A STUDY OF RENEWAL PROCESSES WITH IMRL AND DFR INTERARRIVAL TIMES

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MARK BROWN

TECHNICAL REPORT NO. 6
JUNE 7, 1977

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A Study of Renewal Processes with IMRL and DFR Interarrival Times

Section 1.

Introduction and Summary.

A random variable X is said to have an IMRL (increasing mean residual life) distribution on $[0,\infty)$ if $\Pr(X < 0) = 0$, $\Pr(X > t) > 0$ for all t, $EX < \infty$, and $E(X-t|X > t)$ is increasing in $t \geq 0$. A random variable X is said to have a DFR (decreasing failure rate) distribution on $[0,\infty)$ if $\Pr(X < 0) = 0$, $\Pr(X > t) > 0$ for all t, and the conditional distribution of $X-t$ given $X > t$ is stochastically increasing in $t \geq 0$.

Consider a renewal process with interarrival time distribution F, an IMRL distribution on $[0,\infty)$. Let $M(t)$ be the expected number of renewals in $[0,t]$ including an initial renewal at time zero. If $\mu_2 = \int_0^\infty x^2 dF(x) < \infty$ define $U(t) = t/\mu_1 + \mu_2/2\mu_1^2$. We show (theorem 4) that if $\mu_{k+2} = \int_0^\infty x^{k+2} dF(x) < \infty$ for an integer $k \geq 0$ then:

$$U(t) \geq M(t) \geq U(t) - \min_{0 \leq i \leq k} c_i t^{-i} . \tag{1}$$

In (1) c_i is an explicitly computed function of $\mu_1, \ldots, \mu_{i+2}$, $i = 0, \ldots, k$. Moreover the numbers $v_j = c_{j+1}/c_j$ are increasing and for $v_{j-1} \leq t \leq v_j$, $c_j t^{-j} = \min_{0 \leq i \leq k} c_i t^{-i}$; thus for $t \in [v_{j-1}, v_j]$ the lower bound in (1) reduces to $U(t) - c_j t^{-j}$.

We also show (theorem 5) that if $\psi_F(a_o) = \int_0^\infty e^{a_o t} dF(t) < \infty$ for an $a_o > 0$ then for $0 < a \leq a_o$:

(2) $$U(t) \geq M(t) \geq U(t)-(e^{at}-1)^{-1}[(\mu_1 a)^{-1}-(\mu_2/2\mu_1^2)-(\psi_F(a)-1)^{-1}]$$

where as before $U(t) = t/\mu_1+\mu_2/2\mu_1^2$.

The bounds (1) and (2) give intervals for $M(t)$ whose lengths rapidly approach 0 as $t \to \infty$.

We also derive several monotonicity results for IMRL and DFR renewal processes (theorems 2 and 3). If F is IMRL then the expected forward recurrence time $EZ(t)$ is increasing in t as is $M(t)-t/\mu_1$, and $M(t+h)-M(t) \geq h/\mu_1$ for all $t \geq 0$, $h > 0$. If F is DFR then $A(t)$ (the renewal age at time t) and $Z(t)$ are stochastically increasing in t; $N(t+h)-N(t)$ is stochastically decreasing in t for all $h > 0$; $M(t+h)-M(t) \downarrow h/\mu_1$ as $t \to \infty$; $M(t)$ is concave and if F is absolutely continuous then the renewal density decreases to μ_1^{-1} as $t \to \infty$.

Our results follow from a representation theorem (theorem 1) for stationary IMRL renewal processes, and general delayed DFR renewal processes. This representation seems to be an ideal tool for the study of IMRL and DFR renewal processes.

The theory of DFR distributions is developed in Barlow, Marshall and Proschan [5], Barlow and Marshall ([2],[3]), Barlow [1], and Barlow and Proschan [4]. IMRL processes are studied by Bryson and Siddiqui [6], and Haines and Singpurwalla [8].

Section 2.

Definitions.

A probability distribution F is said to be DFR on $[0,\infty)$ if $F(0^-) = 0$, $F(t) < 1$ for all t, and $\bar{F}(t+s)/\bar{F}(t)$ is increasing in $t \geq 0$ for all $s > 0$ ($\bar{F}(x)=1-F(x)$). By increasing (decreasing) we mean monotone non-decreasing (non-increasing). If $F(0^-) = 0$, $F(t) < 1$ for all t, and F is absolutely continuous with pdf f, then F is DFR on $[0,\infty)$ if and only if there exists a version of f for which $h(x) = f(x)/\bar{F}(x)$ is decreasing ([5]p.378). The function h is called the hazard function.

If $\mu_1 = \int_0^\infty x dF(x) < \infty$ then define $G(x) = \mu_1^{-1} \int_0^x \bar{F}(y)dy$. G is the stationary forward recurrence time and renewal age distribution for a renewal process with interarrival time distribution F. A probability distribution F is said to be IMRL on $[0,\infty)$ if $F(0^-) = 0$, $\bar{F}(t) > 0$ for all t, $\mu_1 < \infty$, and $E(X-t|X > t) = \mu_1\bar{G}(t)/\bar{F}(t)$ is increasing in t. Since $\bar{F}(t)/\mu_1\bar{G}(t)$ is the hazard function of the distribution G we see that F IMRL $\Longleftrightarrow$ G DFR. Also note that if F is DFR and $\mu_1 < \infty$ then F is IMRL. It is easy to construct examples for which F is IMRL but not DFR.

