Technical Note

1975-50

I. S. Reed

The Use of Finite Fields and Rings to Compute Convolutions

6 June 1975

Prepared for the Advanced Research Projects Agency under Electronic Systems Division Contract F19628-73-C-0002 by

Lincoln Laboratory

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Lexington, Massachusetts

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The work reported in this document was performed at Lincoln Laboratory, a center for research operated by Massachusetts Institute of Technology. This work was sponsored by the Advanced Research Projects Agency of the Department of Defense under Air Force Contract F19628-73-C-0002 (ARPA Order 2006).

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This technical report has been reviewed and is approved for publication.

FOR THE COMMANDER

Eugene C. Raabe
Eugene C. Raabe, Lt. CoL., USAF
Chief, ESD Lincoln Laboratory Project Office
THE USE OF FINITE FIELDS 
AND RINGS TO COMPUTE CONVOLUTIONS

I. S. REED
Group 24

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ABSTRACT

This note extends briefly the integer transforms of C. M. Rader (1972) to transforms over finite fields and rings. These transforms have direct application to digital filters and make possible digital filtering without round-off error. In some cases, the parameters of such number-theoretic transforms can be chosen so that substantial reductions in hardware are possible over what would be needed using classical digital filtering techniques.
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SUMMARY

This note reports briefly on material found for utilizing finite fields and rings to compute convolutions of finite sequences of integers. The methods described generalize the integer transform methods of Rader to similar transforms over finite fields and rings.

Some fundamentals of finite or Galois fields $GF(p^n)$ are informally introduced. Then, following Pollard, $d$-point Fourier-like transforms are defined and shown to be the only linear transforms in $GF(p^n)$ with the circular convolution property. This generalizes to Galois fields a result due to Agarwal and Burrus for the convolution of integer sequences.

Since the set $G(p)$ of integers modulo a prime number $p$ is always a subfield of $GF(p^n)$, $d$-point transforms over $GF(p^n)$ can be utilized to compute the transform of a sequence of integers $\{a_1, a_2, \ldots, a_d\}$ where $a_n$ lies in the range $-\frac{(p-1)}{2} \leq a_n \leq \frac{(p-1)}{2}$. As a consequence, the circular convolution of two such sequences can be computed using $d$-point transforms over $GF(p^n)$.

An interesting special case occurs if $n = 2$ and $q$ is a Mersenne prime of form $q = 2^p - 1$, where $p$ is a prime. For this case, $GF(q^2)$ is shown to mimic the complex numbers. That is, all elements of $GF(q^2)$ are of the form $a + ib$ where $a, b \in GF(q)$, and $i$ satisfy the equations $x^2 + 1 = 0$.

The $d$-point transforms of $GF(q^2)$ are shown to be candidates for computing convolutions of two sequences of complex integers. Since $d$, the number of points in the transform, must divide the order $q^2 - 1 = 2^p + 1 (2^p - 1 - 1)$ of the multiplicity subgroup of $GF(p^2)$, the number of points in a transform over $GF(q^2)$ can be chosen to be a power of 2. Thus one can utilize the fast Fourier transform (FFT) algorithm to compute convolutions of complex numbers without round-off error.

In the last section of this note, a theorem, stated by Pollard on transforms over a ring of integers modulo $m$, is examined. This leads to the notion of the modular arithmetic transform. The Chinese remainder theorem is used to map modular arithmetic transforms into the transforms of integers modulo $m$. 

I. INTRODUCTION

Recently C. M. Rader showed in Ref. 1 that the convolution of two finite sequences of integers \((a_k)\) and \((b_k)\) for \(k = 1, 2, \ldots, d\) can be obtained as the inverse transform of the product of two transforms which were other than the usual discrete Fourier transform (DFT). Rader defined transforms of the form

\[
A_k = \sum_{n=0}^{d-1} a_n z^{nk} \mod b
\]  

where \(b\) was either a Mersenne number

\[
b = 2^p - 1, \quad p \text{ a prime}
\]

or \(b\) was the Fermat number

\[
b = 1 + 2^{2m}, \quad m \text{ an integer}
\]

The primary advantage of the above Rader transform over the discrete Fourier transform,

\[
F_k = \sum_{n=0}^{d-1} a_n w^{nk}
\]

where \(w\) is a \(d\)th root of unity, lies in the fact that the multiplications by powers of \(w\) are replaced in binary arithmetic by simple shifts. Of course, this advantage must be weighed against the difficulties of computing the answer modulo \(b\) and of the numeric constraints, relating word length, length of sequence \(d\) and compositeness of \(d\), imposed by the above two choices for \(b\), suggested by Rader. Our purpose here is to review the Rader transform first by enlarging the class of transforms, given by (1), and second by presenting more details of the computational algorithm for computing such a convolution with (1).

In the next section, the class of transforms given by (1) is increased to include a Fourier-type transform over an arbitrary finite field, the Galois field. Such a generalization has been discussed recently by J. M. Pollard in 1971, but also much earlier by Reed and Solomon in 1959 in a somewhat different context. The approach used here will follow the more explicit approach of the earlier reference.

II. DFT ON A GALOIS FIELD

The only finite fields are the Galois fields. The number of elements in a Galois field \(GF(p^n)\) is \(p^n\) where \(p\) is a prime number and \(n\) is a positive integer. To construct a Galois field \(GF(p^n)\), one must first find an \(n\)th degree polynomial \(p(x)\) over \(GF(p)\) which is irreducible. The elements of \(GF(p^n)\) are then all polynomials of the form

\[
f(\alpha) = \sum_{i=0}^{n-1} f_i \alpha^i, \quad f_i \in GF(p), \quad (i = 0, 1, 2, \ldots, n - 1)
\]
where \( a \) is a root of \( p(x) \), i.e., \( p(a) = 0 \). The product \( h(a) \) of two elements say \( f(a) \) and \( g(a) \) in \( GF(p^n) \) is the residue of \( f(x) g(x) \) modulo \( p(x) \) with \( a \) substituted for \( x \). That is, \( h(a) \) is found by
\[
h(x) = f(x) g(x) \text{ Mod } p(x)
\]
where \( x = a \). Similarly, the sum \( s(a) \) is found by
\[
s(x) = f(x) + g(x) \text{ Mod } p(x)
\]
where \( x = a \). By taking the sums and products of all polynomials \( f(a) \) in this manner, the addition and multiplication tables of the elements of \( GF(p^n) \) can be found. Let this be illustrated by the following example.

