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ON A CLASS OF BINOMIAL ATTRITION PROCESSES

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June 1974

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This paper provides underlying sets of assumptions and rigorous derivations of certain static combat attrition processes. Considered are a homogeneous point fire process, a heterogeneous point fire process, and an area fire process. Each set of assumptions is complete, concise, unambiguous, and as general--both mathematically and in terms of physical interpretation--as possible. Some attention is also devoted to various approximations and computational simplifications of the models presented.

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I. INTRODUCTION

The purpose of this paper is to summarize work of Karr (1972a, 1972b, 1973) concerning a class of models of combat attrition processes which, in the sense that attrition occurs instantaneously, do not evolve over time (although discrete time dynamic processes can be constructed from these essentially static models as we briefly discuss). We refer the reader to Karr (1974) for a detailed discussion of a class of continuous time parameter stochastic processes analogous to F. W. Lanchester's differential equation models.

Implementation of these static models in computerized combat simulations is sometimes by means of an exponential approximation to an equation for expected attrition which is binomial in form (hence the title of the paper). We discuss the desirability and applicability of such approximations, but only after first giving rigorous derivations, from carefully and precisely stated probabilistic assumptions, of the expected attritions in homogeneous and heterogeneous cases of point fire combat (where a target must be detected before it can be attacked) and for area fire directed at a homogeneous set of targets. The meaning and appropriateness of the families of underlying assumptions may then be discussed in a useful manner. In addition to exponential approximations we consider some computational simplifications in the heterogeneous point fire case and difficulties which occur when, as happens in an iterative computerized simulation, random variables are replaced in computations by their expectations.

The attrition equations we derive here are one-sided in the sense of expressing attrition to each force in terms of its initial strength, the initial strength of the opposition, and parameters describing the physical situation. All attrition is suffered instantaneously. Two-sided models can be constructed but only in a rather contrived manner.

II. HOMOGENEOUS POINT FIRE MODEL

1. Basic Derivation

Consider a one-sided combat between two homogeneous forces, a force of R indistinguishable "targets" and a force of B indistinguishable "searchers". We make the following assumptions concerning this combat.

A1) At a fixed time all R targets become vulnerable to detection and attack by the B searchers;

A2) The probability that the ith searcher detects the jth target is d for all $i = 1, \ldots, B$ and $j = 1, \ldots, R$. Each particular searcher detects different targets; independently of one another;

A3) A searcher who makes no detections makes no attack. A searcher who makes one or more detections chooses one target to attack according to a uniform distribution over the set of targets he has detected, independent of his detection process;

A4) The conditional probability that a searcher kills a target given detection and attack is k, for all searchers and targets;

A5) No searcher may attack more than one target;

A6) Detection and attack processes of different searchers are mutually independent.

We begin by computing the expected number of targets killed.

(1) PROPOSITION. Assume A1) - A6) are satisfied. Then if K denotes the number of targets killed, we have

$$E[K] = R\left\{1 - \left(1 - \frac{k}{R}\left[1 - (1 - d)^R\right]\right)^B\right\} .$$

PROOF. By elementary conditional probability arguments,

$P\{\text{searcher } i \text{ kills target } j\}$

$= P\{\text{searcher } i \text{ detects and attacks and kills target } j\}$

$= P\{\text{searcher } i \text{ kills target } j \mid \text{searcher } i \text{ detects and attacks target } j\} \cdot P\{\text{searcher } i \text{ detects and attacks target } j\}$

$= k\, P\{\text{searcher } i \text{ detects and attacks target } j\}$

$= k\, P\{\text{searcher } i \text{ attacks target } j \mid \text{searcher } i \text{ detects target } j\} \cdot P\{\text{searcher } i \text{ detects target } j\}$

$= kd\, P\{\text{searcher } i \text{ attacks target } j \mid \text{searcher } i \text{ detects target } j\}$.

By the Law of Total Probability,

$P\{\text{searcher } i \text{ attacks target } j \mid \text{searcher } i \text{ detects target } j\}$

$= \sum_{m=0}^{R-1} P\{\text{searcher } i \text{ attacks target } j \text{ and detects } m \text{ other targets} \mid \text{searcher } i \text{ detects target } j\}$

$= \sum_{m=0}^{R-1} P\{\text{searcher } i \text{ attacks target } j \mid \text{searcher } i \text{ detects target } j \text{ and exactly } m \text{ other targets}\} \cdot P\{\text{searcher } i \text{ detects exactly } m \text{ other targets} \mid \text{searcher } i \text{ detects target } j\}$

by the identity $P\{A \cap B \mid C\} = P\{A \mid B \cap C\} P\{B \mid C\}$.

By assumption A3)

$P\{\text{searcher } i \text{ attacks target } j \mid \text{searcher } i \text{ detects target } j \text{ and exactly } m \text{ other targets}\} = \frac{1}{m+1}$

while by A2)

$= P\{\text{searcher } i \text{ detects exactly } m \text{ of the other R-1 targets}\}$

$= \binom{R-1}{m} d^m (1-d)^{R-1-m}$.

