OPTIMAL SIMPLEX TABLEAU CHARACTERIZATION OF UNIQUE AND BOUNDED ETC (U)

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OPTIMAL SIMPLEX TABLEAU
CHARACTERIZATION OF UNIQUE
AND BOUNDED SOLUTIONS OF
LINEAR PROGRAMS

O. L. Mangasarian

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53706
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O. L. Mangasarian

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ABSTRACT

Uniqueness and boundedness of solutions of linear programs are characterized in terms of an optimal simplex tableau. Let $M$ denote the submatrix in an optimal simplex tableau with columns corresponding to degenerate optimal dual basic variables. A primal optimal solution is unique if and only if there exists a nonvacuous nonnegative linear combination of the rows of $M$ corresponding to degenerate optimal primal basic variables which is positive. The set of primal optimal solutions is bounded if and only if there exists a nonnegative linear combination of the rows of $M$ which is positive. When $M$ is empty the primal optimal solution is unique.

AMS (MOS) Subject Classification: 90C05

Key Words: Linear programming, simplex method, uniqueness, boundedness

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SIGNIFICANCE AND EXPLANATION

Linear programming problems are fundamental to operations research and related areas. The simplex method and its variants are the basic tools for solving these problems. In this report we characterize those linear programming problems that have unique solutions and those that have bounded solutions in terms of information available once the problem is solved by the simplex method.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.
OPTIMAL SIMPLEX TABLEAU CHARACTERIZATION
OF UNIQUE AND BOUNDED SOLUTIONS OF LINEAR PROGRAMS

O. L. Mangasarian

1. Introduction

In [4] the author gave a number of equivalent characterizations of the uniqueness of a solution of a general linear programming problem. These characterizations did not include an explicit characterization which could be applied directly to the final optimal simplex tableau for the standard linear programming problem in order to determine whether the particular optimal solution represented by the tableau is unique or not. Such a characterization, given in Theorem 1 below, follows after some nontrivial algebra from Theorem 2(v) [4]. However a simple direct proof of this characterization is also possible and is given here for the sake of completeness. Theorem 2 characterizes the uniqueness of a dual optimal solution in terms of an optimal simplex tableau also.

In [6] Williams gave characterizations of a bounded solution set of a linear program in terms of the initial data of the linear program. In Theorems 4 and 5 we characterize the boundedness of the primal and dual optimal solution sets respectively in terms of an optimal simplex tableau. As expected the boundedness characterizations impose less stringent conditions than the corresponding uniqueness characterizations. The possible and impossible combinations of uniqueness, boundedness and degeneracy of primal and dual optimal solutions are summarized in Table 1. Examples following Table 1 illustrate all the possible combinations.

We introduce now the standard linear program in canonical form [1]

Maximize $z = c^T y$ subject to $Ay = b, y \geq 0$ \hspace{1cm} (1)

where $c$ and $b$ are given vectors in $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, $A$ is a given real $m \times n$ matrix and the superscript $T$ denotes the transpose. We note immediately that uniqueness of a solution $\bar{y}$ to (1) is equivalent to uniqueness of a solution $(\bar{y}, \bar{s})$ in $\mathbb{R}^{n+m}$ to the

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equivalent problem with slack variable \( s \) in \( \mathbb{R}^m \)

Maximize \( z = \mathbf{c}^T \mathbf{y} \) subject to \( \mathbf{s} = -\mathbf{A} \mathbf{y} + \mathbf{b}, \mathbf{y} \geq 0, \mathbf{s} \geq 0 \).

Define \( \mathbf{x} = (\mathbf{y}, \mathbf{s}) \) and assume that after a finite number of pivots the following standard optimal simplex tableau \([1,5]\) has been obtained after a column and row rearrangement if necessary.

\[
\begin{array}{cccccc}
\mathbf{s}_+ & \mathbf{s}_0 & \mathbf{x}_+ & \mathbf{x}_0 & \mathbf{z} \\
\hline
\mathbf{I} & \mathbf{0} & \mathbf{L}_+ & \mathbf{M}_+ & + & \mathbf{x}_B^+ \\
\mathbf{0} & \mathbf{I} & \mathbf{L}_0 & \mathbf{M}_0 & 0 & \mathbf{x}_B^0 \\
\mathbf{0} & \mathbf{0} & + & 0 & 0 & \mathbf{z} \\
\end{array}
\]

