ELECTRICALLY SMALL LOOP ANTENNA LOADED BY A HOMOGENEOUS AND ISOTROPIC FERRITE CYLINDER—PART II

By

D. V. Girl and R. W. P. King

July 1978

Technical Report No. 668

Division of Applied Sciences
Harvard University Cambridge, Massachusetts
magnetic current on finite ferrite-rod antenna
approximate 3-term solution
numerical solution of coupled integral equations
experimental verification

Two theoretical approaches are developed to determine the magnetic current distribution on a ferrite cylinder of finite length that is center-driven by an electrically small loop antenna carrying a constant current. The first method makes use of the analogy between the ferrite rod antenna and the conducting cylindrical dipole antenna to derive an integral equation for the magnetic current on an infinitely permeable ($\mu_r=\infty$) ferrite antenna that corresponds to the integral equation for the electric current on a perfectly conducting electric dipole.
pole antenna. In the limit h → h, this integral equation is shown to agree with that obtained previously in Part I for the infinite ferrite rod antenna. Continuing to parallel the treatment of the electric dipole antenna, the integral equation is modified by the introduction of an internal impedance per unit length of the magnetic conductor to account for values of p that are large but not infinite, and finally an approximate, three-term expression is derived for the current on an 'imperfectly conducting' magnetic conductor. The second, more rigorous theoretical approach obtains two coupled integral equations in terms of the tangential electric field and the tangential electric surface current from independent treatments of the interior (ferrite) and exterior (free space) problems. The coupled equations are then solved numerically by means of the moment method. Finally the results of the two theories are compared with experimental measurements made on eleven different antenna configurations. The agreement is good.
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July 1978

The research reported in this document was made possible through support extended the Division of Engineering and Applied Physics, Harvard University by the U. S. Army Research Office, the U. S. Air Force Office of Scientific Research, and the U. S. Office of Naval Research under the Joint Services Electronics Program by Contracts N00014-67-A-0298-0005 and N00014-75-C-0648.

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ELECTRICALLY SMALL LOOP ANTENNA LOADED BY A
HOMOGENEOUS AND ISOTROPIC FERRITE CYLINDER — PART II

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ABSTRACT

The problem of a finite, ferrite-rod antenna has been treated theoretically by recognizing an analogy between the ferrite antenna and the conducting cylindrical dipole antenna which has been studied extensively. Initially the ferrite is idealized to be a perfect magnetic conductor and an Hallén type of integral equation [1] is obtained for the magnetic current. By allowing the antenna height to approach infinity, the formulation is shown to be consistent with previously obtained results for the infinitely long ferrite antenna [2]. Subsequently, the integral equation is modified appropriately to treat the ferrite as an imperfect magnetic conductor, and the current is obtained in the three-term form of King and Wu [3]. Because this treatment relies rather heavily on a mathematical equivalence of the two problems under idealized driving conditions, an alternative, more rigorous formulation is presented. The result is a pair of coupled integral equations in the tangential electric field (or magnetic current) and the circumferential electric current. The coupled integral equations are solved numerically. An experimental apparatus was fabricated to verify the solutions. Good agreement is obtained for a range of parameters. The experiments were performed for three values of \( \Omega = 2 \ln(2h/a) = 8.5534, 7.4754 \) and 6.089. The electrical radius \( a_k \) ranged from .00132 to .01662.
1. INTRODUCTION

In an earlier report on this subject [2] the magnetic current on a ferrite-rod antenna was derived explicitly in the form of an inverse Fourier integral. The driving loop loaded by an infinitely long, homogeneous and isotropic ferrite rod was assumed to be electrically small so that it carried an essentially constant current $I_0$. When the ferrite rod is assumed to be of infinite length, the magnetic current is equal to a definite integral which is suitable for numerical evaluation. Two values of electrical radii, viz., $a_0 = 0.05$ and 0.1, were considered and for one of the cases the magnetic current was plotted [2] for several values of the permeability of the ferrite rod ranging from 10 to 200. The total magnetic current can be interpreted in terms of a sum of transmission and radiation currents. If $\mu_r$ and $\epsilon_r$ of the ferrite rod are assumed to be real, the transmission current can be associated with an unattenuated, rotationally symmetric TE surface wave. It was further found that the cutoff condition for this wave is that the electrical radius $a_0$ be greater than 2.405.

In a practical situation, however, the antenna is of necessity finite and electrically short as well, so that a new mathematical formulation along with an experimental investigation is needed for the problem of a finite ferrite-rod antenna. Sections 2 through 8 present the two different theoretical approaches used to determine the magnetic current distribution on the finite ferrite antenna; Section 9 describes the experimental apparatus and results.

2. PROBLEM OF A FINITE FERRITE-ROD ANTENNA

The present formulation is based on the analogy between the cylindrical dipole antenna and the ferrite-rod antenna. The dipole antenna is made up of a wire, rod or tube of high electrical conductivity and may be driven by a
two-wire line. Equivalently, a monopole antenna fed by a coaxial line corresponds to a dipole antenna through its image in a ground plane. In either configuration, the driving source is represented by an idealized voltage or electric field generator which mathematically takes the form of a delta function. Similarly, the ferrite rod antenna is fabricated from a material of high magnetic permeability and is driven by an electrically small loop antenna carrying a constant current. The loop is, correspondingly, represented by an idealized current or magnetic field generator and takes the form of a delta function. These similarities suggest approaching the problem of the ferrite antenna by treating the ferrite rod as a good magnetic conductor. Initially, however, the ferrite is idealized to be a perfect magnetic conductor \((\mu_r = \infty)\) and, later, appropriate changes are made to account for the finiteness of the value of the permeability of the ferrite material.

3. FERRITE AS A PERFECT MAGNETIC CONDUCTOR

The analogy between the ferrite antenna and the dipole antenna is based on the dual property of electric and magnetic field vectors in Maxwell's equations

\[
\begin{align*}
\nabla \times \mathbf{E} &= -\mathbf{\dot{B}} \\
\nabla \times \mathbf{H} &= \mathbf{J} + \mathbf{\dot{D}} \\
\n\nabla \cdot \mathbf{B} &= 0 \\
\n\nabla \cdot \mathbf{D} &= \rho
\end{align*}
\]

(1)

Figure 1(a) shows an electrically small loop antenna of diameter \(2a\). The loop carries a constant current and is assumed to be made up of a wire of infinitesimally small radius. The wire loop is loaded by a ferrite cylinder of height \(2h\). The ferrite is assumed to have an infinite permeability, in which
FIG. 1 (a) ELECTRICALLY SMALL LOOP ANTENNA OF DIAMETER \('2a'\) LOADED BY A FERRITE CYLINDER OF HALF HEIGHT \('h'\) AND SURROUNDED BY FREE SPACE.

(b) MATHEMATICALLY EQUIVALENT BUT PHYSICALLY UNAVAILABLE MODEL FOR THE ANTENNA SHOWING THE IDEALIZED CURRENT GENERATOR \(\phi I_0\) \(\delta(\rho-a) \delta(z)\).
case the value of its dielectric constant $\varepsilon_r$ is immaterial in view of the nature of the driving source. Figure 1(b) shows the mathematical model of the antenna. Region I is the ferrite with parameters $\mu_r$, $\varepsilon_r$, $k_1$ and region II is free space with constitutive parameters $\epsilon_0$, $\mu_0$ and wave number $k_0$. Because of the nature of the driving source and azimuthal symmetry, the non-zero components of the fields are $H_z$, $H_\rho$, and $E_\phi$. A time dependence of the form $\exp(-i\omega t)$ is assumed. Because of the assumption $\mu_r = \infty$, $H_z$ and $H_\rho$ vanish in region I. The ferrite is also assumed to be homogeneous and isotropic. Thus the idealized driving source is taken into account by setting

$$H_z = -i_0^e(z) \quad \text{(for } \rho = a \text{ and } |z| \leq h) \quad (2)$$

Since both regions I and II have vanishing electrical conductivity and there is no free charge, Maxwell's equations reduce to

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad (3a)$$
$$\nabla \times \vec{H} = \vec{J} \quad (3b)$$
$$\nabla \cdot \vec{B} = 0 \quad (3c)$$
$$\nabla \cdot \vec{D} = 0 \quad (3d)$$

It is required to solve (3a-d) for the fields subject to the condition (2) which states that the tangential component of $\vec{H}$ is discontinuous by the true electric surface current at $\rho = a$ and for $|z| \leq h$. In order to obtain an integral equation for the magnetic current on the antenna, an electric vector potential $A^e$ and a magnetic scalar potential $\phi^e$ are defined and used.

$$\vec{D} = -\nabla \times \vec{A}^e \quad (4)$$

The definition of $A^e$ is incomplete unless its divergence is also specified. Using (4) in (3b) gives $\nabla \times (\vec{H} + \vec{J}^e) = 0$, from which the scalar magnetic
potential is defined by setting
\[ \vec{\nabla} + \vec{A} = -\nabla \phi^* \]  

(5)

In terms of the potentials, the fields are now given by
\[ \vec{E} = (-1/\varepsilon)\nabla \times \vec{A} \]  
\[ \vec{H} = -\nabla \phi^* - \vec{A} \]  

(6a, 6b)

Substitution of (6a,b) into Maxwell's equations (3c) and (3a) gives
\[ \nabla^2 \phi^* + \nabla \cdot \vec{A} = 0 \]  
\[ \nabla^2 \vec{A} - \mu \varepsilon \frac{\partial}{\partial t} \vec{A} = \nabla \left[ \nabla \cdot \vec{A} + \mu \varepsilon \nabla \phi^* \right] \]  

(7a, 7b)

Equation (7) is a set of coupled equations for the potentials which may be

decoupled by defining the dual Lorentz gauge
\[ \nabla \cdot \vec{A} + \mu \varepsilon \nabla \phi^* = 0 \]  

(8)

Upon using (8), (7) becomes
\[ \nabla^2 \phi^* - \mu \varepsilon \nabla \phi^* = 0 \]  
\[ \nabla^2 \vec{A} - \mu \varepsilon \frac{\partial}{\partial t} \vec{A} = 0 \]  

(9a, 9b)

If (9) is solved for the potentials, subject to suitable boundary conditions, then the electromagnetic field is known everywhere by making use of (6).

However, for the problem at hand, a \( \hat{z} \)-component of electric vector potential is adequate for a complete solution so that \( \vec{A}^0 = \hat{z} A^0_z \). On the surface \( r = a, |z| < h \) of the antenna, (6b) then becomes
\[ \nabla \cdot \phi^* = \nabla \cdot \phi^* = 0 \]  
\[ \nabla \cdot \vec{A} = -\frac{\partial}{\partial z} A^0_z + i \omega A^0_z \]  

(10)

Using (8), one can rewrite (10) as
\[ \left( \frac{d^2 A_e^z}{dz^2} + k_0^2 A_e^z \right) = i(k_0/v_0)I_0^e \delta(z) \]  

(11)

where \( k_0 \) is the free space wave number and \( v_0 \) the velocity of light in free space. This equation is identical to that for the \( z \)-component of the magnetic vector potential in the case of the dipole antenna [1, eq.(3.2.4)] and, therefore, has a complete solution - like [1, eq.(3.2.12)] - which is given by

\[ A_e^z(z) = (i/v_0)[C \cos(k_0 z) + (I_0^e/2)\sin(k_0 z)] \]  

(12)

Equation (12) is an expression for the \( z \)-component of electric vector potential in terms of the driving current \( I_0^e \). However, the general formula for \( A^e(r) \) due to an arbitrary distribution of magnetic surface current \( K^s(r) \) can be written as

\[ A^e(r) = (\epsilon_0/4\pi) \int \frac{\vec{K}^s(r')}{R(r)} dS' \]  

In general, \( \vec{K}^s(r) = \vec{K}^s(r) + \hat{n} \times \vec{P}(r) \), but because of the nature of the driving source \( \vec{P}(r) = 0 \), so that \( \vec{K}^s(r) = \vec{K}^s(r) = (\hat{n} \times \vec{P}) = \) magnetic surface current. Since rotational symmetry obtains, the total axial magnetic current can be introduced with thin antenna approximation so that \( I^e_z(z) = 2\pi aK^s_z(z) \)

\[ A^o_z(z) = A^o_z(r) = (\epsilon_0/2\pi) \int_{-h}^{h} I^s(z') \left[ \int_{-\pi}^{\pi} \frac{ik_0 R}{R_S} \right] d\theta'/2\pi \]

where

\[ R_S = [(z - z')^2 + (2a \sin \theta'/2)^2]^{1/2} \]

Letting

\[ K^s_z(z, z') = \int_{-\pi}^{\pi} \frac{ik_0 R}{R_S} d\theta'/2\pi \]

gives

\[ A^e_z(z) = (\epsilon_0/4\pi) \int_{-h}^{h} I^s(z')K^s_z(z, z') dz' \]  

(13)
$A_2^e(z)$ was previously obtained in (12). Equations (12) and (13) together lead to the required integral equation,

$$
\int_{-h}^{h} I_s(z')K_s(z,z')\,dz' = 4\pi \zeta \left[ C \cos k_0 z + (I_0^e/2) \sin k_0 |z| \right]
$$

with $\zeta = (\mu_0/\varepsilon_0)^{1/2}$ = the free space characteristic impedance.

The integral equation (14) for the magnetic current on a finite, infinitely permeable, ferrite rod antenna can be identified formally with the similar integral equation [1, eq.(3.223)] for the electric current on a finite dipole antenna made up of a perfect metallic conductor. Comparing the integral equations for the two cases, one finds that the driving voltage $V_e^0$ and the free space characteristic impedance $\zeta_0$ in the electric dipole case are replaced by the driving current $I_e^0$ and the free space characteristic admittance $(1/\zeta_0)$ in the magnetic case. Commonly used metals like copper and brass are found to have sufficiently large electrical conductivities to justify the assumption of vanishing electric field inside the material of the dipole antenna so that an integral equation of the form (14) is adequate and has been used to obtain the electric current distributions. Furthermore, if more accuracy is required, theories do exist for imperfectly conducting cylindrical transmitting antennas. However, it is questionable whether the integral equation (16) is directly applicable to the practical ferrite rod antenna due to the relative permeability ranges of available ferrites. Whereas the treatment of the imperfectly conducting dipole antenna is done for reasons of improved accuracy, a similar treatment ($\mu_r$ large but not infinite) for the ferrite antenna appears to be a necessity. This forms the subject of Section 5.
4. MAGNETIC CURRENT ON AN INFINITE ANTENNA

The magnetic current \( I^*_z(z) \) obtained by solving the integral equation (14) may be called a zeroth-order solution because of the assumption \( \mu_T = \infty \). The integral equation (14) is for a finite antenna from which the zeroth-order solution \( I^*_w(z) \) for an antenna of infinite length may be obtained. For the sake of convenience, the integral equation is rewritten as

\[
\int_{-h}^{h} I^*_w(z') K_g(|z - z'|) dz' = (4\pi/\varepsilon_0) A^e_z(z) = i4\pi I_0^e [C \cos k_0 z + \frac{I_0^e}{2} \sin k_0 |z|]
\]

As \( h \to \infty \), the vector potential \( A^e_z(z) \) is a traveling wave which may be obtained by setting \( C = I_0^e/2i \). In this case,

\[
\int_{-\infty}^{\infty} I^*_w(z') K_g(|z - z'|) dz' = 2\pi I_0^e e^{ik_0 |z|}
\]

Taking Fourier transforms of both sides of (15), one obtains

\[
\int_{-\infty}^{\infty} e^{-iz'} dz \int I^*_w(z') K_g(|z - z'|) dz' = 2\pi I_0^e \int_{-\infty}^{\infty} e^{-iz} e^{ik_0 |z|} dz
\]

The left side of the above equation is a convolution integral and on the right side, the integration may be performed to obtain

\[
\tilde{K}(\xi) \tilde{I}^*_w(\xi) = 2\pi I_0^e \left[ 2\Im k_0 / (k_0^2 - \xi^2) \right]
\]

With \( \gamma^2 = k_0^2 - \xi^2 \)

\[
\tilde{I}^*_w(\xi) = 4\pi I_0^e k_0 / \gamma_0 \tilde{K}(\xi)
\]

By recalling

\[
K_g(|z - z'|) = \int (e^{-i \Phi_0} K_g) d\Phi'/2\pi
\]

with \( \Phi_a = [(z - z')^2 + (2a \sin \theta'/2)^2]^{1/2} \), it can be shown [3] that the Fourier transform \( \tilde{K}(\xi) \) of the kernel \( K_g(z,z') \) is given by
\[
\bar{K}(\xi) = \int_{-\infty}^{\infty} e^{-i\xi z} K_0(z) \, dz = i\pi J_0(ay_0)H_0^{(1)}(ay_0) \tag{17}
\]

Using (17) in (16), one obtains

\[
\bar{I}_m^*(\xi) = 4\pi i\gamma_0^2 e^{ik_0} / y_0 J_0(ay_0)H_0^{(1)}(ay_0)
\]

The inverse Fourier transform may now be taken.

\[
I_m^*(z) = \frac{1}{2\pi} \int \bar{I}_m^*(\xi) e^{i\xi z} \, d\xi
\]

\[
I_m^*(z) = \frac{2}{\pi} I_0^e i^e k_0 \int \frac{e^{i\xi z} \, d\xi}{\gamma_0^2 J_0(ay_0)H_0^{(1)}(ay_0)} \text{ volts} \tag{18}
\]

Equation (18) is thus an explicit expression in the form of an infinite integral for the current on an infinitely long, infinitely permeable, ferrite rod antenna.

The problem of infinitely long ferrite rod antennas was formulated previously [2] in terms of differential equations and the Fourier transform of this current was obtained, from [2, eq.(23)], to be

\[
\bar{I}_m^*(\xi) = -i\omega a_\xi^2 \left[ \frac{J_0(ay_0)H_0^{(1)}(ay_0) - J_1(ay_1)H_0^{(1)}(ay_0)}{\lambda_0 J_0(ay_1)H_1^{(1)}(ay_1) - \lambda_1 J_1(ay_1)H_0^{(1)}(ay_0)} \right] \tag{19}
\]

where \( \gamma^2 = k_1^2 - \ell^2 \) and \( \gamma^2 = k_0^2 - \ell^2 \), \( \xi \) is the Fourier transform variable, and \( k_1 \) and \( k_0 \) are the wave numbers in the ferrite and the surrounding free space medium, respectively. The zeroth-order current on the infinite antenna may be obtained from (19) by taking the limit \( \nu = \infty \).

First, (19) may be rewritten as

\[
\bar{I}_m^*(\xi) = -i\omega a_\xi^2 \left[ \frac{\gamma_0 J_0(ay_0)}{(\nu - 1)J_1(ay_1)} - \frac{\gamma_0 H_0^{(1)}(ay_0)}{(\nu - 1)H_1^{(1)}(ay_0)} \right]^{-1}
\]
As \( \mu_r \rightarrow \infty \), the ratio \( [J_0(\alpha y_0)/J_1(\alpha y_0)] \) is finite so that the first term within the brackets approaches zero. In this case

\[
I^*_o(\xi) = \frac{4\xi_0 I_0^e k_0}{Y_0 J_0(\alpha y_0) H_0^{(1)}(\alpha y_0)} \left[ (\alpha y_0) J_0(\alpha y_0) \right]
\]

Furthermore, for a thin antenna, a small argument approximation may be used for the Bessel functions in the numerator so that \( I^*_o(\xi) \) reduces to

\[
I^*_o(\xi) = \frac{4\xi_0 I_0^e k_0}{Y_0 J_0(\alpha y_0) H_0^{(1)}(\alpha y_0)}
\]

from which

\[
I^*_o(z) = \frac{2}{\pi} I_0^e k_0 \int_{-\infty}^{\infty} \frac{e^{i\xi z}}{\sqrt{2(\alpha y_0) H_0^{(1)}(\alpha y_0)}} \, d\xi \quad \text{volts} \quad (20)
\]

Thus, equations (20) and (18) are both independently derived explicit expressions for the zeroth-order \( (\mu_r = \infty) \) magnetic currents on an infinitely long ferrite rod antenna. In the limit of infinite permeability the two formulations give the same result. This limit is, however, physically unrealizable since a magnetic material with \( \mu_r = \infty \) does not exist and, hence, a modification of the formulation which treats the ferrite as an imperfect magnetic conductor is required. This modification has, once again, an analogue in the electric case in the treatment of the imperfectly conducting cylindrical transmitting antenna [3], [4].