Therefore

$$P\{\text{searcher } i \text{ attacks target } j \mid \text{searcher } i \text{ detects target } j\}$$

$$= \sum_{m=0}^{R-1} \frac{1}{m+1} \binom{R-1}{m} d^m (1-d)^{R-m-1}$$

$$= \sum_{m=0}^{R-1} \frac{1}{m+1} \frac{(R-1)!}{m!(R-1-m)!} d^m (1-d)^{R-m-1}$$

$$= \frac{1}{dR} \sum_{m=0}^{R-1} \frac{R!}{(m+1)!(R-(m+1))!} d^{m+1} (1-d)^{R-(m+1)}$$

$$= \frac{1}{dR} [1 - (1-d)^R] \quad .$$

We thus obtain

$$P\{\text{searcher } i \text{ kills target } j\} = \frac{k}{R} [1 - (1-d)^R]$$

which, as one expects, is independent of i and j. It follows by the independence assumption A6) that

(2) $P\{\text{target } j \text{ is killed}\}$

$$= 1 - \{P \text{ target } j \text{ is not killed}\}$$

$$= 1 - P\{\bigcap_{i=1}^{B} \{\text{searcher } i \text{ does not kill target } j\}\}$$

$$= 1 - \prod_{i=1}^{B} \{P \text{ searcher } i \text{ does not kill target } j\}$$

$$= 1 - \prod_{i=1}^{B} (1 - \frac{k}{R} [1 - (1-d)^R])$$

$$= 1 - (1 - \frac{k}{R} [1 - (1-d)^R])^B$$

so that we finally have

$$E[K] = \sum_{j=1}^{R} P\{\text{target } j \text{ is killed}\}$$

$$= \sum_{j=1}^{R} \{1 - (1 - \frac{k}{R}[1 - (1 - d)^R])^B\}$$

$$= R\{1 - (1 - \frac{k}{R}[1 - (1 - d)^R])^B\}$$

as asserted. ∎

The possibility that two or more searchers "kill" the same target is not excluded.

2. Probability Distribution of the Number of Kills

To compute the probability distribution of the number of targets killed, which is needed to extend this static model to a discrete time dynamic model, we take an alternative approach. It is not true, as one might naively conjecture, that the number of targets killed is binomially distributed with parameters R and q, where q is the probability given on the right hand side of (2). The reason for this is that even though searchers operate independently of one another, different targets do not die independently of one another. Since each searcher can attack at most one target, knowing that a particular target was killed means that some other target was not attacked by the searcher that killed the former target and is hence less likely to have been killed.

(3) PROPOSITION. For m such that $m \leq R$ and $m \leq B$ we have

$$(4) \qquad P\{K = m\} = \binom{R}{m} \sum_{r=0}^{m} (-1)^{m-r} \binom{m}{r} [(1-q_R) + \frac{q_R r}{R}]^B$$

where

$$q_R = k[1 - (1 - d)^R] \quad .$$

PROOF. To begin, the Law of Total Probability implies that

$$P\{K = m\} = \sum_{\ell=m}^{B} P\{K = m, \quad \ell \text{ "fatal" attacks are made}\}$$

$$= \sum_{\ell=m}^{B} P\{K = m \mid \ell \text{ "fatal" attacks are made}\}$$
$$P\{\ell \text{ "fatal" attacks are made}\}$$

where the sum is zero if the lower limit exceeds the upper. Here a "fatal" attack is one which would kill the target attacked if no other attacker attacked the same target. The fact that two or more searchers may simultaneously attack and "kill" the same target means that there may be more "fatal" attacks than targets killed.

Now by (2), since each searcher can kill at most one target,

$$P\{\text{searcher } i \text{ makes a "fatal" attack}\}$$
$$= \sum_{j=1}^{R} P\{\text{searcher } i \text{ kills target } j\}$$

$$= \sum_{j=1}^{R} \frac{k}{R} [1 - (1 - d)^R] \tag{5}$$

$$= k[\quad - (1 - d)^R]$$

$$\equiv q_R \quad .$$

But by the independence assumption A6) different searchers make "fatal" attacks independently of one another, so

$$P\{\text{exactly } \ell \text{ "fatal" attacks are made}\}$$
$$= \binom{B}{\ell} (q_R)^{\ell} (1 - q_R)^{B-\ell} \quad .$$

Next, one notes that computing the conditional probability $P\{K = m \mid \text{exactly } \ell \text{ "fatal" attacks are made}\}$ is equivalent to the problem in combinatorial probability of finding the probability that if ℓ indistinguishable balls are placed into R indistinguishable boxes, independently and according to a uniform

distribution, exactly R-m boxes remain empty. This is so in view of the fact that

$$P\{\text{searcher } i \text{ kills target } j \mid \text{searcher } i \text{ makes a "fatal" attack}\} = \frac{1}{R}$$

for all i and j, which follows at once from expressions above. The occupancy probability is well-known and by Feller (1967, p. 60) is

$$\frac{1}{R^{\ell}}\binom{R}{m} \sum_{\nu=0}^{m} (-1)^{\nu} \binom{m}{\nu}(m-\nu)^{\ell} \quad .$$

The Proposition now follows by some elementary computations.∎

We may use Proposition (3) to construct a two-sided, discrete-time, dynamic version of the unilateral static attrition process studied in Proposition (1). We begin by noting that we can use hypotheses A1) - A6) to compute attrition to the B searchers if roles were reversed and these were subject to detection, attack, and destruction by the R targets. Detection and kill probabilities are in general a function of the side attacking, so if d', k' are the parameters of the complementary process and L denotes the number of searchers killed, then by Proposition (3)

$$(6) \qquad P\{L = n\} = \binom{B}{n} \sum_{\ell=0}^{n} (-1)^{n-\ell}\binom{n}{\ell}[(1-q_B) + \frac{q_B \ell}{B}]^R$$

where

$$q_B = k'[1 - (1 - d')^B] \quad .$$

One way to realize a two-sided attrition process from our one-sided model is to suppose that each side is vulnerable to the opposition's initial strength, and that the two sides independently attrit each other according to equations (4) and (6). This is unsatisfying from a physical standpoint because it essentially requires assuming that all detections and target choices are made

and then all shots fired simultaneously.

