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D. R. Fulkerson, et al

Cornell University

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**Authors:** D. R. Fulkerson & Gary Harding

**Performing Organization:**
Department of Operations Research
College of Engineering, Cornell University
Ithaca, New York 14853

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**Abstract:**
Edmonds has given a complicated algorithmic proof of a theorem characterizing directed graphs that contain edge-disjoint branchings having specified root sets. Tarjan has described a conceptually simple and good algorithm for finding such branchings when they exist. Tarjan's algorithm is based on a lemma implicit in Edmonds' results. A simple direct proof of this lemma is given, thereby providing a simpler proof of Edmonds' theorem and a simpler proof that Tarjan's algorithm works.
DEPARTMENT OF OPERATIONS RESEARCH
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CORNELL UNIVERSITY
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by

D. R. Fulkerson\(^1\) and Gary Harding

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1. **Introduction.** In [1] Edmonds has given a proof of a theorem (Theorem 3.1 below) characterizing those directed graphs that contain k mutually edge-disjoint branchings (spanning arborescences) having specified root sets. His proof is based on a complicated algorithm for constructing such branchings when they exist. While it is not known whether this algorithm is good (runs in polynomial time), Tarjan has described a conceptually simple and good algorithm for finding k mutually edge-disjoint branchings, when they exist [4]. Tarjan's algorithm is based on a lemma (Lemma 2 of [4]; slightly generalized below as Theorem 3.2) and network flow routines. Tarjan's proof of this lemma invokes Edmonds' theorem and algorithm; indeed, as Tarjan notes, his lemma is implicit in Edmonds' results. This poses the problem, pointed out by Tarjan in [4], of finding a simple direct proof of his lemma, one that avoids invoking Edmonds' theorem and its complicated algorithmic proof. The purpose of this note is to give such a proof, thereby providing a simpler proof of Edmonds' theorem and a simpler proof that Tarjan's algorithm works.

2. **Definitions and notation.** A directed graph \( G = [V,E] \) consists of a finite set of vertices \( V \) and a finite set of edges \( E \) such that each edge \( e \in E \) has a head \( h(e) \in V \) and a tail \( t(e) \in V \). We sometimes denote an edge \( e \) by the ordered pair \( (t(e), h(e)) \), even though there may be multiple edges in \( G \) having the same tail and head. A subgraph \( G' = [V',E'] \) of \( G \) is a directed graph having vertex-set \( V' \subseteq V \) and edge-set \( E' \subseteq E \) such that for all \( e \in E' \), we have \( t(e) \in V' \) and \( h(e) \in V' \). A directed path in \( G \) from \( u \in V \) to \( v \in V \) is a sequence \( u = t(e_1), e_1, h(e_1) = t(e_2), e_2, \ldots, h(e_{n-1}) = t(e_n), e_n, h(e_n) = v \), composed alternately of vertices and edges of \( G \). A vertex \( u \) is itself a directed path from \( u \) to \( u \) having no edges.
For any non-empty subset \( R \) of vertices of \( G \), a branching \( B \) of \( G \), rooted at \( R \), is a subgraph of \( G \) such that for every vertex \( v \) of \( G \), there is precisely one directed path in \( B \) from a vertex in \( R \) to \( v \).

For any \( X \subseteq V \), let

\[
\delta^+_G(X) = \{ e \in E : t(e) \in X \text{ and } h(e) \in X \},
\]

\[
\delta^-_G(X) = \{ e \in E : h(e) \in X \text{ and } t(e) \in X \}.
\]

For \( X \subseteq V \), \( Y \subseteq V \), we use the notation

\[
(X,Y) = \{ e \in E : t(e) \in X, h(e) \in Y \}.
\]

Thus \( \delta^+_G(X) = (X,\overline{X}) \), \( \delta^-_G(X) = (\overline{X},X) \). If \( X \) is a non-empty proper subset of \( V \), the set \( \delta^+_G(X) = \delta^-_G(\overline{X}) = (X,\overline{X}) \) is a cut in \( G \); it separates any vertex \( x \in X \) from any vertex \( y \in \overline{X} \), i.e., any directed path in \( G \) from \( x \in X \) to \( y \in \overline{X} \) contains at least one edge of \( \delta^+_G(X) \).