5. FERRITE AS AN IMPERFECT MAGNETIC CONDUCTOR

In order to account for the fact that the relative permeability is large but not infinite, the concept of 'internal impedance' is useful and suffi-
cient. With reference to Fig. 2, the internal impedance per unit length of a cylindrical magnetic conductor of circular cross section of radius \( a \) with its axis along the z-axis of a system of cylindrical coordinates \((\rho, \phi, z)\) may be defined by

\[
H_z (\rho = a) / I_z^* = (r_m^1 - i x_m^1) = H_z (\rho = a) / I_z^*
\]

(21)

where \( H_z (\rho = a) \) is the tangential magnetic field at the surface, \( \rho = a \), of the conductor and \( I_z^* \) is the total axial magnetic current. Recalling the expressions of electric and magnetic fields in terms of the potentials

\[
\begin{align*}
\mathbf{H} &= -\nabla \phi^* - A^e \\
\mathbf{D} &= -\nabla \phi^* \\
\nabla \times \mathbf{A}^e &= 0
\end{align*}
\]

(22a, 22b, 22c)

the electric vector potential satisfies

\[
(v^2 + k^2) A^e = 0
\]

(23)

where \( k \) is replaced by \( k_1 \) and \( k_0 \) for the two regions I and II shown in Fig. 2. For the problem of a thin cylindrical conductor, the axial component of electric potential is sufficient to satisfy Maxwell's equations and the relevant boundary conditions. Thus, the electromagnetic fields everywhere can be obtained by setting \( A^o = \tilde{z} A_z^0 \). With the vector potential being entirely axial and also because of azimuthal symmetry, (23) becomes

\[
\left[ \frac{3^2}{3 z^2} + \frac{1}{\rho} \frac{3}{3 \rho} \right] \Lambda_z^0 (\rho, z) = 0
\]

(24)

A product solution to (24) is sought in the following form:

\[
\Lambda_z^0 (\rho, z) = f_z (z) R_z (\rho)
\]

(25)
FIG. 2 A CYLINDRICAL MAGNETIC CONDUCTOR CARRYING A TOTAL AXIAL MAGNETIC CURRENT OF $I_z^*$ AND IMMERSED IN FREE SPACE.
Substitution of (25) into (24) leads to

\[ R_z \frac{d^2 f_z}{dz^2} + f_z \frac{1}{\rho} \frac{d}{d\rho} \left( \rho \frac{dR_z}{d\rho} \right) + k^2 \frac{1}{\rho} \frac{d^2 R_z}{d\rho^2} = 0 \quad (26) \]

Equation (26) may be rewritten as

\[ \frac{1}{\rho} \frac{d^2 f_z}{dz^2} + k^2 = -\frac{1}{R_z} \frac{1}{\rho} \frac{d}{d\rho} \left( \rho \frac{dR_z}{d\rho} \right) \quad (27) \]

The left side of (27) is a function of \( z \) alone, while the right side is a function of \( \rho \) alone. Hence, they can be equal to each other for all possible values of \( \rho \) and \( z \) only if they are both equal to a constant (say \( \zeta^2 \)) which may, however, be multivalued. Therefore,

\[ \frac{d^2 f_z}{dz^2} + (k^2 - \zeta^2) f_z = 0 \quad \text{and} \quad \frac{1}{\rho} \frac{d}{d\rho} \left( \rho \frac{dR_z}{d\rho} \right) + \zeta^2 R_z = 0 \]

After solving the foregoing differential equations for \( f_z \) and \( R_z \), the axial vector potential in the two regions can be written down as

\[ A_{z1}^e(\rho, z) = C_{11} J_0(\zeta_0 \rho) \exp(i \sqrt{k_1^2 - \zeta_1^2} z) \quad \text{in region I} \]

\[ A_{z2}^e(\rho, z) = C_{21} H_0^{(1)}(\zeta_0 \rho) \exp(i \sqrt{k_0^2 - \zeta_0^2} z) \quad \text{in region II} \]

Boundary conditions that are useful in determining the unknown constants in the solution require the continuity of tangential \( \mathbf{E} \) and \( \mathbf{H} \) across the surface \( \rho = a \); that is

\[ E_z^1(\rho = a) = E_z^2(\rho = a) \]

\[ B_z^1(\rho = a) = B_z^2(\rho = a) \]

In terms of the vector potential, the boundary conditions at the surface \( \rho = a \) are
Application of the boundary conditions yields

\[ \frac{1}{\varepsilon_1} \frac{\partial A_{z1}}{\partial p} = \frac{1}{\varepsilon_0} \frac{\partial A_{z2}}{\partial p} \]

\[ \frac{i\omega \zeta_1^2}{\nu_1 k_1^2} A_{z1} = \frac{i\omega \zeta_0^2}{\nu_0 k_0^2} A_{z2} \]

Since (28) and (29) are valid for all values of \( z \) at all times, it follows that

\[ \sqrt{k_1^2 - \zeta_1^2} = \sqrt{k_0^2 - \zeta_0^2} = q \] (say)

so that \( \zeta_1 = \sqrt{k_1^2 - q^2} \) and \( \zeta_0 = \sqrt{k_0^2 - q^2} \). Dividing (29) by (28) and rearranging, one obtains

\[ \zeta_0 \frac{H_0^{(1)}(\zeta_0 a)}{H_0^{(1)}(\zeta_0 a)} = \frac{\zeta_1 v_0}{\nu_1 k_1^2} \frac{k_1^2}{J_0(\zeta_1 a)} \]

Although, in general, an explicit solution is not possible, equations (30) and (31) are theoretically sufficient to determine the unknowns \( \zeta_1 \) and \( \zeta_0 \).

The two unknowns will be determined here by two methods.

Method 1:

An approximate solution is possible by allowing \( k_1 \) to become very large.

Since \( q \) is finite, \( \zeta_1 \sim k_1 \) and, therefore, \( \zeta_1 \) is also large. Using this on the right side of (31) gives
\begin{align*}
1 & \frac{\varepsilon_1 u_0 k_0^2}{\varepsilon_0 u_1 k_1^2} \xi_1 a \to 0
\end{align*}

Therefore,

\begin{equation}
\xi_0 a = \frac{H_0^{(1)}(\xi_0 a)}{H_0^{(1)}(\xi_0)} \sim 0
\end{equation}

Equation (32) is satisfied by \( \xi_0 = 0 \) since it can be shown that the ratio \( H_0^{(1)}(\xi_0 a)/H_0^{(1)}(\xi_0) \) remains finite as \( \xi_0 \) approaches the value of zero. It then follows from (30) that

\begin{equation}
\xi_1 = \sqrt{k_1^2 - k_0^2} = k_1/1 - (1/\mu_k) \sim k_1
\end{equation}

Thus the solutions are \( \xi_1 \sim k_1 \) and \( \xi_0 \sim 0 \).

Method 2:

Method 1 may seem to be an oversimplification and, hence, a slightly more rigorous method may be needed in some cases. It is observed that (31) may be identified with a similar equation obtained by Sommerfeld [5] in the problem on 'waves on wires.' Sommerfeld has developed an iterative form of solution which may be used here.

In the limit of large \( \xi_1 \), the right side of (31) becomes

\begin{equation}
1 \frac{\varepsilon_1 u_0 k_0^2}{\varepsilon_0 u_1 k_1^2} \xi_1 a = \frac{\varepsilon_1}{u_k} \frac{k_0^2}{k_1^2} \sqrt{k_1^2 - q^2} \sim \frac{c_r}{u_k} \frac{k_0^2}{k_1^2} \sim \frac{c_r}{u_k} \frac{k_0^2}{k_1^2} \sim \frac{ak_0}{u_k} \sqrt{c_r/\mu_k}
\end{equation}

and is small. Since the left side of (31) also has to be small, we have

\begin{equation}
(\xi_0 a)^2 \ln(\gamma_0 a/21) = -\frac{2}{\gamma^2} u \ln u \quad \text{with} \quad u = (\gamma_0 a/21)^2
\end{equation}

where \( \gamma = 1.781 \).

Equation (31) finally becomes
\[ u \ln u = v \quad \text{with} \quad v = -\frac{i\gamma}{2} \frac{ak_0}{\nu r} \sqrt{\varepsilon_r/\mu_r} \]

Since \( u \ln u \) varies slowly in comparison with \( u \), it is possible to write

\[ u_{n+1} = u_n - \frac{1}{v} \]

where \( u_n \) is the \( n \)th approximation to \( u \). The method is best illustrated by an example. In the later part of this report, eleven different antenna configurations were used in the experimental determination of the magnetic current.

The example chosen here (antenna \( \ell 1 \)) corresponds to the lowest value of the \( |\varepsilon_r\nu_r| \) product for the eleven cases.

**Example:** \( ak_0 = 0.00166, \nu_r = (18 + 1.036), \varepsilon_r = 11.0 \).

Let \( u_0 = v = -\frac{i\gamma}{2} \frac{ak_0}{\nu r} \sqrt{\varepsilon_r/\mu_r} \approx 1.1 \times 10^{-4} \)

This gives

\[ u_1 = \frac{v}{u_0} = \frac{-11.1 \times 10^{-4}}{(-9.115 + 16.712)} = -10^{-4} (.049 - 1.095) \]

\[ u_2 = \frac{v}{u_1} = \frac{-11.1 \times 10^{-4}}{(-11.445 + 12.047)} = -10^{-4} (.0167 - 1.0931) \]

Continuing the iteration

\[ u_3 = \frac{v}{u_2} = \frac{-11.1 \times 10^{-4}}{(-11.5684 + 11.7478)} = -10^{-4} (.0140 - 1.0930) \]

\[ u_4 = \frac{v}{u_3} = \frac{-11.1 \times 10^{-4}}{(-11.5748 + 11.7208)} = -10^{-4} (.0138 - 1.0930) \]

and finally

\[ u_5 = \frac{v}{u_4} = \frac{-11.1 \times 10^{-4}}{(-11.5748 + 11.7184)} = -10^{-4} (.0138 - 1.0930) \]

It is seen that this iterative procedure is rapidly converging and using the above value of \( u \),
\[(\zeta_0 a)^2 = -\frac{e^2}{\gamma^2} u^2 = 10^{-10}(1.062 + 1.3225)\]

Furthermore, from (30)
\[(a\zeta)^2 = (ak_0)^2 - (ar_0)^2\]
\[= 2.7556 \times 10^{-6} - 10^{-10}(1.062 + 1.3225)\]
from which \(a_1\) may be calculated using
\[\begin{align*}
(a_1)^2 &= (ak_1^2 - (a\zeta)^2 = (ak_1)^2 - (a\zeta_0)^2 + (a\zeta_1)^2 \\
&= ak_1^2 \left[ 1 - \frac{1}{\nu^2} + \left(\frac{a_0}{ak_1}\right)^2 \right] \\
&= ak_1^2 \left[ 1 - (0.0056 - 1.00001) + 10^{-7}(2.159 + 1.651) \right]
\end{align*}\]

From the calculated values of \((a\zeta_0)^2\) and \((a_1)^2\), it is seen that the approximate solutions, i.e., \(\zeta_0 = 0\) and \(\zeta_1 \approx k_1\), are quite satisfactory even when \(|\nu_1\zeta_1|\) is as low as 180.

Therefore, the vector potential in the interior of the conductor is given by
\[
\Lambda_{x1}(\rho, z) = \Lambda_{x1}(\rho)\Lambda_{z1}(z) = C_1 J_0(k_1 \rho) \exp(i\sqrt{k_1^2 - \zeta_1^2} z)
\]
where \(\zeta_1\) in the argument of the Bessel function is replaced by \(k_1\) in view of the calculations of "Method 2." The constant \(C_1\) can be written in terms of the potential on the surface so that
\[
\Lambda_{x1}(\rho) = \Lambda_{x1}(\rho) J_0(k_1 \rho)/J_0(k_1 a)
\]
Since \(H_z\) is proportional to \(\Lambda_z\),
The magnetization is then given by

\[ \hat{M}_z(p) = (\mu_r - 1) \hat{H}_{z1}(p) = (\mu_r - 1) \hat{H}_{z1}(a) J_0(k_1p) / J_0(k_1a) \]

from which the magnetic current can be obtained as

\[ I_z^*(\rho) = 2\pi \int_0^\rho \mu_0 \hat{M}_z(\rho') \rho' d\rho' \]

Performing the integration gives

\[ I_z^*(\rho) = \frac{2\pi\mu_0 (\mu_r - 1) \hat{H}_{z1}(a)}{J_0(k_1a)} \frac{\rho}{k_1} J_1(k_1\rho) \]  

The total magnetic current carried by the conductor is, however, given by

\[ I_z^*(a) = 2\pi \int_0^a \mu_0 \hat{M}_z(\rho) \rho d\rho \]

which becomes

\[ I_z^*(a) = \frac{2\pi\mu_0 (\mu_r - 1) \hat{H}_{z1}(a)}{J_0(k_1a)} \frac{a}{k_1} J_1(k_1a) \]  

From (33) and (34) the radial distribution of the magnetic current in the interior of the conductor is given by

\[ I_z^*(\rho) = I_z^*(a) \frac{\rho}{J_1(k_1\rho)} \frac{J_1(k_1\rho)}{J_1(k_1a)} \]  

Furthermore, having obtained the total axial magnetic current of (34), the
The internal impedance per unit length defined in (21) can be written as

\[ z_m = \frac{1}{m} - i x_m = H_{21}(p = a)/r_a(a) \]

\[ = \left[ \frac{i}{2\omega} \frac{1}{\mu_0} \frac{1}{m} \frac{1}{\pi a^2} k_1^a \frac{J_0(k_1 a)}{J_1(k_1 a)} \right] \frac{1}{m} \]  

(36)

where

\[ x_m = (\mu_r - 1) = \text{magnetic susceptibility}; \]
\[ \omega = \text{radian frequency}; \]
\[ \mu_0 = 4\pi \times 10^{-7} \text{H/m} = \text{permeability of free space}; \]
\[ \pi a^2 = \text{cross-sectional area of the conductor}; \]

\[ k_1 = \beta_1 + i \alpha_1 = k_0 (\delta_{1N} + i n_{1N}) = \text{wave number in the material of the} \]
\[ \text{conductor}. \]

With \[ v_r = (v_r' + i v_r'') = |v_r| e^{i \delta} \] and \[ \epsilon_r = (\epsilon_r' + i \epsilon_r'') = |\epsilon_r| e^{i \theta}, \]

\[ k_1 = k_0 |v_r \epsilon_r|^{1/2} \exp[i(\delta_r + \theta_r)/2] \]

so that

\[ \beta_1 = k_0 |v_r \epsilon_r|^{1/2} \cos[(\delta_r + \theta_r)/2] \]  

(37a)

and

\[ \alpha_1 = k_0 |v_r \epsilon_r|^{1/2} \sin[(\delta_r + \theta_r)/2] \]  

(37b)

One may now use the internal impedance per unit length of a magnetic conductor carrying an axial magnetic current to find an integral equation for the magnetic current on a finite ferrite rod antenna. The axial component \[ H_0^0(z) \] on the surface of a cylindrical antenna that has an internal impedance per unit length \[ z_m \], carries an axial current \[ I_m^a(z) \], and is driven at \[ z = 0 \] by a delta-function generator with an maf of \( I_0^a \), satisfies the following differential equation:
If the antenna were made of a perfect magnetic conductor, $z_m^i = 0$ because $\chi^m = \infty$ so that (38) will reduce to (11). If the radius $a$ of the antenna and the free-space wave number $k_0 = \omega/v_0 = 2\pi/\lambda_0$ satisfy the inequality

$$ak_0 \ll 1$$

then the vector potential is given approximately by

$$A_0^z(z) \sim \frac{c_0}{\lambda_0} \int_{-h}^{h} I_0^e(z') K(z, z') \, dz'$$

(39)

If the equations (38) and (39) are formally identified with those for the imperfectly conducting, electric dipole antenna [3, eqs. (7) and (9)], it is observed that $v_0$ and $I_0^e$ play the roles of $\varepsilon_0$ and $V_0^e$. King and Wu [3] have developed a three-term solution for the electric current on the imperfectly conducting dipole antenna which can be well applied to the present problem of the ferrite as an imperfect magnetic conductor. The procedure used to obtain the three-term solution will be described here briefly; for a detailed analysis the reader is referred to [3].

The approximate kernel in (39) may be separated into real and imaginary parts,

$$K(z, z') = K_R(z, z') - iK_I(z, z') = e^{ik_0r}/r$$

so that

$$K_R(z, z') = \frac{\cos k_0r}{r}, \quad K_I(z, z') = -\frac{\sin k_0r}{r}$$

with $r = [(z - z')^2 + a^2]^{1/2}$. The vector potential may also be divided into two parts,

$$A_0^e(z) = A_R^e(z) - iA_I^e(z)$$
where

\[
A_R^e(z) = \frac{k_0'c_0}{4\pi} \int_{-h}^{h} I_z(z') \cos \frac{k_0'r}{k_0'z'} dz'
\]

(40)

\[
A_+^e(z) = -\frac{k_0'c_0}{4\pi} \int_{-h}^{h} I_z(z') \sin \frac{k_0'r}{k_0'z'} dz'
\]

(41)

The properties of the two integrals are quite different. The kernel in (40) has a sharp peak at \(k_0'|z - z'| = 0\) and thus greatly magnifies the contribution to the integral due to current elements near \(z = z'\). The current vanishes at the end but the vector potential \(A^e_R(h)\) has a small finite value so that the difference in vector potential should vary closely like \(I^*_z(z)\).

Therefore,

\[
(4\pi/\epsilon_0)[A_R^e(z) - A_R^e(h)] = \Psi(z)I^*_z(z) \pm \Psi I^*_z(z)
\]

(42)

where \(\Psi\) is the approximately constant value of \(\Psi(z)\) defined at a suitable reference value of \(z\). However, in the second integral in (41) the rather flat behavior of \((\sin k_0'r)/k_0'r\) with \(k_0'r\) allows the following approximation:

\[
\frac{\sin k_0'r}{k_0'r} = 2 \sin \frac{k_0'r}{2} \cos \frac{k_0'r}{2} \cos \frac{k_0'r}{2}
\]

which is useful over a range \(k_0'r \leq \pi\). This approximation leads to

\[
A^e_1(z) = A^e_1(0) \cos \frac{k_0'z}{2}
\]

(43)

where \(A^e_1(0)\) is a constant given by

\[
A^e_1(0) = \frac{k_0'c_0}{4\pi} \int_{-h}^{h} I_z(z') \cos \frac{k_0'z'}{2} dz'
\]

(44)

If equation (42), rearranged in the form \(I^*_z(z) = (4\pi/\epsilon_0)[A_R^e(z) - A_R^e(h)]\), is substituted in the differential equation (38), one obtains:
A complex constant $k$ may now be defined by

$$k^2 = (\beta + i\alpha)^2 = k_0^2 \left( 1 + \frac{i4\pi z^2}{k_0^2 m} \right)$$

Using (43) and (46), (45) becomes

$$\left( \frac{d^2}{dz^2} + k^2 \right) [A^e_z(z) - A^e_z(h)] = -14\pi e_0 k_0 z^2 [A^e_R(z) - A^e_R(h)] - k_0^2 A^e(h)$$

$$+ \frac{i}{\omega} k_0^2 e_0 \delta(z)$$

The integral equation of (39) may now be written in the form

$$[A^e_z(z) - A^e_z(h)] = \frac{e_0}{4\pi} \int_{-h}^{h} I^* (z') K_d(z, z') \, dz'$$

where the difference kernel $K_d$ is given by

$$K_d(z, z') = K(z, z') - K(h, z') = \frac{e_0 k_0 r}{r} - \frac{e_0 k_0 r_h}{r_h}$$

with $r = [(z - z')^2 + a^2]^{1/2}$ and $r_h = [(h - z')^2 + a^2]^{1/2}$. If the differential equation (47) is solved for the vector potential difference and the solution is substituted for the left-hand side of (48), an integral equation for the magnetic current on the ferrite antenna is obtained, viz.,

$$\int_{-h}^{h} I^* (z') K_d(z, z') \, dz' = \frac{14\pi e_0 k_0}{k \cos kh} \left( \frac{1}{2} I^*_{0'} F_{h} + U^*_{0'} F_{kz} - D \cos kh F'_{0z} \right)$$

where for ease of reference, the same notation as in King and Wu [3] is employed and the various factors on the right side are given by

$$M_{kz} = \sin k(h - |z|)$$
Following the procedure as in [3], an approximate formal solution to the integral equation may be written in the form

\[ I_z^*(z) = I_{Vz}^* + I_{Uz}^* F_{kz}^* + I_{Dz}^* F_{0z}^* \]

(50)

where the coefficients \( I_{Vz}^* \), \( I_{Uz}^* \) and \( I_{Dz}^* \) are obtained by a numerical procedure.