This assumption does, however, lead to a dynamic model based on the stated assumptions. The model is a discrete time stochastic process constructed on the assumption that at each time point an interaction occurs as described in the preceding paragraph, involving only those searchers and targets that have survived previous interactions and which is otherwise independent of the prior history of the process. Let $P_2(B,R;\ k)$ be the probability that exactly k targets survive a single interaction if there are B searchers and R targets present; that is $P_1(B,R;\ k)$ is $P\{K = R-k\}$ as given by (4). Similarly, let $P_2(R,B;\ j)$ be $P\{L = B-j\}$ as given by (6). Let T_n denote the number of targets surviving after n interactions and S_n the corresponding number of surviving searchers.

The preceding statements together imply the following result.

(7) THEOREM. Under the preceding hypotheses, the process $((S_n,T_n))_{n\geq 1}$ is a two-dimensional Markov process with state space $\{0,1,2,\ldots\} \times \{0,1,2,\ldots\}$ and transition matrix P given by

$$P((B,R),\ (j,k)) = P_1(B,R;\ j) \cdot P_2(B,R;\ k)\ .$$

3. Exponential Approximations

Use of the approximation

$$E[K] \sim R[1 - \exp(-\tfrac{Bk}{R}[1 - (1-d)^R])] \tag{8a}$$

or the further approximation

$$E[K] \sim R[1 - \exp(-\tfrac{Bk}{R}[1 - e^{-dR}])] \tag{8b}$$

is valid in the sense that

$$\lim_{n\to\infty,\ a\downarrow 0} |e^{-an} - (1-a)^n| = 0$$

but largely unnecessary, especially within the context of computerized

combat simulations. The correct expression given in (1) can be computed as quickly (or perhaps even more quickly, depending on how a specific computer performs exponentiations) as either (8a) or (8b). Moreover, the approximations may be rather poor for small values of B or R, or moderately large values of k and d.

4. Use of Expectations as Inputs

Another problem involved in the use of this model in computerized simulations is the replacement of random variables in certain expressions by the expectations of those random variables. To be more specific, suppose in the Markov process of Theorem (7) we wanted to compute $E[T_2]$, the expected number of targets surviving two interactions. Let us assume for simplicity that only targets can be killed. Then subject to the initial conditions of B searchers and R targets we have

$$E[T_2] = \sum_j \sum_k k\, P^2((B,R);\ (j,k))$$

where P^2 is the square of the matrix P defined in (7). One might seek, however, especially in simulation models, to use instead the "approximation"

$$E[T_2] \sim R - E[T_1] - E[T_1]\left\{1 - \left(1 - \frac{k}{E[T_1]}\left[1 - (1-d)^{E[T_1]}\right]\right)^B\right\},$$

with $E[T_1]$ computed using Propostion (1). What is done here is to suppose there are exactly $E[T_1]$ targets surviving after the first interaction, even though this number need not be an integer (which doesn't matter too much, since the right-hand side of (1) makes sense for any B and R) and to assess the expected number of kills in the second interaction using $E[T_1]$ as the initial number of targets.

This technique is simply wrong; it is not an approximation because we cannot estimate the error committed by its use, nor give a limiting process leading to no error, even though, as the example below indicates, that error may be rather small.

The following example gives an illustration.

EXAMPLE, Suppose that $B = 1$. Then (T_n) is a Markov process with transition matrix P given by

$$P(R,j) = \begin{cases} k[1 - (1 - d)^R] & j = R - 1 \\ 1 - k[1 - (1 - d)^R] & j = R \\ 0 & \text{otherwise} \end{cases} .$$

Suppose

$$\begin{aligned} d &= .1 \\ k &= .5 \\ T_0 &= 3 \end{aligned} .$$

Restricted to $\{0,1,2,3\}$ P is given by

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ .05 & .95 & 0 & 0 \\ 0 & .095 & .915 & 0 \\ 0 & 0 & .1405 & .8595 \end{bmatrix} .$$

By direct computation

$$P^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ .10 & .90 & 0 & 0 \\ .005 & .185 & .81 & 0 \\ 0 & .014 & .246 & .74 \end{bmatrix} .$$

Thus

$$E[T_1] = 2P(3,2) + 3P(3,3) = 2.85$$

and

$$E[T_2] = P^2(3,1) + 2P^2(3,2) + 3P^2(3,3) = 2.72 \quad .$$

If we try to compute $E[T_2]$ as

$$E[T_1](1 - \frac{k}{E[T_1]}[1 - (1 - d)^{E[T_1]}])$$

we obtain 2.70 which is near to the correct value 2.72 in absolute terms and roughly one percent different in relative terms.

Hence in this case the error is small, but the error grows with n and cannot, so far as we have been able to investigate, be estimated or bounded <u>a priori</u>. Thus there remains the possibility of substantial error if (1) is used as an assessment equation in a dynamic combat model.

III. HETEROGENEOUS POINT FIRE MOREL

1. The Basic Model

In this section we extend the model of Section II to the case where there are several types of targets and searchers, with detection and kill probabilities dependent on the type of target and type of searcher. The physics of the process, however, remains unchanged.

Let us assume that there are M types of searchers, B_i searchers of type i ($i = 1, \ldots, M$), N types of targets, and R_j targets of type j ($j = 1, \ldots, N$). We will impose the following hypotheses:

A1) At a fixed time all targets become vulnerable to detection and attack;

A2) The probability that a given, fixed searcher of type i detects a given, fixed target of type j is d_{ij}.

A3) Of the targets (of all kinds) detected by a given searcher, he chooses one to fire upon according to a uniform distribution.

