PARALLEL EXECUTION OF A SEQUENCE OF
TASKS ON AN ASYNCHRONOUS MULTIPROCESSOR

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June 1977

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of
COMPUTER SCIENCE

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This research is supported in part by the National Science Foundation under Grant MCS 75-222-55, the Office of Naval Research under Contract N00014-76-C-0370, NR 044-422, and a research grant from the Institut de Recherche d'Informatique et d'Automatique (IRIA), France.
ABSTRACT

Given a sequence of tasks to be performed serially, a parallel algorithm is proposed to accelerate the execution of the tasks on an asynchronous multiprocessor by taking advantage of fluctuations in the execution times.

A parallel program requiring no critical section is given to implement the algorithm and its correctness is proved. A spacewise more efficient implementation is also given but requires the use of critical sections.

An analysis is presented for both implementations to estimate the speed-up achievable with the parallel algorithm. When the execution times are exponentially distributed, and no critical section is used, the algorithm with k processes yields a speed-up of order $\sqrt{k}$. 
1. INTRODUCTION

We are interested in the design and analysis of parallel algorithms for asynchronous multiprocessors such as C.mmp [6]. For any given task, the task execution time on such a system is dependent upon the properties of the operating system, effects of other users, processor-memory interference, and many other factors. As a result, it is necessary to assume that task execution times are random variables rather than constants. In this paper we propose a novel way of using asynchronous multiprocessors, which takes advantage of fluctuations in task execution times. We shall present our result as a solution to the problem of execution a sequence of \( n \) tasks \( w_1, \ldots, w_n \) under the following conditions:

C1. For \( i = 2, \ldots, n \), task \( w_i \) cannot be started before the completion of \( w_{i-1} \) (i.e., the tasks are linearly ordered).

C2. For \( i = 1, \ldots, n \), no parallelism can be utilized in the execution of \( w_i \) (i.e., we are not allowed to decompose a task).

C3. The execution time of a task is a random variable rather than a constant. (This condition corresponds to the asynchronous nature of the multiprocessor.)

We will view a parallel algorithm for asynchronous multiprocessors as a collection of asynchronous processes which communicate among each other through the use of global variables. Such an algorithm will be defined by giving the procedure each of its processes executed when assigned to a processor. While analyzing the algorithm, we shall always assume that a processor is available for any of the runnable processes of the algorithm.
In Section 2 we give an algorithm which uses \( k \geq 1 \) asynchronous processes to solve the problem. The algorithm is interesting because at most one process is doing useful work at any given time. Nevertheless, by taking advantage of condition C3, the mean execution time is less for \( k > 1 \) than for \( k = 1 \), i.e., a speedup is achieved.

As an example, consider the computation of \( x_1, \ldots, x_n \) defined by

\[
x_{i+1} = \psi(x_i, \ldots, x_{i-d}),
\]

where \( x_0, x_{-1}, \ldots, x_{-d} \) are given and \( \psi \) is some iteration function. Let \( w_{i+1} \) be the task of computing \( \psi(x_i, \ldots, x_{i-d}) \). Our algorithm could be used to execute tasks \( w_1, \ldots, w_n \), which is equivalent to evaluating \( x_1, \ldots, x_n \).

The speed-up ratio \( S_k(n) \) of a parallel algorithm using \( k \) processes is defined in Section 3, and some preliminary results are proved there. In Section 4 we give programs to implement our algorithm both with and without using critical sections and prove informally their correctness. In Section 5 we consider the implementation without critical sections, and obtain an analytic expression for the speed-up under certain assumptions (A1 and A2 of Section 5). For large \( n \) and \( k \), our result is \( S_k(n) \sim \sqrt{\frac{2k}{n}} \).

In Section 6 we consider the implementation which uses critical sections. Here the analysis is more difficult, and we can obtain analytic results only for \( k \leq 2 \). Some conclusions and open problems are stated in Section 7.
2. THE ALGORITHM

For each positive integer k, we define an algorithm with k processes for executing tasks \( w_1, \ldots, w_n \) under conditions C1 and C2 stated in the preceding section. The algorithm is specified as follows:

Whenever a process, \( P \), is ready to execute a task,

(i) if no task has yet been completed by any process, process \( P \) starts executing task \( w_1 \),

(ii) otherwise, if the last task \( w_n \) has not yet been completed by any process, process \( P \) starts executing a task which is unfinished and ready for execution.