If \( S \) is a set, we let \( |S| \) denote the cardinality of \( S \). As above, we use the symbol "\( \subseteq \)" for set inclusion; henceforth we use "\( \subset \)" for proper inclusion.

3. Main theorems. Let \( G = [V,E] \) be a directed graph with designated non-empty root-sets \( R_1, R_2, \ldots, R_k \), and suppose that \( G \) contains \( k \) mutually edge-disjoint branchings \( B_1, B_2, \ldots, B_k \), where \( B_i \) is rooted at \( R_i \), \( i = 1, 2, \ldots, k \). Then it is clear that for every proper subset \( X \) of \( V \), we must have

\[
|\delta^+_G(X)| \geq |\{i : 1 \leq i \leq k \text{ and } R_i \subseteq X\}|.
\]
Theorem 3.1 (Edmonds). For any directed graph $G = [V, E]$ and any sets $R_i$, $\emptyset \neq R_i \subseteq V$, $1 \leq i \leq k$, there exist mutually edge-disjoint branchings $B_i$, $1 \leq i \leq k$, rooted respectively at $R_i$, if and only if, for every proper subset $X$ of $V$, we have

$$|\delta^+(X)| \geq |\{i: 1 \leq i \leq k \text{ and } R_i \subseteq X\}|.$$  

Theorem 3.2. Suppose given a directed graph $G = [V, E]$ and a class of non-empty subsets $R_1, R_2, \ldots, R_k$ of $V$ such that (3.1) holds for all $X \neq V$. Let $B_1 = [V_1, E_1]$ be a subgraph of $G$ such that $R_1 \subseteq V_1$ and on the subgraph $G' = [V, E - E_1]$ we have

$$|\delta^+(X)| \geq |\{i: 2 \leq i \leq k \text{ and } R_i \subseteq X\}|, \text{ all } X \neq V.$$  

Then if $V_1 \neq V$, there is an edge $e \in \delta^+(V_1)$ and for all $X \neq V,$

$$e \in \delta^+(X) \Rightarrow |\delta^+(X)| \geq |\{i: 2 \leq i \leq k \text{ and } R_i \subseteq X\}| + 1.$$  

In applying Theorem 3.2 and Tarjan's algorithm to prove Theorem 3.1, the subgraph $B_1$ of Theorem 3.2 would be taken to be a branching rooted at $R_1$ of some subgraph of $G$. In this instance, Theorem 3.2 reduces to Lemma 2 of [4].

4. Proof of Theorem 3.2. We begin the proof of Theorem 3.2 with some preliminary lemmas. It is convenient first to extend $G$ by adding a "source" vertex $s$ and vertices $r_1, r_2, \ldots, r_k$ corresponding to the root-sets $R_1, R_2, \ldots, R_k$. We also add the edges $(s, r_1)$, together with the sets of edges
(r_i, R_i), i = 1, 2, ..., k, thereby obtaining an enlarged directed graph \( H = [N, A] \) containing \( G \) as a subgraph. Note that all edges joining \( N-V \) and \( V \) in \( H \) are directed into \( V \), i.e. \( (V, N-V) = \emptyset \). Corresponding to the subgraph \( B_1 = [V_1, E_1] \) of \( G \) there is an "\( s \)-rooted subgraph" \( \tilde{B}_1 \) of \( H \) having vertex-set \( V_1 \cup \{ s, r_i \} \) and edge-set \( A_1 = E_1 \cup (s, r_1) \cup (r_1, R_1) \); hence, corresponding to the subgraph \( G' = [V, E-E_1] \) of \( G \) there is the subgraph \( H' = [N, A-A_1] \) of \( H \).

