second-order term in the objective function contains, not \(f''(x_0)\), but \(L_{xx}(x_0, u_0)\)). However, it has been found to be an appropriate approximation of (1.1) for numerical purposes [9, 25]. Since the derivation through linearization of (1.2) is straightforward, we shall refer to (2.2) as the linearization, or linearized form, of (1.1).

If we are looking for a condition to impose on a stationary point \((x_0, u_0)\) of (1.1) to ensure that \(x_0\) will be a strict local minimizer of (1.1), an obvious approach to take is to impose some such conditions on (2.2). In fact, the simplest way of obtaining a
plausible set of conditions is to consider feasible directions of (2.2) at \( x_0 \) and to place some restriction on these. Thus, if we consider vectors \( h \in \mathbb{R}^3 \) such that 
\( x_0 + h \in C \) and \( g(x_0) + g'(x_0)h \in Q^* \), we know (and can prove easily from the fact that 
\( (x_0,u_0) \) satisfies (1.2); see below) that \( f'(x_0)h > 0 \). If \( f'(x_0)h > 0 \), then if \( h \) is small enough the first-order term in the objective of (2.2) will be dominant and the objective value at \( x_0 + h \) will be strictly greater than that at \( x_0 \). On the other hand, if \( f'(x_0)h = 0 \) then the only way to obtain strict increase in the objective function is to resort to the second-order term and to require that \( (h,\varepsilon'(x_0,u_0)h) > 0 \).

We can now use these observations to formulate an appropriate set of conditions. To do so, we note that the requirements on \( h \) involving the objective function and the Lagrangian of (2.5) are independent of the scale of \( h \), but those involving the constraints (i.e., that \( x_0 + h \in C \) and \( g(x_0) + g'(x_0)h \in Q^* \)) are not. To simplify the latter, we enlarge slightly the class of vectors \( h \) considered by requiring that 
\( h \in T_C(x_0) \) and \( g'(x_0)h \in T_Q(g(x_0)) \), where \( T_C(x_0) \) denotes the tangent cone to \( C \) at \( x_0 \), and similarly for \( Q^* \). Note that, since \( C \) and \( Q \) are convex, any \( h \) satisfying the earlier requirements will necessarily satisfy the latter. For small enough \( h \), the converse is true if \( C \) and \( Q \) are polyhedral, but generally not otherwise.

The steps just described therefore lead us to the following general second-order sufficient condition:

**Definition 2.1:** Suppose \( (x_0,u_0) \) is a point satisfying (1.2). The second-order sufficient condition holds at \( (x_0,u_0) \) with modulus \( \varepsilon > 0 \) if for each \( h \in T_C(x_0) \) with 
\[ g'(x_0)h \in T_Q(g(x_0)), \quad f'(x_0)h = 0, \]
one has \( (h,\varepsilon'(x_0,u_0)h) > \varepsilon \| h \|^2 \).

We remark that it would not have changed Definition 2.1 if we had written 
\( f'(x_0)h \leq 0 \), since that inequality implies \( f'(x_0)h = 0 \) in the presence of (1.2). To see this, note that \( h \) and \( -[f'(x_0) + g'(x_0)u_0] \) belong respectively to the tangent and normal cones to \( C \) at \( x_0 \). Thus,
\[ 0 \leq (\partial f(x_0) + g(x_0)^* u_0, h) \]
\[ = f'(x_0)h + (u_0, g'(x_0)h) \quad . \]

However, since \( g(x_0) \) belongs to the normal cone to \( Q \) at \( u_0 \), \( u_0 \) also belongs to the normal cone to \( Q^* \) at \( g(x_0) \) [22, Cor. 23.5.4]. As \( g'(x_0)h \) belongs to the corresponding tangent cone, we have \( (u_0, g'(x_0)h) \leq 0 \) and thus \( f'(x_0)h \geq 0 \). Because of this, in what follows we shall sometimes just show that a vector \( h \) satisfies \( f'(x_0)h \leq 0 \) before applying Definition 2.1. In this form, the condition has been discussed for the standard problem (1.4) by Han and Mangasarian, who have shown [7, Th. 3.5] that for that case it is equivalent to the standard second-order sufficient condition as given in, e.g., [3]. It is thus reasonable to suppose that the condition of Definition 2.1 will enable us to show that some positive definite quadratic form minorizes the objective function of (1.1) on that portion of the feasible set near \( x_0 \), and takes a minimum equal to \( f(x_0) \) at \( x_0 \), since such results have been established for other forms of the condition (cf. [8, Th. 7.3], [24, Th. 4.43], [13, Th. 5.2]). Since we shall need this result in Section 3, we establish it in the following theorem.

**Theorem 2.2:** Suppose that \((x_0,u_0)\) satisfies (1.2), that \( f \) and \( g \) are twice Frechet differentiable at \( x_0 \), and that the second-order sufficient condition holds there with modulus \( \rho \). Then for each \( \epsilon \in (0,\rho) \) there exists a neighborhood \( V \) of \( x_0 \) such that if \( x \in C \cap V \) and \( g(x) \in Q^* \), then \( f(x) > f(x_0) + \frac{1}{2} \epsilon x - x_0 \|^2 \) or \( x = x_0 \).

**Proof:** Suppose that for some \( \epsilon > 0 \) there is a sequence \((x_n)\) in \( C \) converging to \( x_0 \) with \( x_n \neq x_0 \), \( g(x_n) \in Q^* \) and \( f(x_n) < f(x_0) + \frac{1}{2} \epsilon x_n - x_0 \|^2 \) for each \( n \). We shall show that \( \epsilon > \rho \). We can suppose without loss of generality that \( (x_n - x_0)/(x_n - x_0) \) converges to some \( h \in \mathbb{R}^q \); evidently \( \|h\| = 1 \) and \( h \in T_Q(x_0) \). We have for each \( i \)
\[ Q^* g(x_n) = g(x_0) + g'(x_n)(x_n - x_0) + o(\|x_n - x_0\|), \quad (2.3) \]
so
\[ g'(x_n)h = \lim_{i \to \infty} (x_n - x_0)^{-1} [g(x_n) - g(x_0)] \epsilon T_Q(x_0) \epsilon . \]
We have previously remarked that in such a case \( f'(x_0)h \geq 0 \), but since for each \( i \) we have
\[ f(x_0) + \frac{1}{2} c_1 x_1 - x_0 \leq 0 \quad \Rightarrow \quad f(x_1) = f(x_0) + f'(x_0)(x_1 - x_0) + o(|x_1 - x_0|), \]

it follows upon division by \( |x_1 - x_0| \) and passage to the limit that also \( f': x \to -c_1 \), and therefore that \( h \) satisfies the hypothesis of the second-order sufficient condition.