Let $X_0 = 0$, and $X_1, X_2, \ldots$ be i.i.d. with distribution F. Define $S_0 = 0$, $S_n = \sum_1^n X_i$, $n=1,2,\ldots$, $N(t) = \{\#S_i \leq t\} = \min\{i:S_i > t\}$, and $M(t) = EN(t)$. Define $\bar{F}(t) = 1-F(t)$, $q = \bar{F}(0)$, $\mu_k = \int_0^\infty x^k dF(x)$ $L(t) = M(t)-t/\mu_1-\mu_2/2\mu_1^2$, $Z(t) = \sum_1^{N(t)} X_i - t$ (the forward recurrence time at t) and $A(t) = t - \sum_1^{N(t)-1} X_i$ (the renewal age at t). The process

$\{N(t),\ t \geq 0\}$ is called an ordinary F renewal process. If $X_1', X_2', \ldots$ are independent, $\Pr(X_1' > t) = \int_0^\infty (\overline{F}(t+y)/\overline{F}(y))dH(y)$ where H is a probability distribution on $[0,\infty)$, $X_j' \sim F$ for $j \geq 2$, $S_n' = \sum_1^n X_i'$, $N'(t) = \{\# S_i' \leq t\} = \max\{i\colon S_i' \leq t\}$, then $\{N'(t),\ t \geq 0\}$ is called a delayed F renewal process with initial age distribution H. For a delayed F renewal process $Z'(t)$, $Z'(t)$, $N'(t)$, $M'(t)$, X_i', S_i' will denote the anologues of $Z(t)$, $A(t)$, $N(t)$, $M(t)$, X_i, S_i. A delayed F renewal process with initial age distribution $G(x) = \mu_1^{-1} \int_0^x \overline{F}(y)dy$ is called a stationary F renewal process. A stationary F renewal process satisfies $EN'(t) = t/\mu_1$ and $Z'(t) \sim A'(t) \sim G$ for all $t \geq 0$.

Section 3.

Representations.

We will either assume F IMRL on $[0,\infty)$ or DFR on $[0,\infty)$. We will construct two dependent renewal processes. Process 1 will be an ordinary F renewal process. Process 2 will be a stationary F renewal process in the IMRL case, and delayed F renewal process with arbitrary initial age distribution in the DFR case. The special feature of process 2, under this construction, is that $S_{N+i-1} = S'_i$, $i=1,2,\ldots$ for a random integer N (the distribution of N will depend on the initial age distribution). Processes 1 and 2 differ only in that process 2 has zero renewals in $[0,S_N)$ while process 1 has N renewals in this interval. The simple nature of this difference is exploited to obtain our results.

The construction is based on a simple idea which is obscured by the details of the construction and proof. We decompose the hazard into 2 components, the first component causing failure for both processes, the second component only causing failure for process 2. The construction uses the following lemma:

Lemma 1. Let X be distributed as F where F will either be assumed IMRL on $[0,\infty)$ or DFR on $[0,\infty)$. Set $\overline{K}(t) = \overline{G}(t)$ in the IMRL case and $\overline{K}(t) = \int_0^\infty (\overline{F}(t+y)/\overline{F}(y))dH(y)$, where H is an arbitrary probability distribution on $[0,\infty)$, in the DFR case. Define $\overline{K}_v(t) = \overline{K}(t+v)/\overline{K}(v)$, $\overline{J}_v(t) = \overline{F}(t)/\overline{K}_v(t)$. Then $\overline{J}_v$ is the survival function of a perhaps defective distribution on $[0,\infty)$.

Proof. In the IMRL case $\overline{J}_v(t) = (\overline{F}(t)/\overline{G}(t))(\overline{G}(t)/\overline{G}(t+v))\overline{G}(v)$, and since G is DFR both $\overline{F}(t)/\overline{G}(t)$ and $\overline{G}(t)/\overline{G}(t+v)$ are decreasing. Thus $\overline{J}_v$

is decreasing. In the DFR case $\bar{J}_v(t) = \bar{K}(v)[\int_0^\infty (\bar{F}(t+v+y)/\bar{F}(t))(\bar{F}(y))^{-1} dH(y)]^{-1}$. Since F is DFR the denominator is increasing, and then since the numerator is constant, $\bar{J}_v$ is decreasing. Thus in both cases $\bar{J}_v$ is decreasing. In addition $\bar{J}_v$ is right continuous, equals 1 for $t < 0$ and is always between 0 and 1. It is thus the survival function of a perhaps defective distribution on $[0,\infty)$. ||

We proceed with the construction. Again $\bar{K}(t) = \bar{G}(t)$ when F is assumed IMRL, and $\bar{K}(t) = \int_0^\infty (\bar{F}(t+y)/\bar{F}(y))dH(y)$ with H an arbitrary probability distribution on $[0,\infty)$, when F is assumed DFR.

Construct Z_1 and W_1 independent with $Z_1 \sim K$, $W_1 \sim J$ where $\bar{J}(t) = \bar{F}(t)/\bar{K}(t)$. If $Z_1 \leq W_1$ set $X_1 = X_1' = Z_1$ and $X_j = X_j' = Y_{j-1}$, $j=2,3,\ldots$ where $\{Y_i, i \geq 1\}$ is an i.i.d. sequence with distribution F independent of (Z_1,W_1). If $Z_1 > W_1$, set $X_1 = W_1$ and go to stage 2. At stage 2 construct Z_2 and W_2 conditionally independent of each other and of (Z_1,W_1) given W_1, with $Z_2|W_1 = v$ having distribution $\bar{K}_v(t) = \bar{K}(t+v)/\bar{K}(v)$, and $W_2|W_1 = v$ distribution $\bar{J}_v(t) = \bar{F}(t)/\bar{K}_v(t)$. If $Z_2 \leq W_2$ then set $X_2 = Z_2$, $X_1' = W_1+Z_2$ and $X_j = X_{j-1}' = Y_{j-2}$, $j=3,4,\ldots$ where $\{Y_i, i=1,2,\ldots\}$ is i.i.d. with distribution F and independent of (Z_1, W_1, Z_2, W_2). If $W_2 < Z_2$ set $X_2 = W_2$ and go to stage 3. We reach stage m if and only if $W_i < Z_i$, $i=1,\ldots,m-1$, in which case $X_i = W_i$, $i=1,\ldots,m-1$. At stage m we construct Z_m and W_m conditionally independent of each other and of $(Z_1,W_1)\cdots(Z_{m-1},W_{m-1})$ given $\sum_1^{m-1} W_i$, with $(Z_m|\sum_1^{m-1} W_i = v) \sim K_v$, $(W_m|\sum_1^{m-1} W_i = v) \sim J_v$. If $Z_m \leq W_m$ set $X_m = Z_m$, $X_1' = \sum_1^{m-1} W_i+Z_m$, $X_j = X_{j-m+1}' = Y_{j-m}$, $j=m+1,\ldots$ where $\{Y_i, i \geq 1\}$ is i.i.d. with distribution F and independent of

$(Z_1,W_1,Z_2,W_2,\ldots,Z_{m-1},W_{m-1})$. If $Z_m > W_m$ we go to stage $m+1$ and repeat.