This is equivalent to the following condensed or Tucker tableau \([2,7]\)

\[
\begin{array}{ccccccc}
\mathbf{s}_+ & \mathbf{s}_0 & \mathbf{x}_+ & \mathbf{x}_0 & \mathbf{z} & 1 \\
\hline
\mathbf{u}_N & \mathbf{x}_B^+ & = & \mathbf{L}_+ & \mathbf{M}_+ & + & \mathbf{x}_B^+ \\
\mathbf{u}_N & \mathbf{x}_B^0 & = & \mathbf{L}_0 & \mathbf{M}_0 & 0 & \mathbf{x}_B^0 \\
1 & \mathbf{z} = & + & 0 & 0 & \mathbf{w} \\
\end{array}
\]

For convenience define

\[
\mathbf{L} = \begin{bmatrix} \mathbf{L}_0 & \mathbf{x}_0 \end{bmatrix} \quad \text{and} \quad \mathbf{M} = \begin{bmatrix} \mathbf{M}_+ \\ \mathbf{M}_0 \end{bmatrix}.
\]

In the above tableaus the symbols are defined as follows:

\( \mathbf{x}_B^+ \) = primal optimal positive basic variables (with values denoted by + in rightmost column of tableau (3))
\( x_B = \) primal optimal zero basic variables (with values denoted by 0 in rightmost column of tableau (3))

\( u_B = \) dual optimal positive basic variables (with values denoted by + in bottom row of tableau (3))

\( u_0 = \) dual optimal zero basic variables (with values denoted by 0 in bottom row of tableau (3))

\( x^* = \) primal optimal (zero) nonbasic variables corresponding to \( u_B \)

\( x^*_0 = \) primal optimal (zero) nonbasic variables corresponding to \( u_0 \)

\( x^*_0 = \) dual optimal (zero) nonbasic variables corresponding to \( x^*_B \)

\( v_0 = \) dual optimal (zero) nonbasic variables corresponding to \( x^*_0 \)

\( I = \) identity matrix of appropriate dimension

\( M_+ = \) matrix in tableau (3) with rows corresponding to \( x_B \) and columns corresponding to \( u_0 \)

\( M_0 = \) matrix in tableau (3) with rows corresponding to \( x_0 \) and columns corresponding to \( u_0 \)

\( L_+ = \) matrix in tableau (3) with rows corresponding to \( x_B \) and columns corresponding to \( u_0 \)

\( L_0 = \) matrix in tableau (3) with rows corresponding to \( x_0 \) and columns corresponding to \( u_0 \)

\( x_B = (x_B^*, x_B_0) \)

\( x_N = (x_N^*, x_N_0) \)

\( u_B = (u_B^*, u_B_0) \)

\( u_N = (u_N^*, u_N_0) \)

\( w = \) dual objective function

\( Q = \) maximum value of the primal objective function on the feasible region.
Our principal results, contained in Theorems 1 to 5, are given in terms of the sets
\( \mathcal{M}_0, \mathcal{M}, \) and \( \mathcal{L} \) of the optimal tableau (3) and can be summarized as follows:

1. **Primal uniqueness** \( \mathcal{B}_0 \neq \emptyset \) whenever \( \mathcal{B}_0 \neq \emptyset \) and \( u^T \mathcal{M}_0 > 0 \) for some \( u \).

2. **Dual uniqueness** \( u^T \neq 0 \) whenever \( x^T \neq 0 \) and \( M_0 q < 0 \) for some \( q \).

3. **Primal and dual uniqueness** \( x^T = 0 \) and \( u^T = 0 \).

4. **Primal boundedness** \( r^T \mathcal{M} > 0 \) for some \( r \).

5. **Dual boundedness** \( L t < 0 \) for some \( t \).
2. **Uniqueness of Solution**

With the aid of the optimal tableau (3) it is possible to characterize the uniqueness of a primal optimal solution as follows.

**Theorem 1** (Uniqueness of primal optimal solution). The primal optimal solution to the linear program (1), $x_B > 0, x_B = 0, x_{N+} = 0, x_{N0} = 0$, is unique if and only if $u_B$ is nonvacuous whenever $u_B^0$ is nonvacuous and there exists a $p > 0$ such that $p^T u = 0$.