However, we have from (2.3) and (2.4) that since \( u_0, q(x_1) \leq 0 \),

\[
\begin{align*}
\frac{1}{2} (x_1 - x_0)^2 & \geq f(x_1) - f(x_0) \\
& = f(x_0, u_0) + f'(x_0, u_0)(x_1 - x_0) \\
& \quad + \frac{1}{2} (x_1 - x_0, L_{xx}(x_0, u_0)(x_1 - x_0)) + o(|x_1 - x_0|^2) \\
& \geq f(x_0) + \frac{1}{2} (x_1 - x_0, L_{xx}(x_0, u_0)(x_1 - x_0)) + o(|x_1 - x_0|^2).
\end{align*}
\]

Subtracting \( f(x_0) \), dividing by \( \frac{1}{2} (x_1 - x_0)^2 \) and passing to the limit, we find that

\[
\epsilon \geq (h, L_{xx}(x_0, u_0)h) \geq \rho \delta h^2 = 0,
\]

and the theorem now follows by contraposition.

Theorem 2.2 shows that a stationary point of (1.1) satisfying the second-order sufficient condition is a strict local minimizer. A reasonable question to ask is whether it is also an isolated local minimizer: that is, whether there is some neighborhood of \( x_0 \) containing no other local minimizer of (1.1). As we shall now see, this is not so, even for very simple problems. Consider, for instance, the problem

\[
\begin{align*}
\text{minimize} & \quad \frac{1}{2} x^2 \\
\text{subject to} & \quad x^6 \sin(1/x) = 0. \quad [\sin(1/0) := 0].
\end{align*}
\]

Here \( C = \mathbb{R} = Q \); the feasible region is \( \{0\} \cup \{(n)^{-1}|n = \pm 1, \pm 2, \ldots\} \). The second-order sufficient condition is satisfied at the origin; however, the origin is a cluster point of the feasible region and every feasible point is a local minimizer.

One might reply that such bad behavior is not very surprising since (2.5) is a bad problem. This is quite true, and we shall see that a result of the type we seek can indeed be proved if we exclude bad problems: that is, if we impose a constraint qualification.

The qualification we shall use is the requirement that the constraints

\[
g(x) \in Q^n \quad \text{and} \quad x \in C
\]
of (1.1) be regular at $x_0$ in the sense of [18]; that is, that

$$0 \in \text{int}[g(x_0) + g'(x_0)(C - x_0) - Q].$$

(2.6)

We have shown in [17] that (2.6) is a sufficient qualification for the derivation of optimality conditions, and in fact for the standard problem (1.4), the condition (2.6) is equivalent to the well-known constraint qualification of Mangasarian and Fromovitz [12].

We shall now show that if the constraints of (1.1) are regular at a point $x_0$ which, together with some $u_0$, satisfies (1.2), then $x_0$ is locally unique as a solution of (1.2).

THEOREM 2.3: Assume that $f$ and $g$ are twice Frechet differentiable at $x_0$, and suppose that the regularity condition (2.6) holds there. Define $U_0 = \{ u \in \mathbb{R}^n \mid (x_0, u) \}$ satisfies (1.2) (possibly empty), and suppose that for each $u_0 \in U_0$, the second-order sufficient condition is satisfied at $(x_0, u_0)$ with some positive modulus. Then there exists a neighborhood $W$ of $x_0$ such that if $(x, u)$ satisfies (1.2) and $x \in W$, then $x = x_0$.

PROOF: By contraposition. Suppose that there is a sequence $\{ (x_n, u_n) \}$ such that $x_n$ converges to $x_0$, and such that for each $n$, $x_n \neq x_0$ and $(x_n, u_n)$ solves (1.2). Assume without loss of generality that $\{ x_n - x_0 \}$ converges to $h$.

It follows from Theorem 3.2 with $P = \{ p_0 \}$ that the set $U_0$ is bounded (note: the applicable portion of the proof of Theorem 3.2 makes no use of the present theorem nor of any result depending on it). Accordingly, with no loss of generality we can suppose that $\{ u_n \}$ converges to some $u_0$. Taking the limit in (1.3) with $(x_n, u_n)$ in place of $(x, u)$, and using continuity and the closedness of $\partial f_C$ and $\partial g_Q'$, we conclude that $(x_0, u_0)$ satisfy (1.3). Thus $u_0 \in U_0$.

Now from (1.3) we find that for each $i$, $g(x_n) \in Q^*$ and $x_n \in C$; reasoning as in the proof of Theorem 2.2 we can establish that $h \in \mathcal{T}_C(x_0)$, $g'(x_0)h < \mathcal{T}_Q^*(g(x_0))$, and $f'(x_0)h \geq 0$.

Since $(x_n, u_n)$ satisfies (1.3) for each $i$, we have

$$0 = (u_n, g(x_n)) = (u_n, g(x_0)) + (u_n, g'(x_0)(x_n - x_0)) + o(|x_n - x_0|),$$

where we have used the fact that the $u_n$ are uniformly bounded. However, as $u_n \in Q$
and \( g(x_0) \in G^p \) we have \( \langle u_1, g(x_0) \rangle \leq 0 \) and therefore
\[
0 \leq \langle u_1, g'(x_0)(x_1 - x_0) + o(\|x_i - x_0\|^2) \rangle .
\]
Dividing by \( \|x_i - x_0\|^2 \) and passing to the limit, we obtain
\[
\langle u_0, g'(x_0)h \rangle \leq 0 .
\] (2.7)

Now from (1.3) we again have for each \( i \)
\[
f'(x_i) + g'(x_i)u_i + 2'\lambda(x_i) ,
\]
so that
\[
0 \geq f'(x_i)(x_i - x_0) + \langle u_i, g'(x_i)(x_i - x_0) \rangle ,
\]
and upon division by \( \|x_i - x_0\|^2 \) and passage to the limit we find, using (2.7), that
\[
f'(x_0)h \leq 0 \quad \text{and therefore that } h \quad \text{satisfies the hypothesis of the second-order sufficient condition.}
\]