**Example 1**

Consider the integers modulo 3. This is the prime field or \( GF(3) = \{0, 1, 2\} \) where \( 2 = -1 \). Let
\[
p(x) = x^2 + x + 2
\]
Since \( p(0) = 2 \), \( p(1) = 1 \), and \( p(2) = 2 \), \( p(x) \) is irreducible over the coefficient field \( GF(3) \). A root to \( p(x) = 0 \) can only be found in some field containing \( GF(3) \), some extension field. If \( a \) is such a root, then \( a \) satisfies
\[
p(a) = a^2 + a + 2 = 0
\]
Starting with the element \( a \), one computes \( a^2 \) by computing \( x^2 \text{ Mod } p(x) \) as follows:
\[
\begin{array}{c|c}
1 & x^2 + x + 2 \\
\hline
x^2 + x + 2 & \frac{x^2 + x + 2}{-x - 2}
\end{array}
\]
This \( -x - 2 = 2x + 1 \) is the residue of \( x^2 + x + 2 \), and
\[
a^2 = 2a + 1
\]
is the reduced expression for \( a^2 \). Similarly, one can compute \( a^3 \) by computing the residue of \( (x) \ (x^2) = (x) \ (2x + 1) = 2x^2 + x \), i.e.,
\[
\begin{array}{c|c}
2 & 2x^2 + x \\
\hline
2x^2 + x + 1 & \frac{2x^2 + 2x + 1}{2x + 2}
\end{array}
\]
Thus
\[
a^3 = 2a + 2
\]
Continuing in this manner one gets the results shown in Table 1.

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( \alpha^2 )</th>
<th>( \alpha^3 )</th>
<th>( \alpha^4 )</th>
<th>( \alpha^5 )</th>
<th>( \alpha^6 )</th>
<th>( \alpha^7 )</th>
<th>( \alpha^8 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha )</td>
<td>( 2\alpha + 1 )</td>
<td>( 2\alpha + 2 )</td>
<td>2</td>
<td>( 2\alpha )</td>
<td>( \alpha + 2 )</td>
<td>( \alpha + 1 )</td>
<td>1</td>
</tr>
</tbody>
</table>
In this particular case, \( \alpha \) and its powers \( \alpha^i \) (for \( i = 1, 2, \ldots, 8 \)) generate the eight non-zero elements of \( \text{GF}(3^2) \). If an element \( \alpha \) and its powers generate the non-zero elements of a field, \( \alpha \) is called a primitive element. If \( \alpha \) is a primitive element, and a root of \( p(x) \), which it is in this example, then the relation \( p(\alpha) = 0 \) can be used to compute the non-zero elements of \( \text{GF}(p^2) \). This is done for this example as follows: \( p(\alpha) = 0 \) is the relation \( \alpha^2 + \alpha + 2 = 0 \). Solving for \( \alpha^2 \), yields

\[
\alpha^2 = 2\alpha + 1 
\]

Then

\[
\begin{align*}
\alpha^3 &= \alpha(\alpha^2) = \alpha(2\alpha + 1) = 2\alpha^2 + \alpha \\
&= 2(2\alpha + 1) + \alpha = 2\alpha + 2 
\end{align*}
\]

and so forth, thereby obtaining Table 1.

The above example illustrates the following facts about a Galois field. All the elements of \( \text{GF}(p^n) \) satisfy the equation

\[ x^{p^n} = x \quad (3) \]

There exists a primitive element \( \alpha \in \text{GF}(p^n) \) which generates the non-zero elements of \( \text{GF}(p^n) \). The non-zero elements \( \text{GF}(p^n) \) compose a cyclic group.

In general, there always exists an \( \alpha \in \text{GF}(p^n) \) such that \( \text{GF}(p^n) \) is the set \( \{0, \alpha, \alpha^2, \ldots, \alpha^{p^n-2}, \alpha^{p^n-1}\} \). \( \alpha \) is called \( (p^n - 1) \)-th root of unity.

If in (1), \( b \) is a prime \( p \), then the Rader transform \( ^{1} \) can be regarded as a mapping of a subset of \( \text{GF}(p) \) into \( \text{GF}(p) \). To see this, consider the mapping

\[
A(x) = \sum_{k=0}^{d-1} a_k x^k \text{ Mod } p 
\]

Then the elements of the subset

\[
\{1, 2, 2^2, \ldots, 2^{d-1}\} \text{ Mod } p
\]

of \( \text{GF}(p) \) have, successively, the images

\[
\{A(1), A(2), A(2^2), \ldots, A(2^{d-1})\} \text{ Mod } p
\]

also a subset of \( \text{GF}(p) \) where \( a_k \in \text{GF}(p) \). Hence, \( A(x) \) as given by (4) is a mapping of a subset of \( \text{GF}(p) \) into \( \text{GF}(p) \). \( A(x) \) is called a polynomial mapping.

More generally, let \( a_n \) and \( x \) be elements of an arbitrary Galois field, say \( \text{GF}(p^n) \), and consider the mapping of subset of \( d \) distinct non-zero elements

\[
O_d = \{\tau_0, \tau_1, \ldots, \tau_{d-1}\} \tau_k \in \text{GF}(p^n)
\]

into \( \text{GF}(p^n) \) with the polynomial mapping

\[
A(x) = \sum_{k=0}^{d-1} a_k x^k 
\]
This is the most general possible mapping of GF(p^n) into GF(p^n) (see Ref. 3). This mapping can be displayed as a system of linear equations in the coefficients \( a_i \) as follows.

\[
A(\tau_1) = a_0 + a_1 \tau_1 + a_2 \tau_1^2 + \ldots a_{d-1} \tau_1^{d-1}
\]
\[
A(\tau_2) = a_0 + a_2 \tau_2 + a_2 \tau_2^2 + \ldots a_{d-1} \tau_2^{d-1}
\]
\[\vdots\]
\[
A(\tau_d) = a_0 + a_d \tau_d + a_2 \tau_d^2 + \ldots a_{d-1} \tau_d^{d-1}
\]

This system can be written further in matrix form as

\[
\Lambda = Ta
\]

where \( a \) and \( \Lambda \) are the column matrices

\[
a = \begin{bmatrix}
a_0 \\
a_1 \\
\vdots \\
a_{d-1}
\end{bmatrix}
\]

and

\[
\Lambda = \begin{bmatrix}
A(\tau_1) \\
A(\tau_2) \\
\vdots \\
A(\tau_d)
\end{bmatrix}
\]

and

\[
T = \begin{bmatrix}
1 & \tau_1 & \tau_1^2 & \ldots & \tau_1^{d-1} \\
1 & \tau_2 & \tau_2^2 & \ldots & \tau_2^{d-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \tau_d & \tau_d^2 & \ldots & \tau_d^{d-1}
\end{bmatrix}
\]

is a \( d \times d \) matrix of elements in GF(p^n).

