UTILIZATION OF THE BEND TEST FOR DETERMINING TENSILE PROPERTIES OF A BRITTLE MATERIAL

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**Title:** Utilization of the Bend Test for Determining Tensile Properties of a Brittle Material

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**Distribution Statement:** Approved for public release; distribution unlimited.

**Keywords:** Silicon nitride, Ceramic materials, Fracture, Bend tests, Tension tests, Probability theory

**Abstract:** This project has been accomplished as part of the U.S. Army Materials Testing Technology Program, which has for its objective the timely establishment of testing techniques, procedures or prototype equipment (in mechanical, chemical, or nondestructive testing) to insure efficient inspection methods for materiel/material procured or maintained by AMC.
ABSTRACT

A limited number of end as well as tension tests were performed at room temperature on specimens made of Norton HS-130 grade silicon nitride. By application of the two-parameter Weibull analysis for a material governed by volumetric flaw distribution, tensile properties of the specimen, based upon the data of both types of testing, were determined and compared. The results show, at least for the specimens tested, that the bend test tends to predict fracture stresses of the tension specimens approximately 8% higher than those obtained in the actual tension tests. (Author)
# CONTENTS

| I.  | INTRODUCTION                                  | 1 |
| II. | PROBABILITY OF FRACTURE                      |   |
|     | Three-Parameter Analysis                     | 2 |
|     | Two-Parameter Analysis                       | 2 |
| III.| MODIFIED RISK OF RUPTURE                    | 3 |
| IV. | WEIBULL MODULUS                              | 5 |
| V.  | SCALE PARAMETER                              | 6 |
| VI. | RELATIONSHIPS BETWEEN BEND AND TENSILE STRESSES | 7 |
| VII.| DISCUSSION AND RESULTS                       | 8 |
|     | APPENDIX A. DERIVATION OF MODIFIED RISK OF RUPTURE FOR THIRD-POINT LOADING BEND TEST SPECIMENS | 15 |
|     | APPENDIX B. DERIVATION OF MODIFIED RISK OF RUPTURE FOR DUMBELL-SHAPED TENSION TEST SPECIMENS | 19 |
I. INTRODUCTION

Interest in the bend test has gained considerable attention in recent years because of the greater use of high strength materials with little ductility and because of the development and exploitation of such brittle materials as ceramics and carbides.

If tensile properties of a material are sought, it is only natural to think of manufacturing and testing a tension specimen made of the material in question. Because of size limitations, however, even when the material is easily machinable, the task of manufacturing a tension specimen sometimes becomes impossible. On the other hand, even when sufficient material is available, the cost of manufacturing a tension specimen may become prohibitive as, for example, machining a dumbbell-shaped tension specimen made of an extremely brittle material as those mentioned above. In such cases, a bend test with its major advantage of employing a simple specimen with a rectangular cross section becomes a welcome substitute provided, of course, that such a test does yield reliable predicted tensile results.

It has been shown by Nadai\(^1\) that it is theoretically possible to apply a bend test to determine the tensile and compressive stress-strain curves of a material, and experimental verification of this hypothesis has been accomplished with 4340 steel heat treated to various strength levels.\(^2\) Results indicate that close agreement exists, at least to strains to 1-1/2 to 2%, between the stress-strain curves predicted from the bend tests and those determined from the actual tension and compression tests.

In order to include the gamut of material variation, i.e., ductile to brittle, bend specimens were designed and tested according to Reference 2 and tension specimens were designed and tested essentially according to Reference 3. Sufficient expressions have been derived and presented from which properties of tension specimens can be predicted from bend test results. Predicted and actual properties of tension specimens were subsequently obtained and compared.

In determining fracture stresses of ceramic specimens, recourse was made to a statistical approach, and in this study the Weibull statistical theory of fracture,\(^4\) the most widely accepted theory, was chosen. This theory uses two basic criteria of failure: size and normal stress. In the Weibull three-parameter analysis, fracture is predicted in terms of the three-material parameters: zero probability strength, flaw density exponent, and a scale parameter. In the Weibull two-parameter analysis, on the other hand, the first of these material parameters is assumed to be zero, and fracture is predicted in terms of the two remaining parameters.

II. PROBABILITY OF FRACTURE

Three-Parameter Analysis

For a stress field in a homogeneous isotropic material governed by volumetric flaw distribution, the probability of fracture at a given stress \( \sigma \) is given:

\[
P_f = \begin{cases} 
1 - \exp \left( -\int_0^\infty \left( \frac{\sigma - \sigma_u}{\sigma_0} \right)^m \, dV \right) & \text{if } \sigma > \sigma_u \\
0 & \text{if } \sigma < \sigma_u
\end{cases}
\]  

(1)

where

\[
R = \int_0^\infty \left( \frac{\sigma - \sigma_u}{\sigma_0} \right)^m \, dV
\]  

(2)

is the risk of rupture, and

- \( \sigma_u \) = zero probability strength (strength below which there is no fracture)
- \( m \) = Weibull modulus or flaw density exponent
- \( \sigma_0 \) = scale parameter.