**Proof.** The condition that there exists a $p$ such that $p > 0$ and $p^T u > 0$ is equivalent by Motzkin's theorem of the alternative [3] to

$$-M_0 q > 0, \ 0 \neq q > 0 \text{ has no solution.} \quad (4)$$

We establish now the necessity and sufficiency of condition (4) for the uniqueness of the solution $(x_B, x_N)$.

**(Necessity)** Let $(x_B, x_N)$ be a unique solution of (1). If $u_B^0$ is empty then condition (4) is vacuously satisfied because $M_0$ is vacuous. Suppose now that $u_B^0$ is nonvacuous and that $x_B$ is nonvacuous, else for sufficiently small positive $\lambda$ and for a vector $e$ of ones the point $\tilde{x}_B = x_B - \lambda M e, \tilde{x}_{N+} = 0, \tilde{x}_{N0} = \lambda e$ is primal feasible and distinct from $(x_B, x_{N+}, x_{N0})$ and the corresponding value of the objective function is $\tilde{z} = Q$ contradicting the uniqueness of $(x_B, x_{N+}, x_{N0})$. Hence $M_0$ is nonvacuous. Suppose now that there exists a $q$ such that $-M_0 q < 0$ and $0 \neq q < 0$, thus violating (4). We will show that this contradicts the uniqueness of $(x_B, x_N)$. For a sufficiently small positive number $\lambda$, the point

$$\tilde{x}_B = x_B - \lambda M q > 0$$

$$\tilde{x}_{B0} = -\lambda M q > 0$$

$$\tilde{x}_{N+} = 0$$

$$\tilde{x}_{N0} = \lambda q > 0, \ \lambda q \neq 0$$

-5-
is primal feasible and distinct from \((x_B^+, x^+_N, x^0, x_N^0)\) but the corresponding value of the objective function is \(\hat{z} = Q\) thus contradicting the uniqueness of \((x_B^+, x^+_N, x^0, x_N^0)\).

(Sufficiency) If \(u^0_B\) is empty then for any other primal feasible point \((\hat{x}_B, \hat{x}_N)\) distinct from \((x_B^+, x_N^0)\) at least one component of \(\hat{x}_N^0\), say \((\hat{x}_N^0)_k\), must be positive while \(\hat{x}_N^0 > 0\) in which case the corresponding value of the objective function is
\[
\hat{z} = -u_B^{T} \hat{x}_N + Q = -\left( u_{B_k} \right) (\hat{x}_N^0)_k + Q < Q
\]
and hence \((\hat{x}_B, \hat{x}_N)\) cannot be primal optimal and so \((x_B^+, x_N^0)\) is unique. Suppose now that \(u^0_B\) is nonvacuous then \(x^0_B\) and consequently \(M^0_N\) are nonvacuous and suppose that (4) holds. We will now show that if \((x_B^+, x_N^0)\) is not unique a contradiction ensues. For a distinct optimal solution \((\hat{x}_B, \hat{x}_N)\) to exist we need to have \(0 \neq (\hat{x}_N^0, \hat{x}_N^0) \geq 0\). If \(0 \neq \hat{x}_N^0 \geq 0\), then \(\hat{z} = -u_B^{T} \hat{x}_N + Q < Q\) and hence the point cannot be optimal. So \(\hat{x}_N^0 = 0\) and \(0 \neq \hat{x}_N^0 \geq 0\).

Now if for some k-th component of \(\{x^0_B, x^0_N\}\), say \(x^0_N/k \neq 0\), it follows that \(\hat{x}_N^0 < 0\) making the point infeasible. Hence \(x^0_N \geq 0\) and \(0 \neq \hat{x}_N^0 \geq 0\), which contradicts (4).

Remark 1. In [1,p.95] Dantzig established the sufficiency of the emptiness of \(u^0_B\) for the uniqueness of the primal optimal solution. This is a special case of Theorem 1 above.

Uniqueness of a solution to the dual linear program

\[
\text{Minimize } w = b^T \bar{v} \text{ subject to } A\bar{v} \geq c, \bar{v} \geq 0 \quad (5)
\]

associate\(\)d with the linear program (1) can also be obtained by means of the optimal tableau (3). We again note that uniqueness of a solution \(\bar{v}\) to (5) is equivalent to uniqueness of a solution \((\bar{v}, \bar{t})\) in \(\mathbb{R}^{m+n}\) to the equivalent linear program with slack variable \(t\) in \(\mathbb{R}^n\)

\[
\text{Minimize } w = b^T \bar{v} \text{ subject to } t = A\bar{v} - c, \bar{v} \geq 0, t \geq 0 \quad (6)
\]

The combined dual variables \(v\) and \(t\) are defined as \(u = (v, t)\) and appear in tableau (3).