We know that
\[
\partial^2_{XX}(x_i) \approx \hat{\partial}^2_{XX}(x_0) = \hat{\partial}^2_{XX}(x_0) + o(\|x_i - x_0\|^2) ,
\] (2.8)
and
\[
\partial^2_{X}(x_0) \approx \hat{\partial}^2_{X}(x_0) .
\] (2.9)

Also,
\[
\partial^2_{X}(u_i) \approx g(x_i) = g(x_0) + g'(x_0)(x_i - x_0) + g''(x_0)(x_i - x_0)^2 + o(\|x_i - x_0\|^2) ,
\] (2.10)
and
\[
\partial^2_{X}(u_0) \approx g(x_0) .
\] (2.11)

Using (2.8)-(2.11) together with the monotonicity of \( \partial^2_{XX} \) and \( \partial^2_{X} \), we find that
\[
0 \leq (\hat{\partial}^2_{XX}(x_0, u_0)(x_i - x_0), x_i - x_0) + o(\|x_i - x_0\|^2) ,
\] (2.12)
and
\[
0 \leq \langle u_i - u_0, g'(x_0)(x_i - x_0) + g''(x_0)(x_i - x_0)^2 + o(\|x_i - x_0\|^2) \rangle .
\] (2.13)

Adding (2.12) and (2.13), we obtain
\[
0 \leq (\hat{\partial}^2_{XX}(x_0, u_0)(x_i - x_0), x_i - x_0) + o(\|x_i - x_0\|^2) ,
\]
and division by \( \|x_i - x_0\|^2 \) and passage to the limit yields a contradiction to the second-order sufficient condition. This completes the proof.

We have thus shown that when regularity is added to the second-order sufficient condition, the stationary points \( x_0 \) which, with some \( u_0 \), satisfy (1.2), are isolated.
It is an easy consequence of this fact that if the conditions of Theorem 2.3 hold in some compact region of $R^n$, that region contains only finitely many stationary points (perhaps none). However, this result says nothing about the behavior of stationary points, or minimizers, of problems which are "near" (1.1) in the sense that their problem functions are slight perturbations of those of (1.1). In the next section, we use the second-order sufficient condition and the regularity condition to analyze the behavior of such problems.

3. Local solvability of perturbed nonlinear programming problems. In this section we shall be concerned with the problem

$$\text{minimize } f(x,p)$$

$$\text{subject to } g(x,p) \in Q^0,$$

$$x \in C,$$

where $p$ is a perturbation parameter belonging to a topological space $P$, and $x$ is the variable in which the minimization is done. The functions $f$ and $g$ are defined from $\Omega \times P$ to $R$ and $R^n$ respectively, where $\Omega$ is an open set in $R^q$; $Q$ and $C$ are as previously. We shall identify (1.1) with the particular case of (3.1) arising when $p$ is some fixed $p_0 \in P$; our interest here will be in predicting, from information in (1.1), aspects of the behavior of (3.1) when $p$ varies near $p_0$, such as solvability, location of minimizers, etc. Throughout Sections 3 and 4 we shall make the blanket assumption that $f(\cdot,p)$ and $g(\cdot,p)$ are Fréchet differentiable on $\Omega$, that $f$, $g$, $f'$ and $g'$ are continuous on $\Omega \times P$, and that $f(\cdot,p_0)$ and $g(\cdot,p_0)$ are twice Fréchet differentiable at $x_0$, a point in $C \cap \Omega$ which is a stationary point of (3.1) for $p = p_0$. The stationary-point conditions for (3.1) are

$$0 \in f'(x,p) + g'(x,p)u + \partial C(x)$$

$$0 \in -q(x,p) + \partial q_0(x) ,$$

and we define $U_0 := \{ u \in R^n \mid (x_0,u) \text{ satisfies (3.2) for } p = p_0 \}$. We shall denote the
distance from a point $a$ to a set $A$ by $d(a, A) := \inf\{d(a, a') | a' \in A\}$. 

Our first theorem shows that if the second-order sufficient condition holds at a local minimizer at which the constraints are regular, then that local minimizer persists under small perturbations.

**THEOREM 3.1:** Suppose that for $p = p_0$, (3.1) satisfies the second-order sufficient condition at $x_0$ and some $u_0 \in U_0$ and its constraints are regular at $x_0$.

Then for each neighborhood $M$ of $x_0$ in $\Omega$ there is a neighborhood $N$ of $p_0$ such that if $p \in N$ then (3.1) has a local minimizer in $M$.

**PROOF:** By hypothesis, for $p = p_0$ the constraints of (3.1) are regular at $x_0$. By [18, Th. 1] there are neighborhoods $M_0$ and $N_0$ of $x_0$ and $p_0$ respectively, and a constant $\zeta$, such that for each $x \in C \cap M_0$ and $p \in N_0$,

$$d(x, F(p)) \leq \zeta d(g(x, p), Q^*)$$

where $F(p) := \{x \in C | g(x, p) \in Q^*\}$. Let $\epsilon_0$ be the modulus for the second-order sufficient condition at $(x_0, u_0)$; let $\epsilon \in (0, \epsilon_0)$ and choose a positive $\delta$ so that $(x_0 + 2\delta B) \subset M_0 \cap N_0 \cap V$, where $V$ is the neighborhood of $x_0$ given by Theorem 2.2 for the chosen $\epsilon$. Select a neighborhood $N_0'$ of $p_0$ with $N_0' \subset N_0$, and some positive $\alpha$, such that if $p \in N_0'$ and if $x_1, x_2 \in x_0 + 2\delta B$ with $x_1 - x_2 \leq \alpha$, then $f(x_1, p) - f(x_2, p_0) < \alpha^2 / 16 = \delta$. Next, find a neighborhood $N$ of $p_0$ with $N \subset N_0'$ and such that if $p \in N$ then

$$\zeta \sup\{d(g(x, p) - g(x, p_0) | x \in x_0 + \delta B\} \leq \min(\alpha, \delta / 2).$$

Denote $f(x_0, p_0)$ by $\sigma_0$.