For simplicity, we shall assume that no two tasks are completed at the same time. Then due to the linear ordering of the tasks, (ii) defines without ambiguity a unique task to be executed by process \( P \).

Let \( t_1, t_2, t_3, \ldots \) with \( t_1 < t_{i+1} \) be the time instants of completions of tasks by the processes. The diagram in Figure 2.1 illustrates a possible scheduling of the tasks when they are executed by the algorithm with three processes.

Figure 2.1. A possible task scheduling with three processes.
Note that when process $P_3$ finishes task $w_3$ at time $t_8$, process $P_2$ has already completed $w_4$. Thus, after $P_3$ completes $w_3$, it starts executing $w_5$ rather than $w_4$. Task $w_4$ is skipped by $P_3$. Similarly, tasks $w_5$ and $w_7$ are skipped by $P_1$, and tasks $w_2$ and $w_7$ by $P_2$. After any one of the three processes has executed six tasks, tasks $w_1$ through $w_8$ rather than tasks $w_1$ through $w_6$ are completed. A speed up has been achieved!

Observe that at any given time at most one process is doing work useful for later computation. With respect to the scheduling given by Figure 2.1, the time intervals on which processes are doing useful computations are indicated in Figure 2.2.

![Diagram](image)

Figure 2.2. Time intervals on which processes are doing useful work.

Thus the speed up is not achieved by sharing work among processes but is achieved by taking advantage of fluctuations in the execution times.
3. A SPEED-UP MEASURE

Consider the algorithm with \( k \) processes as specified in the preceding section. The algorithm is said to be the sequential algorithm if \( k = 1 \) and to be a parallel algorithm if \( k > 1 \). Let

\[
T_k(n) = \text{the time to execute tasks } w_1, w_2, \ldots, w_n \text{ by the algorithm with } k \text{ processes.}
\]

Let \( \tilde{T}_k(n) \) be the mean of the random variable \( T_k(n) \). We define the speed-up ratio of the parallel algorithm with \( k \) processes to be

\[
S_k(n) = \frac{\tilde{T}_1(n)}{\tilde{T}_k(n)}.
\]

For each \( k \) and for each execution of the algorithm with \( k \) processes, we define \( s_{k,i} \) to be the time instant of the first completion of task \( w_i \), and define \( s_{k,0} = 0 \). For example, with respect to the scheduling of Figure 2.1, with \( k = 3 \), we have

\[
s_{3,1} = t_1, s_{3,2} = t_2, s_{3,3} = t_5, s_{3,4} = t_7, s_{3,5} = t_9, s_{3,6} = t_{12}, s_{3,7} = t_{13}, \ldots.
\]

The following theorem describes the relation between \( \{s_{k,i}\} \) and \( \{t_i\} \) in terms of the scheduling of the tasks. This theorem is important in sections 5 and 6 for computing speed-up ratios.
Theorem 3.1

Suppose that \( s_{k,i} = t \_j \) with \( 1 \leq i \leq n-1 \). Then \( s_{k,i+1} = t \_j \) for some \( 1 \leq j \leq k \) if and only if

(a) the \( j \) processes completing tasks at times \( t \_j, t \_j+1, \ldots, t \_j \) are all distinct, and

(b) the process completing task \( w_{i+1} \) at time \( t \_j \) is one of the \( j \) processes mentioned in (a).

Proof.

\((\Rightarrow)\) Suppose that some process \( P \) completes two tasks at times \( t \_j + h \) and \( t \_j + m \) for \( 0 \leq h < m < j-1 \). Then, since at time \( t \_j + h \) task \( w_i \) has already been completed, the task completed at time \( t \_j + m \) by process \( P \) must be \( w_{i+1} \). This contradicts the fact that \( w_{i+1} \) is completed for the first time at time \( t \_j \), since \( t \_j + m < t \_j \). This proves (a).

Let \( P \) be the process completing task \( w_{i+1} \), for the first time, at time \( t \_j \). Suppose that \( P \) does not complete any task in the time interval \( [t \_j, t \_j + j-1] \). Then the task completed by \( P \) at time \( t \_j \) must be started before time \( t \_j \). But at any time before \( t \_j \), task \( w_i \) is not completed yet. Hence any task started before time \( t \_j \) cannot be \( w_{i+1} \). In particular, the task completed by \( P \) at time \( t \_j \) cannot be \( w_{i+1} \). This contradiction proves (b).

\((\Leftarrow)\) The proof is omitted, since it is similar to the one above.