The last three values are material parameters only.

Two-Parameter Analysis

If it is assumed that \( \sigma_u = 0 \) (and certainly there can be no fracture at zero stress level), then Equations (1) and (2) become:

\[
P_f = 1 - \exp \left( -\int_0^\infty \left( \frac{\sigma}{\sigma_0} \right)^m \, dV \right) = 1 - \frac{1}{e^{R'}}
\]  

(3)

where

\[
R' = \int_0^\infty \left( \frac{\sigma}{\sigma_0} \right)^m \, dV
\]  

(4)

is the modified risk of rupture.

Application of the two-parameter analysis yielded values of material parameters that described the test data very well and the results of this analysis are herein reported.

III. MODIFIED RISK OF RUPTURE

It should be apparent that the value of modified risk of rupture of a specimen— in terms of maximum tensile stress, for example, within the specimen—is the result of integration of Equation (4) throughout the total volume. In addition, if the stress throughout the specimen is not constant, this integration may become rather cumbersome.

Generally, however, a specimen is so designed that the greatest risk of rupture occurs in the middle or gage length section, and, therefore, the value of modified risk of rupture determined by integrating throughout the volume of the gage length section will be only negligibly smaller than that determined by integrating throughout the total volume.

Equation (4) may always be expressed as:

\[ R' = K V \left( \frac{\sigma_{\text{max}}}{\sigma_0} \right)^m \]  

(5)

where

\( K \) = a load factor determined by carrying out portions or all of the integration indicated in Equation (4), \( K = K(m) \).

\( V \) = volume of all or gage length section only of the specimen, as selected

\( \sigma_{\text{max}} \) = maximum stress (tensile in this study) in specimen

\( \sigma_0, m \) = material parameters.

Shown below are listed values of \( R' \) for the cases pertinent to this study (see Appendixes A and B for actual derivations):

a. Third-Point Loading Bend Specimen

1. Total volume considered

\[ R' = \frac{V_{\text{total}} (m+3)}{6(m+1)^2} \left( \frac{\sigma_b}{\sigma_0} \right)^m \]

(6)

where in Equation (5) \( K \) is taken as

\[ K = K_b = \frac{(m+3)}{6(m+1)^2} \]  

and subscript \( b \) refers to bend specimen.
2. Only volume of gage length section considered

\[ R' = \frac{V_{b_{GL}}}{2(m+1)} \left( \frac{\sigma_b}{\sigma_0} \right)^m \]  

(7)

where in Equation (5) \( K \) is taken as \( K = \frac{1}{2(m+1)} \) and again subscript \( b \) refers to bend specimen

and where

\[
\begin{align*}
V_{b_{total}} & = \text{total volume of bend specimen} \\
V_{b_{GL}} & = \text{volume of gage length section of bend specimen} \\
\sigma_b & = \text{maximum tensile stress in bend specimen} \\
\sigma_0 & = \text{tensile stress in outer fiber in gage length section of bend specimen.}
\end{align*}
\]

b. Dumbbell-Shaped Tension Spec

1. Total volume considered

\[ R' = K_t V_{t_{GL}} \left( \frac{\sigma_t}{\sigma_0} \right)^m \]  

(8)

where in Equation (5) \( K \) is taken as \( K = K_t \) a function of \( m \) (see Table 3 for pertinent values of \( K_t \)) and subscript \( t \) refers to tension specimen.

2. Only volume of gage length section considered

\[ R' = V_{t_{GL}} \left( \frac{\sigma_t}{\sigma_0} \right)^m \]  

(9)

where in Equation (5) \( K \) is taken as \( K = K_t = 1.0 \) and again subscript \( t \) refers to tension specimen and where

\[
\begin{align*}
V_{t_{GL}} & = \text{volume of only gage length section of tension specimen}
\end{align*}
\]
\[ c_t = \text{maximum tensile stress in tension specimen} \]
\[ = \text{tensile stress in gage length section of specimen.} \]

IV. WEIBULL MODULUS

According to Reference 6, the value of \( m \), Weibull modulus or flaw density exponent, may be determined by use of the expression listed below:

\[
\frac{\Gamma \left( \frac{1+1}{m} \right)}{\left[ \Gamma \left( \frac{1+2}{m} \right) - \Gamma^2 \left( \frac{1+1}{m} \right) \right]^{\frac{1}{2}}} \cdot \frac{\bar{\sigma}_{\text{fract}}}{\sigma_{\text{fract}}} \]

(10)

where

\[ \bar{\sigma}_{\text{fract}} = \text{mean fracture stress} \]
\[ S_{\sigma_{\text{fract}}} = \text{standard deviation of mean fracture stress} \]

\[ \Gamma = \text{gamma function} \]

and

\[ \sigma_{\text{fract}} = \text{individual fracture stress} \]
\[ \sigma_b_{\text{fract}} \text{ for a bend specimen} \]
\[ \sigma_t_{\text{fract}} \text{ for a tension specimen.} \]

As may be seen, the value of \( m \) is determined from test results.