By casting (5) into the equivalent format of problem (1)
we can characterize uniqueness of its solution by means of tableau (3) as follows.

**Theorem 2. (Uniqueness of dual optimal solution)** The dual optimal solution to the linear program (1), \( u_{B_0} > 0, u_B = 0, u_{N_0} = 0, u_{N} = 0, \) is unique if and only if \( u_{B_0} \) is nonvacuous whenever \( x_{B_0} \) is nonvacuous and there exists a \( q \geq 0 \) such that \( M_0 q < 0 \).

By combining Theorems 1 and 2 we can characterize the simultaneous uniqueness of both primal and dual optimal solutions as follows.

**Theorem 3. (Uniqueness of primal and dual optimal solutions)** The primal and dual optimal solutions to the linear program (1) are both unique if and only if both are nondegenerate, that is \( x_{B_0} \) is empty and \( u_{B_0} \) is empty.

**Proof.** If both the primal and dual optimal solutions are nondegenerate then the dual optimal solution is unique by Theorem 2 and the primal optimal solution is unique by Theorem 1. Suppose now that both primal and dual optimal solutions are unique and that one of them is degenerate. We will exhibit a contradiction. If the primal (dual) optimal solution is degenerate then by Theorem 2 (Theorem 1) the dual (primal) optimal solution is also degenerate. Hence both primal and dual optimal solutions are degenerate. By Theorem 1 then there exists a \( p \neq 0 \) such that \( p^T M_0 > 0 \) and by Theorem 2 there exists a \( q \geq 0 \) such that \( M_0 q < 0 \). Since both \( p \) and \( q \) are nonzero this then leads to the contradiction

\[
0 < (p^T M_0) q = p^T (M_0 q) < 0 .
\]
3. Boundedness of Solution

Again with the aid of the optimal tableau (3) it is possible to characterize the boundedness of a primal optimal solution set as follows.

**Theorem 4.** (Boundedness of the primal optimal solution set) The primal optimal solution set to the linear program (1) is bounded if and only if for some or all optimal simplex tableaus such as (3) there exists an \( r > 0 \) such that \( r^T M > 0 \).

**Proof.** Again as in the proof of Theorem 1, the condition that there exists an \( r > 0 \) that \( r^T M > 0 \) is equivalent by Motzkin's theorem of the alternative [3] to

\[-Ms > 0, 0 \neq s > 0 \text{ has no solution } s.\]

We establish now the necessity and sufficiency of condition (8) for the boundedness of the solution set of (1).

**(Necessity)** Let (3) be some optimal tableau for problem (1) and let there exist a nonzero \( s \) satisfying \( s > 0 \) and \( -Ms > 0 \). We will show that this implies that the primal optimal solution set is unbounded. For any positive \( \lambda \) the point

\[
\begin{align*}
\tilde{x}_B &= (x_B^1, \ldots, x_B^n, x_{n_A^0}) - \lambda M s \\
\tilde{x}_N &= 0 \\
\tilde{x}_N^0 &= \lambda s > 0
\end{align*}
\]

is primal feasible, the corresponding value of the objective function is \( Q \) and hence is primal optimal. However \( \|x_N^0\| = \lambda \|s\| \) is unbounded as \( \lambda \to \infty \) because \( s \neq 0 \). Hence the primal optimal solution set is unbounded.

**(Sufficiency)** If for some optimal tableau (3) \( u_{B_0} \) is empty then by Theorem 1, \( (x_B, x_N) \) is a unique solution of problem (1). So suppose now that \( u_{B_0} \) is nonempty for all optimal tableaus of problem (1) and let (3) be any such optimal tableau. We will show that if (1) has an unbounded primal optimal solution set then there exists a nonzero \( s \) such that \( s \geq 0 \) and
Since the primal optimal solution set is unbounded there exists a sequence of nonnegative optimal vectors \( \{x^i_B, x^i_N\}_i \), \( i = 1,2,\ldots \), such that

\[
\lim_{i \to \infty} ||x^i_B - x^{i*}_B, x^i_N - x^{i*}_N|| = \infty.
\]