For \( i = 1, 2, \ldots, n \), let \( \tau_k(i) \) be the random variable representing the quantity \( s_{k,i} - s_{k,i-1} \). Then since \( T_k(n) = s_{k,n} \), we have
(3.1) \[ T_k(n) = \tau_k(1) + \tau_k(2) + \ldots + \tau_k(n). \]

(3.1) will be used later to compute \( T_k(n) \), which is needed for evaluating the speed-up ratio \( S_k(n) \).
4. PARALLEL PROGRAMS FOR THE ALGORITHM AND THEIR CORRECTNESS

We give two programs to implement the algorithm with k processes: one without critical sections and one with critical sections.

4.1. A Program without Critical Sections

Program A:

**global** (integer or real) array \( U[1:n] \);

**global** Boolean array \( M[1:n+1] \);

Initialization: \( \text{begin} \)

\[ \text{for } m = 1 \text{ to } n+1 \text{ do } M[m] \leftarrow \text{false}; \]

\( \text{start processes } P_1, \ldots, P_k \)

\( \text{end} \)

Process \( P_j \):

\( \text{begin integer } m_j \);

\( m_j \leftarrow 1; \)

\( (4.1) \text{ while } M[m_j] \text{ do } m_j \leftarrow m_j+1; \)

\( (4.2) \text{ while } m_j \leq n \text{ do} \)

\( \text{begin} \)

\( (4.3) \text{ perform task } w_{m_j}; \)

\( (4.4) \text{ write the output of task } w_{m_j} \text{ on } U[m_j]; \)

\( (4.5) M[m_j] \leftarrow \text{true}; \)

\( (4.6) \text{ while } M[m_j] \text{ do } m_j \leftarrow m_j+1 \)

\( \text{end} \)

\( \text{end} \)
Assume that the tasks are not allowed to alter the array M and integers m_j. We shall prove that Program A is correct in the following sense:

P1. For m = 2, ..., n, task w_m is executed only if task w_{m-1} has been finished and its output has been written on U[m-1].

P2. For j = 1, ..., k, process P_j can execute the loops at (4.1), (4.2) and (4.6) at most n times.

P3. All the tasks w_1, ..., w_m will have been completed at the time when any one of the processes P_1, ..., P_k terminates its execution.

Property P2 guarantees that the program will terminate. (Note that there is no possibility of deadlocks in the program.) Property P1 ensures that the linear ordering requirement of the executions of the tasks is maintained, and property P3 implies that when the program terminates all the tasks are completed.

Lemma 4.1

(i) For m = 1, ..., n, if M[m] is set to true, it remains true afterwards.

(ii) After being initialized to false, M[n+1] is never modified.

Proof

After initialization, M can only be modified through statement (4.5) executed by some process P_j. But when entering the main while-loop (starting at (4.2)), m_j satisfies the condition m_j ≤ n and is not modified before the execution of (4.5). Therefore M[n+1] can never be modified. ■

Lemma 4.2

For j = 1, ..., k, if m_j has the value m ≥ 2, then M[m-1] is true.
Proof

Suppose that \( m_j = m \) with \( m \geq 2 \) at time \( t \). If \( m_j \) was incremented by one to the value \( m \) inside the while statement (4.1) or (4.6), then the test of \( M[m_j] \) being \textit{true} with \( m_j = m-1 \) must have been satisfied. Hence \( M[m-1] \) was \textit{true} at some time before \( t \). Thus, by Lemma 4.1 \( M[m-1] \) is \textit{true} at time \( t \).

Lemma 4.3

For \( m = 2, \ldots, n \), if \( M[m] \) is \textit{true}, then \( M[m-1] \) is \textit{true}.

Proof

Suppose that \( M[m] \) is \textit{true}. Then \( M[m] \) must have been assigned to \textit{true} through instruction (4.5) by some process \( P_j \) with \( m_j \) having the value \( m \). Therefore, by Lemma 4.2, \( M[m-1] \) is \textit{true}.

Lemma 4.4

For \( m = 1, \ldots, n \), if \( M[m] \) is \textit{true}, then task \( w_m \) is completed and its output is on \( U[m] \).

Proof

Suppose that \( M[m] \) is \textit{true}. Then \( M[m] \) must have been assigned to \textit{true} through instruction (4.5) by some process \( P_j \) with \( m_j \) having the value \( m \). Since \( P_j \) executes instruction (4.5) only after the completion of task \( w_{m_j} \) and since \( m_j \) is not modified in between, we conclude that task \( w_m \) is completed.
We are now able to prove the following theorem.