<u>Theorem 1</u>. Under the above construction:

(i) $\{X_i, i=1,2,\ldots\}$ are i.i.d. with distribution F.

(ii) $\{X_i', i \geq 1\}$ are independent, $X_1' \sim K$, $X_j' \sim F$ for $j=2,3$.

(iii) $X_i' = X_{N+i-1}$ for $i=1,2,\ldots$ where $N = \min\{i: Z_i \leq W_i\}$ and $\Pr(N < \infty) = 1$.

<u>Proof</u>. (i) Define $N = \min\{i: Z_i \leq W_i\}$, $N = \infty$ if $W_i < Z_i$ for all i. Now $(X_i \mid N \geq i, (W_j,Z_j) = (w_j,z_j), j=1,\ldots,i-1) \sim \min(Z_v^*,W_v^*)$ where $v = \sum_1^{i-1} w_j$, $Z_v^* \sim K_v$, $W_v^* \sim J_v$ and Z_v^* and W_v^* are independent. Since $\overline{K}_v(t)\overline{J}_v(t) = \overline{F}(t)$, $(X_i \mid N \geq i, (W_j,Z_j) = (w_j,z_j)\ j=1,\ldots,i-1) \sim F$. Moreover $(X_i \mid N=j < i) \sim Y_{i-j} \sim F$. Thus $X_i \sim F$ independent of $X_1,\ldots,X_{i-1}$. Since this holds for all i, $\{X_j, j \geq 1\}$ is i.i.d. with distribution F.

(ii) In our construction we generated $W_1,\ldots,W_N$. It will now be convenient to continue constructing W_j's for $j > N$. At stage j construct W_j to be conditionally independent of $W_1,\ldots,W_{j-1}$ given $\sum_1^{j-1} W_i$, with $(W_j \mid \sum_1^{j-1} W_i = v) \sim J_v$. We know that $\Pr(\sum_1^{\infty} W_i = \infty) = 1$ because K is DFR, so $\inf_i \Pr(W_i > t) \geq \inf_v \Pr(W_v > t) = \overline{F}(t)/\lim_{v\to\infty} \overline{K}_v(t) > 0$. Thus, given t, for almost all $\{W_i = w_i, i=1,2,\ldots\}$ we can find j so that $\sum_1^{j-1} w_i < t \leq \sum_1^{j} w_i$. Then

$$\Pr(X_1' > t \mid w_1,w_2,\ldots) = \Big(\prod_{\ell=1}^{j-1} \overline{K}_{\sum_1^{\ell-1} w_i}(w_\ell)\Big)\, \overline{K}_{\sum_1^{j-1} w_i}\Big(t - \sum_1^{j-1} w_i\Big) = \overline{K}(t) .$$

Thus $X_1' \sim K$.

(iii) Since $X_1' \sim \sum_1^N X_i$ and process 1 can have only finitely many renewals in a finite time interval, and since $\Pr(X_1' < \infty) = 1$, it follows that $\Pr(N < \infty) = 1$. $\|$

Section 4.

Some properties of IMRL and DFR renewal processes.

Theorems 2 and 3 below present several properties of IMRL and DFR renewal processes. Theorem 2 is extensively used in section 5 to obtain our bound for $M(t)$.

We will need a simple result which is well known, but for which we have no reference.

Lemma 2. Let F be a distribution on $[0,\infty)$. If $\mu_1 < \infty$ define $\bar{F}_1(t) = \int_t^\infty \bar{F}(x)dx$; if $\mu_2 < \infty$ define $\bar{F}_2(t) = \int_t^\infty \bar{F}_1(x)dx$. Then:

(i) $\mu_k < \infty \Rightarrow t^k\bar{F}(t) \to 0$ as $t \to \infty$; for $k \geq 0$, $\mu_{k+1} < \infty \Rightarrow \int_0^\infty t^k\bar{F}(t)dt = \mu_{k+1}/k+1 < \infty$ and $t^k\bar{F}_1(t) \to 0$ as $t \to \infty$; for $k \geq 0$, $\mu_{k+2} < \infty \Rightarrow \int_0^\infty t^k\bar{F}_1(t)dt = \mu_{k+2}/(k+1)(k+2) < \infty$ and $t^k\bar{F}_2(t) \to 0$ as $t \to \infty$; for $k \geq 0$, $\mu_{k+3} < \infty \Rightarrow \int_0^\infty t^k\bar{F}_2(t)dt = \mu_{k+3}/(k+1)(k+2)(k+3) < \infty$.

(ii) For $a > 0$, $\psi_F(a) = \int_0^\infty e^{at}dF(t) < \infty$ implies $e^{at}\bar{F}(t)$, $e^{at}\bar{F}_1(t)$ and $e^{at}\bar{F}_2(t)$ converge to 0 as $t \to \infty$. Moreover $\int_0^\infty e^{at}\bar{F}(t)dt = a^{-1}(\psi_F(a)-1) < \infty$, $\int_0^\infty e^{at}\bar{F}_1(t)dt = a^{-2}(\psi_F(a)-a\mu_1-1) < \infty$, and $\int_0^\infty e^{at}\bar{F}_2(t)dt = a^{-3}(\psi_F(a)-a^2(\mu_2/2)-a\mu_1-1) < \infty$.