Now choose any $p \in N$. If $F(p) \cap \{x | \|x - x_0\| = \delta\}$ is not empty, let $x'$ be any point in it. One has $x' \in C \cap (x_0 + \delta B)$ and $g(x', p) \in Q^*$, so

$$d(x', F(p_0)) \leq \zeta d(g(x', p_0), Q^*) \leq \zeta d(g(x', p_0) - g(x', p) \| \leq \min(\alpha, \delta / 2).$$

Thus there is some $x' \in F(p_0)$ with $\|x' - x_0\| \leq \min(\alpha, \delta / 2)$, and therefore with $\|x_0 - x_0\| \geq \|x' - x_0\| - \|x' - x_0\| \geq \delta / 2$. Accordingly, by Theorem 2.2 we have
\[
f(X_0'p_0) \geq \delta' + \frac{1}{2} \varepsilon(\delta/2)^2 = \delta' + 28.
\]
Also, \(1x_0' - x_0 \leq 1x_0' - 28\) \(1x_0' - 28(\delta + \delta/2) = 36\), so \(x_0' \approx x_0 + 28\); as \(1x_0' - x_0 \leq \alpha\) we have
\[
1f(x',p) - f(x_0',p_0)\| \geq 8.
\]
Therefore
\[
f(x',p) \geq f(x_0',p_0) - 8f(x',p) - f(x_0',p_0)\|
\]
\[
\geq \delta' + 28 + 8 = \delta' + 8.
\]
It is also true that
\[
d(x_0,F(p)) \leq \delta[g(x_0,p),\mathcal{Q}] < \mathcal{I}g(x_0,p) - g(x_0,p_0)\|
\]
\[
\leq \min(a,\delta/2).
\]
Accordingly, there is some \(x'' \in F(p)\) with \(1x_0 - x'' \leq \min(a,\delta/2)\). Thus
\[
1f(x'',p) - f(x_0,p_0)\| \leq \delta, \quad \text{so}
\]
\[
f(x'',p) \leq f(x_0,p_0) + \delta f(x'',p) - f(x_0,p_0)\| < \delta' + 8.
\]  
Putting (3.3) and (3.4) together, we see that a minimizer of the function \(f(\cdot,p)\) on the set \(F(p) \cap (x_0 + \delta B)\) exists (by compactness and the fact that \(x''\) belongs to \(F(p) \cap (x_0 + \delta B)\)), but that no such minimizer lies on \(F(p) \cap \{x \mid 1x - x_0 \leq \delta\}\).
Therefore each such minimizer is in fact a local minimizer of \(f(\cdot,p)\) on \(F(p)\): that is,
\[
a local minimizer of (3.1) which belongs to \(M\). This completes the proof.
\]
Our next results will concern the continuity behavior of local minimizers and of stationary points of (3.1). For ease of reference, we proceed now to define certain multifunctions which display these points. Assume that the hypotheses of Theorem 3.1 hold for (3.1) with \(p = p_0\), then it follows from the results of [16] that there are neighborhoods \(M_1\) of \(x_0\) and \(N_1\) of \(p_0\) such that for any \(p \in N_1\) and any \(x \in M_1\) satisfying
\[
g(x,p) \in \mathcal{Q},
\]
the system (3.5) is regular at \(x\). 

\[
g(x,p) \in \mathcal{Q},
\]
\[
x \in \mathcal{C},
\]
\[
-13-
\]
At this point it will be useful to note a property of the multipliers which we shall shortly use. We shall show that if regularity holds at \( x_0 \) for \( p = p_0 \), then the multipliers in (3.2) are uniformly bounded, and in fact the set of all such multipliers is an upper semicontinuous multifunction of \( (x,p) \). This extends a result in [11], also given in [5].

**Theorem 3.2:** If the system (3.5) is regular at \( x_0 \) for \( p = p_0 \), then there exist neighborhoods \( N_2 \) of \( x_0 \) and \( N_2 \) of \( p_0 \), such that if \( U : M_2 \times N_2 \to \mathbb{R}^n \) and \( SP : N_2 \to M_2 \) are multifunctions defined by

\[
U(x,p) := \{ u \in \mathbb{R}^n \mid (x,u,p) \text{ satisfies (3.2)} \} \quad \text{for} \quad (x,p) \in M_2 \times N_2,
\]

\[
SP(p) := \{ x \in N_2 \mid \text{for some } u(x,u,p) \text{ satisfies (3.2)} \} \quad \text{for} \quad p \in N_2,
\]

then \( U \) and \( SP \) are upper semicontinuous.

**Proof:** We first show that \( U \) is locally bounded at \( (x_0,p_0) \). Assume, on the contrary, that there are sequences \( \{x_i\} \subset U \) and \( \{p_i\} \subset P \) converging to \( x_0 \) and \( p_0 \) respectively, and a sequence \( \{u_i\} \subset \mathbb{R}^n \) with \( \lim_{i \to \infty} u_i = u \), such that for each \( i \) the triple \( (x_i,u_i,p_i) \) satisfies (3.2). With no loss of generality we can suppose that \( u_i/u_i \) converges to some \( y \). For each \( i \), (3.2) implies that \( u_i \in Q \) (a cone) and \( (u_i,g(x_i,p_i)) = 0 \). Dividing by \( u_i \) and passing to the limit we find that \( y \in Q \) and \( (y,g(x_0,p_0)) = 0 \). Again from (3.2), we have for each \( i \),

\[
f'(x_i,p_i) + g'(x_i,p_i)u_i \in -2 \mathbb{C}(x_i);
\]

again dividing by \( u_i \) and passing to the limit, using the fact that \( 3 \mathbb{C} \) is a closed multifunction whose values are cones, we obtain \( g'(x_0,p_0)y \in -3 \mathbb{C}(x_0) \). By regularity, for some \( \varepsilon > 0 \) there is a point \( \varepsilon \in C \) with \( g(x_0,p_0) + g'(x_0,p_0)(x_0 - x_0) + cy \in \mathbb{C} \).