By (7) the polynomial mapping (5) can also be regarded as a linear mapping of the vector \( a \) onto vector \( \Lambda \). Such a mapping is one to one or is invertible if matrix \( T \) has an inverse, that is, if the determinant \( |T| \) of \( T \) is non-zero. Since the determinant of \( T \) is a Vandermonde determinant, it can be evaluated as

\[
|T| = \prod_{i<j} (\tau_i - \tau_j) \neq 0
\]

since the \( \tau_j \)'s are all distinct. Thus \( T^{-1} \) exists and (7) can be solved as

\[
a = T^{-1} \Lambda
\]

the inverse "transform."
Next let us impose on (7) the constraint that it can be used to compute circular convolution $s_n$ of sequences $a_n$ and $b_n$.

$$S_n = \sum_{k=0}^{d-1} a_k b_{(n-k)} \quad (9)$$

where $(n-k)$ is the residue of $(n-k)$ modulo $d$. One wants the transform of $S_n$, namely, $\mathcal{S}$ to be given by

$$\mathcal{S} = \begin{bmatrix} S(\tau_1) \\ S(\tau_2) \\ \vdots \\ S(\tau_d) \end{bmatrix} = \begin{bmatrix} A(\tau_1) & B(\tau_1) \\ A(\tau_2) & B(\tau_2) \\ \vdots & \vdots \\ A(\tau_d) & B(\tau_d) \end{bmatrix} = A \bigotimes B \ .$$

Equating components

$$S(\tau_k) = A(\tau_k) B(\tau_k) \quad \text{for } k = 1, 2, \ldots d$$

or

$$\sum_{n=0}^{d-1} s_n \tau_k^n = \sum_{t=0}^{d-1} \sum_{m=0}^{d-1} a_t b_m \tau_k^{t+m} \ .$$

Substituting (9) in the left side,

$$\sum_{n=0}^{d-1} \sum_{p=0}^{d-1} a_p b_{(n-p)} \tau_k^n = \sum_{t=0}^{d-1} \sum_{m=0}^{d-1} a_t b_m \tau_k^{t+m} \ .$$

Next if one substitutes $t$ for $p$ and $m$ for residue of $(n-p)$ Mod $d$ in the left side, then

$$\sum_{t=0}^{d-1} \sum_{m=0}^{d-1} a_t b_m \tau_k^{(m+t)} = \sum_{t=0}^{d-1} \sum_{m=0}^{d-1} a_t b_m \tau_k^{t+m} \ .$$

Finally, equating coefficients of $a_t b_m$, one gets

$$\tau_k^{(m+t)} = \tau_k^{t+m} \quad (10)$$

for $(k, t, m = 0, 1, 2, \ldots d - 1)$ where $(m + t)$ is the residue of $(m + t)$ modulo $d$.

In order to satisfy (10), suppose $m + t$ is an integer $r$ in the interval $d \leq r < 2d$, then

$$m + t = r = d + (r)$$

where $(r)$ is the residue. In this notation (10) becomes

$$\tau_k^{(r)} = \tau_k^{d+(r)} = \tau_k^d \cdot \tau_k^{(r)} \quad .$$

(11)
Since by assumption $\tau_k \neq 0$, the inverse element $[\tau_k^{(r)}]^{-1}$ in $\text{GF}(p^n)$ of $\tau_k^{(r)}$ exists. Multiplying both sides of (11) by this inverse yields

$$
\tau_k^d = 1 \quad \text{for} \quad k = 1, 2, \ldots d \quad \text{(12)}
$$

That is, for transform (7) to yield circular convolutions, $\tau_k$ must be a $d^{th}$ root of unity for $k = 1, 2, \ldots d$ in $\text{GF}(p^n)$. This is essentially the same result Agarwal and Burrus got in Ref. 4 for the circular convolution of integer sequences.

Since the non-zero elements of $\text{GF}(p^n)$ form a cyclic group of order $p^n - 1$, the truth of (12) for an element $\tau_k \in \text{GF}(p^n)$ implies integer $d$ divides $p^n - 1$. That is, $d | p^n - 1$ if transform (7) is to yield a circular convolution. Moreover, since the set of elements $(\tau_1, \tau_2, \ldots \tau_d)$ are distinct and are all $d^{th}$ roots of unity, this set must be a cyclic subgroup of the cyclic subgroup of the non-zero elements of $\text{GF}(p^n)$. Thus the set, $(\tau_1, \tau_2, \ldots \tau_d)$, equals the subgroup $(\alpha, \alpha^2, \ldots \alpha^{d-1}, 1) = \varphi_d^1$, i.e.,

$$
(\tau_1, \tau_2, \ldots \tau_d) = (\alpha, \alpha^2, \ldots \alpha^{d-1}, 1) = \varphi_d
\text{(13)}
$$

in some order where $\alpha \in \text{GF}(p^n)$ is a generator of the subgroup.

If the group $\varphi_d = (\alpha, \alpha^2, \ldots \alpha^{d-1}, 1)$ is substituted for $(\tau_1, \tau_2, \ldots \tau_d)$ in transform (7), the transform becomes

$$
A_k = \sum_{n=0}^{d-1} \alpha_n \alpha^{kn} \quad \text{for} \quad (k = 0, 1, 2, \ldots, d - 1) \quad \text{(14)}
$$

To invert (14), observe first that all elements of $\varphi_d$ satisfy the equation

$$
x^d - 1 = 0
$$

But since $x^d - 1$ factors as

$$
x^d - 1 = (x - 1) \sum_{n=0}^{d-1} x^k
$$

one has

$$
\sum_{k=0}^{d-1} x^k = 0 \quad \text{for} \quad x \neq 1 \quad \text{and} \quad x \in \varphi_d \subset \text{GF}(p^n)
$$

$$
\sum_{k=0}^{d-1} x^k = 1 + 1 + \ldots + 1 = (d) \quad \text{for} \quad x = 1 \quad \text{(15)}
$$

where $(d)$ denotes the residue of $d$ modulo $p$. This formula is given by Pollard [Ref. 2, Eq. (8)] and earlier by Reed and Solomon [Ref. 3, Eq. (3)].