Letting \( T_U = I_{Uz}^*/I_{Vz}^* \), \( T_D = I_{Dz}^*/I_{Vz}^* \) and evaluating \( I_{Vz}^* \), one may write (50) as

\[ I_z^*(z) = \frac{-i\omega k_0 c_0 I_e^c}{k V^2} \cos kh \sin k(h - |z|) + T_U^*(\cos kZ - \cos kh) \]

\[ + T_D^*(\cos \frac{k_0 z}{2} - \cos \frac{k_0 h}{2}) \]

(51)

where \( k \) is redefined by

\[ k^2 = (\beta + i\alpha)^2 = k_0^2 \left( 1 + \frac{-4\pi z \gamma}{k_0 dR} \right) \]

(52)

and \( \psi_{dR} \) is given by the integral expression

\[ \int_{-h}^{h} \sin k(h - |z'|)K_{dR}(z,z') \, dz' + \int_{-h}^{h} \sin k(h - |z|) \psi_{dR} \]

(53)

Thus, equation (51) is the required expression for the total magnetic current on the antenna from which the admittance can be obtained to be
\[
Y = G - iB = \frac{I^+(z)/I^0(z)}{\text{ohms}}
\]
\[
= \frac{-12\pi k_0 k}{k \cos kh} \left[ \sin kh + T_U^*(1 - \cos kh) + T_D^*(1 - \cos \frac{k_0 h}{2}) \right]
\]

Note that, because of an earlier approximation of the imaginary part of the kernel, equations (51) and (54) are valid representations for the magnetic current and admittance only when \( k_0 h < \frac{5\pi}{4} \).

The existing computer programs for the imperfectly conducting dipole antenna due to King, Harrison and Aronson \cite{bib4} have been modified for use on the IBM 370/155 computer system of the Joint Harvard/M.I.T. Batch Processing Center. Appendix B includes a listing of the Fortran IV programs that compute the magnetic current distribution and the admittance of the finite ferrite-rod antenna when the ferrite is treated as an imperfect magnetic conductor.

6. THE LIMITATIONS OF THE THEORETICAL FORMULATION

The present formulation is based on an analogy between the ferrite-rod antenna and the conducting cylindrical dipole antenna. Because of the symmetry in Maxwell's equations, a set of scalar magnetic \( (\phi^*) \) and electric vector \( (A^\theta) \) potentials was defined and used in formulating the finite ferrite-rod antenna problem. It is considered useful to determine the existence of these potentials for the infinite antenna and thus provide some justification for their use in the finite antenna problem.

The electromagnetic fields in both regions for the case of the infinitely long antenna were determined previously \cite{bib2} to be:

Region 1, \( 0 < \rho < a \):

\[
\vec{E}_\phi(\rho, \xi) = i \omega \mu_1 a I_0^0 H_1^{(1)}(\gamma_0 a) J_1(\gamma_1 \rho) / D(\xi)
\]
\[ \tilde{H}_z(r, \xi) = \frac{1}{i \omega_1} \left[ \beta E_{\phi 1}(r, \xi) / \partial \rho + \tilde{E}_{\phi 1}(r, \xi) / \rho \right] = \alpha I_0 Y_1^1(y_0 \rho J_0(y_1 \rho) / D(\xi) \]

\[ \tilde{H}_\rho(r, \xi) = (\xi / \omega_1) \tilde{E}_{\phi 1}(r, \xi) = i \alpha I_0 Y_1^1(y_0 \rho J_1(y_1 \rho) / D(\xi) \]

\[ \tilde{E}_{\phi 1}(r, \xi) = \tilde{E}_{\rho 1}(r, \xi) = \tilde{E}_{z1}(r, \xi) = 0 \] (55)

**Region II, \( \rho > a \):**

\[ \tilde{E}_{\phi 2}(r, \xi) = i \omega_1 \alpha I_0 Y_1^1(y_1 \rho) H_1^0(y_0 \rho) / D(\xi) \]

\[ \tilde{H}_z(r, \xi) = \alpha I_0 Y_0 Y_1^1(y_1 \rho) H_0^1(y_0 \rho) / D(\xi) \]

\[ \tilde{H}_\rho(r, \xi) = i \alpha I_0 Y_1^1(y_1 \rho) H_1^1(y_0 \rho) / D(\xi) \]

\[ \tilde{E}_{\phi 2}(r, \xi) = \tilde{E}_{\rho 2}(r, \xi) = \tilde{E}_{z2}(r, \xi) = 0 \] (56)

where

\[ D(\xi) = a [y_1 Y_1^1(y_1 \rho) H_1^1(y_0 \rho) - y_0 Y_0 Y_1^1(y_1 \rho) H_1^1(y_0 \rho)] \]

The actual field quantities may be obtained by applying the Fourier inverse formula to the above transformed fields. It can be verified easily that the above field quantities satisfy the following transformed boundary conditions:

i) Tangential \( \tilde{E} \): \( \tilde{E}_{\phi 2}(a^+, \xi) = \tilde{E}_{\phi 1}(a^-, \xi) \) (57a)

ii) Tangential \( \tilde{H} \): \( \tilde{H}_{z2}(a^+, \xi) - \tilde{H}_{z1}(a^-, \xi) = -\gamma_0 \) (57b)

iii) Normal \( \tilde{B} \): \( \tilde{B}_{\rho 2}(a^+, \xi) = \tilde{B}_{\rho 1}(a^-, \xi) \) (57c)

iv) Normal \( \tilde{D} \): Zero in both regions

\( \psi \) is a scalar magnetic potential and has a non-zero value in both regions. For the infinitely long antenna, the only non-zero component of \( \tilde{A}^e \) is
the z-component so that \( \hat{\mathbf{A}}^e = \hat{\mathbf{A}}^e_z \). The potentials may be derived either from the already known electromagnetic fields or from an independent solution of the following wave equations with suitable boundary conditions:

\[
(V^2 + k^2) A^e(\rho, z) = 0 , \quad (V^2 + k^2) \phi(\rho, z) = 0
\]

The equations reduce to

**Region I,** \( 0 < \rho < a \):

\[
\left[ \frac{\alpha^2}{\partial z^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} + k_1^2 \right] A^e_{z1}(\rho, z) = 0
\]

**Region II,** \( \rho > a \):

\[
\left[ \frac{\alpha^2}{\partial z^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} + k_0^2 \right] A^e_{z2}(\rho, z) = 0
\]

Using a Fourier transform pair, the above equations become

\[
\left[ \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + (k_1^2 - \xi^2) \right] A^e_{z1}(\rho, \xi) = 0
\]

\[
\left[ \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + (k_0^2 - \xi^2) \right] A^e_{z2}(\rho, \xi) = 0
\]

With a change of variable the above equations can be recognized as Bessel equations with the following solutions,

\[
A^e_{z1}(\rho, \xi) = P J_0(\gamma_1 \rho) \quad \text{for} \ 0 < \rho < a
\]

\[
A^e_{z2}(\rho, \xi) = Q H_0^{(1)}(\gamma_0 \rho) \quad \text{for} \ \rho > a
\]
where \( \gamma_0 = (k_0^2 - \xi^2)^{1/2} \) and \( \gamma_1 = (k_1^2 - \xi^2)^{1/2} \)

The boundary conditions \((57a,b)\), expressed in terms of the electric vector potential, become

\[
\begin{align*}
(1/\epsilon_1)A_z^e(a_-,\xi)/\partial \rho &= (1/\epsilon_0)A_z^e(a_+^*,\xi)/\partial \rho \\
(\omega \gamma_0^2/k_0^2)A_z^e(a_+,\xi) - (\omega \gamma_1^2/k_1^2)A_z^e(a_-,\xi) &= -i \omega \Gamma_0^e
\end{align*}
\]

(58a)

By applying the boundary conditions and determining \( P \) and \( Q \), the electric vector potential can be written as:

\[
\begin{align*}
\bar{A}_z^e(\rho,\xi) &= -i \omega \mu_1 \epsilon_1 a_1 \gamma_0^e J_0(\gamma_1 \rho)/\gamma_1 \psi(\xi) \quad \text{for } 0 \leq \rho \leq a \\
\bar{A}_z^e(\rho,\xi) &= -i \omega \mu_0 \epsilon_0 a_1 \gamma_0^e J_1(\gamma_1 \rho)/\gamma_1 \psi(\xi) \quad \text{for } \rho > a
\end{align*}
\]

(59)

Similarly, by solving the wave equation for the scalar magnetic potential \( \phi^* \), the solution can be obtained as:

\[
\begin{align*}
\bar{\phi}_1^*(\rho,\xi) &= i a_1 \gamma_0^e \gamma_0^e J_0(\gamma_1 \rho)/\gamma_1 \psi(\xi) \quad \text{for } 0 \leq \rho \leq a \\
\bar{\phi}_2^*(\rho,\xi) &= i a_1 \gamma_0^e \gamma_0^e J_1(\gamma_1 \rho)/\gamma_1 \psi(\xi) \quad \text{for } \rho > a
\end{align*}
\]

(60)

The boundary conditions satisfied by \( \bar{\phi}^*(\rho,\xi) \) at the surface \( \rho = a \) are:

\[
\begin{align*}
(\omega \mu_1/\xi)\partial \bar{\phi}_1^*(a_-,\xi)/\partial \rho &= (\omega \mu_0/\xi)\partial \bar{\phi}_2^*(a_+^*,\xi)/\partial \rho \\
\gamma_0^2 \bar{\phi}_2^*(a_+,\xi) - \gamma_1^2 \bar{\phi}_1^*(a_-,\xi) &= -i \omega \Gamma_0^e
\end{align*}
\]

(61a)

(61b)

It can also be verified that the potentials satisfy the gauge condition,

\[
\partial A_z^e(\rho,z)/\partial z - i \omega \mu_0 \psi^*(\rho,z) = 0 \quad \text{in both regions}
\]
The potentials of (59) and (60) can also be obtained from the electromagnetic fields of (55) and (56) by making use of the following relationships in both regions:

\[ E_z(\rho, z) = (1/\epsilon) \partial \Phi(\rho, z)/\partial z \quad ; \quad H_z(\rho, z) = -\partial \Phi(\rho, z)/\partial z + i\omega \Phi(\rho, z) \]

and

\[ \partial \Phi(\rho, z)/\partial z - i\omega \mu \Phi(\rho, z) = 0 \]

The above analysis verifies that when the antenna is infinitely long, both the scalar magnetic and electric vector potentials exist. They are both discontinuous across the antenna surface and satisfy respective wave equations, appropriate boundary conditions, and the gauge condition.

In the case of the finite antenna, however, a precise knowledge of the vector potential in the two regions is not necessary to derive an approximate integral equation for the magnetic current. What is required is the electric vector potential on the surface of the antenna. To determine this, an internal impedance per unit length is defined and used to obtain the three-term solution for the magnetic current. Using the computer programs described and listed in Appendix B, the magnetic current was evaluated for a range of parameters. The current distribution was studied as a function of the four independent parameters, viz., \( \mu_\tau \); \( \mu'' \); \( Q = \mu_\tau /\mu'' \); \( \hbar /\lambda_0 \) or \( k_0 \hbar \); and \( ak_0 \) or \( Q \). In this study the value of the dielectric constant of the ferrite was fixed at 10.

The ranges of the four parameters were as follows: \( \mu_\tau = 10, 100, 1000 \); \( Q = 1 \) to \( Q = 100 \); \( \hbar /\lambda_0 = .1 \) to \( \hbar /\lambda_0 = .5 \); and \( ak_0 = .001 \) to \( ak_0 = .1 \). Typical results of the computations are shown plotted in Fig. 3. The quantities \( \mu_\tau \), \( ak_0 \), \( \hbar /\lambda_0 \) and \( Q \) are varied, respectively, in Fig. 3a-d, while in each case the remaining three parameters are kept constant.
FIG. 3 PLOT OF THE MAGNITUDE OF NORMALIZED MAGNETIC CURRENT ($|I_m(Z)/I_0|$) AS A FUNCTION OF NORMALIZED DISTANCE ($z/h$) FOR VARIOUS PARAMETER RANGES. ($\varepsilon_r = 10 + i0$ FOR ALL THE CASES)
In Fig. 3a for fixed height, radius, and ratio $\theta$, the magnetic current on the antenna is seen to increase with the real part of the relative permeability. A similar behavior is observed in Fig. 3b for increasing antenna radius and fixed height, permeability and $\theta$. A comparison between Fig. 3a and Fig. 3d shows that a large value of $\mu'_r$ produces a greater increase in the magnetic current than a high $Q$ ratio; in fact, an increase in $Q$ for $Q < 50$ is seen to reduce the magnitude of the magnetic current. To interpret Fig. 3c, it is useful to examine the behavior of the propagation constant $k$ on the antenna, given by

$$k = \beta + i\alpha = k_0 \left(1 + i \left(4 \pi \frac{z_m^2}{\kappa_0} \psi / dR \right) \right)^{1/2}$$

If the dimensionless parameter $\Phi_1 = \left(4 \pi \frac{z_m^2}{\kappa_0} / k_0 \right)$ is introduced, this expression becomes

$$k = \beta + i\alpha = k_0 \left(1 + i \frac{\Phi_1}{\psi} dR \right)^{1/2}$$

Despite the fact that $\psi dR$ is itself a function of $k$, an efficient iterative method can be used to determine the value of the propagation constant. By substituting for $z_m^2$ from (36) the following expression for $\Phi_1$ is obtained:

$$\Phi_1 = \frac{2 i a k_1 \psi}{(\kappa_0^2 - 1) \psi^2} \frac{J_0(ak_1)}{J_1(ak_1)}$$

$\Phi_1$ becomes positive imaginary for the cases plotted in Fig. 3c where $ak_1$ is real. This makes the propagation constant $k$ on the antenna pure imaginary which leads to an exponentially decreasing magnetic current. For most practical ferrites the positive imaginary part of $\Phi_1$ dominates, which makes the attenuation constant $\alpha$ significantly larger than the phase constant $\beta$. This can also be seen in the experimental results reported in Section 9.
At this stage it is considered useful to summarize all the approximations and assumptions involved in the derivation of the integral equation in (38) with (39). The ferrite was first treated as a perfect magnetic conductor \( \mu_r = \infty \) and the integral equation in (14) was obtained. This expression was later modified by adding an intrinsic impedance per unit length for a practical ferrite that is an imperfect magnetic conductor and finite. The basic assumption that the radius be small, i.e., \( ak_0 \ll 1 \), was made. An implied approximation was introduced when the impedance per unit length \( z_m^i \), derived originally for the infinitely long magnetic conductor, was used for the finite antenna. Its use can be justified as follows. For an infinitely long magnetic conductor, the transverse distribution of electric vector potential is independent of the axial distribution. It is reasonable to assume that this remains the case when the conductor length is large compared to the radius, so that the intrinsic impedance per unit length derived for the infinitely long conductor can be used directly for antennas of finite length. A further question arises concerning the discontinuity of the electric vector potential across the antenna surface. It has been established that the electric vector potential is discontinuous across the antenna surface when the antenna is infinitely long. It is reasonable to conclude that the discontinuity exists even when the length of the antenna is finite. The derivation of the integral equation for the magnetic current or the tangential electric field requires a knowledge of the electric vector potential on the surface \( \rho = a \), which has apparently two values. This problem is not peculiar to the ferrite-rod antenna but also exists in the analogous resistive electric dipole antenna. In either case, the value of the vector potential used is that obtained by approaching the antenna surface from the surrounding medium. It is believed, however, that the discontinuity in the vector potential is a
consequence of the way in which the vector potential was defined and can be overcome with the introduction of a suitable scale factor in the definition.

The approach of treating the ferrite as an imperfect magnetic conductor relies on the mathematical equivalence of the two analogous problems. One cannot escape the fact, however, that while there are two pieces of conductor separated by a slice voltage or electric field generator in the case of an electric dipole, the magnetic conductor in the ferrite problem is a single continuous rod driven on the outside surface. A delta function, although unphysical, is a mathematical convenience in either case.

In view of the above discussion, a more rigorous analysis which does not invoke the analogy with the electric dipole is developed and presented in the following section.

7. A MORE RIGOROUS TREATMENT OF THE FINITE ANTENNA

Since the total magnetic current \( I_z^* (z) \) is linearly related to the tangential electric field \( E_{\phi} (a, z) \) by the relation

\[
I_z^* (z) = -2\alpha a F_{\phi} (a, z),
\]

the following procedure seeks to derive an integral equation for \( E_{\phi} (a, z) \) by solving the ferrite-interior and free-space-exterior problems.

**Interior Problem.** The interior problem consists of a ferrite cylinder of height \( 2h \) driven at the center by a constant-current loop. The driving condition will be accounted for after the interior and exterior problems are solved. The diameter of the rod and of the loop is \( 2a \) and the restriction \( ak_0 \ll 1 \) is satisfied in order to maintain a constant current \( I_0 \) in the driving loop. Given a cylindrical coordinate system \((\rho, \phi, z)\) and after eliminating \( \vec{H} \) from Maxwell's curl equation and imposing azimuthal symmetry, one obtains
for the electric field

\[
\left[ \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \left( k_1^2 - \frac{1}{\rho^2} \right) + \frac{3}{\partial z^2} \right] E_\phi(\rho, z) = 0
\]  

(62)

with \( k_1 = k_0 (\mu_x e_x) \frac{1}{2} \), \(|z| \leq h\), \( 0 \leq \rho \leq a \), and \( E_\phi(\rho, z) = E_\phi(\rho, -z) \).

Solving (62) by a separation-of-variables technique gives

\[
E_\phi(\rho, z) = \sum_{n=-\infty}^{\infty} A_n \cos[(n + 1/2)\pi z/h] J_1(\rho [k_1^2 - (n + 1 + 2/2) \pi^2/h^2]^{1/2})
\]  

(63)

with the coefficients \( A_n \) given by

\[
A_n = \frac{1}{h J_1(a [(n + 1/2) \pi^2/h^2]^{1/2})} \int_{-h}^{h} E_\phi(a, z') \cos[(n + 1/2)\pi z'/h] \, dz'
\]

This procedure aims to determine the tangential magnetic field \( H_z(a, z) \) from independent treatments of both the interior and exterior problems and then to require that their difference equal \(-i \delta_0(z)\), the true electric surface current. Thus, \( H_z(a, z) \) can be obtained from the above by using

\[
H_z(\rho, z) = (1/i\omega_1) [\partial E_\phi(\rho, z)/\partial \rho + E_\phi(\rho, z)/\rho]
\]

\[
= (1/i\omega_1 h) \left[ \sum_{n=-\infty}^{\infty} \left( \int_{-h}^{h} E_\phi(a, z') \cos[(n + 1/2)\pi z'/h] \right) \right.
\]

\[
\times \cos[(n + 1/2)\pi z/h] \frac{J_0(\rho [k_1^2 - (n + 1 + 2/2) \pi^2/h^2]^{1/2})}{J_1(a [k_1^2 - (n + 1 + 2/2) \pi^2/h^2]^{1/2})}
\]

\[
\times [k_1^2 - (n + 1 + 2/2) \pi^2/h^2]^{1/2}
\]  

(64)

It should be pointed out that, as a first approximation, \( E_\phi(a, z) \) is made
to vanish on the top and bottom surfaces defined by $|z| = h$ and $0 \leq \rho \leq a$, thus neglecting all the fringing fields at the ends of the antenna. In practice, this condition nearly prevails for antennas with heights large compared to the radius ($h \gg a$).