Using our information about \( y \), we find that

\[
\exists \varepsilon > 0, \quad \|y\|^2 > 0,
\]

a contradiction. It follows that there must exist neighborhoods \( N_2 \) of \( x_0 \) in \( U \) and...
\( N_2 \) of \( p_0 \) in \( P \), and a compact set \( K \subset \mathbb{R}^3 \), such that if \( (x,p) \in \mathbb{N}_2 \times \mathbb{N}_2 \) then \( u(x,p) \subset K \). Without loss of generality we can suppose that \( \mathbb{N}_2 \) and \( \mathbb{N}_2 \) are small enough so that \( \mathbb{N}_2 \) is bounded and that for \( (x,p) \in \mathbb{N}_2 \times \mathbb{N}_2 \) the values \( i g(x,p) \), \( I f'(x,p) \) and \( I g'(x,p) \) all satisfy some uniform bound.

Now let

\[
\begin{bmatrix}
    f'(x,p) + g'(x,p)u \\
    -g(x,p)
\end{bmatrix}
\]

we note that there is some compact set \( L \) such that if \( (x,p,u) \in \mathbb{N}_2 \times \mathbb{N}_2 \times K \) then \( h(x,p,u) \in L \).

Now denote by \( G \) the graph of the multifunction \( \mathbb{G}_C \times \mathbb{G}_G \), and define

\[
H := \{ (x,p,u,v) | (x,p) \in \mathbb{N}_2 \times \mathbb{N}_2, (x,p,u,v) \in \text{graph } h, (x,u,-v) \in G \} .
\]

The continuity assumptions and the properties of \( \mathbb{G}_C \) and \( \mathbb{G}_G \) imply that \( H \) is closed in \( \mathbb{N}_2 \times \mathbb{N}_2 \times \mathbb{R}^m \times \mathbb{R}^m \). However, if \( (x,p,u,v) \in H \) then \( (x,p) \in \mathbb{N}_2 \times \mathbb{N}_2 \) and \( (x,u,v) \) satisfies (3.2), so that \( u \in K \) and therefore \( v \in L \). As \( K \) and \( L \) are compact it follows that the projection of \( H \) on the space of the first three, or of the first two, components of \( (x,p,u,v) \) is closed. The first of these projections is the graph of the multifunction \( U \), and the second is that of \( SP \). Thus \( U \) and \( SP \) are closed; however, as the image of \( U \) is contained in the compact set \( K \) and that of \( SP \) in the precompact set \( \mathbb{N}_2 \), it follows that \( U \) and \( SP \) are actually upper semicontinuous. This completes the proof.

Our main result on continuity of local minimizers and stationary points is given in the next theorem. For the multifunction \( SP \) and the set \( \mathbb{N}_3 \) in the theorem, \( SP \cap \mathbb{N}_3 \) denotes the multifunction defined by \( (SP \cap \mathbb{N}_3)(p) := SP(p) \cap \mathbb{N}_3 \).

**THEOREM 3.3:** Suppose that for \( p = p_0 \) for some \( x_0 \in \mathbb{R} \) and each \( u \in U_0(p_0) \), (3.1) satisfies the second-order sufficient condition at \( (x_0,u) \), and that its constraints are regular there.
Then there exist neighborhoods \( N_3 \) of \( x_0 \) and \( N_3 \) of \( p_0 \) such that if the multifunction \( \text{LM} : N_3 \to N_3 \) is defined by

\[
\text{LM}(p) := \{ x \mid x \text{ is a local minimizer of (3.1)} \},
\]

then \( \text{SP} \cap N_3 \) is continuous \( \forall \) and for each \( p \in N_3 \) one has \( \partial \text{LM}(p) \subseteq \text{SP}(p) \).

**PROOF:** Let \( N_2 \) and \( W \) be the neighborhoods of \( x_0 \) produced by Theorems 3.2 and 3.3 respectively; define \( N_3 := N_2 \cap W \); suppose in addition that \( N_3 \) is bounded and is small enough so that for some neighborhood \( N_4 \) of \( p_0 \), any \( p \in N_4 \) and any \( x \in N_3 \) satisfying (3.5), the system (3.5) is regular at \( x \). Let \( N_2 \) be the neighborhood of \( p_0 \) given by Theorem 3.2; let \( N_5 \) be the neighborhood obtained by applying Theorem 3.1 to (3.1) with \( M = M_2 \) and define \( N_3 := N_2 \cap N_4 \cap N_5 \). Define \( \text{LM} \) as in (3.6).

We shall first show that \( \text{LM} \) and \( \text{SP} \cap N_3 \) are lower semicontinuous at \( p_0 \). By Theorem 2.3, \( \text{SP}(p_0) \cap N_3 = \{ x_0 \} \) and also, for any \( p \in N_3 \) if \( x \in \text{LM}(p) \) then since regularity of (3.5) is a sufficient constraint qualification [17] we have \( x \in \text{SP}(p) \cap N_3 \). Thus if \( p \in N_3 \) then \( \partial \text{LM}(p) \subseteq \text{SP}(p) \cap N_3 \). However, since \( x_0 \in \text{LM}(p_0) \) by Theorem 2.2, we actually have \( \text{LM}(p_0) = \text{SP}(p_0) \cap N_3 = \{ x_0 \} \). Let \( S \) be any open set in \( N_3 \) with \( S \cap \text{LM}(p_0) = \emptyset \); then obviously \( (\text{SP} \cap N_3)(p_0) \subseteq S \). Now by applying Theorem 3.1 to \( S \) we can find some neighborhood \( N_6 \) of \( p_0 \) such that \( x_0 \subseteq N_6 \) and if \( p \in N_6 \) then

\[
\text{LM}(p) \cap S = \emptyset.
\]

Hence \( \text{LM} \) is lower semicontinuous at \( p_0 \), but since \( \text{LM} \subseteq \text{SP} \cap N_3 \) and \( \text{LM}(p_0) = (\text{SP} \cap N_3)(p_0) \), we see that \( \text{SP} \cap N_3 \) is also lower semicontinuous at \( p_0 \). But \( \text{SP} \) is upper semicontinuous from \( N_2 \) to \( N_2 \), so \( \text{SP} \cap N_3 \) is upper semicontinuous from \( N_3 \) to \( N_3 \), and thus \( \text{SP} \cap N_3 \) is actually continuous at \( p_0 \). This completes the proof.