From (15) we now derive the discrete "delta" function needed to invert (14). Consider the sum of $x^n$ over all the elements of the multiplicative subgroup $\varphi_d$, defined by (13). This is

$$
\sum_{x \in \varphi_d} x^n = \sum_{k=0}^{d-1} (\beta^k)^n = \sum_{k=0}^{d-1} (\beta^n)^k
$$

where $\beta$ is a generator of $\varphi_d$. This formula is given by Pollard [Ref. 2, Eq. (8)].
But this is in the form of (15) and $\beta^n$ is an element of $\varphi_d$, thus

$$\sum_{x \in \varphi_d} x^n = \sum_{k=0}^{d-1} (\beta^n)^k = 0 \quad \text{for } n \not\equiv 0 \text{ Mod } d$$

$$= (d) \quad \text{for } n \equiv 0 \text{ Mod } d$$

$$= (d) \, \delta_d(n) \quad \text{(16)}$$

where $\delta_d(n)$ is the delta function

$$\delta_d(n) = 0 \quad \text{for } n \not\equiv 0 \text{ Mod } d$$

$$= 1 \quad \text{for } n \equiv 0 \text{ Mod } d$$

Since $(d)$ is an element of field $\text{GF}(p^n)$, the inverse $(d)^{-1}$ exists in $\text{GF}(p^n)$. Now, multiply $A_k$ by $(d)^{-1} \alpha^{-km}$ and sum on $k$ for $(k = 0, 1, 2, \ldots, d - 1)$. This yields by (14) and (16).

$$(d)^{-1} \sum_{k=0}^{d-1} A_k \alpha^{-km} = (d)^{-1} \sum_{k=0}^{d-1} \sum_{n=0}^{d-1} a_n \alpha^{kn} \alpha^{-km}$$

$$= (d)^{-1} \sum_{n=0}^{d-1} a_n \left( \sum_{k=0}^{d-1} \alpha^{k(n-m)} \right) = (d)^{-1} (d) \sum_{n=0}^{d-1} a_n \delta_d(n - m)$$

$$= a_m$$

Thus,

$$A_k = \sum_{n=0}^{d-1} a_n \alpha^{kn}$$

and

$$a_n = (d)^{-1} \sum_{k=0}^{d-1} A_k \alpha^{-kn} \quad \text{(17)}$$

where $a_n$ and $A_k$ are elements of $\text{GF}(p^n)$ and $\alpha$ is a generator of $d$ element subgroup $\delta_d$, the multiplicative subgroup of $\text{GF}(p^n)$.

To show the circular convolution property of (17), let

$$A_k = \sum_{n=0}^{d-1} a_n \alpha^{kn}, \quad B_k = \sum_{m=0}^{d-1} b_m \alpha^{km}$$

and

$$C_k = A_k \cdot B_k.$$
Then by (17) the inverse transform of $C_k$ for $k = 0, 1, \ldots, d - 1$ is

$$
(d)^{-1} \sum_{k=0}^{d-1} C_k \alpha^{-kp} = (d)^{-1} \sum_{k=0}^{d-1} \sum_{n=0}^{d-1} \sum_{m=0}^{d-1} a_n b_m \alpha^{k(m+n-p)}
$$

$$
= (d)^{-1} \sum_{n=0}^{d-1} \sum_{m=0}^{d-1} a_n b_m \delta_d(n + n - p) = \sum_{n=0}^{d-1} a_n b_n (p-n)
$$

(18)

where $(p-n)$ denotes the residue of $(p-n)$ modulo $d$.

The result, given by (18), shows finally that the imposition of condition (12) on the transform, given by (7), is both necessary and sufficient for transform (7) to yield circular convolutions. This generalizes a similar result, given by Agarwal and Burrus in Ref. 4, for the field of complex numbers to all fields both finite and infinite. In the next section, we show how to restrict the finite field transform, given by (17), so that it yields circular convolutions over both the integers and complex integers.

III. INTEGER ARITHMETIC PRESERVING FINITE FIELD TRANSFORMS

Suppose $a$ is an integer of magnitude less than or equal $(p - 1)/2$ where $p$ is a prime. Then integer $a$ satisfies

$$
-[(p - 1)/2] \leq a \leq (p - 1)/2
$$

If $a \geq 0$, $a$ is the residue modulo $p$. If $a = -b$ where $b > 0$, then

$$
a \equiv p - b \text{ Mod } p
$$

Thus the set of positive integers

$$
\{(-\frac{p-1}{2}, \ldots, -2, -1, 0, 1, 2, \ldots, \frac{p-1}{2})\}
$$

corresponds in a one-to-one manner with the following set of residues modulo $p$,

$$
\{(p - \frac{p-1}{2}, \ldots, p - 2, p - 1, 0, 1, 2, \ldots, \frac{p-1}{2})\}
$$

Since the latter set exhausts all residues modulo $p$, this set uniquely represents the set of all positive and negative real integers of magnitude less than or equal to $(p - 1)/2$, namely, the set $\{x \mid x \leq (p - 1)/2\}$, $x$ a positive or negative integer. However, the set of residues modulo $p$ composes precisely the Galois or finite field $GF(p)$, hence the above correspondence maps the set of integers less than or equal to $(p - 1)/2$ onto $GF(p)$ in a one-to-one manner.

In order to carry out arithmetic operations in $GF(p)$ which arrive at the correct arithmetic answer, one must often restrict the operating ranges of the integer variables even further. For example, to compute the circular convolution (18) in $GF(p)$ where $a_n$ and $b_n$ are integers, one requires the final convolution to lie in the same "dynamic range" as the integers $a_n$ and $b_n$.

That is, in order to avoid ambiguity
\[- \frac{p - 1}{2} \leq \sum_{n=0}^{d-1} a_n b_{(p-n)} \leq \frac{p - 1}{2}\]

or its equivalent

\[
\sum_{n=0}^{d-1} a_n b_{(p-n)} \leq \frac{p - 1}{2}.
\]  \hspace{1cm} (19)

Since

\[
\left| \sum_{n=0}^{d-1} a_n b_{(p-n)} \right| \leq \sum_{n=0}^{d-1} |a_n||b_{(p-n)}|
\]

where equality holds, if \(a_n\) and \(b_n\) are positive integers, to satisfy (19) for all sequences \(a_n\) and \(b_n\) such that \(|a_n| \leq A\) and \(|b_n| \leq B\), it is necessary that

\[
\sum_{n=0}^{d-1} (\text{Max } |a_n| \text{ [Max } |b_{(p-n)}|]) = dAB \leq \frac{p - 1}{2}.
\]  \hspace{1cm} (20)

\(A\) and \(B\) are the dynamic or operating ranges of integers, \(|a_n|\) and \(|b_n|\), respectively. If \(A = B\), then by (20) the largest value of \(A\) is given by

\[
A = \left\lfloor \frac{p - 1}{2d} \right\rfloor
\]  \hspace{1cm} (21)

where \([x]\) denotes greatest integer less than \(x\), what is often called the principle part of \(x\).