For all bend and tension specimens in this work the fracture stresses were recorded, and these values are listed in Tables 1 and 2. Details of the actual test systems, procedures, and methods of computation, are contained in References 2 and 3.

However, it will be noted here that fracture stress in bending is defined as the maximum tensile stress in the gage length section at fracture, and these values were determined from:

\[
\sigma_{b,\text{fract}} = \frac{M_{\text{max}}}{I} \frac{c}{1}
\]  

(11)

where

\[M_{\text{max}} = \text{maximum bending moment at fracture}\]

\[c = \frac{1}{2} \text{ depth of specimen}\]

\[I = \text{moment of inertia of cross section of specimen.}\]

Fracture stress in the dumbbell-shaped tension specimen is also defined as the maximum tensile stress in the gage length section at fracture and these values were determined from:

\[
\sigma_{t,\text{fract}} = k \frac{P_{i,\text{fract}}}{r_0^2 - r_1^2} \frac{r_2^2 - r_1^2}{r_1^2}
\]  

(12)

where

\[k = \text{constant of calibration} = 0.97\]

\[P_{i,\text{fract}} = \text{internal pressure at fracture}\]

\[r_0 = \text{outside (largest) diameter of specimen}\]

\[r_1 = \text{inside (smallest) diameter of specimen.}\]

Finally, equation (10) was used to determine the values of \(m\) for all bend as well as tension tests, and these values are shown in Table 3.

V. SCALE PARAMETER

The value of \(\sigma_0\), the scale parameter, may be determined as indicated by:

\[
\sigma_0 = \frac{\bar{\sigma}_{\text{fract}} (KV)^{1/m}}{\Gamma \left(1 - \frac{1}{m}\right)}
\]  

(13)

where all terms have already been defined.
For the specimens involved in this work, Equation (13) becomes:

a. Third-Point Loading Bend Specimens

1. Total volumes considered

\[ \sigma_0 = \frac{\bar{\sigma}_{b,\text{fract}} \left[ \frac{V_{b,\text{total}}}{6(m+1)^2} \right]^{1/m}}{\Gamma \left(1 + \frac{1}{m}\right)} \]  
(14)

2. Only volumes of gage length sections considered

\[ \sigma_0 = \frac{\bar{\sigma}_{b,\text{fract}} \left[ \frac{V_{b,\text{GL}}}{2(m+1)} \right]^{1/m}}{\Gamma \left(1 + \frac{1}{m}\right)} \]  
(15)

b. Dumbbell-Shaped Tension Specimens

1. Total volumes considered

\[ \sigma_0 = \frac{\bar{\sigma}_{t,\text{fract}} \left[ K_t V_{t,\text{GL}} \right]^{1/m}}{\Gamma \left(1 + \frac{1}{m}\right)} \]  
(16)

where values of \( K_t \) are those shown in Table 3.

2. Only volumes of gage length sections considered

\[ \sigma_0 = \frac{\bar{\sigma}_{t,\text{fract}} \left( V_{t,\text{GL}} \right)^{1/m}}{\Gamma \left(1 + \frac{1}{m}\right)} \]  
(17)

Values of \( \sigma_0 \), determined by Equations (14) and (15) for bend specimens and by Equations (16) and (17) for tension specimens, are also listed in Table 3.

VI. RELATIONSHIPS BETWEEN BEND AND TENSILE STRESSES

The relationships between bend and tensile stresses depend upon the expressions representing the values of modified risks of rupture. By equating the expressions for modified risks of rupture for either the total or gage length volumes the following relationships are obtained:
a. Total Volumes of Specimens Considered

Equating the value of $R'$ shown in Equation (6) to that shown in Equation (8) leads to:

$$
\sigma_t = \sigma_b \left[ \frac{(m+3) V_{b_{\text{total}}}}{6(m+1)^2 K_t V_{t_{GL}}} \right]^{1/m}
$$

(18)

where values of $K_t$ again are those shown in Table 3.

b. Only Volumes of Gage Length Sections Considered

Equating the value of $R'$ shown in Equation (7) to that shown in Equation (9) leads to:

$$
\sigma_t = \sigma_b \left[ \frac{V_{b_{GL}}}{2(m+1) V_{t_{GL}}} \right]^{1/m}
$$

(19)

Note that in both expressions the values of $\sigma_o$ factors are out.

VII. DISCUSSION AND RESULTS

For convenience, the individual fracture stresses have been presented for all test specimens, both bend and tension. The bend test values have been presented in three groups of 8, 17, and 19, as shown in Table 1, and the tension test values in two groups of 6 and 11, as shown in Table 2. In either type of testing, the smaller number of test results is also included in the larger number.

Based on these experimentally determined values of fracture stresses and the expressions presented in the text, various properties have been determined and shown in Table 3. These properties include mean fracture stresses, standard deviations, coefficients of variance, load factors, Weibull moduli, and scale parameters. The Weibull moduli and scale parameters were determined both by expressions in which consideration was given to total volumes as well as expressions in which consideration was given to volumes of only gage length sections of the specimens.