Theorem 3.3 shows that that portion of the set of stationary points near \( x_0 \) is continuous at \( p_0 \), but it gives no measure of how it depends on \( p \). In the next section, we show that if the sets \( C \) and \( Q \) are polyhedral then the dependence of \( \text{SP} \) and \( U \) upon \( p \) can be measured by the dependence of the problem functions and their derivatives upon \( p \); in particular, if the latter are Lipschitzian then \( \text{SP} \) is "upper Lipschitzian". In the meantime, however, we present an example, taken from [20], to show
that LM may not be single-valued near $p_0$ and that the inclusion of LM in SP may be strict.

Consider the quadratic programming problem

\[
\begin{align*}
\text{minimize} & \quad \frac{1}{2} (x_1^2 - x_2^2) - nx_1 \\
\text{subject to} & \quad -x_1 + 2x_2 \leq 0 \\
& \quad -x_1 - 2x_2 \leq 0,
\end{align*}
\]

where $n$ is the parameter. For $n = 0$ this problem has a unique minimizer at the origin; for any positive $n$ it has local minimizers at $\frac{2}{3} n(2, \pm 1)$ and a saddle point at $(n, 0)$. Thus in this case LM is multivalued and is strictly contained in SP whenever $n > 0$.

4. Upper Lipschitz continuity in the polyhedral case. Throughout this section we shall assume that $C$ and $Q$ are polyhedral convex sets, and we shall see that in this case we can prove stronger results than we did in Section 3. Specifically, we shall prove that the solutions and multipliers of (3.1), regarded as multifunctions of $p$, are locally upper Lipschitzian at $p_0$ under appropriate continuity assumptions on the problem functions. We recall that if $F$ is a multifunction from $\mathbb{R}^n$ to $\mathbb{R}^m$, $F$ is said to be locally upper Lipschitzian at $z_0 \in \mathbb{R}^n$ if there are a neighborhood $Z$ of $z_0$ and a constant $\lambda$ such that for each $z \in Z$,

\[
F(z) \subset F(z_0) + \lambda \mathbb{B},
\]

where $\mathbb{B}$ is the unit ball in $\mathbb{R}^n$. We showed in [21] that any polyhedral multifunction (one whose graph is the union of finitely many polyhedral convex sets) is locally upper Lipschitzian everywhere with a uniform constant.

We begin by formulating a general continuity result for nonlinear generalized equations. It says that if a certain condition is satisfied by the linearizations of the nonlinear problem, then the latter has, at least locally, an upper Lipschitzian inverse (which, of course, may not be single-valued and might even be empty). As in Section 3, for a set $S$ and a multifunction $F$, we define the multifunction $F \cap S$ by

\[
F \cap S = \{ z \in S : z \in F(z) \}.
\]
(F \cdot S)(x) := F(x) \cdot S. The norm on a product space is taken to be the maximum of the norms of the component spaces.

**Theorem 4.1:** Let F be a Fréchet differentiable function from an open set Ω ⊂ R^n to R^m. Let P be any polyhedral multifunction from R^n to R^m, and define H := F + P.

For x_0 ∈ Ω and x ∈ R^n define

$$LF_x(x_0) := F(x_0^* + P)(x_0^* - x)$$

Suppose there is some compact set X_0 ⊂ Ω such that

1) For each x_0 and x_1 in X_0, the restrictions of LF_{x_0} and LF_{x_1} to X_0 are the same.

2) For some positive ε and each x_0 ∈ X_0, 

$$X_0 \cap (LF_{x_0} + R)^{-1}(0) = X_0$$

where $$X_0 = X_0^* + \mathbb{R}^s$$.

3) F' is continuous on X_0, and there is a function δ : R_+ → R_+ with

$$\lim_{t \to 0} \delta(t) = 0 = \delta(0),$$

such that for each x ∈ X_0 and each x_0 ∈ X_0,

$$\|LF_x(x_0)\| \leq \|x_0 - x\| \delta(\|x_0 - x\|).$$

Then there is some positive δ such that the multifunction $H^{-1} \cap X_0$ is locally upper Lipschitzian at x_0, with $(H^{-1} \cap X_0)(0) = X_0$.

We remark that (iii) is satisfied in particular if Ω is convex and F' is Lipschitzian there.

**Proof:** For x_0 ∈ X_0, define $T_{x_0}$ to be $(LF_{x_0} + R)^{-1}$. Let ε := \frac{1}{2} γ. We first show that there exist a constant λ and some positive η such that for any x_0 ∈ X_0,

$$(T_{x_0} \cap X_0)$$ is upper Lipschitzian on ηB with modulus λ. Choose any x_0 ∈ X_0; then the sum LF_{x_0} + R is polyhedral and by [21, Prop. 1] there are a constant λ(x_0) and a positive η(x_0) such that $T_{x_0}$ is upper Lipschitzian on ηB with modulus λ(x_0). If we take η(x_0) small enough, it follows from the fact that $(T_{x_0} \cap X_0)(0) = X_0$ that the multifunction $T_{x_0} \cap X_0$ is upper Lipschitzian on ηB with modulus λ(x_0). Now, using the continuity assumptions, choose a neighborhood $N_0$ of x_0 small enough that for any $x ∈ X_0 \cap N_0$, \[ \|x - x_0\| < \epsilon \]
\begin{itemize}
  \item[a)] \( \lambda(x_0) F'(x) = F'(x_0) \leq \frac{1}{2} \).
  \item[b)] For any \( x' \in X \), \( \| I_{\frac{1}{2}} F'(x') - F'(x) \| \leq \frac{1}{2} n(x) \).
\end{itemize}

Choose any \( x \in X_0 \cap X \), and any \( q \in \frac{1}{2} n(x)B \). If \( w \in (X \cap X_c)(q) \) then \( w \in X_c \) and also \( q \in LF_x(w) + R(w) \), so
\[ q + LF_x(w) - LF_x(w) \in LF_x(w) + R(w) \]
and thus \( w \in (X_0 \cap X_c)[q + LF_x(w) - LF_x(w)] \).

However,
\[ \| w - \lambda(x)q + LF_x(w) - LF_x(w) \| \leq \| w - \lambda(x)q \| + \| LF_x(w) - LF_x(w) \| \]
\[ \leq \frac{1}{2} n(x) + \frac{1}{2} n(x) = n(x) \],
so by the upper Lipschitz continuity we have
\[ w \in X_0 + \lambda(x)q + LF_x(w) = LF_x(w)B \).