Proof. The equalities between integrals follow from interchanging the order of integration. The convergence to zero follows from the equality between the integrals in each pair and integration by parts. ||

Theorem 2. Consider an ordinary F renewal process with F IMRL. Then:

(i) $M(t)-t/\mu_1$ and $EZ(t)$ are increasing in $t \geq 0$; $M(t+h)-M(t) \geq h/\mu_1$

for all $t \geq 0$, $h > 0$ and converges to h/μ_1 as $t \to \infty$; if $\mu_2 < \infty$ then $L(t) = M(t)-t/\mu_1-\mu_2/2\mu_1^2 \uparrow 0$ as $t \to \infty$.

(ii) If $\mu_2 < \infty$ then $0 \geq L(t) \geq -\mu_1^{-1} \int_t^\infty (\bar{G}(x)-q^{-1}\bar{F}(x))dx$ $\geq -\mu_1^{-1} \int_t^\infty \bar{G}(x)dx$, where $q = \bar{F}(0)$.

(iii) For $k \geq 0$, $\mu_{k+2} < \infty \Rightarrow \lim_{t\to\infty} t^k L(t)=0$ and $\mu_{k+3} < \infty$ $\Rightarrow 0 \geq \int_0^\infty t^k L(t)dt > -\infty$; For $a > 0$, $\psi_F(a) = \int_0^\infty e^{at}dF(t) < \infty$ $\Rightarrow \lim_{t\to\infty} e^{at}L(t) = 0$ and $0 \geq \int_0^\infty e^{at}L(t)dt > -\infty$.

(iv) If $\mu_2 < \infty$ and h is measurable, bounded, and $\lim_{t\to\infty} h(t) = 0$ then $\lim_{t\to\infty} [\int_0^t h(t-x)dM(x)-\mu_1^{-1} \int_0^t h(x)dx] = 0$.

<u>Proof</u>. Recall that process 1 is an ordinary F renewal process, process 2 a stationary F renewal process, and $S_i' = S_{N+i-1}$, $i=1,\ldots$. Note that by Wald's identity $EN = \mu_2/2\mu_1^2$ whether or not μ_2 is finite.

(i) $M(t)-t/\mu_1 = E(N(t)-N'(t))$. Since $N(t)-N'(t) \uparrow N$ $M(t)-t/\mu_1 \uparrow EN = \mu_2/2\mu_1^2$, by the monotone convergence theorem. Thus if $\mu_2 < \infty$ then $0 \geq L(t) \uparrow 0$ as $t \to \infty$. Since $Z(t) = \sum_1^{N(t)} X_i-t$ it follows from Wald's identity that $EZ(t) = \mu_1(M(t)-t/\mu_1)$, thus $EZ(t)\uparrow$. Since $M(t+h)-M(t)-h/\mu_1 = (M(t+h)-(t+h)/\mu_1)-(M(t)-t/\mu_1)$ and $M(x)-x/\mu_1$ is increasing, $M(t+h)-M(t) \geq h/\mu_1$. The convergence of $M(t+h)-M(t)$ to h/μ_1 can be proved directly from the construction, but we will simply appeal to Blackwell's theorem noting that if F is a lattice distribution with period ω then $E(X-\omega/2|X > \omega/2) = E(X-\omega/2|X > 0)$ $= E(X|X > 0)-\omega|2 < E(X|X > 0)$ so F is not IMRL.

(ii) By the argument in (i) $L(t) = \mu_1^{-1}E(Z(t)-Z'(t))$. By construction

$$0 \geq Z(t)-Z'(t) = \begin{cases} 0 & \text{if } t \geq X_1' \\ Z(t)-(X_1'-t) & \text{if } t < X_1' \end{cases} .$$

Thus $\mu_1 L(t) = \overline{G}(t)\, E(Z(t)|X_1' > t) - \int_t^\infty \overline{G}(x)dx$. Since $Z(t)|X_1' > t$ is a mixture of distributions of the form $X-v|X > v$ where $X \sim F$ which is IMRL, $E(Z(t)|X_1' > t) \geq E(X|X > 0) = q^{-1}\mu_1$. Thus

$$0 \geq L(t) \geq -\mu_1^{-1} \int_t^\infty (\overline{G}(x)-q^{-1}\overline{F}(x))dx \geq \mu_1^{-1} \int_t^\infty \overline{G}(x)dx .$$

(iii) These results follow from the inequality $0 \geq L(t) \geq -\mu_1^{-1} \int_t^\infty \overline{G}(x)dx$ and lemma 2.

(iv) Consider $\int_0^t h(t-x)dM(x)-\mu_1^{-1} \int_0^t h(x)dx = \int_0^\infty h(t-x)I(x \leq t)$. $d(M(x)-x/\mu_1)$, where $I(x \leq t) \begin{cases} 1 & \text{if } x \leq t \\ 0 & \text{if } x > t \end{cases}$.

By assumption the integrand converges to zero. By part (i) of this theorem $M(x)-x/\mu_1 \uparrow \mu_2/2\mu_1^2$ as $x \to \infty$, thus if $\mu_2 < \infty$ $M(x)-x/\mu_1$ has total finite variation on $[0,\infty)$ equal to $\mu_2/2\mu_1^2-q^{-1}$. Thus $h(t-x)I(x \leq t)$ is dominated by $\sup_s|h(s)|$, and $\int_0^\infty |\sup_s h(s)|d(M(x)-x/\mu_1)$ $= (\sup_s|h(s)|)(\mu_2/2\mu_1^2-q^{-1}) < \infty$. The result follows from the dominated convergence theorem. ||

<u>Theorem 3</u>. Let F be DFR on $[0,\infty)$. Then:

(i) $A(t)$ and $Z(t)$ are stochastically increasing in t.

(ii) $N(t+h)-N(t)$ is stochastically decreasing in t and $M(t+h)-M(t) \downarrow h/\mu_1$ as $t \to \infty$.

(iii) $M(t)$ is concave.

(iv) If F is absolutely continuous then $m(t) \downarrow \mu_1^{-1}$ as $t \to \infty$, where m is the renewal density function.

<u>Proof</u>. (i) Give process 2 initial age distribution $H = F_{A_s}$ where F_{A_s} is the age distribution of process 1 at time s. Then since $A'(t) \geq A(t)$ and $Z'(t) \geq Z(t)$ for all t by construction, $\Pr(A(t+s) > a = \Pr(A'(t) > a) \geq \Pr(A(t) > a)$ and similarly for $Z(t)$.