Theorem 4.1

Program A satisfies properties P1, P2 and P3.

Proof

1. Suppose that process \( P_j \) is executing \( w_m \) with \( m = m_j \geq 2 \). Then by Lemma 4.2 \( M[m-1] \) is true, and hence by Lemma 4.4 \( w_{m-1} \) is completed and its output is on \( U[m-1] \). We conclude that program A satisfies P1.

2. Property P2 follows from (ii) of Lemma 4.1, since \( m_j \) is incremented by one in each execution of a loop.

3. Suppose that a process, say process \( P_j \), terminates. This happens only when \( m_j = m+1 \). Thus by Lemma 4.2 \( M[n] \) is true and hence by Lemma 4.3 \( M[m] \) is true for all \( m = 1, \ldots, n \). Therefore by Lemma 4.4 all tasks are completed. We have shown that Program A satisfies property P3.

Program A is very reliable in the following sense. Property P3 implies that, even if some processes fail (for reasons external to the algorithm: e.g., "crash" of the processors executing the processes) the program may still continue executing tasks and eventually complete all the tasks provided that there remains at least one active process. We shall not pursue this reliability issue any further in this paper, though we believe it is important.

4.2. A Program with Critical Sections

For problems where we are only interested in the output of the last task \( w_n \), the use of the global arrays \( U[1:n] \) and \( M[1:n+1] \) in Program A can be avoided at the expense of using critical sections.
We shall illustrate the idea with the following example. Consider the problem of generating the nth iterate \( x_n \) by \( x_{i+1} = \varphi(x_i) \) given the initial iterate \( x_0 \). Suppose that we use Program A. Then corresponding to the global array \( U[1:n] \) we have the global array \( x[0:n] \) where \( x[i] \) keeps the value of the ith iterate, and (4.3) and (4.4) become

\[
x[m_j] \leftarrow \varphi(x[m_j-1]).
\]

Note that we only need \( x[n] \). The use of the array \( x[0:n] \) is wasteful in space, and might even be impractical (e.g., when \( n \) is large and when the elements \( x[0], x[1], \ldots, x[n] \) are themselves vectors or complicated structures). The following program solves the problem:

**Program B:**

```
global integer m; global real x;

Initialization: begin
    m \leftarrow 1; x \leftarrow x_0;
    start processes \( P_1, \ldots, P_k \)
end

Process \( P_j \): begin integer \( m_j \); real \( y_j \);
    \{ \( m_j \leftarrow m; y_j \leftarrow x \} \)
    while \( m_j \leq n \) do begin
        \begin{align*}
        & \quad \text{begin} \\
        & \quad y_j \leftarrow \varphi(y_j)
        \end{align*}
        \{ \begin{align*}
        & \text{if } m_j = m \text{ then } (m \leftarrow m_j+1; x \leftarrow y_j); \\
        & (m_j \leftarrow m; y_j \leftarrow x)
        \end{align*}
    end
end```

It is crucial to assume that the statements enclosed within a pair of curly brackets (lines (4.7), (4.8) and (4.9)) are programmed as critical sections. (As a matter of fact, the two lines (4.8) and (4.9) can be programmed as one critical section.) With this assumption it is possible to prove the correctness of the above program. The proof is based on the observation that the global variable ci is a non-decreasing function of time which takes on all integer values between 1 and n+1. The proof is relatively easy and hence is omitted here.

Note that, as was already mentioned, x and y may represent a large amount of data. Hence the execution of \( x \leftarrow y \) or \( y \leftarrow x \) inside a critical section may take a significant amount of time. After presenting, in Section 5, an analysis for programs which do not have critical sections, we will give, in Section 6, an analysis for programs which do have critical sections.
5. SPEED-UP RATIOS - IMPLEMENTATIONS WITHOUT CRITICAL SECTIONS

Let $t_{1,j}$ be the random variable representing the time to execute task $w_j$ by process $P_j$. In this and the next section, we assume that the $t_{i,j}$ for $i = 1, \ldots, n$ and $j = 1, \ldots, k$, are independent and identically distributed.

The assumption is reasonable when all tasks are of the same complexity and executed by identical processors. We shall use $T$ to denote any of the random variables $t_{i,j}$, and use $\tau$ to denote the mean of $T$.

It is easy to obtain $\tilde{T}_1(n)$. By (3.1) with $k = 1$, we have

$$T_1(n) = \tau_1(1) + \tau_1(2) + \ldots + \tau_1(n).$$

Since, in this case, the $\tau_1(i)$ are independent and identically distributed with mean $\tau$, we deduce

$$\tilde{T}_1(n) = n\tau.$$ (5.1)

In the rest of the paper, in order to evaluate $\tilde{T}_k(n)$, we impose the following further assumptions:

A1. All processes start at the same time $t = 0$. (I.e., at $t_0$ all the $k$ processes start with the execution of task $w_1$.)

A2. The random variable $T$ is exponentially distributed with mean $\tau$.

We observe that by the independence of the $t_{i,j}$ and by assumption A2 the quantities $\tau_k(i)$, $i = 1, \ldots, n$, are independent random variables. It follows, from equation (3.1) and assumption A2, that

$$\tilde{T}_k(n) = \tilde{\tau}_k(1) + \ldots + \tilde{\tau}_k(n).$$ (5.2)
In addition, by assumption A1, $\tau_k(1)$ is given by the minimum of $k$ random variables distributed as $T$. Since $T$ is exponentially distributed, the minimum has the mean:

$$\tau_k(1) = \frac{1}{k}.$$  

We now consider $\tau_k(i+1)$ for $i = 1, \ldots, n-1$. Define the distribution probability $p_{k,j}$, $j = 1, 2, \ldots$, as follows. (We use here the same notation as in Section 3.) Let $p_{k,j}$ be the probability that $s_{k,i+1} = t_{k,j}$, given that $s_{k,i} = t_{k,l}$ for some $l$. Hence for $1 \leq j \leq k$, $p_{k,j}$ is the probability that conditions (a) and (b) of Theorem 3.1 hold. Using the same argument as used in the proof of Theorem 3.1, it is easy to show that $p_{k,j} = 0$ if $j > k$. In addition, assumption A2 implies that, from the memory-less property of the exponential distribution, $p_{k,j}$ is independent of $i$ and $l$. We have

$$\tau_k(i+1) = \begin{cases} 
\tau_{k+1} - \tau_l, & \text{with probability } p_{k,1}, \\
(t_{k+1} - \tau_l) + (t_{k+2} - t_{k+1}) & \text{with probability } p_{k,2}, \\
\vdots & \\
(t_{k+1} - \tau_l) + \ldots + (t_{k+2} - t_{k+k-1}) & \text{with probability } p_{k,k}.
\end{cases}$$  

Since by assumption A2 the random variables $t_{k+1} - t_{k,l}$, $l = 1, 2, \ldots$, are independent (and identically distributed) random variables with mean $\frac{1}{k}\tau$, we derive from (5.4) that, for $i = 1, \ldots, n-1$, the mean of $\tau_k(i+1)$ is given by:

$$\tau_k(i+1) = \sum_{j=1}^{k} p_{k,j} (J_k) = \frac{1}{k} \sum_{j=1}^{k} j p_{k,j}.$$