Values of mean fracture stresses of tension specimens — both those determined from the actual tension tests as well as those predicted from the bend tests — have been listed in Table 4. Percentage discrepancies, i.e., measures of disagreement between predicted and actual mean fracture stresses, are also shown in this table.
Table 1. BEND TEST RESULTS OF NORTON HS-130 GRADE SILICON NITRIDE SPECIMENS TESTED AT ROOM TEMPERATURES

<table>
<thead>
<tr>
<th>Fractures Near or Under Rollers Omitted</th>
<th>Fractures Under Rollers Omitted</th>
<th>All Fractures Included</th>
</tr>
</thead>
<tbody>
<tr>
<td>n</td>
<td>$b_{fract}$ (psi)</td>
<td>n</td>
</tr>
<tr>
<td>---</td>
<td>-----------------</td>
<td>---</td>
</tr>
<tr>
<td>1</td>
<td>74,167</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>82,778</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>95,556</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>97,222</td>
<td>4</td>
</tr>
<tr>
<td>5</td>
<td>100,000</td>
<td>5</td>
</tr>
<tr>
<td>6</td>
<td>101,389</td>
<td>6</td>
</tr>
<tr>
<td>7</td>
<td>106,667</td>
<td>7</td>
</tr>
<tr>
<td>8</td>
<td>107,611</td>
<td>8</td>
</tr>
<tr>
<td>9</td>
<td>109,444</td>
<td>9</td>
</tr>
<tr>
<td>10</td>
<td>106,444</td>
<td>10</td>
</tr>
<tr>
<td>11</td>
<td>106,667</td>
<td>11</td>
</tr>
<tr>
<td>12</td>
<td>101,389</td>
<td>12</td>
</tr>
<tr>
<td>13</td>
<td>101,389</td>
<td>13</td>
</tr>
<tr>
<td>14</td>
<td>103,194</td>
<td>14</td>
</tr>
<tr>
<td>15</td>
<td>106,667</td>
<td>15</td>
</tr>
<tr>
<td>16</td>
<td>107,611</td>
<td>16</td>
</tr>
<tr>
<td>17</td>
<td>109,444</td>
<td>17</td>
</tr>
<tr>
<td>18</td>
<td>109,444</td>
<td>18</td>
</tr>
<tr>
<td>19</td>
<td>109,444</td>
<td>19</td>
</tr>
</tbody>
</table>

Note: All fractures occurred in gage length sections only.

$n$ = number of specimen when fracture stresses are listed in ascending order of magnitude.

$b_{fract}$ = fracture stress of bend specimen

$t_{fract}$ = maximum tensile stress (in outer fiber of gage length section) of bend specimen at fracture.

Table 2. TENSION TEST RESULTS OF NORTON HS-130 GRADE SILICON NITRIDE SPECIMENS TESTED AT ROOM TEMPERATURE

<table>
<thead>
<tr>
<th>Fractures in Gage Length Sections Only</th>
<th>All Fractures (Inside and Outside Gage Length Sections)</th>
</tr>
</thead>
<tbody>
<tr>
<td>n</td>
<td>$t_{fract}$ (psi)</td>
</tr>
<tr>
<td>---</td>
<td>-----------------</td>
</tr>
<tr>
<td>1</td>
<td>62,880</td>
</tr>
<tr>
<td>2</td>
<td>63,030</td>
</tr>
<tr>
<td>3</td>
<td>65,460</td>
</tr>
<tr>
<td>4</td>
<td>73,540</td>
</tr>
<tr>
<td>5</td>
<td>76,720</td>
</tr>
<tr>
<td>6</td>
<td>78,400</td>
</tr>
<tr>
<td>7</td>
<td>78,400</td>
</tr>
<tr>
<td>8</td>
<td>73,540</td>
</tr>
<tr>
<td>9</td>
<td>76,720</td>
</tr>
<tr>
<td>10</td>
<td>76,720</td>
</tr>
<tr>
<td>11</td>
<td>85,410</td>
</tr>
</tbody>
</table>

Legend

$n$ = number of specimen when fracture stresses are listed in ascending order of magnitude.

$t_{fract}$ = fracture stress of tension specimen

$t_{fract}$ = maximum tensile stress (in gage length section) of tension specimen at fracture.
Table 3. VALUES OF MEAN FRACTURE STRESSES, STANDARD DEVIATIONS, COEFFICIENTS OF VARIANCE, LOAD FACTORS, AND SCALE PARAMETERS FOR NORTON 105-130 GRADE SILICON NITRIDE SPECIMENS TESTED AT ROOM TEMPERATURE.