Now let \( x_1 \) be a point in \( X_0 \) with \( d[w,x_0] = 1w - x_1 \). By hypothesis, since \( x_0, x \) and \( x_1 \) are in \( X_0 \), we have \( LF_{x_0}(x_1) = LF_x(x_1) \). Therefore
\[ \| LF_{x_0}(w) - LF_x(w) \| = \| LF_{x_0}(x_1) - LF_x(x_1) + [F'(x_0) - F'(x)](w - x_1) \| \]
\[ \leq \| F'(x_0) - F'(x) \| \| w - x_1 \| , \]
and thus
\[ d[w,x_0] \leq \lambda(x_0)1q + \lambda(x_0)\| \lambda(x_0)F'(x_0) - F'(x) \| \| w - x_1 \| \]
\[ \leq \lambda(x_0)1q + \frac{1}{2} d[w,x_0] , \]
Therefore
-19-
\[ d[w, x_0] \leq 2 \lambda \langle x_0 \rangle q + 2 \lambda \langle x_0 \rangle q \Phi \]

and so

\[ (T_x \cap X_c)(q) \subseteq (T_x \cap X_c)(0) + 2 \lambda \langle x_0 \rangle q \Phi \]

which proves that \( T_x \cap X_c \) is upper Lipschitzian on \( \frac{1}{2} \gamma(x_0)B \) with modulus \( 2 \lambda \langle x_0 \rangle \). An elementary compactness argument now shows that for some \( \lambda \) and some \( \eta > 0 \), for any \( x \in X_0 \), \( T_x \cap X_c \) is upper Lipschitzian on \( nB \) with modulus \( \lambda \).

Choose a positive \( \delta \leq \min(\varepsilon, \eta) \) so that if \( \delta \leq \delta \) then \( \max(1, \lambda) \delta(\delta) \leq \frac{1}{2} \). Now let \( q \in \frac{1}{2} nB \) and let \( x \in (H^{-1} \cap X_0)(q) \). Then \( q \in F(x) + R(x) \), so if \( x_1 \in X_0 \) with

\[ \| x - x_1 \| = d[x, X_0] \]

we have

\[ q + LF_{x_1}(x) - F(x) \in \Phi_{x_1}(x) + R(x) \]

and therefore \( x \in (T(x_1) \cap X_c)[q + LF_{x_1}(x) - F(x)] \). However,

\[ \| q + LF_{x_1}(x) - F(x) \| \leq \| q \| + \delta(\| x - x_1 \|) \| x - x_1 \| \]

\[ \leq \frac{1}{2} \eta + \frac{1}{2} \eta \leq \eta \]

so by upper Lipschitz continuity of \( T(x_1) \cap X_c \),

\[ d[x_1, x_0] \leq \lambda \| q \| + \frac{1}{2} d[x_1, x_0] \]

implying

\[ d[x_1, x_0] \leq 2 \lambda \| q \| \]

Hence

\[ (H^{-1} \cap X_0)(q) \subseteq X_0 + 2 \lambda \| q \| \Phi \]

for each \( q \in \frac{1}{2} nB \).

Now let \( x_0 \in X_0 \). By hypothesis

\[ 0 \in (LF_{x_0} + R)(x_0) = F(x_0) + R(x_0) = H(x_0) \]

so \( x_0 \in (H^{-1} \cap X_0)(0) \). On the other hand, by applying (4.1) with \( q = 0 \) we see that

-20-
so that \((H^{-1} \cap X)(0) = X\) and \(H^{-1} \cap X\) is upper Lipschitzian on \(\frac{1}{2} nB\) with modulus \(2\lambda\). This completes the proof.

This result will allow us to measure the displacement in a stationary point \(x\) and multipliers \(u\) in terms of perturbations in the problem functions and their derivatives. We show this next, and then proceed to develop a simpler measure of displacement for the case in which the functions involved are Lipschitzian in \(p\). In what follows, we write \(U_0\) for \(U(x_0,p_0)\).

**THEOREM 4.2:** Assume the hypotheses of Theorem 3.3, and suppose also that \(C\) and \(Q\) are polyhedral. Then there are a neighborhood \(N_7\) of \(p_0\) and a constant \(u\) such that for each \(p \in N_7\), each \(x \in SP(p)\) and each \(u \in U(x,p)\),

\[
d([x,u], [x_0] \times U_0) \leq u|F(x,u,p_0) - F(x,u,p)|, \tag{4.2}
\]

where

\[
F(x,u,p) := \begin{bmatrix} f'(x,p) + g'(x,p)^*u \\ -g(x,p) \end{bmatrix}.
\]

**PROOF:** We shall apply Theorem 4.1 to the function \(F(x,u)\) given by \(F(x,u,p_0)\). We take \(R\) to be \(\partial \psi \cap Q\) and as a candidate for \(X_0\) we take the set \([x_0] \times U_0\). Because of the special form of \(X_0\), it is clear that \(F'\) is continuous there; the rest of hypothesis (iii) follows from the differentiability assumptions and the fact that \(U_0\) is bounded.

For (i), we have to show that if \(u_0, u_1, u_2 \in U_0\) then

\[
LF(x_0, u_0)^{-1}(u_2) = LF(x_0, u_1)^{-1}(u_2).
\]

However, an algebraic computation shows that each of these quantities is equal to

\[
\begin{bmatrix} f'(x_0, p_0) + g'(x_0, p_0)^*u_2 \\ -g(x_0, p_0) \end{bmatrix},
\]

so that (i) is satisfied. For hypothesis (ii), we suppose that \(u_0 \in U_0\) and that for some \(x\) and \(u, 0 \in \{LF(x_0, u_0)^{-1} \partial \psi\}(x,u)\). Then \(x\) is a stationary point, and \(u\) is an associated multiplier, for the quadratic programming problem (cf. (2.2)).
minimize \[ f'(x_0, p_0) (x - x_0) + \frac{1}{2} f''(x_0, u_0, p_0) (x - x_0)^2 \]
subject to \[ g(x_0, p_0) + g'(x_0, p_0) (x - x_0) \in \mathcal{Q} \]
\[ x \in C. \]

However, (4.3) satisfies the second-order sufficient condition (at \( x_0 \) and any multiplier), the modulus being independent of the multiplier, and its constraints are regular: both of these properties follow from the corresponding properties of the nonlinear problem (3.1).