**Lemma 4.1.** Condition (3.1) implies that there are at least \( k \) mutually edge-disjoint directed paths in \( H \) from \( s \) to \( v \), for all \( v \in V \).

**Proof.** By the max-flow min-cut theorem and the integrity theorem for network flows [2], it suffices to show that if \( (S, N-S) \) is a cut in \( H \) separating \( s \) from \( v \), then \( |(S, N-S)| = |\delta^+(S)| \geq k \). Let \( (S, N-S) \) be a cut in \( H \) with \( s \in S \), \( v \in N-S = \overline{S} \). Let \( R = \{ r_1, r_2, ..., r_k \} \). We may partition \( (S, \overline{S}) \) as follows:

\[
(s, \overline{S}) = (s, R \cap \overline{S}) \cup (R \cap S, V \cap \overline{S}) \cup (V \cap S, V \cap \overline{S}).
\]

Suppose \( R_i \not\subseteq S \). Then either \( r_i \not\in S \), in which case \( (s, r_i) \notin (s, R \cap \overline{S}) \), or \( r_i \in S \) and there is a vertex \( u \in R_i \cap \overline{S} \), in which case \( (r_i, u) \notin (R \cap S, V \cap \overline{S}) \). Thus

\[
|s, R \cap \overline{S}) \cup (R \cap S, V \cap \overline{S})| \geq |\{ i: 1 \leq i \leq k \text{ and } R_i \not\subseteq S \}|.
\]

Since \( v \in V \cap \overline{S} \), we have \( V \cap S \subseteq V \), and hence condition (3.1) implies

\[
|V \cap S, V \cap \overline{S})| \geq |\{ i: 1 \leq i \leq k \text{ and } R_i \subseteq S \}|.
\]
It follows from (4.1), (4.2), and (4.3) that $|(S,\overline{S})| \geq k$, as was to be shown.

A similar proof establishes

**Lemma 4.2.** Condition (3.2) implies that there are at least $(k-1)$ mutually edge-disjoint directed paths in $H'$ from $s$ to $v$, for all $v \in V$.

We next state two lemmas that are valid for any directed graph with "source" $s$. (Later on they will be applied to the directed graph $H'$.) While these lemmas can be found in a recent paper by Lovász [3], they are consequences of well-known results in network flow theory. In particular, the second of the two (Lemma 4.4 below) is stated explicitly in [2, Chap. I]. We describe these lemmas as in [3], using the following definition. In a directed graph $H = [N,A]$ with "source" $s \in N$, let $m(s,x)$ denote the maximum number of mutually edge-disjoint directed paths from $s$ to $x$, for $x \in N - \{s\}$. Say that a set $X \subseteq N - \{s\}$ is regular with core $x$ if $x \in X$ and $m(s,x) = \delta^+_H(X)$. (In other words, $X$ is the "sink" set of a minimum cut $(X, \overline{X})$ separating $s \in \overline{X}$ from $x \in X$ in $H$.)

**Lemma 4.3.** If $X$ and $Y$ are regular sets with cores $x$ and $y$, respectively, and if $x \in Y$, then $X \cap Y$, $X \cup Y$ are regular with cores $x$, $y$, respectively.

**Lemma 4.4.** For each vertex $x \neq s$, there is a regular set $T_x$ with core $x$ such that whenever $X$ is a regular set with core $x$, then $T_x \subseteq X$.

We continue with the proof of Theorem 3.2. Suppose that (3.1) and (3.2) hold, but that (3.3) does not hold. Let $e_j$, $j \in J$, be an enumeration of the edges of $G$ comprising the set $\delta^+_G(V_1)$. Thus for each $e_j \in \delta^+_G(V_1)$ there is a set $S_j \neq V$ such that $e_j \in \delta^+_G(S_j)$ and
\[ |\delta^+_{G}(s_j)| < |\{i: 2 \leq i \leq k \text{ and } R_i \subseteq S_j\}| + 1. \]

Combined with (3.2), this yields

\[ (4.4) \quad |\delta^+_{G}(s_j)| = |\{i: 2 \leq i \leq k \text{ and } R_i \subseteq S_j\}| \]

for all \( j \in J \).