<table>
<thead>
<tr>
<th>Type of Test</th>
<th>Number of Tests</th>
<th>$S_{\text{fract}}$ (psi)</th>
<th>$V_{\text{fract}}$</th>
<th>$V_{\text{fract}}$</th>
<th>$K_t$</th>
<th>$S'$</th>
<th>$V'$</th>
<th>$S'$</th>
<th>$V'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bend</td>
<td>17</td>
<td>94.13</td>
<td>9.903</td>
<td>11.52</td>
<td>11.52</td>
<td>0.01544</td>
<td>1.7219</td>
<td>61.172</td>
<td>0.03994</td>
</tr>
<tr>
<td>Tension</td>
<td>6</td>
<td>79.913</td>
<td>6.440</td>
<td>9.20</td>
<td>11.77</td>
<td>0.01332</td>
<td>1.2146</td>
<td>58.674</td>
<td>0.03504</td>
</tr>
<tr>
<td></td>
<td>11</td>
<td>60.649</td>
<td>7.477</td>
<td>9.73</td>
<td>11.29</td>
<td>0.01577</td>
<td>1.2345</td>
<td>56.626</td>
<td>0.04068</td>
</tr>
</tbody>
</table>

Legend:
- $S_{\text{fract}}$ = mean fracture stress
- $V_{\text{fract}}$ = coefficient of variance of mean fracture stress
- $V_{\text{fract}} = \frac{S_{\text{fract}}}{\text{mean fracture stress}} \times 100$
- $K_t$ = load factor of tension specimen
- $S'$ = scale parameter, see Equations (14) through (17)

Plots of modified risk of rupture of tension specimens versus maximum tensile stress are shown in Figure 1, and plots of probability of fracture of tension specimens versus maximum tensile stress are shown in Figures 2 and 3. These figures offer a means of comparison of tensile properties predicted from bend tests to those determined from actual tension tests. In addition, Figure 1 shows the effects of consideration of volumes of gage length sections in lieu of total volumes of specimens; Figure 2 shows how well a fit exists between the predicted and experimental probabilities of fracture values, i.e., how well the predicted values fit the data; and both Figures 1 and 3 show the effects of a number of tests on the validity of test data.

In Section III only the expressions for determining values of modified risks of rupture have been listed, but the actual derivations of these expressions are shown in the appendixes.

Although it was planned to test 20 each of both bend as well as tension specimens, it should be noted that 19 bend but only 11 tension test results are reported. Extreme difficulty in machining the tension specimens is the primary cause for this difference. Some tension specimens broke during machining and never could be tested. Those that were completed had machining lines in the
Table 4. VALUES OF MEAN FRACTURE STRESSES OF TENSION SPECIMENS DETERMINED BY TENSION TESTS, MEAN FRACTURE STRESSES OF TENSION SPECIMENS DETERMINED BY BEND TESTS, AND PERCENTAGE DISCREPANCIES FOR NORTON HS-130 GRADE SILICON NITRIDE SPECIMENS TESTED AT ROOM TEMPERATURE.

<table>
<thead>
<tr>
<th>Type of Test</th>
<th>Number of Tests</th>
<th>( t_{\text{fract}}^{\text{act}} ) (psi)</th>
<th>( t_{\text{fract}}^{\text{pred}} ) (psi)</th>
<th>( % \text{Discrepancy}^{*} )</th>
<th>( t_{\text{fract}}^{\text{pred}} ) (psi)</th>
<th>( % \text{Discrepancy}^{*} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tension</td>
<td>6</td>
<td>70,013</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>11</td>
<td>69,640</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Bend</td>
<td>8</td>
<td>75,287</td>
<td>7.5</td>
<td>74,894</td>
<td>7.5</td>
<td></td>
</tr>
<tr>
<td></td>
<td>17</td>
<td>75,765</td>
<td>8.2</td>
<td>75,356</td>
<td>8.2</td>
<td></td>
</tr>
<tr>
<td></td>
<td>19</td>
<td>75,644</td>
<td>7.9</td>
<td>75,144</td>
<td>7.9</td>
<td></td>
</tr>
</tbody>
</table>

*Discrepancy between predicted value and actual value of 70,013 psi.

Legend

- \( t_{\text{fract}}^{\text{act}} \) = actual mean fracture stress of tension specimens determined from tension test results (Equation (12)).
- \( t_{\text{fract}}^{\text{pred}} \) = predicted mean fracture stress of tension specimens determined from bend test results. (Equation (18) when total volumes are considered and Equation (19) when volumes of only gage length sections are considered.)
- Discrepancy = disagreement between actual and predicted mean fracture stresses of tension specimens.

\[
\frac{t_{\text{fract}}^{\text{pred}} - t_{\text{fract}}^{\text{act}}}{t_{\text{fract}}^{\text{act}}} \times 100
\]

circumferential direction and the finishes were poor. Some of these were tested and broke below the expected tensile strength. It was theorized that the low strength values were due to direction and degree of surface finish. Recourse was made to machine lapping each of the remaining tension specimens to a 4 rms finish in the longitudinal direction, the same direction, and degree of surface finish as any of the bend specimens. The effect of this operation was to increase the tensile strength substantially. The only bend test result not reported was one in which premature failure occurred because strain gage wires were inadvertently placed between the specimen and one of the steel rollers (load surfaces) of the test fixture. Needless to add, the cost of machining the tension specimens was unusually high.