A4) Given that he detects and fires upon a target of type j, a given searcher of type i destroys that target with probability k_{ij};

A5) A given searcher detects different targets independently of one another;

A6) No searcher may fire more than once;

A7) The detection and firing processes of all the searchers are mutually independent.

We wish to compute, under these assumptions, the expected number of targets of each type destroyed. First, we give an analytical solution. Let

$$B = B_1 + \ldots + B_M$$

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denote the total number of searchers and

$$R = R_1 + \dots + R_N$$

the total number of targets. Let K_j be the number of type j targets destroyed.

(9) THEOREM. Under the assumptions A1) - A7) the expected number of type j targets destroyed is given by

$$(10) \qquad E[K_j] = R_j\left(1 - \prod_{i=1}^{M} [1 - d_{ij}k_{ij}p(i,j)]^{B_i}\right)$$

where

$$(11) \qquad p(i,j) = \sum_{r_1=0}^{R_1} \dots \sum_{r_{j-1}=0}^{R_{j-1}} \sum_{r_j=0}^{R_j-1} \sum_{r_{j+1}=0}^{R_{j+1}} \dots \sum_{r_N=0}^{R_N}$$

$$\left\{ \frac{1}{1 + \sum_{p=1}^{N} r_p} \binom{R_j-1}{r_j} d_{ij}^{r_j}(1 - d_{ij})^{R_j-1-r_j} \right.$$

$$\left. \times \prod_{\substack{1\le q\le N \\ q\ne j}} \binom{R_q}{r_q} d_{iq}^{r_q}(1 - d_{iq})^{R_q-r_q} \right\} .$$

PROOF. To begin, let us define the following events

$K_{ij}(k,\ell) = \{\ell^{th}$ target of type j is killed by k^{th} searcher of type i$\}$

$A_{ij}(k,\ell) = \{\ell^{th}$ target of type j is attacked by k^{th} searcher of type i$\}$

$D_{ij}(k,\ell) = \{\ell^{th}$ target of type j is detected by k^{th} searcher of type i$\}$

for $i \le M$, $j \le N$, $k \le B_i$ and $\ell \le R_j$.

By A2) and A4), $P\{K_{ij}(k,\ell)\}$ is independent of k and ℓ. Moreover, A7) implies that

$$P\{\ell^{th} \text{ target of type } j \text{ is killed}\}$$

$$= 1 - \prod_{i=1}^{M} [1 - P\{K_{ij}(1,1)\}]^{B_i} .$$

We now need to find the probabilities $P\{K_{ij}(k,\ell)\}$. To begin, since

$$K_{ij}(k,\ell) \subseteq A_{ij}(k,\ell) \subseteq D_{ij}(k,\ell)$$

we have

$$\begin{aligned} P\{K_{ij}(k,\ell)\} &= P\{K_{ij}(k,\ell)|A_{ij}(k,\ell)\} \cdot P\{A_{ij}(k,\ell)\} \\ &= k_{ij} \cdot P\{A_{ij}(k,\ell)\} \\ &= k_{ij}\ P\{A_{ij}(k,\ell)|D_{ij}(k,\ell)\}\, P\{D_{ij}(k,\ell)\} \\ &= k_{ij}\ P\{A_{ij}(k,\ell)|D_{ij}(k,\ell)\}\, d_{ij} \quad . \end{aligned}$$

We may write the remaining conditional probability, which is independent of k and ℓ for fixed i and j, as

$$P\{A_{ij}(k,\ell)|D_{ij}(k,\ell)\}$$

$$= \sum_{r_j=0}^{R_j-1} \left(\sum_{r_1=0}^{R_1} \cdots \sum_{r_{j-1}=0}^{R_{j-1}} \sum_{r_{j+1}=0}^{R_{j+1}} \cdots \sum_{r_N=0}^{R_N} \right)$$

$$P\{A_{i,j}(k,\ell)|D_{ij}(k,\ell) \cap G_{ij}(r_1, \ldots, r_N; k,\ell)\}$$

$$\times P\{G_{ij}(r_1, \ldots, r_N; k,\ell)\} ,$$

where $G_{ij}(r_1, \ldots, r_N; k,\ell)$ is the event "the kth searcher of type i detects, in addition to the ℓth target of type j, r_j other type j targets and r_p targets of type $p \neq j$," whose probability is independent of k and ℓ. By A3)

$$P\{A_{ij}(k,\ell)|G_{ij}(r_1, \ldots, r_N; k,\ell)\} = \frac{1}{1 + \sum_{p=1}^{N} r_p} ,$$

while by A5)

$$P\{G_{ij}(r_1, \ldots, r_N; k,\ell)\} = \binom{R_j-1}{r_j} d_{ij}^{r_j} (1 - d_{ij})^{R_j-1-r_j}$$

$$\times \prod_{\substack{1\leq q\leq N \\ q\neq j}} \binom{R_q}{r_q} d_{iq}^{r_q} (1 - d_{iq})^{R_q-r_q} ,$$

which, as expected, is independent of k and ℓ when i and j are fixed.

Collecting terms now yields the asserted result. ∎

Instead of the assumptions A1) and A3) concerning the physics of the process of detection and attack, we could assume that targets become vulnerable to a given searcher sequentially in a randomly chosen order, with all R! orders equally probable, that each target is either detected or not and that the first detected target is fired upon. This process occurs once for each searcher independently of the processes corresponding to other searchers, but no targets are destroyed until the end of the entire process. This interpretation is useful in considering allocation of fire problems.

2. Fire Allocation

It should be noted that this model has no provision for searcher determination of whether a given detected target is to be attacked or not. In reality a searcher might be tempted to pass over low value targets in the hope of detecting a high value target. Or, if all targets are vulnerable simultaneously, a given searcher would not

choose among them uniformly, but would instead choose the target whose destruction entails the largest reward to him.