**Exterior Problem.** The exterior problem is concerned with the free space surrounding the ferrite rod which extends from $0 \leq \rho \leq a$ and $-h \leq z \leq h$ for all $\phi$. It is equivalent to solving the problem with the ferrite removed but with the tangential electric field on the surface $E_\phi(a, z)$ for $|z| \leq h$ required to be the same as that used in the interior problem. With an assumed $e^{-i\omega t}$ time dependence, the governing equations are:

\begin{align}
\nabla \times \vec{H} &= -i\omega \mu_0 \vec{E} \\
\nabla \times \vec{E} &= i\omega \epsilon_0 \vec{H} \\
\n\nabla \cdot \vec{H} &= 0 \\
\n\nabla \cdot \vec{D} &= 0
\end{align}

From (65d) in free space, one may define an electric vector potential

\[ \vec{A} = -\nabla \times \vec{A}^e \]

so that

\[ \vec{E} = -(1/\epsilon_0) \nabla \times \vec{A}^e \]

This leads to

\[ \vec{H} = -\nabla \phi^* + i\omega \vec{A}^e \]

The exterior problem may be modeled by a cylindrical surface of radius $a$. 
that extends from \( z = -h \) to \( z = h \). This surface, when immersed in free space, has the following boundary conditions valid for \(|z| < h|\):

\[
\begin{align*}
E_\phi(a^+, z) &= f^+(z) = E_\phi(a, z) \\
E_\phi(a^-, z) &= f^-(z) = 0
\end{align*}
\]

Substituting for \( E \) and \( H \) in (65b), one obtains

\[
\begin{align*}
-v \times v \times \mathbf{A}^e + \imath \omega \mu_0 \varepsilon_0 \mathbf{v} \mathbf{\phi}^* + k_0^2 \mathbf{A}^e &= 0 \\
(v^2 + k_0^2)\mathbf{A}^e &= v(v \cdot \mathbf{A}^e + \mu_0 \varepsilon_0 \mathbf{\phi}^*) = v_x
\end{align*}
\]

If the Lorentz gauge is satisfied, the Lorentz factor \( \gamma \) [and the right-hand side of (67b)] is zero. The equations may now be specialized to the problem at hand. There is rotational symmetry in the problem and the non-zero quantities are \( E_\phi \), \( H_\rho \), \( H_z \), \( A^e_z \) and \( \phi^* \). Equations (65a,b) for the different components become

\[
\begin{align*}
\left( \frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial \rho} \right) &= -\imath \omega \mu_0 E_\phi \\
\imath \omega \mu_0 H_\rho &= -\frac{\partial E_\phi}{\partial z} \\
\imath \omega \mu_0 H_z &= \frac{1}{\rho} \frac{2}{\partial \rho} (\rho E_\phi)
\end{align*}
\]

These three equations are true everywhere except on the surface \( \rho = 1 \) and \(|z| < h|\). To make the equations valid on the surface, one has to introduce the surface conditions into the above equations. In addition to the conditions in (66a,b), there is an electric current \( k_\phi(z) \) on the surface as well as a large axial magnetic field. Thus, (68a-c) become
\begin{align}
(\partial H/\partial z - \partial H/\partial \rho) + \delta(\rho - a)K_\phi(z) &= -i\omega \epsilon E_\phi \\
\text{(69a)}
\end{align}
\begin{align}
i\omega \mu_0 H_\rho &= -\partial E_\phi /\partial z \\
\text{(69b)}
\end{align}
\begin{align}
i\omega \mu_0 H_z + \delta(\rho - a)E_\phi &= (\partial E_\phi /\partial \rho + E_\phi /\rho) \\
\text{(69c)}
\end{align}

In terms of the potentials, the fields are given by
\begin{align}H_\rho &= -\partial \phi^*/\partial \rho \\
H_z &= -\partial \phi^*/\partial z + i\omega A^e_z \\
E_\phi &= (1/\epsilon_0)\partial A^e_z /\partial \rho
\end{align}

From the preceding equation,
\[ A_z^e(\rho,z) = \epsilon_0 \int_\rho^\infty E_\phi(\rho',z) \, d\rho' \]
\[ \text{(70)} \]

It is now required to set up an equation for \( A_z^e(\rho,z) \).
\begin{align}(v^2 + k_0^2)A_z^e(\rho,z) &= \left[ \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial z^2} + k_0^2 \right] \epsilon_0 \int_\rho^\infty E_\phi(\rho',z) \, d\rho' \\
&= \epsilon_0 \left[ \partial E_\phi(\rho,z)/\partial \rho + E_\phi(\rho,z)/\rho \right] + \epsilon_0 \frac{3}{\partial z} \int_\rho^\infty \partial E_\phi(\rho',z)/\partial z \, d\rho' \]
&\quad + \epsilon_0 k_0^2 \int_\rho^\infty E_\phi(\rho',z) \, d\rho'
\end{align}

With (69b,c) this becomes
\begin{align}(v^2 + k_0^2)A_z^e(\rho,z) &= \epsilon_0 \left[ i\omega \mu_0 H_z(\rho,z) + \delta(\rho - a)E_\phi(\rho,z) \right] \\
&\quad - i\omega \mu_0 \epsilon_0 \int_\rho^\infty \partial H(\rho',z)/\partial \rho \, d\rho' + \epsilon_0 k_0^2 \int_\rho^\infty E_\phi(\rho',z) \, d\rho'
\end{align}

Using (69a) gives
\[ (v^2 + k_0^2)\Lambda_z^\phi(p, z) \]

\[ = i\omega_0 \varepsilon_0 H_z(p, z) + \varepsilon_0 \delta(p - a) E_\phi(p, z) - i\omega_0 \varepsilon_0 \int^\infty_0 [-i\omega_0 E_\phi(p', z) + \frac{\partial H_z(p', z)}{\partial p}] \]

\[ - \delta(p' - a) K_\phi(z)] \, dp' + \varepsilon_0 k_0^2 \int p_1 P_0 \, dp' \]

\[ = i\omega_0 \varepsilon_0 H_z(p, z) + \varepsilon_0 \delta(p - a) E_\phi(p, z) - \varepsilon_0 k_0^2 \int p_1 P_0 \, dp' \]

\[ + i\omega_0 \varepsilon_0 K_\phi(z) \int^\infty_0 \delta(p' - a) \, dp' - i\omega_0 \varepsilon_0 H_z(p, z) + \varepsilon_0 k_0^2 \int p_1 P_0 \, dp' \]

\[ = \varepsilon_0 \delta(p - a) E_\phi(p, z) + i\omega_0 \varepsilon_0 K_\phi(z) \int^\infty_0 \delta(p' - a) \, dp' \]

The \( p' \) integral may be performed:

\[ \int^\infty_0 \delta(p' - a) \, dp' = \mathcal{H}(a - \rho) = \begin{cases} 1 & \text{if } 0 \leq \rho \leq a^- \\ 0 & \text{if } \rho \geq a^+ \end{cases} \]

Therefore, finally

\[ (v^2 + k_0^2)\Lambda_z^\phi(p, z) = \varepsilon_0 \delta(p - a) E_\phi(p, z) + i\omega_0 \varepsilon_0 K_\phi(z) \mathcal{H}(a - \rho) \quad (71) \]

If (71) is formally identified with (67b), it is seen that the second term on the right in (71) corresponds to the Lorentz factor term. The Lorentz gauge is satisfied \((\chi = 0)\) in the exterior region \((p > a)\) but is not satisfied in the interior region. Furthermore, by differentiating with respect to \( p \)

\[ \chi = \nabla \cdot \Lambda^\phi + \omega_0 \varepsilon_0 \frac{\partial \phi}{\partial z} - i\omega_0 \varepsilon_0 \frac{\partial \phi}{\partial z} \]

it can be shown that \( \chi \) is independent of \( p \) and a function of \( z \) only, i.e.,

\[ \chi = \chi(z), \] which leads to:
\( \chi(z) = 3 \Lambda_e^e(\rho, z)/\partial z - i \omega \mu_0 \epsilon_0 \phi^*(\rho, z) = \begin{cases} 
\frac{\omega \mu_0 \epsilon_0}{2} \int_0^z K_\phi(z') dz' & \text{if } 0 \leq \rho \leq a^- \\
0 & \text{if } \rho \geq a^+ 
\end{cases} \)

It is thus seen that the Lorentz condition is satisfied on the exterior but not in the interior. This is because of the presence of the transverse electric current in the ferrite medium. This situation can be contrasted to an electric dipole antenna (thin or thick), where there are no magnetic currents to make the Lorentz condition invalid.

It is now required to solve (71) for the electric vector potential. The equation becomes

\[
\left( \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial z^2} + k_0^2 \right) \Lambda_e^e(\rho, z) = \epsilon_0 \delta(\rho - a) E_\phi(\rho, z) + i \omega \mu_0 \epsilon_0 K_\phi(z) H(a - \rho)
\]

This equation can be solved with the use of Green's theorem and the principle of superposition. Thus,

\[
\Lambda_e^e(\rho, z) = \Lambda_{ZE}^e(\rho, z) + \Lambda_{zk}^e(\rho, z)
\]

where

\[
\Lambda_{ZE}^e(\rho, z) = -(\epsilon_0/4\pi) \int \frac{d\phi'}{2\pi} \int_0^{2\pi} 2\rho \int_0^\infty dz' E_\phi(\rho, z') \delta(\rho' - a)(e^{-ik_0 R}/R) \]

\[
= - (\alpha \epsilon_0/2) \int_{-h}^h dz' E_\phi(a, z') K(z - z', \rho)
\]

with

\[
K(z - z', \rho) = \int_{-\pi}^{\pi} \frac{d\phi'}{2\pi} e^{ik_0 R}/R
\]

and \( R = ((z - z')^2 + \rho^2 + a^2 - 2a \rho \cos \phi')^{1/2} \). \( \Lambda_{zk}^e(\rho, z) \) will be used later.
to obtain $A_z^e(a,z)$. Similarly,

$$A_z^e(p,z) = -(i\omega_0 \epsilon_0 / 2\pi) \int_{-\pi}^{\pi} d\phi' / 2\pi \int_0^{2\pi} d\rho' \int_{a-\rho}^{a+\rho} dz' K_\phi(z') \mathcal{H}(a - \rho)(e^{-ik_0 R_1 / R_1})$$

where

$$M_1(z - z', \rho) = \int_{-\pi}^{\pi} \frac{d\phi'}{2\pi} \int_0^{2\pi} d\rho' \frac{\rho e^{-ik_0 R_1}}{R_1}$$

and $R_1 = ((z - z')^2 + \rho^2 - 2\rho' \cos \phi' )^{1/2}$.

Thus, the total electric vector potential is

$$A_z^e(p,z) = -(i\omega_0 \epsilon_0 / 2) \int_{-h}^{h} dz' E_\phi(a,z') K(z - z', \rho)$$

$$- (i\omega_0 \epsilon_0 / 2) \int_{-h}^{h} dz' K_\phi(z') M_1(z - z', \rho)$$

By specializing (74) for $\rho = a^+$ and $\rho = a^-$ and by making use of (70) and (66a,b), one obtains

$$\epsilon_0 E_\phi(a,z) = -(i\omega_0 \epsilon_0 / 2) \int_{-h}^{h} dz' E_\phi(a,z') \frac{\partial K(z - z', \rho)}{\partial \rho} \bigg|_{\rho=a^+}$$

$$- (i\omega_0 \epsilon_0 / 2) \int_{-h}^{h} dz' K_\phi(z') \frac{\partial M_1(z - z', \rho)}{\partial \rho} \bigg|_{\rho=a^-}$$

Returning to (74), one may now obtain for the tangential magnetic field

$$H_z(p,z) = (i\omega/k_0^2) \left( \frac{\epsilon_0^2}{2} + k_0^2 \right) A_z^e(p,z)$$
Thus,

\[ H_z(\rho, z) = (a/2)(1/iw_0) \left( \frac{\partial^2}{\partial z^2} + k_0^2 \right) \int_{-h}^{h} dz' \ E_\phi(a, z') K(z - z', \rho) \]

\[ + (1/2) \left( \frac{\partial^2}{\partial z^2} + k_0^2 \right) \int_{-h}^{h} dz' \ K_\phi(z') M_1(z - z', \rho) \]  

(76)

Once again, on the exterior surface \( \rho = a^+ \),

\[ H_z(a^+, z) = (a/2)(1/iw_0) \left( \frac{\partial^2}{\partial z^2} + k_0^2 \right) \int_{-h}^{h} dz' \ E_\phi(a, z') K(z - z', a^+) \]

\[ + (1/2) \left( \frac{\partial^2}{\partial z^2} + k_0^2 \right) \int_{-h}^{h} dz' \ K_\phi(z') M_1(z - z', a^+) \]  

(77)

It now remains to use (77) and (64) to obtain the integral equation.

**Integral Equation for** \( E_\phi(a, z) \) **and** \( K_\phi(z) \). The required integral equations for the unknown quantities may be obtained from the results of (64) and (77) for the interior and exterior problems by requiring that

\[ H_z(a^+, z) - H_z(a^-, z) = -i0^0(\rho) \]

This gives

\[ \left\{ \left( (a/2)(1/iw_0) \left( \frac{\partial^2}{\partial z^2} + k_0^2 \right) \int_{-h}^{h} dz' \ E_\phi(a, z') K(z - z', a) + (1/2) \left( \frac{\partial^2}{\partial z^2} + k_0^2 \right) \right) \right\} \]

\[ \times \int_{-h}^{h} dz' \ K_\phi(z') M_1(z - z', a) + \left\{ \left( i/\omega a \right) \left( \sum_{n=0}^{\infty} \int_{-h}^{h} dz' \ E_\phi(a, z') \right) \right\} \times \cos(pz') \]  

\[ \times \cos(pz) \frac{J_0\left( a(k_1^2 - p^2)^{1/2} \right) J_1\left( a(k_1^2 - p^2)^{1/2} \right)}{J_1(a(k_1^2 - p^2)^{1/2}) - a(k_1^2 - p^2)^{1/2}} = i0^0(\rho) \]  

(78a)

with \( p = (n + 1/2)\pi/h \).
The other equation to be satisfied simultaneously is (75), which is reproduced here for convenience:

\[ E_\phi(a,z) = -(a\omega_0^2/2) \int_{-h}^{h} dz' E_\phi(a,z') \partial K(z - z',\rho)/\partial \rho \bigg|_{\rho = a} + \]

\[ - (i\omega\omega_0\epsilon_0/2) \int_{-h}^{h} dz' K_\phi(z') \partial M_1(z - z',\rho)/\partial \rho \bigg|_{\rho = a} \]  (78b)

The two kernels \( K(z - z',\rho) \) and \( M_1(z - z',\rho) \) appearing in the coupled integral equations above are defined by (72) and (73) respectively.

It can be verified easily that in the limit \( h \to \infty \) the integrals in (78a,b) become convolution integrals, that the two equations decouple and that the expression for \( F_\phi(\rho,z) \) on the surface \( \rho = a \) is in complete agreement with the results presented in Part I [2, Eqs. (17) or (18)].

Returning to the coupled integral equations in (78a,b), it is seen that there are three kernels. First of all, the kernel on the right-hand side of (78a) will be examined carefully. The kernel is made up of an infinite series which is clearly divergent since, for large values of \( n \), it behaves like \( n \).

Although strictly not valid, the operations of summation and integration will be interchanged for the purpose of examining the series. The interchange is reversed at a later stage so that, in effect, all the steps are valid.

It is convenient to define the kernel \( M(z,z') \) on the left-hand side of (78a) as:

\[ M(z,z') = \sum_{n=-\infty}^{\infty} \cos(pz') \cos(pz) \frac{J_0[a(k_1^2 - p'^2)^{1/2}]}{J_1[a(k_1^2 - p'^2)^{1/2}]} \frac{a(k_1^2 - p'^2)^{1/2}}{[a(k_1^2 - p'^2)^{1/2}]} \]

where \( p = (n + 1/2)\pi/h \).

As was pointed out earlier, this series is divergent and, hence, it is useful to write it as the sum of two series, using the first two terms in the
asymptotic form. For large values of \( n \), the series behaves like

\[
\sum \cos(pz') \cos(pz) \frac{J_0(iap)}{J_1(iap)} (iap) \approx \sum \cos(pz') \cos(pz) \frac{I_0(ap)}{I_1(ap)} (ap)
\]

\[
\approx \sum \cos(pz') \cos(pz) \frac{1 + \frac{1}{8ap}}{1 - \frac{3}{8ap}} (ap) \approx \sum \cos(pz') \cos(pz) \left[ 1 + \frac{1}{2ap} \right] (ap)
\]

We now write

\[
M(z,z') = P(z,z') + Q(z,z') \tag{79}
\]

with

\[
P(z,z') = \sum_{n=-\infty}^{\infty} \left[ \frac{J_0[a(k_1^2 - p^2)^{1/2} - ap - \frac{1}{2}]}{J_1[a(k_1^2 - p^2)^{1/2}]} \right] \cos(pz') \cos(pz) \tag{80}
\]

and

\[
Q(z,z') = \sum_{n=-\infty}^{\infty} (ap + 1/2) \cos(pz') \cos(pz) \tag{81}
\]

Equation (79) along with (80) and (81) is exact because it only adds and subtracts the first two terms in the asymptotic form. Now \( P(z,z') \) can be written as:

\[
P(z,z') = \sum_{n=-\infty}^{\infty} A_n \cos(pz') \cos(pz)
\]

with \( A_n \) given by the term in the square brackets in (80).

If all the coefficients \( A_n \) were equal, \( P(z,z') \) would be a delta function; but this is not the case. In view of the differential operator on the left-hand side of (78a), it is helpful to remove a similar factor from \( P(z,z') \). This is easily accomplished by solving an equation of the form:

\[
(\frac{d^2}{dz^2} + k_0^2)f(z) = \cos(pz)
\]
Through the use of Green's function (or by any other method), one can obtain:

\[ f(z) = a_1 \cos(k_0z) + a_2 \sin(k_0z) + \frac{1}{k_0} \int_0^z \sin[k_0(z - z')] \cos(pz') \, dz' \]

Note that, without any loss of generality, the constants \(a_1\) and \(a_2\) can be set equal to zero and the integral on the right performed to obtain:

\[ f(z) = \frac{[\cos(pz) - \cos(k_0z)]}{(k_0^2 - p^2)} \]

so that

\[ P(z, z') = \left( \frac{\alpha^2}{\beta} + k_0^2 \right) \sum_{n=-\infty}^{\infty} A_n \left[ \cos(pz) - \cos(k_0z) \right] \frac{k_0^2}{k_0^2 - p^2} \cos(pz') \quad (82) \]

In (82) it appears that one of the terms in the series will be equal to infinity if \(p = k_0\). This condition is equivalent to \(h/\lambda = 1/4, 3/4, 5/4, \ldots\).

This is not the case, however, because of the numerator and the fact that, as \(p \to k_0\), the term in the square brackets in (82) approaches \([z \sin(k_0z)]/2k_0\).

Returning to (81), it is found that, since \(Q(z, z')\) is an odd series, its most divergent part is identically equal to zero, so that

\[ Q(z, z') = (1/2) \sum_{n=-\infty}^{\infty} \cos(pz') \cos(pz) = (h/2) \delta(z - z') \quad (83) \]

Using (82) and (83) with (79) in the integral equation (78a), one obtains

\[ \left( \frac{\alpha^2}{\beta} + k_0^2 \right) \left[ \int_{-h}^{h} \cos[p_1(a, z')] K(z - z', a) + \frac{\imath \omega K_0}{a} \int_{-h}^{h} K_0(z') M_1(z - z', a) \, dz' \right] \]

\[ = -\frac{2}{a} \left\{ \frac{\imath \omega K_0}{a} \delta(z) - \frac{1}{a} \left[ \frac{1}{2} E_\psi(a, z) - \frac{1}{a} \right] \int_{-h}^{h} \frac{1}{\beta} \left( \frac{\alpha^2}{\beta} + k_0^2 \right) \frac{h}{dz'} F_\psi(a, z') \right\} \]

(Continued)
Rearranging terms gives
\[
\left( \frac{\partial^2}{\partial z^2} + k_0^2 \right) \left[ \int_{-h}^{h} dz' E_\phi(a,z') K_1(z - z') + \frac{i \omega \mu_0}{a} \int_{-h}^{h} K_\phi(z') M_1(z - z', a) dz' \right] = -\frac{2}{a} \left[ i \omega \mu_0 i \phi_{\phi}(z) - \frac{1}{a \mu_r} \frac{1}{2} E_\phi(a,z) \right]
\] (84)

where the combined kernel $K_1(z - z')$ is defined as
\[
K_1(z - z') = \int_{-\pi}^{\pi} \frac{dk}{2\pi} e^{ikR_s} - \frac{2}{a \mu_r \hbar} \sum_{n=-\infty}^{\infty} A_n \left[ \frac{\cos(pz) - \cos(k_0z)}{k_0^2 - p^2} \right] \cos(pz')
\] (85)

with the coefficients $A_n$ given by
\[
A_n = \frac{J_0[a(k_1^2 - p^2)^{1/2}]}{J_1[a(k_1^2 - p^2)^{1/2}]} \left[ a(k_1^2 - p^2)^{1/2} \right] - ap - \frac{1}{2}
\]

and $p = (n + 1/2)\pi/h$.