Assuming (21), which for many practical applications is somewhat pessimistic, one would need to constrain \(a_n\) and \(b_n\) to the interval

\[
-A = -\left\lfloor \frac{p - 1}{2d} \right\rfloor \leq a_n b_n \leq \left\lfloor \frac{p - 1}{2d} \right\rfloor = A
\]  \hspace{1cm} (22)

in order to compute the circular convolution

\[
C_p = \sum_{n=0}^{d-1} a_n b_{(p-n)}
\]  \hspace{1cm} (23)

unambiguously with modulo \(p\) arithmetic, i.e., keep \(c_n\) in the interval

\[-\frac{p - 1}{2} \leq c_n \leq \frac{p - 1}{2}.
\]

To compute convolution (23) when \(a_n\) and \(b_n\) are integers in a Galois field with transforms of the type suggested by Rader [Eq. (1)], one must first represent the integers in such a field.

To preserve the arithmetic operations of addition and multiplication, the representation must necessarily be restricted to \(GF(p)\) in the manner shown above. However, \(GF(p)\) is a subfield of \(GF(p^n)\); in fact, the ground field of \(GF(p^n)\) for all \(n (n = 1, 2, 3, \ldots)\). Thus, convolution (23) can be performed with transforms of type (17) on a Galois field \(GF(p^n)\) if \(a_n\) and \(b_n\) are restricted to \(GF(p)\). In others words, if \(a_n\) and \(b_n\) are \(GF(p^n)\) for \((n = 0, 1, 2, \ldots d - 1)\) and the transforms are

\[
A_k = \sum_{n=0}^{d-1} a_n \sigma^{kn} \quad \text{and} \quad B_k = \sum_{n=0}^{d-1} a_n \sigma^{kn} \quad \text{for } (k = 0, 1, \ldots d - 1)
\]
where \( \alpha \) is a generator of a d-element subgroup \( \varphi_d \) of \( \text{GF}(p^n) - 0 \), then the d-point convolution

\[
C_p = \sum_{n=0}^{d-1} a_n b_n (p-n)
\]

if integers \( a_n \) and \( b_n \) is found by forming

\[
C_k = A_k \cdot B_k \quad \text{for} \quad (k = 0, 1, \ldots, d - 1)
\]

and then taking the inverse transform

\[
C_n = (d)^{-1} \sum_{k=0}^{d-1} C_k \sigma^{-kn}
\]

If an \( \alpha \) can be found so that multiplications by powers of \( \alpha \) are simple in hardware, the above extension might be useful in increasing the number of possible points in the convolution. This follows from the fact that \( d \) is a divisor of \( p^n - 1 \) and the number of divisors of \( p^n - 1 \) is always greater than the number of divisors of \( p - 1 \).

In applications to radar and communications systems, one generally wants to take convolutions of complex numbers. Towards this end set \( a_n = \alpha_n + i\beta_n \) and \( b_n = x_n + iy_n \) where \( \alpha_n, \beta_n, x_n, \) and \( y_n \) are integers, suitably restricted in \( \text{GF}(P) \) so that the real and imaginary parts of

\[
C_p = \sum_{n=0}^{d-1} a_n b_n (p-n) = \gamma_n + i\delta_n
\]

lie in the interval \(-[(p - 1)/2] \leq \gamma_n, \delta_n \leq (p - 1)/2\) for \( n = 0, 1, \ldots, d - 1 \)

\[
a_n b_n = \alpha_n x_n - \beta_n y_n + i(\alpha_n y_n + \beta_n x_n)
\]

Thus one needs four transforms, \( A_k, B_k, X_k, \) and \( Y_k \) of \( \alpha_n, \beta_n, x_n, \) and \( y_n \) respectively, as well as four inverse transforms of the products,

\[
A_k X_k, B_k Y_k, A_k Y_k, B_k X_k
\]

(25)

to find (24), the circular convolution of complex integers. It is of interest to note that, for certain prime numbers \( q \), this computational requirement can be reduced from four to two Rader-type transforms.

To achieve this, prime \( q \) must be such that

\[
X^2 \equiv -1 \text{ Mod } q
\]

is not solvable. But the non-solvability of (26) is the same as the statement, \((-1)\) is a quadratic nonresidue (Ref. 5, p. 82). This is further equivalent to

\[
(-1)^{q-1}/2
\]

where \( (a/q) \) is the Legendre symbol, defined by

\[
\frac{a}{q} = +1 \quad \text{if} \quad a \text{ is quadratic residue Mod } q
\]

\[
= -1 \quad \text{if} \quad a \text{ is quadratic nonresidue Mod } q.
\]
There are two important special cases.

Case I.

Mersenne primes of form $M_p = 2^p - 1$ where $p$ is prime. For this case

$$\left(\frac{-1}{M_p}\right) = (-1)^{(M_p - 1)/2} = (-1)(2^p-2)/2$$

Thus $(-1)$ is a quadratic nonresidue and (26) is not solvable, modulo $M_p$.

Case II.

Fermat primes of form $F_m = 2^m + 1$ for $1 < m < 4$. For this case

$$\left(\frac{-1}{F_m}\right) = (-1)^{(F_m - 1)/2} = (-1)^2^{2^{m-1}} = 1$$

Thus $(-1)$ is a quadratic residue modulo $F_m$ and (26) is solvable.

If (26) is not solvable, which is true when $q$ is a Mersenne prime $M_p = 2^p - 1$, then polynomial

$$P(x) = x^2 + 1$$

is irreducible in $\text{GF}(q)$. By the procedure of the last section (see Example 1) a root, say $\hat{i}$, of

$$P(x) = x^2 + 1 = 0$$

(27)

can be found in the extension field $\text{GF}(q^2)$. $\text{GF}(q^2)$ is composed of the set

$$\text{GF}(q^2) = \{a + \hat{i}b | a, b \in \text{GF}(q)\}$$

(28)

where $\hat{i}$ is a root of (27), satisfying

$$\hat{i}^2 = 1$$

(29)

where $-1 \equiv (q - 1) \text{ Mod } q$.