While the model makes no provision for such a choice mechanism, it could be incorporated in the "simultaneously vulnerable" interpretation. Referring to the proof of (9), we see that the uniform target choice is manifested solely in the conditional probability

$$P\{A_{ij}(k,\ell)|G_{ij}(r_1, \ldots, r_N; k,\ell)\} = \frac{1}{1 + \sum_{p=1}^{N} r_p} .$$

Hence a different rule for target choice can be incorporated simply by changing the form of this conditional probability, subject to some obvious regularity conditions.

EXAMPLE. Suppose the value to a type i searcher of destroying a type j target is u_{ij}. Then

$$v_{ij} = k_{ij}u_{ij}$$

is the expected return from an attack upon type j target. The rule "Of the targets detected fire at one whose destruction is of maximal value" leads to

$$P\{A_{ij}(k,\ell)|G_{ij}(r_1, \ldots, r_N; k,\ell)\}$$

$$= \begin{cases} \dfrac{1}{1 + r_j} & \text{if } v_{ij} \geq v_{im} \text{ for all } m \neq j \text{ such that } r_m > 0 , \\ 0 & \text{otherwise.} \end{cases}$$

Here we have assumed for simplicity that $v_{ij} \neq v_{im}$ whenever $j \neq m$. Other examples can be described similarly.

3. Further Computations

To implement (10) as the attrition equation in a computerized combat model would be laborious, and lead to a cumbersome result.

Let us consider the difficulties more carefully beginning with some simple cases. First suppose that

$$N = 3$$

and

$$R_1 = R_2 = R_3 = 1 \quad .$$

In this case, (11) yields

$$\begin{aligned} p(i,1) &= \sum_{\ell=0}^{1} \sum_{m=0}^{1} \frac{1}{1+\ell+m} \left[\binom{1}{\ell} d_{i2}^{\ell} (1 - d_{i2})^{1-\ell} \binom{1}{m} d_{i3}^{m} (1 - d_{i3})^{1-m} \right] \\ &= 1[(1 - d_{i2})(1 - d_{i3})] \\ &\quad + \frac{1}{2}[(1 - d_{i2}) d_{i3} + d_{i2} (1 - d_{i3})] \\ &\quad + \frac{1}{3}[d_{i2} d_{i3}] \\ &= 1 - \frac{1}{2}(d_{i2} + d_{i3}) + \frac{1}{3}(d_{i2} d_{i3}) \ , \end{aligned}$$

which doesn't look too ugly; analogous expressions for p(i,2) and p(i,3) clearly exist. If

$$N = 4$$

and

$$R_1 = R_2 = R_3 = R_4 = 1$$

we have

$$\begin{aligned} p(i,1) = 1 &- \frac{1}{2}(d_{i2} + d_{i3} + d_{i4}) \\ &+ \frac{1}{3}(d_{i2} d_{i3} + d_{i2} d_{i4} + d_{i3} d_{i4}) \\ &- \frac{1}{4} d_{i2} d_{i3} d_{i4} \quad . \end{aligned}$$

The general pattern is by now evident; we summarize it in the following result, whose inductive proof we omit.

(12) PROPOSITION. For any N, if

$$R_1 = R_2 = R_3 = \dots = R_N = 1 \quad ,$$

then for $1 \le i \le M$ and $1 \le j \le N$,

$$p(i,j) = 1 + \sum_{m=2}^{N} \frac{(-1)^m}{m} \sum_{\substack{i_1 \\ i_1 \neq j}} \sum_{\substack{i_2 > i_1 \\ i_2 \neq j}} \cdots \sum_{\substack{i_m > i_{m-1} \\ i_m \neq j}} d_{i,i_1} d_{i,i_2} \cdots d_{i,i_m}$$

Proposition (12) is not so useless as it first appears: in a computer implementation one could assume that there is only one target of each type--even though some targets have the same probabilities of detection and kill given detection--then carry out the attrition calculation using (10) and (12). If targets were grouped into classes of indistinguishable (except by artificial notation) objects, attrition ΔR_k to objects in the kth class would then be given by

$$\Delta R_k = \sum_{j \in C_k} \left(1 - \prod_{i=1}^{M} [1 - d_{ij} k_{ij} p(i,j)]^{B_i} \right) \tag{13}$$

where C_k is the set of indices j such that the jth object belongs to the kth class. For fixed i and k, d_{ij} and k_{ij} are independent of $j \in C_k$.

4. Approximations and Simplifications

While (13) could be implemented on a computer, its use might involve long and time-consuming computations, especially if R were of the magnitude of the number of soldiers in a moderate-sized land battle. Hence it is worthwhile to seek simplified versions of (10), several of which we now proceed to consider.

The following result, which aids in developing simplified versions of (10), shows that the complicated form of (10) arises from having more than one type of target, rather than from having more than one type of searcher.

(14) PROPOSITION. If $N = 1$, then

$$E[K] = R\left(1 - \prod_{i=1}^{M}\left(1 - \frac{k_i}{R}\left[1 - (1 - d_i)^R\right]\right)^{B_i}\right) \tag{15}$$

where subscripts i denote the type of searcher.