The second integral equation from (78b) is:
\[
-\frac{\alpha}{2} \int_{-h}^{h} dz' E_\phi(a,z') \frac{\partial}{\partial \rho} K(z - z', \rho) \bigg|_{\rho = a^+} - \frac{i \omega \mu_0}{2} \int_{-h}^{h} dz' K_\phi(z') \frac{\partial}{\partial \rho} M_1(z - z', a) \bigg|_{\rho = a^-} = E_\phi(a,z)
\]

The coupled integral equations can now be written in a short-hand notation suitable for numerical evaluation:
\[
\left( \frac{\partial^2}{\partial z^2} + k_0^2 \right) \left[ \int_{-h}^{h} dz' E_\phi(z') K_1(z - z') + C_1 \int_{-h}^{h} dz' I_\phi(z') M_1(z - z') \right] = C_2 \delta(z) + C_3 E_\phi(z)
\] (86a)
\[ C_4 \int_{-h}^{h} E_{\phi}(z') \frac{\partial}{\partial \rho} M_2(z, z') + C_5 \int_{-h}^{h} I_{\phi}(z') K_2(z, z') = \frac{1}{2} E_{\phi}(z) \]  

(86b)

where the electric surface current \( I_{\phi}(z) = 2\pi a \Phi(z) \). Also, the kernel \( K_1(z - z') \) has been defined previously in (85), and

\[ C_1 = i\omega_0/2\pi a^2 \quad ; \quad C_2 = -2i\omega_0\Phi_0/a \quad ; \quad C_3 = 1/a^2 \mu_r \quad ; \]
\[ C_4 = -a/2 \quad ; \quad C_5 = -i\omega_0/4\pi a \]

The factor \((1/2)\) on the right-hand side of (86b) comes from the discontinuity in the derivative of the \( K_2(z - z') \) kernel, viz.,

\[ K_2(z - z') = \frac{3}{\partial \rho} K(z - z', \rho) \bigg|_{\rho=a} + \frac{3}{\partial \rho} K(z - z', \rho) \bigg|_{\rho=-a} \]
\[ M_2(z - z') = \frac{3}{\partial \rho} M_1(z - z', \rho) \bigg|_{\rho=a} + \frac{3}{\partial \rho} M_1(z - z', \rho) \bigg|_{\rho=-a} \]

8. NUMERICAL SOLUTION BY THE MOMENT METHOD OF THE COUPLED INTEGRAL EQUATIONS

The differential equation (86a) can be solved to obtain:

\[ \int_{-h}^{h} E_{\phi}(z') K_1(z, z') + C_1 \int_{-h}^{h} I_{\phi}(z') M_1(z, z') = \frac{1}{2} E_{\phi}(z) \]

\[ = C_6 \cos(k_0 z) + C_7 \sin(k_0 |z|) + C_8 \int_{0}^{z} E_{\phi}(z') \sin(k_0(z - z')) \]

(87a)

Similarly, from (86b)

\[ \int_{-h}^{h} E_{\phi}(z') K_2(z, z') + C_9 \int_{-h}^{h} I_{\phi}(z') M_2(z, z') = \frac{1}{2} E_{\phi}(z) \]

\[ = C_6 \cos(k_0 z) + C_7 \sin(k_0 |z|) + C_8 \int_{0}^{z} E_{\phi}(z') \sin(k_0(z - z')) \]

(87b)

where \( C_6 \) is unknown and determined numerically by employing the end condition \( I_{\phi}(h) = 0 \), and
\[ C_1 = \frac{i\omega_0}{2\pi a^2} \quad C_7 = \frac{C_2}{2k_0} = \frac{-i\omega_0}{4\pi a k_0} \]

\[ C_8 = \frac{C_3}{k_0} = \frac{1}{2\pi a^2} k_0 \quad C_9 = \frac{C_5}{C_4} = \frac{i\omega_0}{2\pi a^2} \quad C_{10} = \frac{1}{2C_4} = -\frac{1}{a} \]

It is now considered useful to examine the four kernels in (87a,b) and to obtain their Fourier transforms. Thus,

\[ K_1(z-z') = K(z-z') = \frac{\cos(\psi) - \cos(k_0\psi)}{k_0^2 - \rho^2} \]

with \( \rho = (n+1/2)\pi/a \). The Fourier transform of \( K(z-z') \) is given by \( \tilde{K}(\xi) = i\pi J_0(\alpha_0)H_0^{(1)}(\alpha_0) \) where \( \beta_0^2 = k_0^2 - \xi^2 \).

\[ \tilde{K}_1(\xi,\rho) = i\pi \int_0^\infty (\rho_\beta \gamma_0)J_0(\rho_\beta \gamma_0) \rho_\beta \rho \]

\[ \tilde{H}_1(\xi,\rho) = i\pi \int_0^\infty (\rho_\beta \gamma_0)J_0(\rho_\beta \gamma_0) \rho_\beta \rho \]

where

\[ \rho_\beta \quad \text{larger of} \quad \rho \quad \text{and} \quad \rho' \]

\[ \rho_\gamma \quad \text{smaller of} \quad \rho \quad \text{and} \quad \rho' \]

which leads to

\[ \tilde{H}_1(\xi,\rho) = \begin{cases} 
(\pm\alpha_0\gamma_0)J_0(\rho_\gamma \gamma_0)H_1^{(1)}(\alpha_0) & \text{if} \ \rho > \rho' \\
(\pm\alpha_0\gamma_0)J_0(\rho_\gamma \gamma_0)J_1(\alpha_0) & \text{if} \ \rho < \rho' 
\end{cases} \]

It is easily seen that

\[ \tilde{H}_2(\xi) = \frac{3}{a} \tilde{H}_1(\xi,\rho) \bigg|_{\rho=a} = \frac{3}{a} \tilde{H}_1(\xi,\rho) \bigg|_{\rho=a} = -i\alpha_0 J_1(\alpha_0)H_1^{(1)}(\alpha_0) \]
Finally,
\[ \tilde{K}_2(\xi) = \frac{\partial}{\partial \rho} \tilde{K}(\xi, \rho) \bigg|_{\rho = \alpha^+} \]

where
\[ \tilde{K}(\xi, \rho) = i\pi J_0(\rho, y_0) H_0^{(1)}(\rho, y_0) \]

with
\[ \rho_\prec = \text{smaller of } \rho \text{ and } a \]
\[ \rho_\succ = \text{larger of } \rho \text{ and } a \]

so that
\[ \tilde{K}(\xi, \rho) = \begin{cases} 
  i\pi J_0(\rho y_0) H_0^{(1)}(\rho y_0) & \text{for } \rho < a \\
  i\pi J_0(\rho y_0) H_0^{(1)}(\rho y_0) & \text{for } \rho > a 
\end{cases} \]

Therefore,
\[ \tilde{K}_2(\xi) = -i\pi y_0 J_0(\rho y_0) H_1^{(1)}(\rho y_0) \]

To begin the numerical procedure, it is recognized that, because of the evenness of \( E_\phi(z) \) and \( I_\phi(z) \), the integrals ranging from \(-h\) to \(h\) may be converted as follows:

\[
\int_{-h}^{h} E_\phi(z') K_{1,2}(z - z') \, dz' = \int_{-h}^{h} E_\phi(z') [K_{1,2}(z - z') + K_{1,2}(z + z')] \, dz' 
\]

Similarly,
\[
\int_{-h}^{h} I_\phi(z') M_{1,2}(z - z') \, dz' = \int_{-h}^{h} I_\phi(z') [M_{1,2}(z - z') + M_{1,2}(z + z')] \, dz' 
\]
Now the interval from 0 to $h$ can be subdivided into $n+1$ panels. Within each panel the unknown quantities $E(z)$ and $I(z)$ are approximated by constants and the constant value is assigned to a location $z$ which corresponds to the center of the panel. With the length $h = 2nt$, each panel is of width $2t$ except the first and last panels which are of width $t$.

The locations at which the unknown quantities are determined are given by $z_i = (2i - 2)t$ with $i = 1, 2, 3, \ldots, (n+1)$. Typically,

\[
\int_0^h E_\phi(z') [K_1(z - z') + K_1(z + z')] \, dz' = \left\{ \begin{array}{l}
\int_0^t + \int_0^{3t} + \int_0^{5t} + \ldots + \int_0^{(2n-1)t} + \int_0^{2nt} \\
\int_0^{(2n-3)t} + \int_0^{(2n-1)t}
\end{array} \right\} E_\phi(z')[K_1(z - z') + K_1(z + z')] \, dz'
\]

In each of these intervals $E_\phi(z)$ is approximated by a constant value so that

\[
K_1(I,J) = \int_{z_{J-1}}^{z_J} [K_1(z_{J-1} + z') + K_1(z_J + z')] \, dz' = \left\{ \begin{array}{l}
\frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{\tilde{r}}(\xi)e^{-i\xi z'} \, d\xi \\
\int_{(2J-3)t}^{(2J-1)t}
\end{array} \right\} \int_{z_{J-1}}^{z_J} \frac{1}{d_2} \int_{a}^{a_1} \left[ \cos(pz) - \cos(k_0 n z) \right] \cos(pz') \, dz'
\]

\[
+ e^{-i\xi z'} d\xi - \frac{1}{a^2 u_h} \int_{a}^{a_1} \left[ \frac{\cos(pz') - \cos(k_0 n z)}{k_0^2 - p^2} \right] \cos(pz') \, dz'
\]
By substituting for $\tilde{K}(\xi)$ and carrying out the $\pi^i$ integration, one obtains

$$K_{1}(I,J) = K(I,J) + B(I,J)$$

where

$$K(I,J) = \frac{4}{\pi} \int \left[ i\pi J_0(\alpha y_0)J_1(\alpha y_0) \right] \left[ (\sin \xi t)/\xi \right] \left[ \cos[2\xi(I+J-2)t] + \cos[2\xi(I-J)t] \right] d\xi$$

$$B(I,J) = -\frac{4}{\pi \mu} \sum_{n=0}^{\infty} \frac{\cos[2\pi t(I-1)] - \cos[2\pi t(I-1)]}{a^2(k_0^2 - p^2)}$$

$$+ \left[ \frac{\sin[\pi t(2J-1)] - \sin[\pi t(2J-1)]}{(h/\pi)p} \right]$$

The integral in $K(I,J)$ is evaluated by suitably deforming the contour from the real axis to a contour that wraps around the branch cut. When this is done, $K(I,J)$ for $I \neq J$ can be written in the form

$$K(I,J) = \int_{0}^{\infty} f(x) e^{-x} \, dx$$

where $f(x)$ is a complex function of a real variable $x$. The integrals are evaluated using a 10-point Gauss-Laguerre quadrature method. The special case of diagonal elements $(I=J)$ can be written in the form

$$K(I,I) = \frac{1}{\pi} \int \frac{2i\xi}{\pi} \left[ \sin \xi t \right] \left[ e^{2i\xi I} + 1 \right] d\xi = (1/2)K(1,1) + T(I)$$

where

$$T(I) = \frac{1}{\pi} \int \frac{2i\xi}{\pi} \left[ \sin \xi t \right] e^{2i\xi I} d\xi$$
with \( z_I = 2(I - 1)t, I = 1, 2, \ldots, (n+1) \). \( K(1,1) \) is evaluated as an integral on the real axis because of the absence of the exponential decay factor, using 10-point Gauss quadrature routines. \( T(I) \) can once again be put in a form suitable for Gauss-Laguerre quadrature by a deformation of the contour that wraps around the branch cut at \( \xi = k_0 \). Care is taken in evaluating the first and last panels' integrations because of their half normal width. What is discussed for kernel \( K(z - z') \) or \( \tilde{K}(\xi) \) is essentially true with the calculation of the elements corresponding to the three other kernels.

Referring back now to the three terms on the right-hand side of (87a), viz.,

\[
C_6 \cos(k_0 z_I) + C_7 \sin(k_0 |z_I|) + C_8 \int_0^{z_I} dz' E_\phi(z') \sin[k_0(z_I - z')] ,
\]

the first and last terms, containing respectively the unknowns \( C_6 \) and \( E_\phi(z) \), are moved to the left-hand side. For example,

\[
C_6 \cos(k_0 z_I) = C_6 \cos[k_0(2I - 2)t] \\
C_8 \int_0^{z_I} dz' [ ] = C_8 \int_0^{(2I-2)t} dz' [ ] = C_8 \left\{ \int_0^t + \int_t^{2t} + \ldots + \int_0^{(2I-3)t} \right\} dz' [ ]
\]

We now define

\[
A(I,P) = \int_{(I-1)t}^{Pt} \sin[k_0(z_I - z')] \, dz' = (1/k_0)(\cos[k_0t(2I - P - 2)] - \cos[k_0t(2I - P - 1)])
\]

It is seen that when the term associated with \( C_8 \) is moved to the left-hand side, it affects only the lower triangle elements of \( K_1(1,1) \) and not the upper triangle elements, thus rendering the \( K_1(1,1) \) matrix elements not equal.
Extending these calculating principles to (87b), and using the fact that \( I_\phi(h) = I_\phi(I=n+1) = 0 \), one can finally set up the following matrix equation:

\[
\begin{bmatrix}
  a_{11} & a_{12} & \cdots & a_{1,n+1} \\
  a_{1,n+2} & a_{1,n+3} & \cdots & a_{1,2n+1} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n+1,1} & a_{n+1,2} & \cdots & a_{n+1,2n+1} \\
  a_{n+2,1} & a_{n+2,2} & \cdots & a_{n+2,2n+1} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{2n+2,1} & a_{2n+2,2} & \cdots & a_{2n+2,2n+1}
\end{bmatrix}
\begin{bmatrix}
  I_1 \\
  I_2 \\
  \vdots \\
  I_n \\
\end{bmatrix}
= \begin{bmatrix}
  E_1 \\
  E_2 \\
  \vdots \\
  E_n \\
\end{bmatrix}
\]

where the elements on the right-hand side are given by \( G(I) = C_7 \sin[k_0(2I-2)t] \) with \( I = 1, 2, \ldots, (n+1) \).

The magnetic current \( I_2^*(z) \) is easily obtained from the solution of the system of linear equations by using \( I_2^*(z) = -2naE_\phi(z) \) volts. The computer programs are included in Appendix C and the results are plotted and discussed in the next section.
9. EXPERIMENTAL MEASUREMENT OF THE MAGNETIC CURRENT

The magnetization current is essentially the time rate of change of the magnetization vector \( \vec{M} \) integrated over the antenna cross section. The experimental procedure, however, determines the total axial magnetic flux with the use of a shielded loop placed coaxially over a driven loop which is loaded by a ferrite cylinder. Suitable modifications to the theory have to be made, therefore, before the computations can be compared with the experimental results. These modifications and the assumption of azimuthal symmetry on which they are based are discussed in detail in Appendix D.

Ferrite materials that are available commercially have been used in this experimental investigation. Table 1 lists the initial permeability \( \mu'_r \) (i.e., the slope of the B-H curve for small \( H \)) and the applicable frequency range for a variety of ferrite materials, grouped under their respective suppliers. Ferrites #C-2050 of Ceramic Magnetics, Inc. and #Q-3 of Indiana General were selected for use in the 5-100 MHz frequency range. Toroidal samples of the #C-2050 material were obtained and its properties (\( \mu'_r \) and \( Q \)) measured as a function of frequency by means of a Q-meter. The quality factor \( Q \) of the ferrite material is defined by

\[
Q = \frac{1}{\text{loss factor}} = \frac{\mu''_r}{\mu'_r} = \frac{2\pi \times \text{stored energy}}{\text{energy dissipated per period, } 2\pi/\omega} \tag{88}
\]

The measured values of \( Q \) and \( \mu'_r \) for the ferrite material #C-2050 are shown plotted in Fig. 4(a) as a function of frequency together with the values supplied by the manufacturer. Fair agreement is observed between the two. The imaginary part \( \mu''_r \) of the relative permeability can be calculated easily using (88) and measured values of \( \mu'_r \) and \( Q \). The manufacturer-supplied values of \( \mu'_r \) and \( Q \) for the ferrite material #Q-3 are shown in Fig. 4(b). The values of the relative permittivity \( \varepsilon'_r \) used in the theoretical calculations were
# TABLE 1. List of Commercially Available Ferrite Materials

**Source #1:** Ceramic Magnetics, Inc., Fairfield, N.J.

<table>
<thead>
<tr>
<th>Type of Ferrite Material</th>
<th>Manufacturer Code</th>
<th>Initial Permeability ( \mu' )</th>
<th>Frequency Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mn-Zn</td>
<td>MN-31 DC-10</td>
<td>2800</td>
<td>Up to 10 MHz</td>
</tr>
<tr>
<td>Mn-Zn</td>
<td>MN-31 DC-20</td>
<td>3300</td>
<td>Up to 10 MHz</td>
</tr>
<tr>
<td>Ni</td>
<td>CN-20</td>
<td>800</td>
<td>300 KHz - 2 MHz</td>
</tr>
<tr>
<td>Ni</td>
<td>CM-2002</td>
<td>1500</td>
<td>1 KHz - 1 MHz</td>
</tr>
<tr>
<td>Mn</td>
<td>MN-30</td>
<td>4000</td>
<td>Up to 500 KHz</td>
</tr>
<tr>
<td>Mn</td>
<td>MN-60</td>
<td>6000</td>
<td>Up to 600 KHz</td>
</tr>
<tr>
<td>Mn</td>
<td>MN-100</td>
<td>9500</td>
<td>Below 1 MHz</td>
</tr>
<tr>
<td>C-2010</td>
<td></td>
<td>200-300</td>
<td>Below 15 MHz</td>
</tr>
<tr>
<td>C-2025</td>
<td></td>
<td>150-200</td>
<td>Below 15 MHz</td>
</tr>
<tr>
<td>C-2050</td>
<td></td>
<td>100-150</td>
<td>Below 20 MHz</td>
</tr>
<tr>
<td>C-2075</td>
<td></td>
<td>25-50</td>
<td>Below 50 MHz</td>
</tr>
<tr>
<td>CHD-5005</td>
<td></td>
<td>1400</td>
<td>Up to 10 MHz</td>
</tr>
<tr>
<td>N-40</td>
<td></td>
<td>15-20</td>
<td>Up to 100 MHz</td>
</tr>
</tbody>
</table>

**Source #2:** Indiana General, Keasbey, N.J.

| Ni-Zn                    | Q-1               | 125                           | Up to 10 MHz   |
| Ni-Zn                    | Q-2               | 40                            | Up to 50 MHz   |
| Ni-Zn                    | Q-3               | 18                            | Up to 200 MHz  |

**Source #3:** Fair-Rite Products Corp., Wallkill, N.Y.

| Ni-Zn                    | 30-61             | 125                           | 200 KHz - 10 MHz|

**Source #4:** Ferroxcube Corp., Saugerties, N.Y.

| Ni-Zn                    | 4C4               | 125                           | Up to 50 MHz   |
| Mn-Zn                    | 3D3               | 750                           | Up to 5 MHz    |
| Mn-Zn                    | 3B9               | 1800                          | Up to 5 MHz    |
| Mn-Zn                    | 3B7               | 2300                          | Up to 1 MHz    |

**Source #5:** National Moldite Co., Inc., Newark, N. J.

| M-Grade                  | 125 @ 1 MHz       | Up to 20 MHz                 |

**Source #6:** Stackpole-Carbon Co., St. Marys, Pa.

| Grade 24                 | 2500              | Up to 100 KHz                |
| Grade 27A                | 1000              | Up to 800 KHz                |
| Grade 9                  | 190               | Up to 2 MHz                  |
| Grade 11                 | 125               | Up to 6 MHz                  |
| Grade 12                 | 35                | Up to 60 MHz                 |
| Grade 2285A              | 7.5               | Up to 300 MHz                |
FIG. 4 MAGNETIC PROPERTIES OF FERRITE MATERIAL AS A FUNCTION OF FREQUENCY

(a) MATERIAL #C-2050; SOURCE — CERAMIC MAGNETICS INC.

(b) MATERIAL #Q-3; SOURCE — INDIANA GENERAL.
also supplied by the manufacturers. It can be seen from Fig. 4(a) that the value of $Q$ for the ferrite material #C-2050 is nearly constant ($\approx 100$) up to about 2 MHz and then falls off rapidly to less than 10% of this value at 20 MHz. Similarly for the material #Q-3, Fig. 4(b) indicates a nearly constant value of $Q$ ($\approx 2500$) up to about 15 MHz, beyond which it decays to $\approx 50$ at 200 MHz.

Three antenna cores were fabricated, as photographed in Fig. 5(a)-(c). Cores (a) and (b) are made of material #C-2050, while core (c) is made of material #Q-3. Cylindrical rods of 5/16" diameter and 5.25" height were used along with adhesive tape to fabricate core (a) of 2" overall diameter and 21" height. Core (b) has the same height as core (a) but is comprised of five cylindrical rods of 1" diameter and varying lengths. Core (c) is formed from three cylindrical rods of .625" diameter and 7.5" length for a total height of 22.5". As was pointed out earlier, cores (a) and (b) are useful for frequencies up to about 20 MHz, core (c) up to 200 MHz.

The three cores were used in various antenna configurations in which an electrically small loop antenna is loaded by a finite cylindrical ferrite rod. The antenna parameters for the eleven different cases are tabulated in Table 2. For antennas numbered 1 through 3, measurements were made at frequencies of 10, 50 and 100 MHz, respectively. The electrical radius $a_{k0}$ of the driven loop ranges from .00166 to .01662. Antennas numbered 4 through 7 were operated at frequencies of 5, 10, 15 and 20 MHz, respectively; the electrical radius ranged from .00132 to .00531. The operating frequencies for antennas numbered 8 through 11 were the same as for the previous set but the radius was doubled.