By (5.2), (5.3) and (5.5), we obtain that

\[ T_k(n) = \frac{1}{k} \tau \left[ 1 + (n-1) \sum_{j=1}^{k} j p_{k,j} \right]. \]

To evaluate \( T_k(n) \), we need to know the following quantity:

\[ N_k = \sum_{j=1}^{k} j p_{k,j}. \]

Lemma 5.1

\[ p_{k,j} = \frac{j!}{(k-j)!}, \quad \text{for } j = 1, \ldots, k. \]

Proof

We first observe that, by assumption A2, for \( \ell = 1, 2, \ldots \), any one of the \( k \) processes is equally likely to complete a task at time \( t_{\ell,\ell} \). Suppose that \( s_{k,i} = t_{\ell,\ell} \) and \( s_{k,i+1} = t_{\ell,\ell+1} \). Then by condition (a) of Theorem 3.1, the \( j \) processes completing tasks at time \( t_{\ell,\ell+1}, \ldots, t_{\ell,\ell+j-1} \) are different. This occurs with probability

\[ \frac{k!}{k^j(k-j)!}. \]

Moreover, by condition (b) of Theorem 3.1, the process completing a task at time \( t_{\ell,\ell+j} \) must be one of the \( j \) processes mentioned above. This occurs with probability \( \frac{j!}{k^j(k-j)!} \). Hence the probability that \( s_{k,i} = t_{\ell,\ell} \) and \( s_{k,i+1} = t_{\ell,\ell+j} \) is

\[ \frac{j!}{k^j(k-j)!}. \]
The problem of computing the leading terms in the asymptotic series for \( N_k \) is rather difficult. Fortunately, some known results can be used here. Define

\[
Q_k = \sum_{j=1}^{k} \frac{k!}{j^j (k-j)!}.
\]

**Lemma 5.2**

\[ N_k = Q_k. \]

**Proof**

We have

\[
N_k = \sum_{j=1}^{k} j_{n+1} = \sum_{j=1}^{k} (k-(k-j)) \frac{k}{j}.
\]

\[
= \sum_{j=1}^{k} \frac{jk!}{j^j (k-j)!} - \sum_{j=1}^{k-1} \frac{jk!}{j^j (k-j+1)!}.
\]

\[
= \sum_{j=1}^{k} \frac{jk!}{j^j (k-j)!} - \sum_{j=1}^{k} \frac{(j-1)k!}{j^j (k-j)!}.
\]

\[ = \sum_{j=1}^{k} \frac{jk!}{j^j (k-j)!}. \]

The leading terms in the asymptotic series for \( Q_k \) are known (Knuth [2, Eq. (25) in §1.2.11.3]):

\[
Q_k = \sqrt{\frac{\pi}{2k}} - \frac{1}{3} + \frac{1}{12} \sqrt{\frac{\pi}{2k}} - \frac{4}{135k} + \frac{1}{288} \sqrt{\frac{\pi}{2k^3}} + O(k^{-2}).
\]
Hence by (5.1), (5.6) and Lemma 5.2, we have the following theorem:

**Theorem 5.1**

Using $k$ processes, the speed-up ratio is given by

$$S_k(n) = \frac{nk}{1 + (n-1)N_k},$$

where

$$N_k = \sqrt{\frac{\pi}{2}k} - \frac{1}{3} + \frac{1}{12} \sqrt{\frac{\pi}{2k}} - \frac{4}{135k} + \frac{1}{288} \sqrt{\frac{\pi}{3}} \frac{1}{2k} + o(k^{-2}).$$

Asymptotically, when both $n$ and $k$ are large we obtain

$$S_k(n) \sim \sqrt{\frac{2}{\pi}} \cdot \sqrt[k]{k} \approx 0.798 \sqrt[k]{k}.$$
6. SPEED-UP RATIOS - IMPLEMENTATIONS WITH CRITICAL SECTIONS

In this section, we analyze speed-up ratios achievable by the algorithms when they are implemented with critical sections.

The diagram of Figure 6.1 illustrates a portion of a possible scheduling of the tasks by the parallel algorithm with two processes.

Figure 6.1. A possible task scheduling with two processes.

In the diagram, the marks '↑' and '→' indicate the sequences of time instants \(u_i\) and \(v_i\), \(i=1,2,...\), when a process completes a task and when the same process completes the subsequent critical section. Since, at any time, only one process can execute the critical section, a process may have to wait before entering the critical section. The periods of waiting times are indicated by the marks '-----'. The time instants \(t_i\) when processes actually enter the critical section are indicated by the marks '-----'.

As in the preceding section, we assume that the time a process takes to execute a task is a random variable independent of the process and of the task. Let \(F\) be its distribution function, and \(f\) its density function. Similarly, we assume that the time a process takes to execute the critical section is a random variable independent of the process. Let \(B\) be its distribution function and \(b\) its density function. Furthermore, let \(\tau\) and \(\beta\) denote the average execution times for a task and for the critical section, respectively.
In the following, we derive a general formula for evaluating the speed-up ratio achievable by the parallel algorithm with two processes for the case when $F$ is an exponential distribution function and $B$ is a general distribution function.

Observe that at time $t_i$ when a process enters the critical section, the second process is necessarily performing some task (possibly just starting a task). Since the distribution function $F$ is exponential, at time $t_i$ the remaining execution time for the task performed by the second process is distributed according to the same distribution function $F$. Therefore the evolution of the processes, from time $t_i$ on, is independent of the past for any distribution $B$. In particular, the random variables $t_{i+1} - t_i$, for $i = 1, 2, \ldots$, are independent and identically distributed, and the same holds for the random variables $\tau_k(i+1)$, for $i = 1, 2, \ldots$, defined in Section 3.