PROOF. From (11) we have for each i

$$p(i) = p(i,1) = \sum_{r=0}^{R-1} \frac{1}{1+r}\binom{R-1}{r} d_i^r (1 - d_i)^{R-1-r}$$

which, as indicated in the proof of Proposition (1), is equal to

$$\frac{1}{d_i R}\left(1 - (1 - d_i)^R\right) ,$$

yielding (15). ∎

We next discuss three simplified versions of (10) for the case $N > 1$. The notation $\Delta R_j = E[K_j]$ is used hereafter.

a. <u>Successive Application of All Searchers to Each Type of Target</u>. If there were only type j targets then by (15) the expected number of such targets destroyed would be

$$\Delta R_j = R_j\left[1 - \prod_{i=1}^{M}\left(1 - \frac{k_{ij}}{R_j}\left[1 - (1 - d_{ij})^{R_j}\right]\right)^{B_i}\right] . \tag{16}$$

One means of attrition assessment would calculate each ΔR_j by (16), but this method clearly gives an advantage to searchers which is unwarranted in terms of assumption A1). In effect, this method

allows each searcher one shot at each type of target, rather than one shot altogether.

b. Calculation With Weighted Probabilities. For each i, the quantity

$$\bar{d}_i = \frac{1}{R} \sum_{j=1}^{N} d_{ij} R_j$$

represents a detection probability for type i searchers which is averaged with respect to the numbers of targets present, while

$$\bar{k}_i = \frac{1}{R} \sum_{j=1}^{N} k_{ij} R_j$$

is a similarly averaged conditional probability of kill given detection.

We could then approximate the original attrition process by one with R targets of a single type and for each i B_i searchers of type i with detection and kill probabilities $\bar{d}_i$ and $\bar{k}_i$, respectively. Using (15), attrition to these R targets would be

$$\Delta R = R\left(1 - \prod_{i=1}^{M}\left(1 - \frac{\bar{k}_i}{R}\left[1 - (1 - \bar{d}_i)^R\right]\right)^{B_i}\right) .$$

Against type j targets would then be assessed the fraction R_j/R of this attrition ΔR, so that

$$(17) \qquad \Delta R_j = R_j\left(1 - \prod_{i=1}^{M}\left(1 - \frac{\bar{k}_i}{R}\left[1 - (1 - \bar{d}_i)^R\right]\right)^{B_i}\right) .$$

In (17) it was unnecessary to use the averaged conditional probabilities of kill given detection and attack, and we could as well (and more accurately) have written

$$(17') \qquad \Delta R_j = R_j\left(1 - \prod_{i=1}^{M}\left(1 - \frac{k_{ij}}{R}\left[1 - (1 - d_i)^R\right]\right)^{B_i}\right) .$$

It is possible, using one further averaging step, to use the homogeneous equation directly. The numbers

$$d = \frac{1}{B} \sum_{i=1}^{M} B_i \bar{d}_i$$

and

$$k = \frac{1}{B} \sum_{i=1}^{M} B_i \bar{k}_i$$

are, respectively, probabilities of detection and kill given detection averaged with respect to both targets and searchers.

We may approximate the original attrition process by an attrition process with R targets of a single type, B searchers of a single type, and parameters d,k as computed just above. The exact attrition in such a process is given by

$$\Delta R = R\left(1 - \left(1 - \frac{k}{R}\left[1 - (1 - d)^R\right]\right)^B\right) .$$

Using the same attrition apportionment as in (17), these three equations yield the following values for the expected attrition to type j targets

$$\Delta R_j = R_j\left(1 - \left(1 - \frac{k}{R}\left[1 - (1 - d)^R\right]\right)^B\right) . \tag{18}$$

c. Prior Apportionment of Searchers. We might also model the attrition process as a number of smaller engagements by, prior to attrition assessment, dividing the searchers among different types of targets on the basis of relative numbers of targets present and vulnerable. That is, type j targets would be vulnerable to only

$$B(i,j) = B_i \cdot \frac{R_j}{R}$$

type i searchers, rather than to all B_i type i searchers. One can interpret this as assigning to each searcher one and only one type of target.

Using (15), we see that attrition to type j targets is given in this case by

$$(19) \qquad \Delta R_j = R_j\left(1 - \prod_{i=1}^{M}\left(1 - \frac{k_{ij}}{R_j}\left[1 - (1 - d_{ij})^{R_j}\right]\right)^{B(i,j)}\right) .$$

But there are other methods of prior searcher allocation which are more closely related to our model and to methods developed by L. B. Anderson (1973). The point is to compute explicitly the searcher allocation for a "typical" target set (we leave the term purposely vague) and to use this allocation to allocate searchers in each attrition computation, as described below.

Suppose that t_j is the proportion of a "typical" target set which are type j targets and that a^t_{ij} is the fraction of the shots fired by a type i searcher which are directed at type j targets when the target set is "typical". If we put

$$(20) \qquad \alpha_{ij} = \frac{a^t_{ij}}{t_j} ;$$

then the fraction of shots fired by a type i searcher which are aimed at type j targets when the target set consists of R_j targets of type j (j = 1, ..., N) is given by

$$\frac{\alpha_{ij}R_j}{\sum_k \alpha_{ik}R_k} ,$$

and we could then define

$$B(i,j) = B_i \frac{\alpha_{ij}R_j}{\sum_k \alpha_{ik}R_k} ,$$

to be used in (19).

An interesting question at this point is, Can the a_{ij}^t be derived from other given quantities using the assumptions set forth above? To begin, we note that the object is to derive a distribution of the fire of each searcher. For the model described by A1) - A7) we have the following result.

(21) PROPOSITION. For each i and j, the conditional probability that a fixed type i searcher attacks some type j target, given that the searcher makes an attack, is

$$\frac{R_j d_{ij} p(i,j)}{1 - \prod_{k=1}^{N} (1 - d_{ik})^{R_k}} , \tag{22}$$

with $p(i,j)$ defined by (11).