An important question arises concerning differences in results obtained from expressions involving total volumes to those involving volumes of only gage length sections. The answer to this question may be observed from the plots of modified risks of rupture versus stress level based on both types of expressions as shown in Figure 1. Since for any stress level the lower the value of modified risk of rupture, the lower the value of probability of failure and, conversely, the greater the chances of survival; a plot of this type is useful for comparison purposes. For the results herein reported, it may be seen that little or no differences exist between values of modified risks of rupture determined from
Figure 1. Modified risk of rupture of tension specimens versus maximum tensile stress of Norton HS-130 grade silicon nitride specimens tested at room temperature.

Figure 2. Probability of fracture of tensile specimens versus maximum tensile stress of Norton HS-130 grade silicon nitride specimens tested at room temperature.

Figure 3. Predicted probability of fracture of tension specimens versus maximum tensile stress of Norton HS-130 grade silicon nitride specimens tested at room temperature.
either set of expressions, i.e., those involving total volumes and those involving volumes of only gage length sections. Where the slight differences do exist (bend test values only), the results derived when consideration is given to total volumes are more conservative, i.e., at any given stress level, the probability of failure is greater. When there is any doubt, therefore, as to whether or not the contributions to the value of modified risk of rupture of a specimen due to the omitted sections really are negligible, the expressions involving total volumes should be used.

Another important question arises concerning the number of tests that should be made before valid data can be expected, and this question may be answered with the help of Figures 1 and 3. The spread between the plots in either group (results based on both bend as well as tension tests) indicates differences due to the number of tests selected while the spread between groups of plots indicates differences between values predicted from bend tests and those determined from tension tests, i.e., how well the bend test replaces the tension test. The latter will be discussed shortly.

From Figure 1, plots of modified risk of rupture versus maximum tensile stress within the tension specimen, it may be seen that considerable spread does exist when the number of tension tests is increased from 6 to 11 and when the number of bend tests is increased from 8 to 17 or from 8 to 19. What is more important is that considerable spread (in the reverse direction in this case) exists even when a relatively high number of tests is increased by only 2 more, from 17 to 19.

These spreads are to be expected, however, since \( R' \), modified risk of rupture, really depends upon \( m \) (see Equations (8) and (9)) where \( K_t \) and \( \sigma_0 \) are both functions of \( m \), and \( m \) in turn depends upon \( \bar{\sigma}_{\text{fract}} \) and \( \sigma_{\text{fract}} \) [see Equation (10)].

These last two terms are experimentally determined, and it should be obvious that the smaller the number of tests involved in determining these terms the greater will be the effect on their values when one or more test results, either excessively high or low, are added.

Fortunately, however, the indicated spreads in the probability of fracture plots of Figure 3 are not too bothersome. Only 3% disagreement exists when the tension tests are increased from 6 to 11 or when the bend tests are increased from 8 to 17. Even less disagreement exists when the bend tests are increased from 8 to 19 or from 17 to 19. The reason for these close agreements is due to the insensitivity of the value of modified risk of rupture on the value of probability of fracture as indicated in Equation (3).

To answer the question of required number of tests for valid data, the greater the number of tests, the more valid are the results likely to be, and certainly 20 tests are not too many.

One final question remains concerning substitution of the bend test for the tension test. As mentioned earlier, the spreads between groups of plots of Figures 1 and 3 are indications of the answer to this question. The smaller the spread in any one group of plots, the closer the agreement. In addition, the percentage discrepancies between mean fracture stresses predicted from bend tests to those determined from tension tests, indicated in Table 4, also bring out the answer. It may be seen from these values that the bend test may be substituted for the tension test, at least for the specimen tested, within an accuracy of about 8%.
APPENDIX A. DERIVATION OF MODIFIED RISK OF RUPTURE FOR THIRD-POINT LOADING BEND TEST SPECIMENS

Determination of the value of modified risk of rupture may be made by carrying out the integration indicated by Equation (4) of the text which is repeated here:

\[ R' = \int \left( \frac{P}{P_0} \right)^m \, dA \]  

(A1)

Diagrams indicating type of loading and distribution of both moment and stress will help in carrying out the indicated integration:

THIRD-POINT LOADING BEND TEST

MOMENT DISTRIBUTION

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In the gage length section (middle third of specimen in this case)

\[ M = \frac{P_x}{2} = \frac{P_x}{2} \frac{h}{3} = \frac{P_x}{b} \]  \hspace{1cm} (A2)

\[ \sigma_b = \frac{M h}{2T} = \frac{P_x h}{6} \frac{h}{2T} = \frac{P_x}{12T} \]  \hspace{1cm} (A3)

\[ \sigma_b = \frac{P_x h}{12} \frac{h}{bh^3} = \frac{P_x}{bh^2} \]  \hspace{1cm} (A4)

\[ \sigma_y = \frac{2y}{h} \frac{\gamma'}{b} \]  \hspace{1cm} (A5)

and in the end sections,

\[ M = \frac{P_x}{2} \]  \hspace{1cm} (A6)

\[ \sigma_b = \frac{M h}{2T} = \frac{P_x h}{2} \frac{h}{6T} = \frac{P_x}{4T} \]  \hspace{1cm} (A7)

\[ \sigma_b = \frac{P_x h}{12} \frac{h}{bh^3} = \frac{3P_x}{bh^3} \]  \hspace{1cm} (A8)

\[ \sigma_y = \frac{2y}{h} \frac{3P_x}{bh^2} = \frac{6P_x y}{bh^3} \]  \hspace{1cm} (A9)