We want to work with the enlarged directed graphs \( H \) and \( H' \), rather than \( G \) and \( G' \). Hence we define

\[ (4.5) \quad T_j = S_j \cup \{s\} \cup \{r_i: R_i \subseteq S_j\}. \]

It follows that

\[ (4.6) \quad |\delta^+_{H}(T_j)| = k-1, \quad j \in J. \]

To see this, note first that if \( R_i \notin S_j \), then \( r_i \notin T_j \), and hence \((s, r_i) \in \delta^+_{H}(T_j)\), whereas no edge \( e \) of \( H' \) with tail \( t(e) = r_i \) belongs to \( \delta^+_{H}(T_j) \). On the other hand, if \( R_i \subseteq S_j \), then no edge incident to \( r_i \) belongs to \( \delta^+_{H}(T_j) \). Thus

\[ |\delta^+_{H}(T_j)| = |\delta^+_{G}(s_j)| + |\{i: 2 \leq i \leq k \text{ and } R_i \notin S_j\}|. \]

By (4.4), we have

\[ |\delta^+_{H}(T_j)| = |\{i: 2 \leq i \leq k \text{ and } R_i \subseteq S_j\}| + |\{i: 2 \leq i \leq k \text{ and } R_i \notin S_j\}| = k-1, \]
verifying (4.6).

Since \( s \in T_j \) and \( h(e_j) \in \overline{T}_j = N - T_j \), the set \( \delta_{H'}^+(T_j) \) is a cut in \( H' \) of size \( k - 1 \) separating \( s \) and \( h(e_j) \), for all \( j \in J \). Lemma 4.2 thus implies that \( \overline{T}_j \) is regular with core \( h(e_j) \). Using Lemma 4.4, we may assume that \( \overline{T}_j \) is minimal with core \( h(e_j) \). (Note that with this assumption, we still have \( e_j \in \delta_{H'}^-(\overline{T}_j) \).

Lemma 4.5. Let the sets \( \overline{T}_j \) be minimal regular sets in \( H' \) with cores \( h(e_j) \), \( j \in J \). Suppose that \( j, k \in J \) with \( h(e_k) \in \overline{T}_j \). Then \( \overline{T}_k \subseteq \overline{T}_j \). If \( \overline{T}_k \subseteq \overline{T}_j \), then \( h(e_j) \notin \overline{T}_k \).

Lemma 4.5 follows from Lemma 4.3, since if \( \overline{T}_k \) and \( \overline{T}_j \) are regular with cores \( h(e_k) \) and \( h(e_j) \), respectively, and if \( h(e_k) \in \overline{T}_j \), then \( \overline{T}_k \cap \overline{T}_j \) is regular with core \( h(e_j) \). Since \( \overline{T}_k \) is minimal with respect to this property, we have \( \overline{T}_k \subseteq \overline{T}_k \cap \overline{T}_j \), and hence \( \overline{T}_k \subseteq \overline{T}_j \). If this inclusion is proper and if \( h(e_j) \in \overline{T}_k \), then \( \overline{T}_k \) would be regular with core \( h(e_j) \), contradicting the minimality of \( \overline{T}_j \). Thus if \( \overline{T}_k \subseteq \overline{T}_j \), then \( h(e_j) \notin \overline{T}_k \).

We apply Lemma 4.5 repeatedly to prove the next lemma.

Lemma 4.6. Let the sets \( \overline{T}_j \) be minimal regular sets in \( H' \) with cores \( h(e_j) \), \( j \in J \). There is at least one \( j^* \in J \) such that if \( j \in J \) and \( h(e_j) \in \overline{T}_{j^*} \), then \( \overline{T}_j = \overline{T}_{j^*} \).