It can be seen in Table 2 that the value of $a_{k0}$ does not exceed 0.017 for any of the eleven antennas. This ensures the validity of the assumption
(a) 
Dia. 2a = 2", height 2h = 21"

(b) 
Dia. 2a = 1", height 2h = 21"

(c) 
Dia. 2a = .625", height 2h = 22/5"

FIGURE 5
FERRITE RODS USED IN THE EXPERIMENT
TABLE 2. Antenna Parameters

Q-3 Material (Supplier: Indiana General)

\[ 2a = 0.625'', \quad 2h = 22.5'', \quad \Omega = 2 \ln(2h/a) = 8.5534 \]

<table>
<thead>
<tr>
<th>Antenna #</th>
<th>( u_r = u'_r - j\mu'_r )</th>
<th>( h/\lambda_0 )</th>
<th>( ak_0 )</th>
<th>( z_m = r_m + jx_m )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>18 - j0.036</td>
<td>0.00952</td>
<td>0.00166</td>
<td>0.00797 - j3.7637</td>
</tr>
<tr>
<td>2</td>
<td>19 - j0.0544</td>
<td>0.04762</td>
<td>0.00831</td>
<td>0.00214 - j0.70979</td>
</tr>
<tr>
<td>3</td>
<td>20 - j0.0890</td>
<td>0.09525</td>
<td>0.01662</td>
<td>0.001577 - j3.3443</td>
</tr>
</tbody>
</table>

QC-2050 Material (Supplier: Ceramic Magnetics, Inc.)

i) \( 2a = 1'' \), \( 2h = 21'' \), \( \Omega = 2 \ln(2h/a) = 7.4754 \)

<table>
<thead>
<tr>
<th>Antenna #</th>
<th>( u_r = u'_r - j\mu'_r )</th>
<th>( h/\lambda_0 )</th>
<th>( ak_0 )</th>
<th>( z_m = r_m + jx_m )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>100 - j1.0</td>
<td>0.00444</td>
<td>0.00132</td>
<td>0.0051 - j5.0479</td>
</tr>
<tr>
<td>5</td>
<td>115 - j2.55</td>
<td>0.00889</td>
<td>0.00265</td>
<td>0.0049 - j2.1892</td>
</tr>
<tr>
<td>6</td>
<td>125 - j12.50</td>
<td>0.01333</td>
<td>0.00398</td>
<td>0.01341 - j1.3270</td>
</tr>
<tr>
<td>7</td>
<td>135 - j67.5</td>
<td>0.01778</td>
<td>0.00531</td>
<td>0.03747 - j0.7394</td>
</tr>
</tbody>
</table>

ii) \( 2a = 2'' \), \( 2h = 21'' \), \( \Omega = 2 \ln(2h/a) = 6.089 \)

<table>
<thead>
<tr>
<th>Antenna #</th>
<th>( u_r = u'_r - j\mu'_r )</th>
<th>( h/\lambda_0 )</th>
<th>( ak_0 )</th>
<th>( z_m = r_m + jx_m )</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>105 - j0.63</td>
<td>0.00444</td>
<td>0.00265</td>
<td>0.01275 - j1.2611</td>
</tr>
<tr>
<td>9</td>
<td>120 - j2.4</td>
<td>0.00889</td>
<td>0.00531</td>
<td>0.001225 - j0.5456</td>
</tr>
<tr>
<td>10</td>
<td>150 - j15.</td>
<td>0.01333</td>
<td>0.007979</td>
<td>0.003352 - j0.3292</td>
</tr>
<tr>
<td>11</td>
<td>140 - j42.</td>
<td>0.01778</td>
<td>0.01064</td>
<td>0.009368 - j0.1815</td>
</tr>
</tbody>
</table>
that the driven loop be electrically thin in the theoretical calculation of antenna currents. The height of the monopole antenna \((h/\lambda_0)\) ranges from \(0.00444\) to \(0.09525\) so that the longest dipole is nearly \((1/5)\)-wavelength long. The value of the relative permeability \(\mu_r = \mu_r' - j\mu_r''\) from Fig. 4 and the internal impedance \(z_m^i\) per unit length calculated using equation (36) are also listed in Table 2. In this section an \(e^{jwt}\) time dependence is implicit and is more convenient. Due account of this change in notation has been taken in using (36) to calculate \(z_m^i\). It is observed that for all antennas considered, the internal impedance is largely reactive.

For each of the three ferrite cylindrical cores described above, a set of driven and measuring loops was fabricated. A photograph and representative line drawing showing the construction of the loops are shown in Fig. 6. The six loops were all constructed from commercially available microcoaxial cables ending in a modified BNC connector. The driven and measuring loops are placed coaxially in the experimental setup, as can be seen in the photograph in Fig. 7 and the block diagram in Fig. 8. The short lengths of microcoaxial transmission lines leading away from the two loops are at right angles to one another in the horizontal plane so that any inductive coupling between the two is minimized. The signal source used in this experiment was either a GR-1001A (5-50 MHz) or an HP-3200B (10-500 MHz) oscillator. When the GR-1001A oscillator was used for measurements with antennas \#4 through \#11, the power amplifier was not needed. The HP-230B power amplifier was used only in conjunction with the HP-3200B oscillator for measurements on antennas \#1 through \#3. The source frequency was accurately measured using an HP-5240 electronic counter. A signal proportional to the total axial magnetic field was induced in the receiving loop. An HP-8405A vector voltmeter was used to detect and record this signal (B). The reference signal (A) to the
FIGURE 6
THREE PAIRS OF DRIVEN AND MEASURING LOOPS ALONG
WITH A LINE DIAGRAM SHOWING THE CONSTRUCTIONAL
DETAILS OF A REPRESENTATIVE LOOP.
FIGURE 7
PHOTOGRAPH OF THE EXPERIMENTAL SET-UP
FIG. 8  BLOCK DIAGRAM OF THE EXPERIMENTAL SETUP FOR MEASUREMENT OF MAGNETIC CURRENT ON THE FERRITE ROD ANTENNA.
vector voltmeter was provided from a coaxial T. The vector-voltmeter readings were recorded as a function of the axial distance \( z \) from the driving loop. In this manner the amplitude and phase of the magnetic current distribution were obtained for the eleven antenna configurations described in Table 2. The unnormalized data are given in Table C-2 of Appendix C.

Computer programs, described and listed in Appendices A and B, were utilized in calculating the magnetic current distributions for the eleven cases. As discussed earlier, the theoretical calculations are based on a treatment of the ferrite rod as an imperfect magnetic conductor. The theoretical and experimental current distributions are shown graphically in Figs. 9 through 11. Also appearing in Figs. 9 - 11 are Tables 3, 4 and 5, respectively; these show the calculated values of input admittance \( Y^* = \frac{I_2^*(0)}{I_0^e} = (G + jB)^* \text{ohms} \) and input impedance \( Z^* = \frac{1}{Y^*} \text{mhos} \). The antenna numbering scheme used in the figures and tables corresponds to that given in Table 2. The values of \( \Omega = 2 \ln(2h/a) \) for the antennas in Figs. 9 - 11 are, respectively, 8.5534, 7.4754 and 6.089.

The agreement between the theory and experiment is seen to be good. The antennas used here are relatively short and, consequently, the current distribution is seen to be nearly triangular. As may be expected, the largest deviation of the experimental values from the theoretical computations occurs at either end of the antenna \( (z/h = 0, 1) \) and especially at the driving point. For this reason the raw experimental data were normalized in most cases to the theoretical calculations at a point nearly a third the distance from the driving point to the end of the antenna. In the case of antennas \#1 and \#2, there appears to be a kink in the experimental values for the phase of the current distribution. This is believed to be due to the stacking of individual ferrite rods by means of adhesive tape. This is not seen in the magnitude
Fig 9 PLOT OF MAGNITUDE AND PHASE OF THE MAGNETIC CURRENT ALONG THE ANTENNA $I_A = 2 \left( \frac{h}{a} \right) = 8.5534$.

### Table 3

<table>
<thead>
<tr>
<th>ANTENNA</th>
<th>X'</th>
<th>X''</th>
<th>Y'</th>
<th>Y''</th>
<th>Z'</th>
<th>Z''</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0130 + 0.0009i</td>
<td>0.0017 + 0.0031i</td>
<td>0.0049 + 0.0049i</td>
<td>0.0049 + 0.0049i</td>
<td>0.0054 + 0.0054i</td>
<td>0.0054 + 0.0054i</td>
</tr>
<tr>
<td>2</td>
<td>0.0047 + 0.0047i</td>
<td>0.0047 + 0.0047i</td>
<td>0.0047 + 0.0047i</td>
<td>0.0047 + 0.0047i</td>
<td>0.0047 + 0.0047i</td>
<td>0.0047 + 0.0047i</td>
</tr>
<tr>
<td>3</td>
<td>0.0073 + 0.0073i</td>
<td>0.0073 + 0.0073i</td>
<td>0.0073 + 0.0073i</td>
<td>0.0073 + 0.0073i</td>
<td>0.0073 + 0.0073i</td>
<td>0.0073 + 0.0073i</td>
</tr>
</tbody>
</table>

---

**THEORETICAL (3-TERM THEORY)**

---

**EXPERIMENTAL**

(FERRITE SOURCE, MATERIAL G#3 SUPPLIED BY INDIANA GENERAL)
TABLE 4
Calculated (3-term theory) \( V_1^* \) and \( Z_1^* \) for the above four antennas.

<table>
<thead>
<tr>
<th>ANTENNA</th>
<th>INPUT ADMITTANCE</th>
<th>INPUT IMPEDANCE</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( y_* )</td>
<td>( Z_{(G+jB)^i} )</td>
</tr>
<tr>
<td>4</td>
<td>0.0506 + j7.631</td>
<td>0.00050 + j1.12769</td>
</tr>
<tr>
<td>5</td>
<td>0.1397 + j6.545</td>
<td>0.00051 + j0.0043</td>
</tr>
<tr>
<td>6</td>
<td>0.9660 + j25.679</td>
<td>0.0146 + j0.3880</td>
</tr>
<tr>
<td>7</td>
<td>6.12 + j36.637</td>
<td>0.0444 + j0.02655</td>
</tr>
</tbody>
</table>

FIG 10 PLOT OF MAGNITUDE AND PHASE OF THE MAGNETIC CURRENT ALONG THE ANTENNA.
\( \Omega = 2 \pi (2h/a) = 7.4754 \)
FIG. 11 PLOT OF MAGNITUDE AND PHASE OF THE MAGNETIC CURRENT ALONG THE ANTENNA.
(\( \Omega = 2\pi (2h/a) = 6.089 \))

<table>
<thead>
<tr>
<th>TABLE 5</th>
<th>Calculated (3-term Theory) ( r^* ) and ( Z^* ) for the above four antennas</th>
</tr>
</thead>
<tbody>
<tr>
<td>ANTENNA</td>
<td>INPUT ADMITTANCE ( r^* )</td>
</tr>
<tr>
<td>#</td>
<td>( r^* )</td>
</tr>
<tr>
<td>8</td>
<td>0.041 + 15.805</td>
</tr>
<tr>
<td>9</td>
<td>0.176 + 32.807</td>
</tr>
<tr>
<td>10</td>
<td>1.160 + 150.373</td>
</tr>
<tr>
<td>11</td>
<td>6.749 + 70.870</td>
</tr>
</tbody>
</table>

THEORETICAL (3-TERM THEORY)

- EXPERIMENTAL

FERRITE SOURCE: MATERIAL C #2050 SUPPLIED BY CERAMIC MAGNETICS
curves because of the relatively low magnitude values. The overall agreement of the theory and experiment was used in deciding the point of normalization.

The coupled integral equations in (86a,b) in the two variables, the tangential electric field $E_\phi(z)$ and circumferential electric current $I_\phi(z)$, were solved numerically by the moment method on a Sigma-7 computer system. The method itself has been discussed in Section 8; the computer programs are listed in Appendix C. Table 6 contains a description of all the subroutines used in this computation. The basic philosophy of this method is to reduce the set of coupled integral equations to a system of linear algebraic equations. The standard routines [6] for solving a system of linear equations were modified to handle complex variables. The results of these computations are plotted in Figs. 12 through 14. As before, the experimental data have been normalized at a point approximately one third the distance from the driving point ($z = 0$) to the end.

The magnetic current $I^*_z(z)$ is easily obtained from the solution for the tangential electric field using the relation $I^*_z(z) = -2\pi a E_\phi(z)$ volts per unit current in the driving loop. The input parameters $Y^*$ and $Z^*$ are also tabulated and the tables are included in the figures showing the magnitude and phase of the magnetic current. In all eleven cases the phase is nearly constant, since the antennas are electrically short in free space, and most of the magnetic current is in phase quadrature. The agreement between the experiment and the theoretical calculations is very good including near the source. This was to be expected because the coupled integral equations (86a,b) in two variables comprise a far more accurate and independent theoretical formulation of the problem than the approximate integral equation (39) which relies rather heavily on an analogy between the ferrite rod antenna and the resistive cylindrical dipole antenna.
<table>
<thead>
<tr>
<th>PROGRAM NAME</th>
<th>PURPOSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>MAIN</td>
<td>Computes $E_{\phi}(z)$ and $I_{\phi}(z)$ by solving the coupled integral equations (86a,b).</td>
</tr>
<tr>
<td>BSLSML</td>
<td>Computes Bessel functions $J_0(z)$ and $J_1(z)$ for $</td>
</tr>
<tr>
<td>BESH</td>
<td>Computes Hankel functions $H_{0}^{(1)}(z)$, $H_{0}^{(2)}(z)$, $H_{1}^{(1)}(z)$ and $H_{1}^{(2)}(z)$.</td>
</tr>
<tr>
<td>QGL10</td>
<td>10-point Gauss-Laguerre quadrature routine.</td>
</tr>
<tr>
<td>QG10</td>
<td>10-point Gauss quadrature routine.</td>
</tr>
<tr>
<td>FCTK</td>
<td>Computes the integrand for $K(z - z')$ for $I \neq J$.</td>
</tr>
<tr>
<td>FKII, FKIII</td>
<td>The same, for $I = J = 1$.</td>
</tr>
<tr>
<td>FKII</td>
<td>The same, for $I = J \neq 1$.</td>
</tr>
<tr>
<td>FCTMII, FMIIIR, FMIIII, FMIII</td>
<td>Computes the integrand for $H_{1}^{(1)}(z - z')$ for $I \neq J$, $I = J = 1$, and $I = J \neq 1$, respectively.</td>
</tr>
<tr>
<td>FKII, FKIII</td>
<td>Computes the integrand for $K_2(z - z')$ for $I \neq J$, $I = J = 1$, and $I = J \neq 1$, respectively.</td>
</tr>
<tr>
<td>FCTMII, FMIIIR, FMIIII, FMIII</td>
<td>Computes the integrand for $M_2(z - z')$ for $I \neq J$, $I = J = 1$, and $I = J \neq 1$, respectively.</td>
</tr>
<tr>
<td>AUX</td>
<td>Auxiliary function used in computing the above integrands.</td>
</tr>
<tr>
<td>SERIES</td>
<td>Computes the infinite series part of the kernel $K_{1}(z - z')$.</td>
</tr>
<tr>
<td>DECOMP, SOLVE, IMPRUV, SING</td>
<td>Programs used in solving the linear system of algebraic equations.</td>
</tr>
<tr>
<td>ANGLE</td>
<td>Computes the phase angle of complex variables $I_{\phi}(z)$ and $E_{\phi}(z)$.</td>
</tr>
</tbody>
</table>
Fig. 12. Plot of magnitude and phase of the magnetic current along the antenna: \( D = 2 \ln(2h/a) = 8.5534 \).
(Uncorrected; see Appendix D.)
Fig. 13. Plot of magnitude and phase of the magnetic current along the antenna; $\theta = 2 \ln(2h/a) = 7.4754$. 

Calculated
(Coupled Integral Equations)

Experimental

(Ferrite source: Material C32050 supplied by Ceramic Magnetics)
TABLE 7
INPUT ADMITTANCES AND IMPEDANCES OF THE ELEVEN ANTENNAS OBTAINED FROM SOLVING THE COUPLED INTEGRAL EQUATIONS (86a,b)

<table>
<thead>
<tr>
<th>Antenna #</th>
<th>Input Admittance (ohms)</th>
<th>Input Impedance (mhos)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.003 + j 4.99</td>
<td>0.00012 - j 0.20040</td>
</tr>
<tr>
<td>2</td>
<td>0.03 + j 30.22</td>
<td>0.00003 - j 0.03309</td>
</tr>
<tr>
<td>3</td>
<td>0.10 + j 62.32</td>
<td>0.00003 - j 0.01605</td>
</tr>
<tr>
<td>4</td>
<td>0.03 + j 9.23</td>
<td>0.00035 - j 0.10834</td>
</tr>
<tr>
<td>5</td>
<td>0.14 + j 20.13</td>
<td>0.00035 - j 0.04967</td>
</tr>
<tr>
<td>6</td>
<td>0.97 + j 29.81</td>
<td>0.00109 - j 0.03351</td>
</tr>
<tr>
<td>7</td>
<td>6.21 + j 45.62</td>
<td>0.00293 - j 0.02152</td>
</tr>
<tr>
<td>8</td>
<td>0.02 + j 23.55</td>
<td>0.00004 - j 0.04246</td>
</tr>
<tr>
<td>9</td>
<td>0.16 + j 44.27</td>
<td>0.00008 - j 0.02239</td>
</tr>
<tr>
<td>10</td>
<td>1.06 + j 60.02</td>
<td>0.00029 - j 0.01666</td>
</tr>
<tr>
<td>11</td>
<td>4.26 + j 75.32</td>
<td>0.00075 - j 0.01323</td>
</tr>
</tbody>
</table>
10. SUMMARY

An electrically small loop that carries a constant current and is loaded by a homogeneous and isotropic ferrite rod has been called the ferrite-rod antenna. In Part I [2] of this report the ferrite rod was assumed to be of infinite length and the problem was treated using a boundary-value approach. In a practical situation, however, the antenna is necessarily finite and often electrically short so that a new mathematical formulation was needed, along with an experimental investigation, for the problem of a finite ferrite rod antenna. With this current distribution known precisely, other quantities of interest can be derived from it.

Although, in terms of physical mechanisms, the ferrite-rod antenna can be compared with the dielectric rod antenna, there exists a complete analogy between the ferrite antenna and the conducting cylindrical dipole antenna. This analogy is based on the dual property of electric and magnetic vectors in Maxwell's equations. The electric dipole antenna has received considerable attention from researchers in the past and, therefore, a treatment of the 'magnetic analog' of the dipole antenna is considered useful. Based on this analogy, an integral equation has been derived for the magnetic current on the finite ferrite-rod antenna. As expected, the integral equation is identical in form to the corresponding equation for the electric current on the dipole antenna. This derivation was based on the assumption that the value of the relative permeability $\mu_r$ of the ferrite material equals infinity. In effect, the ferrite is treated as a perfect magnetic conductor as when in the 'electric case' the antenna material is assumed to have an infinite electrical conductivity $\sigma$. However, in practice, a material with $\mu_r$ equal to infinity does not exist and, furthermore, over a useful frequency range the $\mu_r$ value is not high enough to justify using the perfect conductor approximation.
For this reason, the integral equation had to be modified. The modification was achieved by defining the internal impedance per unit length of the magnetic conductor to be the ratio of the tangential magnetic field to the total magnetic current flowing in the magnetic conductor. An approximate, 3-term expression for the magnetic current was then obtained in a manner paralleling the procedure used by King and Wu to solve for the electric current on the imperfectly conducting dipole antenna. It was found that for commercially available ferrites the internal impedance per unit length was largely reactive so that the propagation constant \( k = \beta + i\alpha \) on the antenna had a large imaginary part. The predominance of the attenuation constant \( \alpha \) makes the magnetic current very small and, thus, one is led to conclude that the practical ferrite-rod antenna is not a very efficient radiator.

The treatment of the ferrite-rod antenna as an analog of the resistive electric dipole antenna relies rather heavily on the mathematical equivalence of the two problems under idealized driving conditions. For this reason, an alternative derivation of the integral equation for the tangential electric field on the ferrite surface was developed. This derivation led to a pair of coupled integral equations in terms of the tangential electric field and tangential electric surface current. The coupled integral equations were solved numerically by the moment method and the magnetic current obtained from the tangential electric field. It was also verified that in the limit \( h \rightarrow \infty \) the equations decouple and are in complete agreement with the results of the theory for the infinite antenna.

Since the magnetic current is proportional to the total axial magnetic field, a simple experimental apparatus was built to measure the magnetic current distribution on several antenna configurations. A graphical comparison of the theoretical and experimental results has been presented. Although the
three-term solution has been shown to give good results for antenna lengths $k_0 h \leq 5\pi/4$, the antennas used in the experiment were much shorter and the near-triangular distribution of currents was verified. The frequency response of the properties of available ferrite materials and the practical limitations on the size of the ferrite rods made it difficult to construct antennas of longer length.

In conclusion, while ferrites have been used extensively at microwave frequencies and up to several Megahertz, the fact that ferrites are now becoming commercially available in the range 30 - 300 MHz should lead to useful applications. In situations in which the physical size of the antenna must be kept small, a loop antenna has limited usefulness because of its low efficiency and radiation resistance. The insertion of a suitable ferrite core offers the advantage of both improved efficiency and increased radiation resistance. Although, in theory, an increased radiation resistance should simplify the problem of matching the antenna to its associated circuit, in practice there remains a severe problem. This has been discussed by Dropkin, Motzer and Cacheris [71] who made measurements of the receiving characteristics of a cylindrical ferrite-rod antenna at a frequency of 75 MHz. They conclude that both ferrite-core and air-core loops can be described by similar equivalent circuits. These circuits have resonant properties and each of the lumped circuit elements can be identified with a physical quantity characterizing the antenna. The improved efficiency and the increased radiation resistance which were determined experimentally can be attributed directly to an increased magnetic flux passing through the loop. They also make the interesting observation that with a dielectric cylinder ($\varepsilon_r = 10$, $\mu_r = 1$), the size of the ferrite used had no effect on the air-loop properties. This was because the loop used was small enough to act as a magnetic dipole.
APPENDIX A.