Evidently $\hat{i}$ plays a similar role over the finite field $\text{GF}(q)$ that $\sqrt{-1} = i$ plays over the field of rational numbers. For example, suppose $a + \hat{i}b$ and $c + \hat{i}d$ are elements of $\text{GF}(q^2)$, then by (29)

$$(a + \hat{i}b) \pm (c + \hat{i}d) = (a \pm c) + \hat{i}(b \pm d)$$

and

$$(a + \hat{i}b)(c + \hat{i}d) = ac + \hat{i}^2bd + \hat{i}bc + \hat{i}ad$$

$$= ac - bd + \hat{i}(bc + ad)$$

the exact analogues of what one might expect if $a + \hat{i}b$ and $c + \hat{i}d$ were complex numbers. Thus if $-1$ is a quadratic nonresidue mod $q$, then the circular convolution (24) of the complex integers, $a_n$ and $b_n$, can be computed, using only two inverse transforms on the terms

$$A_kX_k - B_kY_k, A_kY_k + B_kX_k$$

defined in (25).
In the next section we will show how the transforms, developed by Rader for prime fields and extended here to Galois fields, can be extended further to rings, formed from these fields. Before doing this, however, it is of some independent interest to demonstrate one property of the Galois field $\text{GF}(q^2)$ which the field of complex rational numbers does not have. If $x = a + \hat{i}b \in \text{GF}(q^2)$, $x \neq 0$, then

$$x^{q^2-1} = (a + \hat{i}b)^{q^2-1} = 1.$$  

A true complex number does not have this property.

To prove this, use the binomial theorem

$$(a + \hat{i}b)^{q^2-1} = \sum_{k=0}^{q^2-1} \left( \begin{array}{c} q^2-1 \\ k \end{array} \right) (\hat{i}b)^k a^{q^2-1-k}.$$  

But

$$a^{q^2-1} = (a^{q-1})^{q+1} \quad \text{and} \quad a^{q-1} \equiv 1 \mod q,$$

so that

$$a^{q^2-1} = 1 \mod q.$$  

Also the binomial coefficient is

$$\left( \begin{array}{c} q^2-1 \\ k \end{array} \right) = \frac{(q^2-1)(q^2-2)\ldots(q^2-k)}{1 \cdot 2 \cdot 3 \ldots k},$$

$$\equiv \frac{[q(q-1) + (q-1)] [q(q-1) + (q-2)] \ldots [q(q-1) + (q-k)]}{1 \cdot 2 \cdot 3 \ldots k}$$

$$\equiv \frac{(q-1)(q-2)\ldots(q-k)}{1 \cdot 2 \ldots k} \equiv (-1)^k \mod q.$$  

Thus

$$(a + \hat{i}b)^{q^2-1} = \sum_{k=0}^{q^2-1} (-1)^k (\hat{i}b/a)^k$$

$$= \frac{1 - (-\hat{i}b/a)^{q^2}}{1 + \hat{i}b/a}.$$  

However,

$$\hat{i}^{q^2} = \hat{i}^{q^2-1} = (\hat{i}^{q-1})^{q+1} \hat{i}$$

$$= \left[ (-1)^{(q-1)/2} \right]^{q+1} \hat{i} = \left( \frac{-1}{q} \right)^{q+1} \hat{i}.$$  

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where \( \left( \frac{a}{q} \right) \) is the Legendre symbol. But by assumption \((-1)\) is a quadratic nonresidue and 
\( \left( \frac{-1}{q} \right) = -1 \). Hence,

\[
q^2 = 1
\]

so that finally

\[
(a + i b)^q^2 - 1 = \frac{1 + \frac{ib}{a}}{1 + \frac{b}{a}} = \frac{1 + ib/a}{1 + ib/a} = 1 .
\]

We see above that the Mersenne primes \( M_p \) have an advantage over the Fermat primes \( F_m \) in the computation of convolutions of complex integers. However, as Rader points out in Ref. 1, this advantage must be weighed against the fact that the fast Fourier transform (FFT) algorithm can be applied to the transforms, using Fermat primes, but not to the Mersenne primes.

IV. TRANSFORMS IN MODULAR ARITHMETIC AND MODULO \( m \) RINGS

A transform in the ring of integers modulo \( m \) was considered by Pollard in Ref. 2. It is well known that the set of integers modulo \( m \) is a ring \( R_m \) with respect to addition and multiplication modulo \( m \).

Pollard considered first rings where \( m \) was a power of a prime \( p \), namely, \( m = p^n, p > 0 \). He let \( R_m^p \) denote the set of elements of \( R_m \) prime to \( m \), i.e.,

\[
R_m^p = \{ a \in R_m | (a, m) = 1 \}
\]

where \((a, m)\) denotes the greatest common divisor of integers \( a \) and \( m \).

By Euler’s theorem (Ref. 5, p. 48), if \((a, m) = 1\)

\[
a^{\varphi(m)} = 1 \pmod{m}
\]

(30)

where \( \varphi(m) \) denotes the number of divisors of \( m \) less than or equal to \( m \), Euler’s function.

Thus, since 1 is the multiplicative identity of \( R_m^p \), then

\[
a^{\varphi(m)} = 1
\]

(31)

for all \( a \in R_m^p \).

The order of an element \( a \) in \( R_m^p \) (called the exponent of \( a \) in number theory) is the least power \( e(a) \) such that

\[
a^{e(a)} = 1 .
\]

Also, if \( m = p^n \) the number of elements in \( R_m^p \) prime to \( m \) is

\[
\varphi(m) = p^n - p = p^{n-1}(p - 1) .
\]

Thus by (31) the order of each element \( a \in R_m^p \) divides \( \varphi(m) = p^{n-1}(p - 1) \), i.e., \( e(a) \mid p^{n-1}(p - 1) \) all \( a \in R_m^p \).

It is well known (Ref. 5, p. 107) that an element \( g \in R_m^p \) can be found such that \( e(g) = p^{n-1}(p - 1) \). \( g \) is called a primitive root since

\[
g^{\varphi(m)} = 1 \pmod{m}
\]

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and $\varphi(m) = e(g)$ the order or exponent with $g$ belongs to modulo $m$. The powers of $g$, that is the set

$$G = \{g, g^2, \ldots, g^{p^n-1}(p-1)\}$$

are all distinct. Suppose otherwise that

$$g^k = g^t, \quad k > t$$

where

$$g^k, g^t \in G,$$

then

$$g^k \cdot g^{p^n-1}(p-1)-t = g^{k-t} = 1.$$  

But $k - t < p^{n-1}(p-1) = e(g)$ which is contrary to the assumption that $g$ is a primitive root. Hence the elements of $G$ are distinct. Since the elements of $G$ are prime to $m = p^n$ and since $G$ has the same number of elements as $R_m^*$,

$$G = R_m^*.$$  

Thus $R_m^*$ is a cyclic multiplication group of $p^{n-1}(p-1)$ elements with generator $g$.