From tableau (3), since \( x_N = (x^*_N, x^*_N) = 0 \), this is equivalent to

\[
\lim_{i \to \infty} ||(z^*_N, x^*_N - x^i_N, x^i_N, x^{i*}_N)|| = \infty.
\]

If \( 0 \neq x^i_N = 0 \) then the corresponding value of the objective function is

\[
z^i = -u_B x^i_N + \emptyset < \emptyset
\]

and hence the point \( (x^i_B, x^i_N) \) cannot be primal optimal. So \( x^i_N = 0 \), \( i = 1,2,\ldots \), and hence

\[(9) \text{ it follows that } \lim_{i \to \infty} ||x^i_N|| = \infty. \text{ But,}
\]

\[
x^i_B = -Mx^i_N + x^i_B \geq 0, i = 1,2,\ldots
\]

Hence

\[
\frac{-Mx^i_N}{||x^i_N||} + \frac{x^i_B}{||x^i_N||} \geq 0, i = 1,2,\ldots
\]

Since \( \lim_{i \to \infty} ||x^i_N|| = \infty \) it follows by the Bolzano-Weierstrass Theorem that the bounded sequence

\[
\left\{ \frac{x^i_N}{||x^i_N||} \right\}
\]

has an accumulation point \( s \) such that \( 0 \neq s \geq 0 \) and \(-Ms \geq 0\).

By the symmetry between (1) and (5), the following result characterizes the boundedness of the dual optimal variables associated with (1).
Theorem 5. (Boundedness of the dual optimal solution set) The dual optimal solution set to the linear program (1) is bounded if and only if for some or all optimal simplex tableaus such as (3) there exists a $t \geq 0$ such that $Lt < 0$. 
4. Outcomes and Examples

We now summarize for convenience the possible and impossible combinations of uniqueness, boundedness and degeneracy of primal and dual optimal solutions in Table 1. Examples (1' to 7) illustrating all the possible combinations appearing in Table 1 are given following the table.

Table 1

<table>
<thead>
<tr>
<th>Primal Optimal Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
</tr>
<tr>
<td>---</td>
</tr>
<tr>
<td>B</td>
</tr>
<tr>
<td>B(U)</td>
</tr>
<tr>
<td>D</td>
</tr>
<tr>
<td>D(U)</td>
</tr>
<tr>
<td>UD</td>
</tr>
<tr>
<td>UD</td>
</tr>
</tbody>
</table>

B = Bounded  D = Degenerate  UD = Unique

B = Unbounded  D = Nondegenerate  UD = Nonunique

1(i) = Possible combination illustrated by Example (i)

1(i') = Possible combination illustrated by Example (i)

with the roles of the primal and dual problems interchanged

0 = Impossible combination.

Example 1 (Primal bounded nonunique degenerate/nondegenerate, dual unique degenerate)

Max \( x_1 + x_2 \) s.t. \( x_1 x_2 \leq 1, x_2 \leq 1, x_1 \geq 0, x_2 \geq 0 \)

The primal optimal solution set is \( \{ x_1, x_2 | x_1 x_2 = 1, x_1 \geq 0, x_2 \geq 0 \} \) contains the degenerate vertex \( x_1 = 0, x_2 = 1 \) and the nondegenerate vertex \( x_1 = 1, x_2 = 0 \) which correspond to the following two optimal tableaus respectively where the slacks \( x_3 \) and \( x_4 \) have been introduced:
From the first tableau we observe that both primal and dual solutions are degenerate. The primal solution is nonunique, because \( p \cdot (-1) > 0, p \geq 0 \) has no solution. The primal solution set is bounded because \( (r_1, r_2) \begin{pmatrix} -1 \\ 1 \end{pmatrix} > 0 \) has a nonnegative solution \( r_1 = 0, r_2 = 1 \). The dual solution is unique because \(-1 \cdot q < 0, q \geq 0\) has a solution. From the second tableau we observe that the nondegenerate primal solution is nonunique because it is nondegenerate while the dual solution is degenerate. The primal solution is bounded because \( (r_1, r_2) \begin{pmatrix} 1 \\ 1 \end{pmatrix} > 0 \) has a nonnegative solution \( r_1 = 1, r_2 = 1 \). Furthermore the dual solution is unique because the primal solution is nondegenerate. By interchanging the roles of the primal and dual problems this example can also serve to illustrate the case where the dual optimal solution is bounded, nonunique degenerate/nondegenerate while the primal optimal solution is unique and degenerate.