COMPUTATION OF MAGNETIC CURRENT AND ADMITTANCE OF FERRITE ROD ANTENNA

The approximate magnetic current distribution as given by (51) is

\[ I^*(z) = \frac{-12\pi k_0^2 l e}{kV dR \cos kh} \{ \sin k(n - |z|) + T^*_U(\cos kz - \cos kh) 
+ T^*_D(\cos \frac{k_0^2}{2} - \cos \frac{k_0 h}{2}) \} \]

where \( T^*_U \) and \( T^*_D \) are given by

\[ T^*_U = \frac{(C_U E_D - C_D E_U)}{(C_U E_D - C_D E_U)} \]
\[ T^*_D = \frac{(C_U E_D - C_D E_U)}{(C_U E_D - C_D E_U)} \]

with

\[ C_U = [1 - (k^2/k_0^2)](\psi_{dUR} - \psi_{dR})(1 - \cos kh) - (k^2/k_0^2)\psi_{dUR} \cos kh \]
\[ C_D = \psi_{dD}(\cos \frac{k_0 h}{2}) - [1 - (k^2/k_0^2)]\psi_{dR}(1 - \cos \frac{k_0 h}{2}) + \psi_D(h) \]
\[ C_V = -[i\psi_{dI}(\cos \frac{k_0 h}{2}) + \psi_V(h)] \]
\[ E_U = -(k^2/k_0^2)\psi_{dUR} \cos kh + (1/4)\psi_{dUR} \cos \frac{k_0 h}{2} + \psi_U(h) \]
\[ E_D = -(1/4)\psi_{dD} \cos \frac{k_0 h}{2} + \psi_D(h) \]
\[ E_V = -(1/4)\psi_{dI} \cos \frac{k_0 h}{2} - \psi_V(h) \]

The \( \psi \) functions appearing in the above expressions are defined as follows:

\[ \psi_{dR} = \begin{cases} 
\psi_{dR}(0) & k_0 h \leq \pi/2 \\
\psi_{dR}(h - \lambda/4) & \pi/2 \leq k_0 h \leq 3\pi/2
\end{cases} \]
\[ \psi_{dR}(z) = \csc k(h - |z|) \int_{-h}^{h} \sin k(h - |z'|) \left[ \frac{\cos k_0r}{r} - \frac{\cos k_0r_h}{r_h} \right] dz' \]

\[ \psi_{dUR} = [1 - \cos kh]^{-1} \int_{-h}^{h} (\cos kz' - \cos kh) \left[ \frac{\cos k_0r_0}{r_0} - \frac{\cos k_0r_h}{r_h} \right] dz' \]

\[ \psi_{dD} = [1 - \cos \frac{k_0h}{2}]^{-1} \int_{-h}^{h} (\cos \frac{k_0z'}{2} - \cos \frac{k_0h}{2}) \left[ \frac{\cos k_0r_0}{r_0} - \frac{\cos k_0r_h}{r_h} \right] dz' \]

\[ \psi_{dI} = -(1 - \cos \frac{k_0h}{2})^{-1} \int_{-h}^{h} \sin k(h - |z'|) \left[ \frac{\sin k_0r_0}{r_0} - \frac{\sin k_0r_h}{r_h} \right] dz' \]

\[ \psi_{dUI} = -(1 - \cos \frac{k_0h}{2})^{-1} \int_{-h}^{h} (\cos kz' - \cos kh) \left[ \frac{\sin k_0r_0}{r_0} - \frac{\sin k_0r_h}{r_h} \right] dz' \]

where the propagation constant \( k \) is given by

\[ k = k_0 \left[ 1 + \left( \frac{14\pi \xi_0z^i}{k_0\psi_{dR}} \right) \right]^{1/2} \]

with \( \xi_0 = 376.7 \, \Omega \) the characteristic impedance of free space. However, since \( \psi_{dR} \) is dependent on \( k \), an iteration procedure is used. To begin with, \( k_1 \) as given by

\[ k_1 = k_0 \left[ 1 + \left( \frac{14\pi \xi_0z^i}{k_0\psi_{dR}} \right) \right]^{1/2} \]

is determined. With \( k_1 \) substituted for \( k \), \( \psi_{dR1} \) is computed and then \( k \) is evaluated using

\[ k = k_0 \left[ 1 + \left( \frac{14\pi \xi_0z^i}{k_0\psi_{dR1}} \right) \right]^{1/2} \]

This new value of \( k \) is used in evaluating all the \( \psi \) functions and the current distributions.
APPENDIX B

This appendix contains a listing of the main program and all associated subroutines. The main program accepts as inputs $h/\lambda_0$, $\Omega$ and $z_m$, and computes the input impedance $Z^*$, admittance $Y^*$, and magnetic current distribution as a function of distance $(z/h)$ along the antenna. Subroutine NINTC employs Simpson's rule for integration to evaluate the functions. The various integrands are calculated using the subroutines FCTH(Y), FCTO(Y) and FCT1(Y).
APPENDIX C

This appendix opens with two tables containing experimental information. The first, Table C-1, lists the values of the various antenna constants (dimensions, ferrite characteristics, frequency, etc.) for the eleven antenna configurations studied experimentally. Table C-2 gives the raw measured data (unnormalized) for the magnetic current distributions on the eleven antennas as a function of z.

The appendix concludes with a listing of the computer programs used to solve the coupled integral equations in (86a,b). The procedure used, i.e., the moment method, was discussed briefly in Section 8. The coupled integral equations are reduced to a system of linear algebraic equations which are then solved for the unknown variables. An unknown constant in the integral equation is also determined in the numerical procedure by imposing the end condition at z = h. The magnetic current $I_z^*(z)$ is easily computed from the solution of the tangential electric field $E_\phi(z)$ by using $I_z^*(z) = -2\pi a E_\phi(z)$ volts per unit current in the driving loop.
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**Table C-1** Antenna Constants
TABLE C-2  Unnormalized experimental data (Refer to Table C-1 for the numbering scheme and antenna constants).

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0065  146   CH(l, J) = CHC
0066     C   THE FIRST COLUMN ELEMENTS
         DC  50  = 2, API
0067      50   CH(l, 1) = CH(1, 1) - AA(l, 1) * C3
0068     C   THE DIAGONAL ELEMENTS
0069  146   DC  57  = 2, A
0070  146   CH(l, 1) = CH(l, 1) - AA(l, 1) * (2*J - 1) + C3
0071     C   THE ELEMENTS IN THE LOWER TRIANGLE BUT NOT IN THE FIRST
         C   COLUMN OR THE DIAGONAL.
0072  146   DC  60  = 3, ND1
0073  146   IF  KO  J = 1, ND1
0074      60   C*(1 + J) = CH(l, J) - (3*J - 1)*AA(l, 1) - (2*J - 1)
0075     C   COMPUTING K(l, J) ELEMENTS.
0076  146   DC  62  = 1, NP1
0077  146   IF  EO  J = 1, API
0078      62   CH(l, J) = CH(l, J) + CR(l, J)
0079     C   BY NOW ALL THE ELEMENTS IN REGION I ARE COMPUTED.
0080  146   CALL  UC(l, 10, 50, CH(I11, IY1))
0081  146   CALL  UC(l, 50, 50, CM(I11, IY1))
0082  146   CM(I11, IY1) = CM(I11, IY1) + CR(I11, IY1)
0083      65   CM(l11, l1) = CM(l11, l1) + 5*CR(l11, IY1)
0084  146   DC  65  = 1, API
0085      65   CM(I11, IY1) + CR(l11, IY1)
0086  146   IF  70  J = 1, API
0087      70   CM(I11, J1) + CM(I11, J1)
0088  146   IF  75  J = 1, API
0089      75   CM(l11, J1) + CM(l11, J1)
0090  146   BY NOW ALL THE ELEMENTS IN REGION II ARE COMPUTED.
0091     C   COMPUTING ELEMENTS IN REGION III.
0092  146   CALL  UC(l, 10, 50, CM(I11, IY1))
0093  146   CALL  UC(l, 50, 50, CM(I11, IY1))
0094      70   CM(l11, l1) = CM(l11, l1) + CR(l11, IY1)
0095  146   DC  70  = 1, API
0096      70   CM(I11, J1) + CM(I11, J1)
0097  146   CALL  UC(l, 11, J1, 14, IY1)
FORTran IVG LEVEL 21 MAIN DATE 7526S 14/4/31

007 CKE(1) = CPLEX(TR, 11)
008 CKE(1, 1) = CKE(1, 1) + S * CKE(11)
009 TC 85 1 = 1, N1
010 P1 = 1 + 1
011 C F 0, 1 = P1, N1
012 CALL DFLQ1(1, 1, J, E1, T1, T1)
013 CKE(1, 1) = CPLEX(TR, T1)
014 CKE(1, 1) = CKE(1, 1)
015 DC TC = 1, N1
016 CKE(1, 1) = 0.5 * CKE(11, 11)

BY NOW ALL THE ELEMENTS IN REGION III ARE COMPUTED.

019 CALL DFLQ1(1, 1, J, E2, T1, T1)
020 CKE(1) = CPLEX(E1, E2)
021 CKE(1, 1) = CKE(11, 11) + S * CKE(11, 1)
022 CKE(11, 11) = CKE(11)

BY NOW ALL THE ELEMENTS IN REGION IV ARE COMPUTED.

023 COMPUTING ELEMENTS IN REGION V

026 CKE(1) = CKE(1, 1)
027 CKE(1, 1) = CKE(11, 11) + S * CKE(11, 1)
028 CKE(11, 11) = CKE(1)

BY NOW ALL THE ELEMENTS IN REGION Y AND VI ARE DONE.

029 SETTING UP THE FINAL MATRIX FOR
0141 CALL DFCONP1M,CK,CUL)
0142 CALL 5,1V(14,CK,CUL,CX)
0143 WRITE (6,66) (CXX111,1=1,112)
0144 FORMAT (1P1,10,5)
0145 CALL IPD001(/4,CK,CUL,E1,CA,DIGITS)
0146 WRITE (6,66) (CXX11,1=1,N2)
0147 XL=2*10.5,0.5
0148 (XX11) XJ XJ X11
0149 DO 111 L=1,N1
0150 N1=1,11,AP1
0151 N7 (CXX11) X11 X11

PRINT OUT OF THE SOLUTION.

0152 L=1
0153 XL=X11AL(CXX11)
0154 X11=X11AL(CXX11)
0155 X11=X11AL(CXX11)
0156 X11=X11AL(CXX11)
0157 X11=X11AL(CXX11)
0158 X11=X11AL(CXX11)
0159 X11=X11AL(CXX11)
0159 X11=X11AL(CXX11)
0160 X11=X11AL(CXX11)
0160 X11=X11AL(CXX11)
0163 X11=1
0164 IF (L,1.T,111) GO TO 71
0164 CYSTAR=CXX1
0165 CYSTAR+11CYSTAR
0166 IF (L,1.T,111) GO TO 71
0167 CYSTAR+11CYSTAR
0167 CYSTAR+11CYSTAR
0168 HUNT (X11,A11,X11,A11,X11,A11)
0170 CONTINUE
0171 CONTINUE
SUBROUTINE BSLSPL(XERO,XYO,XYL)
C
C 'BSLSPL' COMPUTES BESSEL J FUNCTION WITH COMPLEX ARGUMENTS
C AND ACCURACY UP TO 6TH DECIMAL PLACE OR MORE IS
C OBTAINED FOR ABS(Z) LESS THAN 20.
C
IPLICIT COMPLEX*16(D), REAL*8(T)
C
COMPLEX XYO,XYL

D2Z*XYU

K=NUMOD

TX=REAL(ZZ)

TY=MAG(ZZ)

X=TX

Y=TY

TX*=50*TX

TYC=50*TY

TRX=XTY+TY

TF=2D+TX

K=K+1

L=(GLT(REAL*10.0*(X*X+Y*Y)>)=0)*ETC

TFH=1.0

TF=UT

L=(L1)*(K+1)

K=(Z+K+2)

D=40*40.41

TF=1.0+G0-TGH*TFH+G0+TF1

400 TF*=((TGH*TFH)+G0+TC)

IF (TF<.0.0) GO TO 401

IF (N+GT0) GO TO 402

CONTINUE

35

SCUSL=DCPPLX(TFR+TF1)

36

XYL=LCUSL

37

RETURN

38

402 TGH=1.0

39

G0=0.0

40

TGH=TGH

41

TGH=TCXY+TG1+TY

42

TGH=TCH+G0+TG1+TC

43

IF (TH+.0) GO TO 403

44

G0=0.0

45

G0=0.0

46

T=TX*W

47

IF (TH+.0) GO TO 403

48

G0=0.0

49

T=TX*W

50

TGH=TGH/TW

51

G0=G0/TW

52

TGH=TFR

53

TF=TFH*TGH*TF1+G0+TF1

54

TF1=GTGF1+

G0+TF1

55

G0=0.0

403

END
SUBROUTINE RESH(XYL,N,KIND,XYL,IER)
C
C RESH COMPUTES HANKEL FUNCTION WITH COMPLEX ARGUMENT.
C
IMPLICIT COMPLEX*16(9),REAL*8(T)
COMPLEX XYL
DIMENSION DT(12),CTT(6),DX(6)

IF(N.EQ.0.AND.KIND.EQ.1) GO TO 300
IF(N.EQ.4.AND.KIND.EQ.2) GO TO 400
IF(N.EQ.1.AND.KIND.EQ.1) GO TO 300
IF(N.EQ.1.AND.KIND.EQ.2) GO TO 400
300 CX*DCMPLX(0.DC*,1.DC*)
GO TO 500
400 C=DCMPLX(1.DC*,1.DC*)
500 TRX=REAL(DX)
17. T=AIMAG(DX)
18. TPT=DSCHT(TRX**2+T*X**2)
19. DBX=UCMPLX(0.DC*,G.DC*)
20. IF(N.EQ.10) T=20
21. IF(IER) RETURN
22. IER=1
23. RETURN
24. IF(N.EQ.1) X=22,22,21
25. IER=3
26. RETURN

28. IF(N.EQ.100) 36,36,25
29. C=C00*(I-DX)
30. OI=1.00/CX
31. DC=CDGT(I+6)
39. CO=CC00/CC
40. CT(I)=CTT(I-1)*DB
41. CTT(I)=CTT(I)*3.75D0
42. CO=CMPLX(0.00,0.00)
43. CT(I)=CTT(I)*CMPLX(1,1)*DTX(1)
44. IF(N.EQ.1) X=27,27,27
45. C=COMPLX(0.00,0.00)
46. DO 10 I=2,12
47. CT(I)=CTT(I-1)*DB
48. C=CMPLX(0.00,0.00)
49. GO TO 10
50. CT(I)=CMPLX(1,1)*DTX(I)
51. IF(N.EQ.1) X=2,2,2
52. IER=2
53. GO TO 20
54. C=COMPLX(0.00,0.00)
55. GO TO 20
56. 628 GO TO 00
57. C=COMPLX(0.00,0.00)
58. 628 GO TO 00
59. C=COMPLX(0.00,0.00)
60. C=COMPLX(0.00,0.00)
61. C=COMPLX(0.00,0.00)
62. C=COMPLX(0.00,0.00)
FROM KO USING RECURRENCE RELATION

FROM KG/K1 COMPUTE KN USING RECURRENCE RELATION

FROM KG/K1 COMPUTE K1 USING POLYNOMIAL APPROXIMATION

FROM KG/K1 COMPUTE K1 USING SERIES EXPANSION
10 DO J=4,6
20 GO TO 90
30 compute k1 using series expansion
40 do xj=ub
50 compute mackel function using k8 and k1
60 go to 130
70 do=2,4,0,0 do+1,0,1 do+1,0/dbk/tpi
80 go to 130
90 continue
100 return
110 end

10 subroutine go10(xl,xu,y)
20 a=s(xu-xl)
30 b=xu-xl
40 c=4.849533*b
50 y=0.3333306*(f(a+c)+f(a-c))
60 c=4.325597*b
70 y+y+0.7472567*(f(a+c)+f(a-c))
80 c=3.3970486*b
90 y+y-1.056342*(f(a+c)+f(a-c))
100 c=2.1669778*b
110 t+y+1.346334*(f(a+c)+f(a-c))
120 c=0.7437176*b
130 t+y+y+1.477621*(f(a+c)+f(a-c))
140 return
150 end
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<td>SUBROUTINE FM1I(H,L,J,K,X,T,L)</td>
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<td>0002</td>
<td>IF(MLT(CMIP,CM1P+L(C),K),X,ML)</td>
<td></td>
</tr>
<tr>
<td>0003</td>
<td>C(MP,J),T(J),LM1P,T(A),CM1P,L1M1P,T(J),LM1P,T(A),CP1P</td>
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<tr>
<td>0004</td>
<td>CY=1,<em>((1,1)</em>(X*TK'T))</td>
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<tr>
<td>0005</td>
<td>CY2=CY**2</td>
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<tr>
<td>0010</td>
<td>164=6+41/4</td>
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<td>SUBROUTINE FM211(X1,X2,X3,X4)</td>
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<tr>
<td>0002</td>
<td>T1=1.0 (COMMON TAKO,TAU,TPY,TA,CK1,CA1,TM,T,KOT,KCO,IPS)</td>
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<tr>
<td>0003</td>
<td>CY1=1.0 (COMMON CY1=1.0,XY/KOT)</td>
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<tr>
<td>0004</td>
<td>CY2=2.0</td>
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<tr>
<td>0005</td>
<td>CY3=CY4=2</td>
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<tr>
<td>0006</td>
<td>CALL ISFHC(C0,CV,CJ)</td>
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<tr>
<td>0007</td>
<td>LJ1=N.J+1</td>
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<tr>
<td>0008</td>
<td>14446+K</td>
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<tr>
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<tr>
<td>0021</td>
<td>CFK=CFK/(1+CFK*(1+CFK*(1+CFK))*CFK)</td>
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<tr>
<td>0022</td>
<td>CFK=CFK*(1+CFK)</td>
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<tr>
<td>0023</td>
<td>CFK=REAL(CFK)</td>
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<tr>
<td>0024</td>
<td>!RETURN</td>
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<td>0025</td>
<td>(END)</td>
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<td>0021</td>
<td>CFK=CFK/(1+CFK*(1+CFK*(1+CFK))*CFK)</td>
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<td>0022</td>
<td>CFK=CFK*(1+CFK)</td>
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<td>0023</td>
<td>CFK=REAL(CFK)</td>
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<td>!RETURN</td>
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<tr>
<td>001 F VX4Y(X)</td>
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<tr>
<td>0003 CX=CMGY+PX+PY</td>
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<tr>
<td>0004 VX=PX+CY+CMG</td>
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<td>0005 CY=CMG+CX+PY</td>
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<td>0006 CX=CMG+PY+CY</td>
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<td>0007 CY=CMG+CX+PY</td>
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<td>0011 RETURN</td>
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<td>0012 END</td>
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<tr>
<td>0010 VX=PX+CY+CMG</td>
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<td>9303</td>
<td>TMQT/XX/CFMXX(X,T,DEADF*)</td>
</tr>
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<td></td>
<td>CMMCN TAKO, CMUX, TPY, TA, CK1, CAK1, TH, T, TKOT, TKO, IPs</td>
</tr>
<tr>
<td>0035</td>
<td></td>
<td>CV=1.40,1.3*X/TKOT</td>
</tr>
<tr>
<td>0036</td>
<td></td>
<td>CV=TAKO+50*1L,1-(1-CY2)</td>
</tr>
<tr>
<td>0037</td>
<td></td>
<td>CALL [5,0.1, 0.1, CV, CY]</td>
</tr>
<tr>
<td>0039</td>
<td></td>
<td>CY+CY*2</td>
</tr>
<tr>
<td></td>
<td></td>
<td>CV=TAKO+50*1L,1-(1-CY2)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>TR=REAL(CFX)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>RETURN</td>
</tr>
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<td>0020</td>
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<td></td>
<td>TR=REAL(CFX)</td>
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<td>RETURN</td>
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<td>DATE = 75205</td>
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<td></td>
</tr>
<tr>
<td>0001</td>
<td>SUBROUTINE FCTK2(IN, IJ, X, T, TI)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
| 0002 | **IMPLICIT *COMPLEX** 
| 0003 | COMMON TAKO, CK0, CTH, T, CAK1, CJK1, TH, T, TK07, TK0, IPS |
| 0004 | CY = 1 + (0. + 1.1) * (X / TK07) |
| 0005 | CY2 = CY**2 |
| 0006 | CV = TAKO * CSGRT(1 + CY2) |
| 0007 | CV1 = CV |
| 0008 | CV0 = CV |
| 0009 | CALL HSLN(0, CVA, CJ0) |
| 0010 | CALL SLSN(T, 1, CV1, CJK1) |
| 0011 | CJ2 = CJ0 * CJ1 |
| 0012 | 12 = 2*MI - 2 |
| 0013 | TIP = FLCAT(12) |
| 0014 | TK07 = TK07 + T12 |
| 0015 | CJK07 = (0. + 1.1) * TK07 |
| 0016 | CF = EXP(CJK07) |
| 0017 | CALL AUX(IN, IJ, X, T, TI) |
| 0018 | C1 = COMPLX(T, 11) |
| 0019 | CFX = - CF * CJ2 * CV * C1 + 1.4 / (TA + TK07 * CY) |
| 0020 | T = IMAG(CFX) |
| 0021 | RETURN |
| 0022 | END |