Pollard next chooses a divisor $d$ of $p-1$ and considers an element $r \in R_m^*$ of order $d$, i.e.,

d is the smallest integer for which $r^d = 1$. The powers of $r$ compose a subgroup $G_d$ of $R_m^*$,

$$G_d = \{1, r, r^2, \ldots, r^{d-1}\}.  

He next shows that the equivalent of (16) holds when $\varphi_d$ is replaced by $G_d$. That is, if $d \mid p-1$,

$$\sum_{X \in G_d} X^m = \sum_{k=0}^{d-1} (r^m)^k = 0 \quad \text{for } m \not\equiv 0 \text{ Mod } d$$

$$= (d) \quad \text{for } m \equiv 0 \text{ Mod } d$$

$$= (d) \delta_d(m) \quad (32)$$

where $\delta_d(m)$ is the delta function

$$\delta_d(m) = 0 \quad \text{for } m \not\equiv 0 \text{ Mod } d$$

$$= 1 \quad \text{for } m \equiv 0 \text{ Mod } d$$

and where $(d)$ is $d$ modulo $p^n$.

To prove this, consider first the following cyclic subgroup of $R_m^*$

$$\left\{g^{p-1}, (g^{p-1})^2, \ldots, (g^{p-1})^{p^{n-1}}\right\} = G_{p^{n-1}}  

of $p^{n-1}$ elements. By Fermat's theorem [Eq. (31) for $m$ a prime], an element $g^{(p-1)k}$ of $G_{p^{n-1}}$ satisfies

$$(g^{p-1})^k \equiv 1^k \equiv \text{Mod } p.$$
However, if we consider an arbitrary element of subgroup,

\[ G_{p-1} = \left\{ g^{p^{n-1}}, \left(g^{p^{n-1}}\right)^2, \ldots, \left(g^{p^{n-1}}\right)^{p-1} \right\} \]  

modulo \( p \), then

\[
g^{p^{n-1}k} \equiv \left(\ldots \left(g^p\right)^{p^{n-1}}\ldots\right)^k \equiv \left(\ldots \left(g^p\right)^k\ldots\right) \equiv g^k \mod p . \]

Since integers \( p-1 \) and \( p^{n-1} \) are relatively prime, i.e., \( (p - 1, p^{n-1}) = 1 \), the subgroups \( G_{p-1} \) and \( G_{p-1}^{n-1} \) in (33) and (34), respectively, have only the unit element, \( 1 \), in common. Also by (33) and (34) every element of \( R_m^\circ \) is to be found in the product of \( G^{n-1}_{p-1} \) and \( G_{p-1}^{n-1} \). Hence \( R_m^\circ \) is the direct product of these two subgroups, i.e.,

\[ R_m^\circ = G_{p-1} \times G_{p-1}^{n-1} . \]

Thus the only elements of \( R_m^\circ \) which are not congruent to \( 1 \) modulo \( p \) are the complement of \( G^{n-1}_{p-1} \) and hence in \( G_{p-1} \).

Let \( h \) be a primitive root modulo \( p \), i.e., \( h \) is an integer \( 1 < h < p - 1 \) such that \( p - 1 \) is the least integer for which

\[ h^{p-1} \equiv 1 \mod p . \]

Then it can be shown (see Ref. 5, p. 107) that a primitive root \( g \) modulo \( p^n \) can always be found of form

\[ g = h + \mu p \]

where \( \mu \) is an integer. From this

\[ g^k \equiv (h + \mu p)^k \equiv h^k \mod p \]

where \( 1 < h < p - 1 \). With (35) this yields

\[ g^{p^{n-1}k} \equiv h^k \mod p . \]

Since \( h \) is a primitive root modulo \( p \), it generates the \( p - 1 \) element group \( \phi_{p-1} \) of the non-zero elements of \( R_p^\circ = \text{GF}(p) \). (36) maps the elements of \( G_{p-1} \) onto \( \phi_{p-1} \) in one-to-one manner. Since

\[ g^{p^{n-1}(k+t)} \equiv h^{k+t} \mod p , \]

this mapping is in fact an isomorphism between groups \( G_{p-1} \) and \( \phi_{p-1} \), i.e., \( G_{p-1} \cong \phi_{p-1} \).

By (36) if some element of \( G_{p-1} \) was congruent to \( 1 \) modulo \( p \), then

\[ g^{p^{n-1}k} \equiv h^k \equiv 1 \mod p . \]

Since \( h \) is primitive this is possible if and only if \( k \) is a multiple of \( p - 1 \). Thus \textbf{none} of the elements of \( G_{p-1} \) is congruent to \( 1 \) modulo \( p \), except the unit element \( 1 \). Since \( d | p - 1 \), \( G_d \) is a cyclic subgroup of \( G_{p-1} \), and likewise no element \( x, x \neq 1 \), of \( G_d \) is congruent to \( 1 \) modulo \( p \). Now for \( m \neq 0 \mod d \)

\[ \left( \sum_{k=0}^{d-1} \left(r^m\right)^k \right) (r^m - 1) \equiv (r^d)^m - 1 \equiv (1)^m - 1 \equiv 0 \mod p^n \]

(37)
where \( r \) is a generator of \( G_d \). From the above, if \( m \neq 0 \mod d \),

\[
r^m \not\equiv 1 \mod p
\]

Thus, the integer \( r^m - 1 \) and \( p \) are relatively prime \((r^m - 1, p) = 1\). But this in turn implies \((r^m - 1, p^n) = 1\) for \( m = 1, 2, \ldots, d - 1 \). Thus

\[
\sum_{k=0}^{d-1} (r^m)^k \equiv 0 \mod p^n
\]

for all \( m \neq 0 \mod d \) and (32) is proved. This is essentially the result proved by Pollard in Ref. 2. Pollard states that more generally one can find a \( d \)-point transform for \( m = p_1^{n_1} \ldots p_t^{n_t} \) if \( d \mid (p_i - 1) \) for all \( i \) \((i = 1, \ldots, t)\) and \( d \) is the order \( \mod m \).