PROOF. In the proof of (9) we showed that the probability that the kth searcher of type i attacks the ℓth target of type j is given by

$$P\{A_{ij}(k,\ell)\} = d_{ij} p(i,j) \quad .$$

The probability that the kth type i searcher makes an attack is one minus the probability that he fails to make a detection; by A5) this latter probability is

$$\prod_{k=1}^{N} (1 - d_{ik})^{R_k} \quad .$$

By elementary conditional probability arguments (22) is now obtained. $\mathbb{I}$

Suppose now that a "typical" target set consists of R_j^t targets of type j, $j = 1, \ldots, N$. We can then define

$$t_j = \frac{R_j^t}{\sum_{k=1}^{N} R_k^t} ,$$

and define, through (22),

$$a_{ij}^t = \frac{R_j^t d_{ij} p^t(i,j)}{1 - \prod_{k=1}^{N} (1 - d_{ik})^{R_k^t}}$$

with $p^t(i,j)$ computed by (11) with the R_k there replaced by R_k^t. Then, to compute attrition in an engagement with arbitrary numbers $R_1, \ldots, R_N$ of targets, we would compute the α_{ik} by (20) and apply equation (19). In use, this procedure would require only one computation of p(i,j)'s--that for the "typical" target set.

The rationale behind this procedure is that in a statistical sense of the long run, the probability that a given type of target is fired upon, given that there is a shot fired, is the same as the fraction of shots fired at type j targets if the attrition process were carried out experimentally a large number of times.

5. A Multiple Shot Model

The preceding discussion shows that the complexity of (10) and (11) results almost entirely from the Assumption A3) that each searcher fires at most one shot. If we replace A3) by

A3)′ Each searcher has sufficient firing capability to fire once and only once at each target he detects,

then we have the following result.

(23) PROPOSITION. Under Assumptions A1), A2), A3)′, A4) - A7),

(24) $$\Delta R_j = R_j \left(1 - \prod_{i=1}^{M} (1 - d_{ij} k_{ij})^{B_i}\right)$$

for each j .

We omit the simple proof. The equation (24) is a reasonable and more easily computable alternative to (10), especially for modeling situations in which the one-shot hypothesis A3) seems unrealistic.

6. Exponential Approximations

As in the homogeneous case various exponential approximations to the attrition equations of this section are possible, some of which we give below. Corresponding to (16) are

$$\Delta R_j \sim R_j\left[1 - \exp\left(-\frac{1}{R_j}\sum_{i=1}^{M} B_i k_{ij}\left[1 - (1 - d_{ij})^{R_j}\right]\right)\right]$$

and

$$\Delta R_j \sim R_j\left[1 - \exp\left(-\frac{1}{R_j}\sum_{i=1}^{M} B_i k_{ij}\left[1 - e^{-d_{ij}R_j}\right]\right)\right] \quad .$$

Approximations of (17) are

$$\Delta R_j \sim R_j\left[1 - \exp\left(-\frac{1}{R}\sum_{i=1}^{M} B_i \bar{k}_i[1 - (1 - \bar{d}_i)^{R_j}]\right)\right]$$

and

$$\Delta R_j \sim R_j\left[1 - \exp\left(-\frac{1}{R}\sum_{i=1}^{M} B_i \bar{k}_i\left(1 - \qquad {}^{R}\right)\right)\right] \quad .$$

Equation (18) is approximated as

$$\Delta R_j \sim R_j\left[1 - \exp\left(-\frac{Bk}{R}[1 - (1 - d)^{R_j}]\right)\right]$$

and

$$\Delta R_j \sim R_j\left[1 - \exp\left(-\frac{Bk}{R}[1 - e^{-dR_j}]\right)\right] \quad .$$

Finally, two approximations to (19) are

$$\Delta R_j \sim R_j\left[1 - \exp\left(-\frac{1}{R}\sum_{i=1}^{M} B_i k_{ij}[1 - (1 - d_{ij})]^{R_j}\right)\right]$$

and

$$\Delta R_j \sim R_j\left[1 - \exp\left(-\frac{1}{R}\sum_{i=1}^{M} B_i k_{ij}\left(1 - e^{-d_{ij}R_j}\right)\right)\right] \quad .$$

Such relations as these should be used with great care, if at all, since they are three- and four-tier approximations to the correct computation given by (10) and (11).

IV. AREA FIRE MODEL

1. The Basic Model

We give in this section an axiomatic treatment of a classical area fire problem, following the format of the preceding sections. Karr (1972b) is the basis for this presentation. In our first model here both defenders and incoming shots are uniformly distributed over a disk in the plane; it is shown that under carefully stated assumptions an independence assertion usually thought to be true fails and we denote some attention to the consequences. The second model is an elementary Markov process analysis of a situation in which the defenders regroup after every shot so as to defend an area proportional to their numerical strength, the constant of proportionality being fixed.

For each $r \geq 0$ let S_r be the disk in $\underset{\sim}{R}^2$ with center 0 and radius r ; we denote the unit disk S_1 by A. Defenders (or targets) located in A are attacked according to the following assumptions.

A1) Each incoming shot has a circular area of (theoretical - see A3) below) lethality whose radius is uniformly distributed on the interval $[r_1, r_2]$, where r_1, r_2 are numbers with

$$0 < r_1 \leq r_2 < 1 \quad ,$$

with the understanding that if $r_1 = r_2$ the radius is r_1 almost surely.

A2) The center of the area of lethality of each shot is uniformly distributed on S_{1-r_2} and is independent of the radius of lethality.

This means that the attacker is able to (and does) aim his shots to the extent that no lethal area falls outside A , but that centers are otherwise uniformly distributed. Note also that no point of A is

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a priori immune, although points near the boundary of A appear safer.