(ii) $\Pr(N(t+h)-N(t) > k) = \int_{x=0}^{h} dF_{Z(t)}(x) \cdot \Pr(N(h-x) > k)$. Since $Z(t)$ is stochastically increasing and $\Pr(N(h-x) > k)$ is decreasing in x, $\Pr(N(t+h)-N(t) > k)$ is decreasing in t. Since $N(t+h)-N(t)$ is stochastically decreasing, $M(t+h)-M(t)$ is decreasing.

(iii) We want to show that for $0 \leq x < y$, $0 < \alpha < 1$, $M(\alpha x+(1-\alpha)y) \geq \alpha M(x)+(1-\alpha)M(y)$, equivalently that $r(x,(1-\alpha)(y-x)) \geq r(x+(1-\alpha)(y-x), \alpha(y-x))$ where $r(t,s) = (M(t+s)-M(t))/s$. Since M is right continuous (M is actually continuous) it will suffice to show that $r(t_1,k_1/n) \geq r(t_2,k_2/n)$ for all n, $0 \leq t_1 < t_1 + k_1/n \leq t_2$. consider the case $0 < k_1 \leq k_2$, the case $k_1 > k_2 > 0$ follows similarly. By part (i) of this theorem $r(t_2,k_1/n) \leq r(t_1,k_1/n)$ and $r(t_2+k_1/n,(k_2-k_1)/n) \leq r(t_2+k_1/n,1/n) \leq r(t_1+(k_1-1)/n,1/n) \leq r(t_1,k_1/n)$, thus $r(t_2,k_2/n) = (k_1/k_2)r(t_2,k_1/n)+[(k_2-k_1)/k_2]r(t_2+k_1/n,(k_2-k_1)/n)$ $\leq (k_1/k_2)r(t_1,k/n)+((k_2-k_1)/k_2)r(t_1,k/n) = r(t_1,k/n)$.

(iv) Let $h(x) = f(x)/\overline{F}(x)$ the decreasing hazard function. Now $m(t) = Eh(A_t)$ with $h\downarrow$ and $A(t)$ stochastically increasing. Thus $m(t)$ is decreasing and therefore has a limit; by the elementary renewal theorem this limit must be μ_1^{-1}.

Section 5.

Bounds for $M(t)$.

Theorem. If F is IMRL on $[0,\infty)$ and $\mu_{k+2} < \infty$ for an integer $k \geq 0$ then:

$$U(t) \geq M(t) \geq U(t) - \min_{0 \leq i \leq k} c_i t^{-i} \tag{3}$$

where $U(t) = t/\mu_1 + \mu_2/2\mu_1^2$, $c_0 = \mu_2/2\mu_1^2 - q^{-1}$, and

$$0 \leq c_i = -i \int_0^\infty s^{i-1} L(s)ds = \int_0^\infty s^i d(M(s) - s/\mu_1) > -\infty \text{ for } i=1,\ldots,k .$$

The term c_i is a function of $\mu_1 \cdots \mu_{i+2}$, $i=1,\ldots,k$ which can be recursively computed from:

$$c_i = \gamma_i - \mu_1^{-1} i! \sum_{s=1}^{i-1} (c_s/s!)\lambda_{i+1-s}, \quad i=1,\ldots,k \tag{4}$$

where $\gamma_i = [\mu_{i+2}/(i+1)(i+2)\mu_1^2] - [\mu_2\mu_{i+1}/2(i+1)\mu_1^3]$

$\lambda_i = \mu_i/i!$

Equation (4) can be explicitly solved yielding:

$$c_i = \gamma_i - i! \sum_{j=1}^{i-1} (\gamma_j/j!) \sum_{\ell=1}^{i-j} (-\mu_1)^{-\ell} \sum_{(i_1,\ldots,i_\ell)\in A_{i-j,\ell}} \left(\prod_1^\ell \lambda_{i_r+1}\right) \tag{5}$$

where $A_{k,\ell} = \{(i,\ldots,i_\ell): i_r \geq 1,\ r=1,\ldots,\ell,\ \sum_1^\ell i_r = k\}$.

Proof. Since $M(t) = U(t) + L(t)$ and $L(t) \leq 0$ (theorem 2, (i)), $M(t) \leq U(t)$. Since $L(t)$ is increasing (theorem 2,(i)), $L(t) \geq L(0) = M(0) - \mu_2/2\mu_1^2 = q^{-1} - \mu_2/2\mu_1^2 = -c_0$. Thus $M(t) \geq U(t) - c_0$.

Since $L(t)$ is increasing it is at least as big as its average over $[0,t]$ with respect to any probability measure on $[0,t]$. Thus $L(t) \geq it^{-i} \int_0^t s^{i-1}L(s)ds$ which since $L \leq 0$ exceeds $it^{-i} \int_0^\infty s^{i-1}L(s)ds = -c_i > \infty$ by theorem 2 part (iii). The equivalence between $-it^{-i} \int_0^\infty s^{i-1}L(s)ds$ and $\int_0^\infty s^i d(M(s)-s/\mu_1)$ follows by integration by parts and part (iv) of theorem 2.

To identify the c_i's, start with the identity $M(t) = 1 + \int_0^t M(t-x)dF(x)$, subtract $U(t)$ from both sides, multiply both sides by t^i and use the identity $t^i-(t-x)^i = \sum_{r=0}^{i-1} \binom{i}{r}(t-x)^r x^{i-r}$. This yields:

$$t^iL(t) = \int_0^t (t-x)^iL(t-x)dF(x) + h(t) \tag{6}$$

where $h(t) = h_1(t)+h_2(t)-h_3(t)$, $h_1(t) = \int_0^t [\sum_{r=0}^{i-1} \binom{i}{r}(t-x)^r x^{i-r}] \cdot L(t-x)dF(x)$, $h_2(t) = t^i\mu_1^{-1} \int_t^\infty \bar{F}(x)dx$, $h_3(t) = t^i(\mu_2/2\mu_1^2)\bar{F}(t)$.