Bonneau in Ref. 6 has proved a converse of Pollard's result which we restate and prove here in our terminology.

**Theorem.**

If \( R_m \) has a \( d \)-point transform and \( m = p_1^{n_1} \ldots p_t^{n_t} \), \( m \) odd, then \( d \mid p_i - 1 \) for all \( i \) and there exists an element \( r \in R_m \) such that \( r \) is of order \( d \) in \( R_{p_i^{n_i}} \) for all \( i \).

**Proof.**

Since \( R_m \) has a \( d \)-point transform, the delta function, given by (32), must exist where here \( m = p_1^{n_1} \ldots p_t^{n_t} \). For the inverse transform to exist the inverse \((d)^{-1}\) of \( (d) \), the residue of \( d \mod m \) must exist. To find this inverse it is necessary the \((d,m) = 1\); \( d \) and \( m \) are relatively prime. But this implies \((d, p_i) = 1\) for each \( i \) \((i = 1, 2, \ldots, t)\).

Consider the mapping \( \psi \) of ring \( R_m \) on to the direct product of rings, \( R_{p_1^{n_1}} \times R_{p_2^{n_2}} \times \ldots \times R_{p_t^{n_t}} \), i.e.,

\[
\psi : R_m \rightarrow \prod_{i=1}^{t} R_{p_i^{n_i}}
\]

which explicitly is

\[
\psi(x) = \left( x \mod p_1^{n_1}, x \mod p_2^{n_2}, \ldots, x \mod p_t^{n_t} \right)
\]

where \( x \in R_m \). By the Chinese remainder theorem (Ref. 7, pp. 94-95), \( \psi(x) \) is a one-to-one mapping. Since \( \psi(x + y) = \psi(x) + \psi(y) \) and \( \psi(xy) = \psi(x) \cdot \psi(y) \), \( \psi(x) \) maps ring \( R_m \) onto ring \( \prod_{i=1}^{t} R_{p_i^{n_i}} \) isomorphically.

The set \( R_m^\circ \) of elements relatively prime to \( m \) is an Abelian group. \( \psi(x) \) maps group \( R_m^\circ \) onto the direct product of cyclic groups \( R_{p_i^{n_i}}^\circ \), isomorphically. That is,

\[
R_m^\circ \simeq \prod_{i=1}^{t} R_{p_i^{n_i}}^\circ
\]

The order of \( R_m^\circ \) in the isomorphism (39) is the number of elements relatively prime to \( m \), namely the number,
$$\varphi(m) = \prod_{i=1}^{r} (p_i - 1) p_i^{n_i-1}$$

whereas the number of elements in the cyclic group \(R^*_n\) is

$$\varphi(p_i^{n_i}) = (p_i - 1) p_i^{n_i-1}$$

In order to have the delta function (32), an element \(r \in R^*_m\) of order \(d\) must exist, i.e.,

$$r^d = 1$$

Since \(r \cdot r^{d-1} = 1\), the inverse of \(r\) exists and equals \(r^{d-1}\). But by an elementary theorem on congruences such an inverse exists if and only if \((r, m) = 1\). This implies \(r \in R^*_m\). Since the order of an element of a group divides the order of the group, \(d | \varphi(m)\) or

$$d \big| \prod_{i=1}^{t} (p_i - 1) p_i^{n_i-1}$$

But by an argument above \((d, p_i) = 1\) for all \(i\). This with (40) yields

$$d \big| \prod_{i=1}^{t} (p_i - 1)^{n_i}$$

In order to have a delta function it is necessary that sum \(s_m\) satisfy,

$$S_m = \sum_{k=0}^{d-1} (r_m)^k \equiv 0 \mod m$$

for \((m = 1, 2, \ldots, d - 1)\). Since \(m = \pi p_i^{n_i}\) and the \(p_i^{n_i}\) are all relatively prime, then

$$S_m = \sum_{k=0}^{d-1} (r_m)^k \equiv 0 \mod p_i^{n_i}$$

for \((i = 1, 2, \ldots, t)\) and \((m = 1, 2, \ldots, d - 1)\).

Now mapping \(\varphi(x)\) in (38) sends \(r \in R^*_m\) into the following vector

$$\varphi(r) = (r \mod p_1^{n_1}, r \mod p_2^{n_2}, \ldots, r \mod p_t^{n_t})$$

where \(r_i\) denotes the residue of \(r\) in \(R^*_n\). Consider now the order of \(r_i\) in \(R^*_n\). Let this order be \(d_i\) so that \(r_i^{d_i} = 1\). Evidently \(d_i\) must at least divide \(d\) so that \(d_i < d\).

Now suppose \(d_i < d\). Then

$$d_i \sum_{k=0}^{d-1} (r_i^{d_i})^k \equiv d \times \sum_{k=0}^{d-1} (r^{d_i})^k \equiv d \times \frac{1 + 1 + \ldots + 1}{d} = d \mod p_i^{n_i}$$

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But, a previous argument above, \((d, p_i) = 1\) for \(i = 1, 2, \ldots t\). Thus (41) for \(m = d_i\) satisfies
\[ S_{d_i} = d \neq 0 \mod p_i^{n_i} . \]

This is a contradiction to (41). Thus the "projection" \(r_i\) of \(r\) in \(R_{n_i}\) has order \(d\) for \(i = 1, 2, \ldots t\). But again since the order of an element divides the order of the group,
\[ d | (p_i - 1) p_i^{n_i - 1} \]
for all \(i (i = 1, 2, \ldots t)\). Finally, since \(d\) and \(p_i\) are relatively prime, all \(i\), \(d | (p_i - 1)\) for \((i = 1, 2, \ldots t)\). This proves the converse of Pollard's theorem.

The mapping \(\psi(x)\) given by (38) represents an integer modulo \(m\) as a vector of residues of relatively prime moduli. The arithmetic associated with this representation has come to be known as modular arithmetic. Also the rings associated with the mapping \(\psi(x)\) in (38) are called modular arithmetic rings. Hence it is reasonable to call transforms of type (1), which are mapped by \(\psi(x)\) into a modular arithmetic ring, modular arithmetic transforms.

REFERENCES

## The Use of Finite Fields and Rings to Compute Convolutions

This note extends briefly the integer transforms of C.M. Rader (1972) to transforms over finite fields and rings. These transforms have direct application to digital filters and make possible digital filtering without round-off error. In some cases, the parameters of such number-theoretic transforms can be chosen so that substantial reductions in hardware are possible over what would be needed using classical digital filtering techniques.