**Example 2 (Primal and dual bounded nonunique degenerate)**

\[
\text{Max } x_1 + x_2 \text{ s.t. } x_1 + x_2 \leq 1, x_1 + x_2 \leq 1, x_1 \geq 0, x_2 \geq 0.
\]

The primal optimal solution set is \{\( x_1, x_2 | x_1 + x_2 = 1, x_1 \geq 0, x_2 \geq 0 \)\} contains the two degenerate vertices \( x_1 = 0, x_2 = 1 \) and \( x_1 = 1, x_2 = 0 \) which correspond to the following two optimal tableaus respectively with slacks \( x_3 \) and \( x_4 \):

\[
\begin{array}{cccc}
\text{x}_1 & \text{x}_2 & \text{x}_3 & \text{x}_4 \\
1 & 0 & 1 & -1 & 0 & x_1 \\
0 & 1 & 0 & 1 & 1 & x_2 \\
0 & 0 & 1 & 0 & 1 & z \\
u_3 & u_4 & u_1 & u_2
\end{array}
\]

\[
\begin{array}{cccc}
\text{x}_1 & \text{x}_2 & \text{x}_3 & \text{x}_4 \\
1 & 0 & 1 & 1 & 1 & x_1 \\
0 & 1 & 0 & 1 & 1 & x_2 \\
0 & 0 & 1 & 0 & 1 & z \\
u_3 & u_2 & u_1 & u_4
\end{array}
\]
From these tableaus we observe that both primal and dual solutions are degenerate. The primal solution is not unique because \( p \cdot 0 < 0, p \geq 0 \) has no solution, but the primal solution is bounded because \( (r_1, r_2) \begin{pmatrix} 1 \\ 0 \end{pmatrix} > 0 \) has a nonnegative solution \( r_1 = 1, r_2 = 1 \). The dual solution is not unique because \( 0 \cdot q < 0, q \geq 0 \) has no solution, but the dual solution is bounded because \( (-1) \begin{pmatrix} 1 \\ t_2 \end{pmatrix} < 0 \) has a solution \( t_1 = 1, t_2 = 1 \).

Example 3 (Primal and dual unique and nondegenerate)

\[
\begin{align*}
\text{Max } & x_1 + x_2 \\
\text{s.t. } & x_1 \leq 1, x_2 \leq 1, x_1 \geq 0, x_2 \geq 0.
\end{align*}
\]

The unique primal optimal solution is \( x_1 = x_2 = 1 \) and the unique dual optimal solution is \( u_1 = u_2 = 1 \). These solutions correspond to the optimal tableau with slacks \( x_3 \) and \( x_4 \):

\[
\begin{array}{ccccccc}
& x_1 & x_2 & x_3 & x_4 & = 1 \\
1 & 0 & 1 & 0 & 1 & x_1 \\
0 & 1 & 0 & 1 & 1 & x_2 \\
0 & 0 & 1 & 1 & 2 & 0
\end{array}
\]

We observe from the tableau that both primal and dual optimal solutions are nondegenerate hence they are both unique.

Example 4 (Primal unbounded nondegenerate, dual unique degenerate)

\[
\begin{align*}
\text{Max } & x_2 \\
\text{s.t. } & x_2 \leq 1, x_1, x_2 \geq 0.
\end{align*}
\]

The primal optimal solution set \( \{ x_1, x_2 \mid x_1 \geq 0, x_2 = 1 \} \) contains the nondegenerate vertex \( x_1 = 0, x_2 = 1 \) which corresponds to the following optimal tableau where the slack \( x_3 \) has been introduced:

\[
\begin{array}{ccccccc}
& x_2 & x_1 & x_3 & = 1 \\
1 & 0 & 1 & 1 & x_2 \\
0 & 0 & 1 & 1 & z \\
u_3 & u_2 & u_1
\end{array}
\]
From the tableau we conclude that the primal solution set is unbounded because \( r \cdot 0 \geq 0 \), \( r \geq 0 \) has no solution. The degenerate dual solution is unique because the primal solution is nondegenerate.

Example 5 (Primal unbounded degenerate, dual unbounded degenerate)

Max \( x_2 \) s.t. \( x_2 \leq 1, x_2 \geq 1, x_1, x_2 \geq 0 \).