In this section, let $T_1(n)$ and $T_2(n)$ denote the time to complete task $w_n$ and the subsequent critical section by the sequential algorithm and the parallel algorithm with two processes, respectively. Let $\bar{T}_1(n)$ and $\bar{T}_2(n)$ denote their means. It follows from the above discussion that, for $k = 1$ and 2, we have:

$$\bar{T}_k(n) = \bar{T}_k(1) + \bar{T}_k(2) + \ldots + \bar{T}_k(n) + \beta$$

where the last term, $\beta$, accounts for the time to execute the last critical section (after the completion of task $w_n$).

Consider first the sequential algorithm. In this case, we simply have $\bar{T}_1(1) = \tau$, and, for $i = 2, \ldots, n$, $\bar{T}_1(i) = \beta + \tau$. Therefore, by equation (6.1):

$$\bar{T}_1(n) = n(\tau + \beta).$$
(Here we ignore the fact that in the sequential algorithm the critical section can be shortened, since there is no need to include synchronization primitives.)

Consider now the parallel algorithm. As (5.3), we have

$$\tau_2(1) = \frac{1}{2}\tau.$$  \hspace{1cm} (6.3)

For $j = 1$ and 2, let $p_j$ be the probability that $s_{2,i+1} = t_{j+1}$, given that $s_{2,1} = t_{\ell}$ for some $\ell$. As in Section 5, by Theorem 3.1, we obtain, for $i = 1, 2, \ldots, n-1$,

$$\tau_2(i+1) = \begin{cases} t_{j+1} - t_{\ell} & \text{with probability } p_1, \\ (t_{j+1} - t_{\ell}) + (t_{j+2} - t_{j+1}) & \text{with probability } p_2. \end{cases}$$  \hspace{1cm} (6.4)

We have already mentioned that the random variables $t_{j+1} - t_{\ell}$, $\ell = 1, 2, \ldots$, are independent and identically distributed. Let $\mu$ denote their mean. It follows from (6.4) that the mean of $\tau_2(i+1)$ is given by

$$\tau_2(i+1) = p_1 \cdot \mu + p_2 \cdot (2 - p_1) \mu,$$  \hspace{1cm} (6.5)

since $p_1 + p_2 = 1$.

The following lemma establishes the values of $\mu$ and $p_1$.

**Lemma 6.1**

$$\mu = B + \frac{T}{2} B^*(1/\tau),$$  \hspace{1cm} (6.6)

$$p_1 = \frac{1}{2} B^*(1/\tau),$$  \hspace{1cm} (6.7)

where $B^*$ is the Laplace transform of the distribution function $B$. 
Proof

We consider transitions for passing from time \( t_i \) to time \( t_{i+1} \). Up to a permutation of the processes, there are three possible transitions as defined by the following diagrams:

where the notation of Figure 6.1 is assumed.

Let \( H_j(t) \), \( j = 1,2 \) and 3, be the probability that transition \( A_j \) takes place and that \( t_{i+1} - t_i \leq t \). We have:

\[
H_1(t) = \int_0^t \left[ 1 - F(x) \right] \int_0^x b(y)f(x-y)dy \, dx,
\]

\[
H_2(t) = \int_0^t f(x) \int_0^x b(y)[1 - F(x-y)]dy \, dx,
\]

\[
H_3(t) = \int_0^t b(x)F(x)dx.
\]

But we observe that \( H(t) = H_1(t) + H_2(t) + H_3(t) \) is the distribution function for \( t_{i+1} - t_i \) and that the same process enters the critical section at both times \( t_i \) and \( t_{i+1} \) only with transition \( A_1 \). Hence:

\[
u = \int_0^\infty tdH(t) = \int_0^\infty [1 - H(t)] dt,
\]

\[
p_1 = \int_0^\infty dH_1(t) = \int_0^\infty [1 - F(x)] \int_0^x b(y)f(x-y)dy \, dx,
\]

from which equations (6.6) and (6.7) follow easily. \( \square \)
By collecting the preceding results, we obtain the following theorem:

**Theorem 6.1:**

\[
S_2(n) = \frac{n(\tau + \beta)}{(n-1)[2 - \frac{1}{2}B^*(1/\tau)](\frac{\tau}{2} + \frac{1}{2}TB^*(1/\tau)) + \frac{1}{2} + \beta}
\]

\[
= \frac{1}{2 - \frac{1}{2}B^*(1/\tau)} \cdot \frac{\tau + \beta}{\frac{\tau}{2} + \frac{1}{2}TB^*(1/\tau)} + O\left(\frac{1}{n}\right)
\]

We give below \(B^*(1/\tau)\) for some distribution functions \(B\).

(i) \(B\) is exponential (with parameter \(1/\beta\)):

\[B^*(1/\tau) = \frac{\tau}{\tau + \beta}.\]

(ii) \(B\) is the Dirac function at the point \(\beta\):

\[B^*(1/\tau) = e^{-\beta/\tau}.\]

(iii) \(B\) is uniform over \([a,b]\):

\[B^*(1/\tau) = \frac{e^{-a/\tau} - e^{-b/\tau}}{(b-a)/\tau}.\]

In Figure 6.2, we have plotted the asymptotic speed-up ratio \(S_2\) as a function of the ratio \(\gamma = \tau/(\tau + \beta)\) for the three distributions mentioned above (in the third case, \(a\) and \(b\) have been chosen as \(\beta/2\) and \(3\beta/2\), respectively).