Applying Theorem 2.3 to (4.3) we conclude that there is some neighborhood \( N_{4} \) of \( x_{0} \) such that \( x \notin N_{4} \) or \( x = x_{0} \). If we take \( \gamma \) to be small enough that \( x_{0} + \gamma B \subset N_{4} \), then if \((x, u) \in X_{\gamma}\) we must have \( x \in N_{4} \), hence \( x = x_{0} \), hence \( u \in U_{0} \) by inspection. Thus hypothesis (ii) of Theorem 4.1 is satisfied, and we can conclude that for some positive \( \delta \), and some neighborhood \( \tilde{W} \) of the origin in \( H^{m,n} \), the multifunction

\[ \{(F^{*,*}, p_0) + \delta \mathcal{C}_Q^{-1} \cap X_5 \} \]

is upper Lipschitz in \( \tilde{W} \), with

\[ \{(F^{*,*}, p_0) + \delta \mathcal{C}_Q^{-1} \cap X_5 \} \cap \{0\} = \{x_0\} \times U_0. \]

Let the upper Lipschitz modulus be \( \mu \).

Since \( SP \) and \( U \) are upper semicontinuous at \( p_0 \) by Theorem 3.2, we can find neighborhoods \( N_{4}(p_0) \) and \( M_{3}(x_0) \) such that \( M_{3} \subset x_{0} + \delta B \) and

(a) For \( x \in M_{3} \) and \( p \in N_{4} \), \( U(x, p) \subset U_0 + \delta B \);

(b) For \( p \in N_{4} \), \( SP(p) \subset M_{3} \);

(c) For \( x \in M_{3} \) and \( p \notin N_{4} \), and any \( u \in U(x, p) \),

\[ F(x, u, p_0) - F(x, u, p) \in \tilde{W}. \]

Let \( p \in N_{4} \); let \( x \in SP(p) \) and \( u \in U(x, p) \). Then \((x, u) \in X_{5}\) by (a) and (b). Also, we have by (3.2)

\[ 0 \in F(x, u, p) + \delta \mathcal{C}_Q(x, u), \]

so that

\[ F(x, u, p_0) - F(x, u, p) \subset F(x, u, p_0) + \delta \mathcal{C}_Q(x, u); \]

that is,

\[ (x, u) \in [(F^{*,*}, p_0) + \delta \mathcal{C}_Q]^{-1} \cap X_{5}. \]

where
However, \( w \in W \) by (c), and therefore by upper Lipschitz continuity

\[
d([x,u],[x_0^*] \times U_0) \leq u[F(x,u,p_0) - F(x,u,p)]
\]

which completes the proof.

The bound (4.2) can be simplified if we assume more structure in the perturbations.

**Corollary 4.3:** Assume the hypotheses of Theorem 4.2. Suppose that \( P \) is a subset of a linear space, and that \( f' \), \( g \) and \( g' \) are Lipschitzian on \( \Omega \times P \). Then for some constant \( \lambda \) the bound (4.2) can be replaced by

\[
d([x,u],[x_0] \times U_0) \leq \lambda |p - p_0| . \tag{4.4}
\]

**Proof:** We have, for the quantities in (4.2),

\[
F(x,u,p_0) - F(x,u,p) = \begin{bmatrix}
n[f'(x,p_0) - f'(x,p)] + [g'(x,p_0) - g'(x,p)]u \\
-[g(x,p_0) - g(x,p)]
\end{bmatrix} .
\]

As \( x \in \Omega \) and as \( u \) is uniformly bounded for \( p \) near \( p_0 \), we find that for some \( L \), \( |F(x,u,p_0) - F(x,u,p)| \leq L |p - p_0| \), and (4.4) then follows from (4.2).

The bound (4.4) applies to the distance from \( x \) to the point \( x_0 \), and from \( u \) to the set \( U_0 \). One might wonder whether, if additional hypotheses were added to guarantee that \( S_P \) would be a single-valued function on a neighborhood of \( p_0 \), one could establish some stronger property such as local Lipschitz continuity. If we are willing to replace the assumption of regularity by a considerably stronger one (corresponding in the case of the standard nonlinear programming problem to linear independence of the gradients of the binding constraints), then this can indeed be done, and a framework for analyzing such problems was developed in [20]. However, in the present case we cannot establish local Lipschitz continuity because it need not be true. This is shown by the following example, which settles in the negative a conjecture of Daniel [2], and also appears to contradict a result stated by Levitin [11, Th. 4].
Let $C = \mathbb{R}^2$, $Q = \mathbb{R}^2_+$, and $P = \{(p_1, p_2) \in \mathbb{R}^2 \mid 0 \leq p_1 \leq 1, |p_2| \leq \frac{1}{2} |p_1|^2\}$. Define

$$f(x, p) := \frac{1}{2} \|x\|^2$$

and

$$g(x, p) := -(A + p_1 I)x + \frac{1}{2} (2 + p_1) a + p_2 b,$$

where

$$A := \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad a := \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad b := \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

It is easy to check that the hypotheses of Corollary 4.3 are satisfied. However, for any $p \in P$ the unique minimizer of the problem (3.1) with the above data is given by

$$x(p) = \begin{cases} \frac{1}{2} a & \text{if } p = (0,0) \\ \frac{1}{2} a + p_1^{-1} p_2 b & \text{if } p \neq (0,0) \end{cases},$$

and $x(\ast)$ is obviously not locally Lipschitzian at $(0,0)$, although it does satisfy the bound (4.4). This example also shows that the continuity result of Kojima [10, Th. 7.2] for the case of regular constraints cannot be strengthened to prove Lipschitz continuity.
REFERENCES


-25-


We prove that if the second-order sufficient condition and constraint regularity hold at a local minimizer of a nonlinear programming problem, then for sufficiently smooth perturbations of the constraints and objective function the set of local stationary points is nonempty and continuous; further, if certain polyhedrality assumptions hold (as is usually the case in applications) then the local minimizers, the stationary points and the multipliers all obey a type of Lipschitz condition. Through the use of generalized equations, these results are obtained with a minimum of notational complexity.