The primal optimal solution set is \( \{x_1, x_2 \mid x_1 = 0, x_2 = 1\} \) and the dual solution set is \( \{u_1, u_2 \mid u_1 - u_2 = 1, u_1, u_2 \geq 0\} \). The primal degenerate vertex solution \( x_1 = 0, x_2 = 1 \) corresponds to the following optimal tableau with slack variables \( x_3 \) and \( x_4 \):

\[
\begin{array}{cccc|c}
 x_3 & x_4 & x_1 & x_2 \\
\hline
 1 & 0 & 0 & 1 \\
 0 & 1 & 0 & 1 \\
 0 & 0 & 0 & 1 \\
\hline
 u_4 & u_2 & u_3 & u_1
\end{array}
\]

From the tableau we conclude that the primal solution set is unbounded because \( (r_1, r_2) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) has no nonnegative solution and that the dual solution is also unbounded because \( (0, 1) \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} < 0 \), has no nonnegative solution.

Example 6 (Primal bounded nonunique degenerate, dual unbounded degenerate)

Max \( x_2 \) s.t. \( x_2 \leq 1, x_2 \geq 1, x_1 \leq 1, x_1, x_2 \geq 0 \).

The primal optimal solution set is \( \{x_1, x_2 \mid 0 \leq x_1 \leq 1, x_2 = 1\} \) and the dual optimal solution set is \( \{u_1, u_2, u_3 \mid u_1 - u_2 = 1, u_3 = 0, u_1, u_2 \geq 0\} \). A primal degenerate vertex solution is \( x_1 = 0, x_2 = 1 \) which corresponds to the following optimal tableau with slack variables \( x_3, x_4 \) and \( x_5 \):
From the tableau we conclude that the primal optimal solution set is bounded because

\[
\begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\]
has a nonnegative solution \( r_1 = r_2 = 0, r_3 = 1 \). However the primal solution is nonunique because \( p \cdot 0 < 0 \) has no nonnegative solution. The dual optimal solution set is unbounded because \( (0 \ 0 \ 0)^T \) has no nonnegative solution.

**Example 7** (Primal unique degenerate, dual unbounded degenerate)

\[
\begin{align*}
\text{Max } x_1 & \quad \text{s.t. } x_1 \leq 1, \quad -x_1 + x_2 \leq -1, \quad x_1, x_2 \geq 0.
\end{align*}
\]

The unique primal optimal solution is the degenerate vertex \( x_1 = 1, x_2 = 0 \) and corresponds to the following optimal tableau with slacks \( x_3 \) and \( x_4 \):

\[
\begin{array}{c|cccc|c}
\hline
x_1 & x_4 & x_3 & x_2 = 1 \\
\hline
1 & 0 & 1 & 0 & 1 & x_1 \\
0 & 1 & 1 & 1 & 0 & x_4 \\
0 & 0 & 1 & 1 & 1 & z \\
\hline
u_3 & u_2 & u_1 & u_4 & \\
\end{array}
\]

From the tableau we conclude that the primal solution is unique because \( p \cdot 1 > 0 \) has the solution \( p = 1 \) whereas the dual optimal solution set is unbounded because \( (1 \ 1)^T \cdot (t_1 \ t_2) < 0 \) has no nonnegative solution.
REFERENCES

OPTIMAL SIMPLEX TABLEAU CHARACTERIZATION OF UNIQUE AND BOUNDED SOLUTIONS OF LINEAR PROGRAMS.

O. L. Mangasarian

Mathematics Research Center, University of Wisconsin
610 Walnut Street
Madison, Wisconsin 53706

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SUPPLEMENTARY NOTES
U.S. Army Research Office and National Science Foundation
P.O. Box 12211
Research Triangle Park
North Carolina 27709

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ABSTRACT
Uniqueness and boundedness of solutions of linear programs are characterized in terms of an optimal simplex tableau. Let \( M \) denote the submatrix in an optimal simplex tableau with columns corresponding to degenerate optimal dual basic variables. A primal optimal solution is unique if and only if there exists a nonvacuous nonnegative linear combination of the rows of \( M \) corresponding to degenerate optimal primal basic variables which is positive. The set of primal optimal solutions is bounded if and only if there exists a nonnegative linear combination of the rows of \( M \) which is positive. When \( M \) is empty the primal optimal solution is unique.