When \(\gamma\) tends to 0 (or \(\beta\) tends to infinity), the algorithm approaches its worst performance, since the evaluations of the two processes tend to be exactly interleaved. When \(\gamma = 1\) (or \(\beta = 0\)), the critical section is non-existent and we have the results of Section 5.
We observe from Figure 6.2 that the best speed-up ratio is always obtained when $B$ is an exponential distribution (the first case). We also note that the results obtained for the two other cases are very close to each other and close to the results obtained with the exponential distribution. This suggests that the results obtained with the exponential distribution could be used as approximations to results obtained with other distributions.

We can observe from Figure 6.2 that, unlike the implementation without critical section, better speed-up is not necessarily achieved by using more processes, though we assume that a processor is always available to each process! More precisely, the figure indicates that (when $B$ is an exponential distribution) in order to achieve the best speed-up when two processors are available, one should create two processes when $\alpha > 0.586$, but only one process when $\alpha \leq 0.586$. Similar results are useful in practice, since they can be used to determine the optimal number of processes to create in order to minimize the overall execution time.
Figure 6.2 - Speed-up ratio with 2 processes for various distributions B.
7. CONCLUSION AND OPEN PROBLEMS

In recent years, research in parallel algorithms has dealt mostly with synchronized array or vector processors such as the ILLIAC IV or the CDC STAR, and there are very few results on the design and analysis of algorithms for asynchronous multiprocessors. In this paper, we have proposed a novel method of using asynchronous multiprocessors which takes advantage of their asynchronous behavior. We have also presented analytic techniques to evaluate the performance of an asynchronous algorithm using the method. The algorithm is expected to achieve a large speed-up when the fluctuations in the task execution times are relatively large. Moreover, as noted in Section 4, the algorithm has a nice reliability property. The idea of the algorithm may also be used to construct other reliable algorithms.

For the implementation with critical sections we obtained analytic results for two processes. The results show that the parallel algorithm using two processes is not necessarily faster than the sequential algorithm, because of the critical section overheads associated with the parallel algorithm. This confirms the practical experience that the speed-up ratio does not necessarily increase as the number of processes increases. It would be interesting to extend our analytic results for more than two processes. We have chosen to deal with a simple problem by imposing the condition that the tasks are linearly ordered. An interesting extension would be to consider a set of tasks (possibly generated dynamically) which are ordered by a directed graph (i.e., partially rather than linearly ordered). Another interesting extension would be to design algorithms where the execution of a task by a process may be interrupted by another
process. We expect that this approach would result in more efficient algorithms, since processes which are not doing useful work can be interrupted. A careful performance analysis including the additional overheads introduced by the interruption mechanism is needed here.

Finally, we note that the results of this paper are not restricted to multiprocessor systems. The ideas can be used to solve any problem in operation research which satisfies conditions similar to C1, C2 and C3.
REFERENCES

(Note: References [1, 4, 5] are not cited in the text but contain recent results on asynchronous multiprocessor algorithms.)


Given a sequence of tasks to be performed serially, a parallel algorithm is proposed to accelerate the execution of the tasks on an asynchronous multiprocessor by taking advantage of fluctuations in the execution times. A parallel program requiring no critical section is given to implement the algorithm and its correctness is proved. A spacewise more efficient implementation is also given but requires the use of critical sections. An analysis is presented for both implementations to estimate the speed-up achievable with the parallel algorithm. When the execution times are exponentially distributed, and no critical section is used, the algorithm with k processes yields a speed-up of order \( \sqrt{k} \).