To prove this lemma, select any \( j_0 \in J \). Define \( J_0 = \{ j \in J : h(e_j) \in \overline{T}_{j_0} \} \). If for all \( j \in J_0 \), we have \( \overline{T}_j = \overline{T}_{j_0} \), then take \( j^* = j_0 \). Otherwise, there is a \( j_1 \in J_0 \) such that \( \overline{T}_{j_1} \neq \overline{T}_{j_0} \), in which case Lemma 4.5 asserts that \( \overline{T}_{j_1} \subseteq \overline{T}_{j_0} \), \( h(e_{j_0}) \notin \overline{T}_{j_1} \). Define \( J_1 = \{ j \in J : h(e_j) \in \overline{T}_{j_1} \} \). If for all \( j \in J_1 \), we have \( \overline{T}_j = \overline{T}_{j_1} \), then take \( j^* = j_1 \). Otherwise, there is a \( j_2 \in J_1 \) such that \( \overline{T}_{j_2} \neq \overline{T}_{j_1} \), in which case Lemma 4.5 asserts that \( \overline{T}_{j_2} \subseteq \overline{T}_{j_1} \), \( h(e_{j_1}) \notin \overline{T}_{j_2} \).
Define \( J_2 = \{ j \in J : h(e_j) \in \overline{T}_{j_2} \} \), and so on. Since \( \overline{T}_{j_0} \supset \overline{T}_{j_1} \supset \overline{T}_{j_2} \supset \ldots \), we must eventually find a \( j^* \) satisfying the conclusion of the lemma.

We show next that \( \overline{T}_{j^*} \cap V_1 = \emptyset \). To this end we examine the set of edges \((\overline{T}_{j^*} \cap V_1, \overline{T}_{j^*} - V_1)\) in \( H' \). Suppose first that \((\overline{T}_{j^*} \cap V_1, \overline{T}_{j^*} - V_1) \neq \emptyset \). In this case there is an edge \( e \) with \( t(e) \in \overline{T}_{j^*} \cap V_1 \), \( h(e) \in \overline{T}_{j^*} - V_1 \), and hence \( e \) is one of the edges \( e_j, j \in J \). Since \( h(e_j) \in \overline{T}_{j^*} \), Lemma 4.6 implies \( \overline{T}_j = \overline{T}_{j^*} \). But we have \( t(e_j) \notin \overline{T}_j \), \( t(e_j) \in \overline{T}_{j^*} \), contradicting \( \overline{T}_j = \overline{T}_{j^*} \).

Thus \((\overline{T}_{j^*} \cap V_1, \overline{T}_{j^*} - V_1) = \emptyset \). But then \( \overline{T}_{j^*} - V_1 \) is regular with core \( h(e_{j^*}) \) and hence, since \( \overline{T}_{j^*} \) is minimal with respect to this property, we must have \( \overline{T}_{j^*} = \overline{T}_{j^*} - V_1 \), which implies \( \overline{T}_{j^*} \cap V_1 = \emptyset \).

Thus we have established the existence of \( j^* \in J \) and \( \overline{T}_{j^*} \) such that

\[
|\delta_H^-(\overline{T}_{j^*})| = k - 1 \quad \text{and} \quad \overline{T}_{j^*} \cap V_1 = \emptyset.
\]

It follows from (4.7) that

\[
|\delta_H^-(\overline{T}_{j^*})| = |\delta_H^-(\overline{T}_{j^*})| = k - 1.
\]

Thus \((T_{j^*}, \overline{T}_{j^*})\) is a cut in \( H \) separating \( z \) from \( h(e_{j^*}) \in V \) having only \( k - 1 \) members, contradicting Lemma 4.1. Hence our assumption that (3.3) does not hold is untenable. This completes the proof of Theorem 3.2.
References.