Carrying out the integration throughout the total volume of the specimen leads to:

\[ R^* = b \left[ \int_0^{\frac{h}{3}} \int_0^{\frac{h}{2}} \left( \frac{2y}{h} \frac{\gamma'}{b} \sigma_o \right)^m dydx + 2 \int_0^{\frac{h}{3}} \int_0^{\frac{h}{2}} \left( \frac{6P_x y}{bh^3 \sigma_o} \right)^m dydx \right] \]  \hspace{1cm} (A10)
where \( \sigma_{p} \) = maximum tensile fiber stress (in gage length section) of specimen

or

\[
\sigma_{b} = \frac{3P}{bh^{2}} = \frac{P_{f}}{bh^{2}}.
\]

Continuing with the integration leads to:

\[
R' = b \left[ \left( \frac{2\sigma_{b}}{h} \right)^{m} \int_{0}^{h/2} y^{m} dy + (2) \left( \frac{6P}{bh^{3}\sigma_{o}} \right)^{m} \int_{0}^{h/2} x^{m} y^{m} dy \right].
\]

\[
R' = b \left[ \frac{2}{3} \int_{0}^{h/2} y^{m} dy + \frac{2^{m+1}}{3m+1} \frac{3^{m+1}}{m+1} \int_{0}^{h/2} y^{m} dy \right].
\]

\[
R' = \frac{b h^{m+1}}{3(m+1)} \left[ \frac{2^{m} \sigma_{b}^{m}}{h} + \frac{2^{m+1} 3m+1}{m+1} \right]
\]

\[
R' = \frac{b h^{m+1}}{3^{m+1} m+1} \left[ \frac{2^{m} \sigma_{b}^{m}}{h} + \frac{2^{m+1} 3m+1}{m+1} \right]
\]

\[
R' = \frac{b h^{m+1}}{(2)(3)(m+1)} \left( \frac{\sigma_{b}}{\sigma_{o}} \right)^{m} \left[ 1 + \frac{2}{m+1} \right]
\]

\[
R' = \frac{V_{b} (m+3)}{6(m+1)} \left( \frac{\sigma_{b}}{\sigma_{o}} \right)^{m}
\]

and finally

17
\[ R' = K_b \left( \frac{\sigma_b}{\sigma_o} \right)^m \]  \hspace{1cm} (A19)

where

\[ K_b = \frac{m+3}{6(m+1)^2}. \]

And listing only the first half of the expression shown in Equation (A16) leads to that portion contributed to the value of \( R' \) from only the gage length section of the specimen:

\[ R' = \frac{b \cdot h}{(3)(2)(m+1)} \left( \frac{\sigma_b}{\sigma_o} \right)^m \]  \hspace{1cm} (A20)

\[ R' = V_{b_{GL}} \left( \frac{\sigma_b}{\sigma_o} \right)^m \]  \hspace{1cm} (A21)

or

\[ R' = K_b \cdot V_{b_{GL}} \left( \frac{\sigma_b}{\sigma_o} \right)^m \]  \hspace{1cm} (A22)

where

\[ K_b = \frac{1}{2(m+1)}. \]
APPENDIX B. DERIVATION OF MODIFIED RISK OF RUPTURE FOR DUMBELL-SHAPED TENSION TEST SPECIMENS

Determination of the modified risk of rupture of any specimen may be made by carrying out the integration indicated by Equation (4) of the text, and again this expression is repeated here:

\[ R^* = \int_V \left( \frac{\sigma}{\sigma_0} \right)^m dV \]  \hspace{1cm} (B1)

A loading diagram and information concerning stress distribution that will help in carrying out the indicated integration are:
1. Gage Length Section

\[ r_x = r_i = 0.100 \]  

\[ \sigma_x = \tau = \frac{(0.498^2 - 0.100^2)}{0.100^2} P_i = 23.80 \text{ } P_i \]  

2. Elliptical Section

\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \]

\[ \frac{x^2}{0.75^2} = \frac{y^2}{0.14^2} = 1 \]

\[ r_x = 0.24 - 0.14 \sqrt{1 - \left(\frac{x}{0.75}\right)^2} \]  

\[ \frac{\sigma_x}{23.80 \text{ } P_i} = \frac{0.01}{r_x^2} \]

\[ \sigma_x = \frac{0.238 \text{ } P_i}{r_x^2} \]  

3. 45° Section

\[ r_x = x + 0.2143 \]  

\[ \sigma_x = \frac{0.238 \text{ } P_i}{r_x^2} \]  