Now (6) is the renewal equation $g = h + g*F = h*M$, with $g(t) = t^iL(t)$. By part (iv) of theorem 2, if we can show that $h(t)$ is bounded, integrable and that $\lim_{t\to\infty} h(t) = 0$, then we can conclude that $\lim_{t\to\infty} t^iL(t) = \mu_1^{-1} \int_0^\infty h(t)dt$. But for $i=1,\ldots,k$, $\lim_{t\to\infty} t^iL(t) = 0$ by part (iii) of theorem 2. Thus the conclusion will reduce to $\int_0^\infty h(t)dt = 0$, which will provide us with a useful identity.

To show that h is bounded, integrable and convergent to zero, we do so separately for each h_i. By lemma 2, h_2 and h_3 are convergent to zero and integrable with:

(7) $$\int_0^\infty h_2(t)dt = \mu_{i+2}/(i+1)(i+2)\mu_1$$

(8) $$\int_0^\infty h_3(t)dt = \mu_{i+1}\mu_2/2(i+1)\mu_1^2$$

The boundedness of h_2 and h_3 follows from the boundedness on finite intervals and the convergence to 0 as $t \to \infty$.

Defining $L(y) = 0$ for $y < 0$ we write $h_1(t) = \int_0^\infty (\sum_{r=0}^{i-1} \binom{i}{r} x^{i-r}((t-x)^r L(t-x)))dF(x)$. Since $(t-x)^r L(t-x) \to 0$ as $t \to \infty$ by part (iii) of theorem 2, the integrand converges pointwise to 0. Moreover, since $s^r|L(s)|$ is bounded on finite intervals and converges to 0, $\sup_s s^r|L(s)| < \infty$, thus the integrand is dominated by the integrable function $\sum_{r=0}^{i-1} \binom{i}{r}(\sup_s s^r|L(s)|)x^{i-r}$. Thus by the dominated convergence theorem $h_1(t) \to 0$. The above argument also shows that $|h_1(t)| \leq \sum_{r=0}^{i-1} \binom{i}{r}\mu_{i-r}(\sup_s s^r|L(s)|) < \infty$, thus h_1 is bounded. h_1 is integrable by part (iii) of theorem 2 since $\int_0^\infty |h(t)|dt = \sum_{r=0}^{i-1} \binom{i}{r}\mu_{i-r} \int_0^\infty s^r|L(s)|ds < \infty$. Moreover:

(9) $$\int_0^\infty h_1(t)dt = \sum_{r=0}^{i-1} \binom{i}{r}\mu_{i-r} \int_0^\infty s^r L(s)ds = -i! \sum_{r=1}^{i} (c_r/r!)\lambda_{i+1-r}$$

The identity $\int_0^\infty h(t)dt = 0$ is equivalent to $-\int_0^\infty h_1(t)dt = \int_0^\infty h_2(t)dt - \int_0^\infty h_3(t)dt$. Using (7), (8), and (9) this gives us $i! \sum_{r=1}^{i} (c_r/r!)\lambda_{i+1-r} = \mu_1\gamma_i$ which reduces to (4).

Define $d_0 = 0$, $d_i = c_i/i!$, $i=1,\ldots,k$, $\delta_0 = 0$, $\delta_i = \gamma_i/i!$, $i=1,\ldots,k$, $\beta_0 = 0$, $\beta_i = \mu_1^{-1}\lambda_{i+1}$, $i=1,\ldots,k$. Then rewrite (4) as:

$$d_i = \delta_i - \sum_{s=0}^{i} d_s \beta_{i-s} . \tag{10}$$

(10) is a discrete renewal equation. Its solution is $d_i = \sum_{j=0}^{i} \delta_j M_{i-j}$ where $M_i = \sum_{s=0}^{\infty} \beta_i^{(s)}$, where $\beta^{(s)}$ is the s-th convolution of β. Since $\beta_0 = 0$, $\beta_i^{(s)} = 0$ for $s > i$. We thus obtain $d_i = \sum_{j=1}^{i} \delta_j M_{i-j} = \delta_i + \sum_{j=1}^{i-1} \delta_j M_{i-j}$ $= \delta_i + \sum_{j=1}^{i-1} \delta_j \sum_{(i,\ldots,i_\ell)\in A_{i-j,\ell}} (\prod_1^{\ell} \beta_{i_r})$, which is equivalent to (5). ||

At first glance it may seem that we need to compute $c_i t^{-i}$ for several i in order to compute $\inf_i c_i t^{-i}$. Fortunately this is not the case.

<u>Lemma 3</u>. Assume that F is IMRL on $[0,\infty)$. Then either F is quasi-exponential $(\overline{F}(t) = qe^{-\lambda t}, 0 < q \leq 1, \lambda > 0, t \geq 0)$ in which case $c_i \equiv 0$ or F is not quasi-exponential in which case $c_i > 0$ for all i. Define $v_{-1} = 1$, $v_i = c_{i+1}/c_i$, $i=0,1,2,\ldots$, with $0/0 = 0$, $\infty/\infty = \infty$. Then v_i is increasing and for $v_{j-1} \leq t \leq v_j$, $c_j t^{-j} = \inf_{i=0,1,\ldots} c_i t^{-i}$.

<u>Proof</u>. If $\mu_2 = \infty$ then $\int_0^\infty d(M(t)-t/\mu_1) = \infty$, so $c_i \equiv \infty$, $i=0,1,\ldots$. If $\mu_2 < \infty$ then by theorem 2 part (ii) $L(t)$ is the distribution function of a positive measure. Thus $c_i = 0$ if and only if L is a constant. Since $\lim_{t\to\infty} L(t) = 0$, that constant must be 0. Let $Z'(t)$ denote the forward recurrence time of the stationary process at time t. By our construction $Z'(t) \geq Z(t)$ with equality for all t with probability one iff $X_1' \sim X|X > 0$. Since $L(t) = \mu_1^{-1}E(Z(t)-Z'(t))$ we see that $c_i = 0 \Longleftrightarrow \overline{G}(t) = q^{-1}\overline{F}(t) \Longleftrightarrow h^*(t) \equiv q\mu_1^{-1}$ where h^* is the hazard

function for the distribution G. But, under this last condition $\overline{G}(t) = e^{-q\mu_1^{-1}t}$, so $\overline{F}(t) = qe^{-q\mu_1^{-1}t}$, thus F is quasi-exponential.