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<table>
<thead>
<tr>
<th>FORTRAN IV G. LEVEL 21</th>
<th>FCTK2</th>
<th>DATE = 75205</th>
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<tbody>
<tr>
<td>0001</td>
<td>SUBROUTINE FCTK2(IN, IJ, X, T, TI)</td>
<td></td>
</tr>
</tbody>
</table>
| 0002 | **IMPLICIT *COMPLEX** 
| 0003 | COMMON TAKO, CHM, T, CAK1, CJK1, TH, T, TK07, TK0, IPS |
| 0004 | CY = 1 + (0. + 1.1) * (X / TK07) |
| 0005 | CY2 = CY**2 |
| 0006 | CV = TAKO * CSGRT(1 + CY2) |
| 0007 | CV1 = CV |
| 0008 | CV0 = CV |
| 0009 | CALL HSLN(0, CVA, CJ0) |
| 0010 | CALL SLSN(T, 1, CV1, CJK1) |
| 0011 | CJ2 = CJ0 * CJ1 |
| 0012 | 12 = 2*MI - 2 |
| 0013 | TK07 = TK07 + T12 |
| 0014 | CJK07 = (0. + 1.1) * TK07 |
| 0015 | CF = EXP(CJK07) |
| 0016 | CALL AUX(IN, IJ, X, T, TI) |
| 0017 | C1 = COMPLX(T, 11) |
| 0018 | CFX = - CF * CJ2 * CV * C1 + 1.4 / (TA + TK07 * CY) |
| 0019 | T = IMAG(CFX) |
| 0020 | RETURN |
| 0021 | END |
SUBROUTINE DECOMP(NN, A, UL)

DIMENSION SCALES(35), IPS(35)

COMPLEX A(35,35), UL(35,35), PIVOT, EM

COMMON TAKO, TPY, TKOT, IPS

N=NN

DO 5 I=1,N

IPS(I)=I

ROWNRM=0.0

DO 2 J=1,N

UL(I,J)= A(I, J)

IF(ROWNRM-CABS(UL(I,J))) 1,2,2

1 ROWNRM= CABS(UL(I,J))

2 CONTINUE

IF(ROWNRM) 13,12,13

3 SCALES(I)=1.0/ROWNRM

GOTO 6

CALL SING(I)

4 SCALES(I)=0.0

5 CONTINUE

C GAUSSIAN ELIMINATION WITH PARTIAL PIVOTING

NML=N-1

DO 17 K=1,NM1

11 IPS(K)=IPS(K)

SIZE= CABS(UL(IP,K))SCALES(IP)

IF(SIZE-BIG) 11,11,10

10 BIG=SIZE

INXPIV=1

11 CONTINUE

IF(BIG) 13,12,13

12 CALL SING(2)

GOTO 17

13 IF(INXPIV-K) 14,15,14

14 J=IPS(K)

15 IPS(K)=IPS(INXPIV)

16 IPS(INXPIV)=J

17 KP = IPS(K)

KP1=K+3

18 IF(KP1=K) 21,19,19

PIVOT=UL(KP, K)

19 RETURN

END
SUBROUTINE SOLVE(NN,UL,9,X)
DIMENSION IPS(35)
COMPLEX UL(35,35),B(35),X(35),SUM
COMMON TAKO,TYP,TKOT,IPS
N=NN
NP1= N+1
IP=IPS(1)
X(1) =B(IP)
DO 2 I=2,N
IP=IPS(I)
IM1=I-1
SUM=(0.,0.)
DO 1 J=1,IM1
1 SUM=SUM+UL(I,J)*X(J)
2 X(I)=R(IP)-SUM
IP=IPS(N)
X(N)= X(N)/UL(IP,N)
DO 4 TRACK=2,N
3 SUM=(0.,0.)
DO 3 J=1,IP1,N
3 SUM=SUM+UL(IP,J)*X(J)
4 X(I)=(X(I)-SUM)/UL(IP,I)
RETURN
END

FUNCTION ANGLE(X,Y)
IF(X)550,500,450
300 ANGLE =0.0
RETURN
550 IF(Y) 350,300,250
300 ANGLE =0.0
RETURN
450 IF(Y) 455,454,453
454 ANGLE =0.0
RETURN
550 ANGLE =-270.0+C
RETURN
250 ANGLE =-90.0
RETURN
450 IF(Y) 455,454,453
454 ANGLE =0.0
RETURN
550 ANGLE =270.0+C
RETURN
450 IF(Y) 455,454,453
454 ANGLE =0.0
RETURN
550 ANGLE =360.0
RETURN
550 XN=X
450 IF(Y) 554,553,552
553 ANGLE =180.0
RETURN
552 ANGLE =180.0-CORFF*TATAN(Y/X)
RETURN
552 ANGLE =180.0+CORFF*TATAN(-Y/X)+360.0
RETURN
550 XN=X
450 IF(Y) 554,553,552
553 ANGLE =180.0
RETURN
552 ANGLE =180.0-CORFF*TATAN(Y/X)
RETURN
552 ANGLE =180.0+CORFF*TATAN(-Y/X)
RETURN
END
SUBROUTINE IMPRUV(NN, A, UL, R, DX, ITS)

COMMON TAK0, TPY, TKOT, TPS

COMPLEX A(35, 35), UL(35, 35), R(35), DX(35), T

CUSES ABS(), AMAX1(), ALOG10()

N = NN

EPS = 1.0E-8
ITMAX = 16

**EPS AND ITMAX ARE MACHINE DEPENDENT.**

C

XNORM = 0.0
DO 1 I = 1, N
1 XNORM = AMAX1(XNORM, ABS(A(I, I)))

IF(XNORM) 3, 2, 3

2 DIGITS = -ALOG10(EPS)
GO TO 10

C

3 DO 9 ITER = 1, ITMAX

DO 5 I = 1, N
5 SUM = 0.0

DO 4 J = 1, N
4 SUM = SUM + A(I, J) * X(J)

30 X = X(I)

R(I) = SUM

C **IT IS ESSENTIAL THAT A(I, J) * X(J) YIELD A DOUBLE PRECISION
RESULT AND THAT THE ABOVE + AND - BE DOUBLE PRECISION.**

CALL SOLVE(N, UL, R, DX)

CALL SOLVE(N, UL, R, DX)

C

DXNORM = 0.0

DO 6 I = 1, N
6 T = X(I)

X(I) = X(I) + DX(I)

DO 28 XNORM = AMAX1(DXNORM, ABS(X(I) - T))

6 CONTINUE

IF(ITER - 1) 8, 7, 8

7 DIGITS = -ALOG10(AMAX1(DXNORM/XNORM, EPS))

8 IF(DXNORM - EPS * XNORM) 10, 10, 9

9 CONTINUE

C ITERATION DID NOT CONVERGE

CALL SING(3)

10 RETURN

END
SUBROUTINE GING(LWHY)
I 11 FORMAT (54HOMATRIX WITH O ROW IN DECOMPOSE
2 12 FORMAT (54HOMATRIX WITH O COLUMN IN DECOMPOSE Z NO DIVIDE IN SOLVE
3 13 FORMAT (54HOMATRIX CONVERGENCE IN IMPV. MATRIX IS NEARLY SINGULAR
4 GO TO (1,1,3)IWHY
5 1 WRITE (6,11)
6 2 GO TO 10
7 3 WRITE (5,12)
8 4 GO TO 10
9 5 WRITE (6,13)
10 6 WRITL (6,13)
11 7 PTURN
12 8 END
APPENDIX D

BASIS FOR MORE ACCURATE COMPARISON OF THEORETICAL AND EXPERIMENTAL RESULTS

With reference to Fig. 8, the voltage $V(z)$ induced in the receiving loop is proportional to the tangential electric field, viz.,

$$V(z) = \oint \frac{\mathbf{E}}{\mathbf{S}} = -2\pi a E_z(a, z)$$

This relation, however, assumes that there is azimuthal symmetry and that the current in the receiving loop is negligible. These assumptions are investigated in detail.

**Azimuthal Symmetry.** The value of $ak_0$ for the eleven cases studied in the experiment ranges from 0.00133 to 0.01662, whereas $|ak_1|$ has values between .02 to about .41. It is well known that a loop in free space has nearly constant current if $ak_0 < 1$. If this criterion is applied, all of the eleven cases are rotationally symmetric. However, if $|ak_1| < 1$ is the criterion, rotational symmetry is clearly absent in some of the cases. Because of the unique location of the driving loop on the surface $\rho = a$, a simple analytical criterion on the radius is not possible to ensure rotational symmetry. For this reason, rotational symmetry was ensured experimentally.

As shown in Fig. D-1, a shielded loop of 3/16 in. diameter was fabricated and used to measure the voltage induced by the radial magnetic field as a function of azimuthal coordinate for eight of the eleven cases. The largest deviation from a constant value was found to be less than 5%. The three antennas not tested had lower values of $|ak_1|$ than those tested. The experimental measurements established rotational symmetry conclusively for all of the cases considered in the experimental study.

Measurements were then made of the total axial magnetic current on a
Fig D-1 Photograph showing the driven loop and the two receiving loops. a) Air core
b) Largest radius ferrite core c) Teflon core.
dielectric rod made of Teflon, as shown in Fig. D-1(c). The experimental parameters are summarized below:

Diameter of the driven loop = 1.0 inch
Diameter of the Teflon rod = 1.0 inch
Diameter of loop for axial measurements = 1.0 inch
Diameter of loop for radial measurements = 3/16 inch
Frequency = 20 MHz

The axial magnetic current distribution was measured with and without the Teflon rod present. The results are tabulated in Table D-1 and plotted in Fig. D-2. As one might expect, the dielectric rod \((\varepsilon_r \geq 2.2)\) has very little effect, and the measurements taken with it present do not differ significantly from those taken with it absent. In both cases, rotational symmetry was confirmed experimentally. The characteristics of the Teflon rod antenna are similar to those of the air-core antenna because the loop used in the experiment was small enough to act as a magnetic dipole.

**Correction to Experimental Data.** An answer was then sought to the important question, what is the voltage \(V(z)\) induced at the terminals of the receiving loop? The receiving loop is loaded by the vector voltmeter which has a nominal impedance of 100 K-ohms shunted by a 2.5 pf capacitor. The frequency in this experiment varies from 5 to 150 MHz so that the vector voltmeter impedance has a range of values. An analysis based on circuit theory can be carried out to determine accurately the voltage \(V_R(z)\) measured by the vector voltmeter. A diagram of the two coupled circuits is in Fig. D-3. The various quantities shown in the figure are:

\[V_{e^{-i\omega t}} = \text{Oscillator voltage}\]
TABLE D-1: Unnormalized experimental data of total axial magnetic current with an air core and also a teflon rod. (This data is plotted in Figure D-2).

<table>
<thead>
<tr>
<th>z (cms)</th>
<th>Air Core $ak_0 = 0.00532$</th>
<th>Teflon, $\varepsilon_r = 2.2$, $ak_0 = 0.00792$</th>
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<tr>
<td></td>
<td>Mag</td>
<td>Phase</td>
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<tr>
<td>0.3</td>
<td>11.0</td>
<td>35</td>
</tr>
<tr>
<td>0.5</td>
<td>8.4</td>
<td>35</td>
</tr>
<tr>
<td>1.0</td>
<td>4.5</td>
<td>35</td>
</tr>
<tr>
<td>1.5</td>
<td>2.4</td>
<td>35</td>
</tr>
<tr>
<td>2.0</td>
<td>1.5</td>
<td>35</td>
</tr>
<tr>
<td>2.5</td>
<td>0.98</td>
<td>35.4</td>
</tr>
<tr>
<td>3.0</td>
<td>0.65</td>
<td>35.5</td>
</tr>
<tr>
<td>3.5</td>
<td>0.45</td>
<td>35.6</td>
</tr>
<tr>
<td>4.0</td>
<td>0.33</td>
<td>35.6</td>
</tr>
<tr>
<td>4.5</td>
<td>0.245</td>
<td>36</td>
</tr>
<tr>
<td>5.0</td>
<td>0.18</td>
<td>36</td>
</tr>
<tr>
<td>5.5</td>
<td>0.15</td>
<td>36</td>
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<tr>
<td>6.0</td>
<td>0.135</td>
<td>36.2</td>
</tr>
<tr>
<td>6.5</td>
<td>0.105</td>
<td>36.2</td>
</tr>
<tr>
<td>7.0</td>
<td>0.085</td>
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<td>7.5</td>
<td>0.07</td>
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<td>0.06</td>
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<td>0.05</td>
<td>36</td>
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<tr>
<td>9.0</td>
<td>0.04</td>
<td>36</td>
</tr>
<tr>
<td>10.0</td>
<td>0.03</td>
<td>36</td>
</tr>
</tbody>
</table>
FIG. D-2 PLOT OF MAGNITUDE AND PHASE OF THE RECEIVED VOLTAGE WITH AND WITHOUT A DIELECTRIC ROD. (THE DATA IS IN TABLE D-1)
FIG. D-3  THE TWO COUPLED CIRCUITS
\[ Z_g \] = Generator impedance

\[ I_T \] = Current in the transmitting loop

\[ V_{21} = V_{12} \] = Fictitious generator due to coupling

\[ I_R \] = Current in the receiving loop

\[ Z_L \] = Vector voltmeter impedance

The two mesh equations can be written as

\[ V = I_T(Z_g + Z_g) - I_RZ_M \]  \hspace{1cm} \text{(D-1)}

\[ 0 = I_T(-Z_M) + I_R(Z_g + Z_L) \]  \hspace{1cm} \text{(D-2)}

where \( Z_M \) is the mutual impedance, \( Z_g \) the self-impedance of the two loops.

From (D-1),

\[ I_T = \frac{(V + I_RZ_M)}{(Z_g + Z_g)} \]

Substituting this into (D-2) gives

\[ 0 = \frac{(V + I_RZ_M)(-Z_M)}{(Z_g + Z_g)} + I_R(Z_g + Z_L) \]

or

\[ I_R = \frac{VZ_M}{(Z_g + Z_g)} \frac{1}{[(Z_g + Z_L) - Z_M^2/(Z_g + Z_g)]} \]

The measured voltage \( V_R(z) = I_RZ_L \) in Fig. D-3 is, therefore,

\[ V_R(z) = \frac{VZ_MZ_L}{(Z_g + Z_g)} \frac{1}{[(Z_g + Z_L) - Z_M^2/(Z_g + Z_g)]} \]

\[ = \frac{VZ_M}{Z_g \left(1 + \frac{1}{Z_R/Z_g}\right) \left(1 + \frac{1}{Z_L/Z_g}\right)} \left[\frac{1}{1 - Z_M^2/(Z_g + Z_R)(Z_g + Z_L)}\right] \]  \hspace{1cm} \text{(D-3)}
The theoretical computations from the integral equation account only for the mutual coupling and ignore the generator impedance \( Z_g \), the loading of the receiver by the vector voltmeter impedance \( Z_L \), and the change in the current in the transmitting loop due to the nearness of the receiving loop. The correction terms can be identified in (D-3) as follows:

\[
V_R(z)_{\text{measured}} = V_R(z)_{\text{calculated}} \times C_1 \times C_2 \times C_3(z) \quad (D-4)
\]

where

- \( C_1 \) = Correction due to generator impedance.
- \( C_2 \) = Correction due to the loading of the vector voltmeter.
- \( C_3(z) \) = Correction due to secondary coupling (Lenz's Law).

It is observed that the correction terms \( C_1 \) and \( C_2 \) are independent of the separation between the two loops. Since the measured \( V_R(z) \) is only relative in the present study, \( C_3(z) \) is the only correction factor which is significant. It is [from the right-hand term in (D-3)]:

\[
C_3(z) = \left[ \frac{1}{1 - \frac{Z_M^2(z)}{(Z_g + Z_r)(Z_g + Z_L)}} \right] \quad (D-5)
\]

As a first approximation the generator impedance \( Z_g \) is set equal to zero. \( Z_L \) is the impedance of the vector voltmeter transferred to the gap in the receiving loop. The impedance of the vector voltmeter is composed of a 100 k-ohm resistor shunted by a 2.5 pf capacitance, viz.,

\[
Z_{VVM} = R/(1 + j\omega CR)
\]

with \( R = 10^5 \) ohms and \( C = 2.5 \times 10^{-12} \) farads. \( Z_L \) can be calculated using

\[
Z_L = R_c \left( \frac{Z_{VVM} + j\omega C \tan \beta d}{R_c + jZ_{VVM} \tan \beta d} \right) \quad (D-6)
\]
where

\[ R_c = 50 \text{ ohms} = \text{Characteristic impedance of the line} \]

\[ \beta = \beta_0 \varepsilon_{\text{rt}}^{1/2} \]

\[ \varepsilon_{\text{rt}} = \text{Dielectric constant of Teflon FEP} \]

\[ d = \text{Distance from the vector voltmeter output terminals to the gap in the receiving loop} \]

An approximate value for \( C_3(z) \) can now be computed using (D-5) with \( Z_g \) set equal to zero. However, \( Z_M \) - the mutual impedance between two loops that are in the near zone of one another - is still unknown. An excellent analysis of the mutual impedance of two loops in air is available [8]. In the present case, however, the mutual impedance is required when the ferrite is present. If the permeability \( \mu_0 \) in King's analysis [8] is replaced by the permeability \( \mu_1 \) of the ferrite, an approximate value for the mutual impedance is obtained, viz.,

\[
Z_M(z) = (j\omega \mu_1 a^2/2)((4/(z^2 + 4a^2)^{1/2})[(2/A^2 - 1)K(\pi/2, a) - (2/A^2)E(\pi/2, a)] - j\pi^2 A^2 k_1^3/3)
\]

(D-7)

where

\( \omega = \text{Angular frequency; } k_1 = \text{Propagation Constant in the ferrite} = \omega \mu_1 / \varepsilon_1 \)

\( \mu_1 = \text{Permeability of the ferrite; } \varepsilon_1 = \text{Permittivity of the ferrite} \)

\( a = \text{Radius of the two loops} \)

\( z = \text{Distance separating the two loops} \)

\( A^2 = \sin^2 \alpha = 4a^2/(z^2 + 4a^2) \)

\[
K(\pi/2, a) = \int_0^{\pi/2} \frac{d\psi}{(1 - \sin^2 \alpha \sin^2 \psi)^{1/2}} = \text{Elliptic integral of the first kind}
\]
and

\[ F(\pi/2, \alpha) = \int_0^{\pi/2} \left( 1 - \sin^2 \alpha \sin^2 \psi \right)^{1/2} d\psi = \text{Elliptic integral of the second kind} \]

Fortran IV computer programs for computing the elliptic functions were available in the Scientific Subroutine Pack of IBM 360. Thus, the correction factor \( C_3(z) \) was computed as a function of the separation distance \( z \) using equations (D-5), (D-6) and (D-7). This factor was then used to correct the experimental data for a comparison with the theoretical results. A typical comparison of the theory with corrected and uncorrected experimental data is shown in Fig. D-4 for the specific case of antenna \( \theta 6 \). The uncorrected experimental data depart from the theoretical curve near the driving point, 0 < \( z/h < .25 \). The vector voltmeter impedance \( Z_L \) at the gap for the antenna configuration under consideration (antenna \( \theta 6 \)) is \( Z_L = 2.88 - j534.7 \) ohms.

The corrected experimental data obtained when this value of \( Z_L \) is used to compute the correction factor are plotted in Fig. D-4 and are seen to deviate less from the theory than the uncorrected values. The correction factor does not account for the entire discrepancy, however. This is in part because of the approximations made in computing the correction factor and in part because of the lack of an accurate value for \( Z_L \). For this reason, the correction factor was also computed for a range of real and imaginary parts of \( Z_L \). Two representative cases are shown in Fig. D-4. They illustrate that a precise knowledge of \( Z_L \) could improve the accuracy of the correction factor applied to the experimental data and, thus, minimize the discrepancy with theory near the driving point. Away from the driving point (\( z/h > .25 \)) the agreement is seen to be very good.
FIG. D-4 MAGNITUDE OF THE VOLTAGE RECEIVED BY THE MEASURING LOOP
ACKNOWLEDGMENT

The authors wish to thank Dr. G. S. Smith and Mr. Neal Whitman for their help in the design and construction of the experimental apparatus. Thanks are also due Dr. D. H. Preis for painstakingly reading the manuscript and Ms. Margaret Owens for her assistance in the preparation of this report.

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