With the above information, the integration may now be carried out in three parts:

\[ R' = 2\pi \int_0^{0.7500} \left( \frac{\sigma_x}{\sigma_0} \right) r_x^2 \text{ } dx + 2\pi \int_0^{0.7573} \left( \frac{\sigma_x}{\sigma_0} \right)^m r_x^2 \text{ } dx + 2\pi \int_0^{0.2837} \left( \frac{\sigma_x}{\sigma_0} \right)^m r_x^2 \text{ } dx \]  

where the values of \( \sigma_x \) and \( r_x \) are those listed in the stress distribution information.
Carrying out the required integration leads to:

1. Contribution to $R'$ from Gage Length Section

$$R' = (2\pi) \left( \frac{23.80\pi_i}{\sigma_o} \right)^m \int_0^{0.7500} (0.0100) \, dx$$  \hfill (B9)

$$R' = (0.015\pi) \left( \frac{23.80\pi_i}{\sigma_o} \right)^m$$  \hfill (B10)

or

$$R' = K_t V_{GL} \left( \frac{\sigma_t}{\sigma_o} \right)^m$$  \hfill (B11)

i.e., the value of $K_t = 1.0$ regardless of the value of $m$.

2. Contribution to $R'$ from Elliptical Sections

$$R' = (2\pi) \left( \frac{0.258\pi_i}{\sigma_o} \right)^m \int_0^{0.7375} \frac{r^2 - 2m}{2x} \, dx$$  \hfill (B12)

$$R' = 0.015\pi \left( \frac{23.80\pi_i}{\sigma_o} \right)^m \left[ \frac{(2)(0.258)^m}{(0.015)(23.80)^m} \right] \int_0^{0.7375} \frac{r^2 - 2m}{2x} \, dx$$  \hfill (B13)

or

$$R' = (K_t) \left( V_{GL} \right) \left( \frac{\sigma_t}{\sigma_o} \right)^m$$  \hfill (B14)

and by numerical integration for the pertinent values of $m$ the values of $K_t$ are:

<table>
<thead>
<tr>
<th>$m$</th>
<th>$K_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10.94</td>
<td>0.2386</td>
</tr>
<tr>
<td>11.27</td>
<td>0.2548</td>
</tr>
<tr>
<td>11.29</td>
<td>0.2545</td>
</tr>
<tr>
<td>11.52</td>
<td>0.2519</td>
</tr>
<tr>
<td>13.27</td>
<td>0.2146</td>
</tr>
</tbody>
</table>
3. Contributions to $R'$ by 45° Sections

$$R' = (2\pi) \left( 0.238 \right)^m \left( \frac{p_1}{\sigma_0} \right)^m \int_0^{0.2837} (x + 0.2143)^{2-2m} dx \quad (B15)$$

$$R' = 0.2837 \left( \frac{23.80 p_1}{\sigma_0} \right)^m \left[ (2) (0.238)^m \right] \left[ \frac{0.2837}{(0.015)(23.80)^m} \right] \int_0^{0.2837} (x + 0.2143)^{2-2m} dx \quad (B16)$$

But

$$\int_0^{0.2837} (x + 0.2143)^{2-2m} = \left[ \frac{x + 0.2143}{3-2m} \right]_0^{0.2837} \quad (B17)$$

$\therefore$ $R' = 0.2837 \left( \frac{23.80 p_1}{\sigma_0} \right)^m \left[ (2) (0.238)^m \right] \left[ \frac{0.2837}{(0.015)(23.80)^m} \right] \left[ \frac{(x + 0.2143)^{3-2m}}{3-2m} \right] \quad (B18)$

$$R' = 0.2837 \left( \frac{23.80 p_1}{\sigma_0} \right)^m \left[ (2) (0.238)^m \right] \left[ \frac{(0.498)^{3-2m} - (0.2143)^{3-2m}}{3-2m} \right] \quad (B19)$$

$$R' = 0.2837 \left( \frac{23.80 p_1}{\sigma_0} \right)^m \left[ \frac{2}{(0.015)(3-2m)} \right] \left[ \frac{0.498^{3-2m} - 0.2143^{3-2m}}{100^m} \right] \quad (B20)$$

Or

$$R' = (K_t) \left( \frac{\sigma_t}{\sigma_0} \right) \left( \frac{\sigma_t}{\sigma_0} \right)^m \quad (B21)$$

Where for the pertinent values of $m$ the values of $K_t$ are:

$$\begin{array}{cc}
  m & K_t \\
  10.94 & 3.974 \times 10^{-9} \\
  11.27 & 2.322 \\
  11.29 & 2.247 \\
  11.52 & 1.547 \\
  13.27 & 0.091 \\
\end{array}$$
Contribution to $R'$ from Total Specimen

The values of $R'$, then, for the total specimen is simply the sum of the values contributed by its parts, or

$$R' = K_t \cdot V_{tGL} \left( \frac{\sigma_t}{\sigma_o} \right)^m$$

(B22)

where for the pertinent values of $m$ the values of $K_t$ are:

<table>
<thead>
<tr>
<th>$m$</th>
<th>$K_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10.94</td>
<td>1 + 0.2386 = 1.2386</td>
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<td>1 + 0.2146 = 1.2146</td>
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