Next consider the Hilbert space $L^2[[0,\infty),\beta,L]$ where β is the collection of Borel sets on $[0,\infty)$. Recall again that L is the distribution function of a positive measure. Now

$$c_j^2 = (t^{(j+1)/2}, t^{(j-1)/2}) \leq \|t^{(j+1)/2}\|^2 \, \|t^{(j-1)/2}\|^2$$
$$= c_{j-1}c_{j+1}$$

by Schwartz's inequality. Note that the inequality trivially holds in the quasi-exponential case, and also holds in the alternative case since $c_{j-1} > 0$ and $c_j = \infty \Rightarrow c_{j+1} = \infty$. The inequality is equivalent to $v_{j+1} = c_{j+1}/c_j \geq c_j/c_{j-1} = v_j$ (again $0|0 = 0$, $\infty|\infty = \infty$). Thus the v_i's are increasing.

We next show that $t \leq v_j$ implies $c_j t^{-j} \leq c_{j+m} t^{-(j+m)}$ for $m=1,2,\ldots$. Thus for $t \leq v_j$, $c_j t^{-j}$ is better to use than $c_k t^{-k}$ with $k > j$. A similar argument which we delete shows that $t \geq v_{j-1}$ implies $c_j t^{-j} \leq c_k t^{-k}$ for $k < j$. Together they show that $c_j t^{-j}$ is optimal in $[v_{j-1}, v_j]$. Suppose that $t \leq v_j$ and $m > 1$. Then $c_j t^{-j} \leq c_{j+m} t^{-(j+m)}$ if and only if

$$t^m \leq c_{j+m}/c_j = \prod_{i=0}^{m-1} v_{j+i} \; .$$

Since $t \leq v_j$ and each of the m terms of the product are equal to or greater than v_j (since $v_j \uparrow$) it follows that

$$t^m \leq \prod_{i=0}^{m-1} v_{j+i} \quad \| \; .$$

Remark. The above proof shows that if we only consider $c_0,\ldots,c_k$, where k is finite then $c_k t^{-k} = \min_{0 \le i \le k} c_i t^{-k}$ for all $t \ge v_{k-1}$.

Theorem 5. If F is IMRL on $[0,\infty)$ and $\psi_F(a_0) = \int_0^\infty e^{a_0 t} dF(t) < \infty$ for an $a_0 > 0$, then for $0 < a \le a_0$:

$$(11) \qquad U(t) \ge M(t) \ge U(t) - (e^{at}-1)^{-1}[(\mu_1 a)^{-1} - (\mu_2/2\mu_1^2) - (\psi_F(a)-1)^{-1}] .$$

Proof. The proof is very similar to that of theorem 4. Choose $a \in (0,a_0]$. Using $L(t) \le 0$, $L(t)\uparrow$ as in the proof of theorem 3 we obtain:

$$(12) \qquad U(t) \ge M(t) \ge U(t) + a(e^{at}-1)^{-1} \int_0^\infty e^{as} L(s)ds$$

where $0 \ge \int_0^\infty e^{at} L(t)dt > -\infty$ by part (iii) of theorem 2.

To evaluate $\psi_L(a) = \int_0^\infty e^{as} L(s)ds$ we start with $M(t) = 1 + \int_0^t M(t-x)dF(x)$, subtract $U(t)$ from each side and multiply both sides by e^{at}. This gives:

$$(13) \qquad e^{at} L(t) = \int_0^t e^{a(t-x)} L(t-x)dF(x) + \ell(t)$$

where

$$\ell(t) = \ell_1(t) + \ell_2(t) - \ell_3(t), \quad \ell_1(t) = \int_0^t (e^{ax}-1)e^{a(t-x)} L(t-x)dF(x),$$

$$\ell_2(t) = \mu_1^{-1} e^{at} \int_t^\infty \bar{F}(x)dx, \quad \ell_3(t) = -(\mu_2/2\mu_1^2)e^{at}\bar{F}(t) .$$

We verify the conditions of part (iv) of theorem 2 in a similar manner as in the proof of theorem 3, making heavy use of lemma 2 and part (iii) of theorem 2. The conclusion of part (iv) of theorem 2, in light of $\lim_{t \to \infty} e^{at}L(t) = 0$ (part (iii) of theorem 2 becomes:

$$(14) \qquad (\psi_F(a)-1)\psi_L(a)=(\mu_2/2\mu_1^2)[(\psi_F(a)-1)/a]-(\mu_1 a^2)^{-1}(\psi_F(a)-\mu_1 a-1) .$$

Since $\psi_F(a)-1 \neq 0$ for $a \neq 0$ we can divide both sides of (14) by $\psi_F(a)-1$ and solve for $\psi_L(a)$. This gives:

$$(15) \qquad \psi_L(a) = \mu_2/2\mu_1^2-(\mu_1 a^2)^{-1}+[a(\psi_F(a)-1)]^{-1} .$$

Substituting (15) into (12) gives us (11). ||

<u>Remark</u>. (11) and (15) will hold for $a < 0$ whether or not $\psi_F(a) < \infty$ for an $a > 0$. If $\mu_3 < \infty$ and we let $a \uparrow 0$ in (11) then we obtain $M(t) \geq U(t)-c_1 t^{-1}$ where c_1 is given in (4). In general $\mu_{k+3} < \infty$ implies $\psi_L^{(k)}(0^-)$ exists and equals $-(k+1)^{-1} c_{k+1}$. Thus $\psi_L(a)$ can be considered a generating function for the c_i's. However unless the particular form of $\psi_F(a)$ leads to a simple expression for $\psi_L(a)$, expressions (4) and (5) of theorem 3 will be preferable for computing the c_i's.

Section 6.

Improved bounds when F is DFR.

The bounds given in theorem 4, corollary 3, and theorem 5 for IMRL distributions can be improved for DFR distributions. Define $\alpha_0 = 1$, $\alpha_i = (i/i+1)^i$, $i \geq 1$, $c_i^* = \alpha_i c_i$ where c_i is given in theorem 3, and $v_i^* = c_{i+1}^*/c_i^* = (\alpha_{i+1}/\alpha_i)v_i$. Also define $g_a(t) = a(e^{at}-1)^{-1}\int_0^t se^{as}ds = (te^{at}/e^{at}-1)-a^{-1}$, and $\psi(a) = -a\psi_L(a) = [(\mu_1 a)^{-1}-(\mu_2/2\mu_1^2)-(\psi_F(a)-1)^{-1}]$.

Corollary 4. Assume that F is DFR on $[0,\infty)$. Then:

(i) If $\mu_{k+2} < \infty$ then $U(t) \geq M(t) \geq U(t) - \min_{0 \leq i \leq k} c_i^* t^{-i}$

(ii) $v_i^* \uparrow$ and for $v_{j-1}^* \leq t \leq v_j^*$, $c_j^* t^{-j} = \inf_i c_i^* t^{-i}$;

thus for $v_{j-1}^* \leq t \leq v_j^*$ the bound in (i) is given by $U(t)-c_j^* t^{-j}$.

(iii) If $\psi_F(a_0) < \infty$ for an $a_0 > 0$ then for $0 < a \leq a_0$:

$$U(t) \geq M(t) \geq U(t)-(e^{ag_a^{-1}(t)}-1)^{-1}\psi(a) \geq U(t)-(e^{at+1}-1)^{-1}\psi(a) .$$

Proof. (i) L is concave by theorem 3 part (iv). Thus $L(jt^{-j}\int_0^t s\cdot s^{j-1}ds) = L((j/j+1)t) \geq jt^{-j}\int_0^t s^{j-1}L(s)ds$ $\geq jt^{-j}\int_0^\infty s^{j-1}L(s)ds = -c_j t^{-j}$. Thus $L(t) \geq -c_j[(j+1/j)t]^{-j}$ $= c_j^* t^{-j}$. The argument now proceeds as in theorem 3.

(ii) A simple differentiation argument shows that $\alpha_{i+1}/\alpha_i \uparrow$. Since $v_{i+1}/v_i \uparrow$ by corollary 3 and $v_{i+1}^*/v_i^* = (\alpha_{i+1}/\alpha_i)(v_{i+1}/v_i)$ we see that $v_{i+1}^*/v_i^* \uparrow$.

The argument now proceeds as in corollary 3.

(iii) The concavity argument in (i) shows that $L(g_a(t)) \geq a(e^{at}-1)^{-1}\int_0^t e^{as}L(s)ds \geq -(e^{at}-1)^{-1}\psi(a)$ thus $L(t) \geq -(e^{ag_a^{-1}(t)}-1)^{-1}\psi(a)$. Since $g_a(t) \geq t-a^{-1}$ and both are increasing $g_a^{-1}(t) \geq t+a^{-1}$, thus $-(e^{ag_a^{-1}(t)}-1)^{-1}\psi(a) \geq -(e^{at+1}-1)^{-1}\psi(a)$. ||

<u>Example</u>. $f(x) = (\Gamma(1/2))^{-1}x^{-1/2}e^{-x}$, $x > 0$; this is the $\Gamma(1/2,1) = \chi_1^2/2$ distribution which is DFR ([5],p. 378). The moment bounds given in theorem 3 and corollary 4 for IMRL distributions apply, as well as the improved moment bounds for DFR distributions given in corollary 4. Using our recursive formula (4) we compute $c_0 = 1/2$, $c_1 = c_2 = 1/8$, $c_3 = 15/64$, $c_4 = 21/32$, $c_5 = 315/128$, $c_6 = 1485/128$. Next $v_{-1} = 0$, $v_0 = 1/4$, $v_1 = 1$, $v_2 = 15/8$, $v_3 = 14/5$, $v_4 = 15/4$, $v_5 = 33/7$. Denoting the lower bound given in theorem 4 by $B(t) = U(t) - \min_{0 \leq i \leq 6} c_i t^{-i}$ we obtain:

$$B(t) = \begin{cases} 2t+1, & 0 \leq t \leq 1/4 \\ 2t+3/2-(8t)^{-1}, & 1/4 \leq t \leq 1 \\ 2t+3/2-(8t^2)^{-1}, & 1 \leq t \leq 15/8 \\ 2t+3/2-(15/64t^3), & 15/8 \leq t \leq 14/5 \\ 2t+3/2-(21/32t^4), & 14/5 \leq t \leq 15/4 \\ 2t+3/2-(315/128t^5), & 15/4 \leq t \leq 33/7 \\ 2t+3/2-(1485/128t^6), & 33/7 \leq t < \infty \end{cases}$$

The lower bound given in corollary 4 which we denote by $B^*(t) = U(t) - \min_{0 \leq i \leq 6} c_i^* t^{-i}$, can be similarly written.

The table below gives a few values of t along with the corresponding intervals $[B(t),U(t)]$ and $[B^*(t),U(t)]$ for $M(t)$.

t	$(B(t),U(t))$	$(B^*(t),U(t))$
.1	[1.2,1.7]	[1.2,1.7]
.5	[2.25,2.5]	[2.375,2.5]
1	[3 .375,3.5]	[3.444,3.5]
1.5	[4.444,4.5]	[4.475,4.5]
2	[5.471,5.5]	[5.488,5.5]
3	[7.492,7.5]	[7.497,7.5]
4	[9.4976,9.5]	[9.4990,9.5]
5	[11.49926,11.5]	[11.49971,11.5]

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We consider renewal processes with IMRL (increasing mean residual life) and DFR (decreasing failure rate) interarrival times. We develop monotonicity properties for these processes as well as two sided bounds for the renewal function.

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