A3) There is a measurable function g from $A \times S_{1-r_2} \times [0, r_2]$ into $[0, 1]$ with the interpretation that $g(x,u,r)$ is the probability that a defender at $x \in A$ is killed by a shot with lethal radius r centered about $u \in S_{1-r_2}$.

A4) Each defender's position is uniformly distributed on A and is independent of all incoming shots.

The purpose of the function g is to take account of the effect of terrain, shelters, etc., in decreasing the lethality of a shot. Thus one may suppose that the lethality of A1) is hypothetical in the sense of representing lethality on a treeless plain, so that g has the effect of transforming a "potential" lethality which is a property only of the weapon into an "actual" lethality which depends on both the weapon and the physical situation.

Some examples are helpful at this point.

EXAMPLES. 1) the function

$$g(x,u,r) = \begin{cases} 1 & \text{for } |x-u| \leq r \\ 0 & \text{otherwise,} \end{cases} \tag{25}$$

where $|\cdot|$ denotes Euclidean distance on $\underset{\sim}{R}^2$, is the classical "cookie cutter" damage function: anything in the lethal range of the shot is annihilated with probability one and everything not in the lethal range is unaffected.

2) Suppose the subset T of A is sheltered in the sense that no defender in the shelter can be killed except by a direct hit on the shelter. We would then have for $x \in T$

$$g(x,u,r) = \begin{cases} 1 & \text{if } u \in T \text{ and } |x-u| \leq r \\ 0 & \text{otherwise} \end{cases},$$

while for $x \notin T$, $g(x)$ could be defined by (1) or by some other recipe.

Note that A4) does not require that positions of different defenders be independent of one another, but only that each position be uniformly distributed on A. The independence assumption will be imposed in the second model.

Let us consider the probability $p(1)$ that a single defender with position X survives a single shot with (center, radius) pair (C,R). Writing S for S_{1-r_2}, we have

(26) $1 - p(1) = E[g(X,C,R)] =$

$$= [|A| \cdot |S| \cdot (r_2 - r_1)]^{-1} \int_A dx \int_S du \int_{r_1}^{r_2} dr\, g(x,u,r) \quad .$$

where $|F|$ is the two-dimensional Lebesgue measure (= area) of $F \subset \underset{\sim}{R}^2$.

Of course, $|A| = \pi$ and $|S| = \pi(1 - r_2)^2$, but we leave these quantities unevaluated for the sake of easier interpretation. It should be noted that throughout this section u and x are two-dimensional variables, while r is one-dimensional.

(27) PROPOSITION. If R is any nonnegative random variable bounded above by r_2, if

$$P\{C \in S_{1-r_2}\} = 1 \quad ,$$

and if g is of the form (25), then

(28) $$1 - p(1) = \frac{E[\pi R^2]}{|A|} \quad .$$

PROOF. Let F be the distribution of R and G that of C; then with the assumption on g, and an application of Fubini's Theorem, we have

$$1 - p(1) = \frac{1}{|A|} \int_0^{r_2} F(dr) \int_S G(du) \int_A dx\, I_{[0,r]}(|x-u|)$$

(here I_B is the indicator function of the set B: $I_B(y) = 1$ if $y \in B$ and 0 otherwise)

$$= \frac{1}{|A|} \int_0^{r_2} F(dr) \int_S G(du) \left| \{x \in A : |x - u| \leq r\} \right| .$$

But, since for $u \in S$ and $r \leq u$, $S_r(u) \subset A$,

$$\left| \{x \in A : |x - u| \leq r\} \right| = \pi r^2$$

so that the last expression is equal to

$$\frac{1}{|A|} \int_0^{r_2} F(dr) \, \pi r^2 \int_S G(du) = \frac{E[\pi R^2]}{|A|} . \qquad \P$$

When R has the distribution specified in A1),

$$E[\pi R^2] = \frac{\pi}{r_2 - r_1} \int_{r_1}^{r_2} r^2 \, dr$$

$$= \frac{\pi}{3} (r_2^2 + r_1 r_2 + r_1^2) .$$

As a special case of (27) we may take

$$G(\{0\}) = 1 ,$$

and notice that the probability the defender survives a single shot whose center is fixed at the origin is the same as the probability that he survives a shot with the same radius of lethality distribution whose center is uniformly distributed on S .

Let $p(n)$ be the probability that a single defender survives n independent and identically distributed shots. Then it is not true that

(29) $$p(2) = p(1)^2$$

since the events

$$E_1 = \{\text{defender survives first shot}\}$$

and

$$E_2 = \{\text{defender survives second shot}\}$$

are not independent, but only <u>conditionally independent given the defender's position</u> X .

Thus

$$\begin{aligned} p(2) &= P\{E_1 \cap E_2\} \\ &= E[P\{E_1 \cap E_2 | X\}] \\ &= E[P\{E_1 | X\} \cdot P\{E_2 | X\}] \quad . \end{aligned}$$

It follows from (26) and (27) that for $i = 1, 2$

$$P\{E_i | X\} = h(X)$$

where

$$h(x) = (|S| \cdot (r_2 - r_1))^{-1} \int_S du \int_{r_1}^{r_2} dr \, I_{(r,\infty)}(|x - u|)$$

$$= \frac{1}{|S|}\left[|\{u \in S : |x - u| > r_2\}| + \frac{1}{r_2 - r_1} \int_{\substack{r_1 \le |x-u| \le r_2 \\ u \in S}} (|x - u| - r_1) du\right] \quad .$$

Note that $h(x)$ is the probability that a defender at $x \in A$ survives a shot which is uniformly distributed on S . It follows that

(30) $$p(2) = |A|^{-1} \int_A h(x)^2 \, dx$$

and, by induction, that

$$p(n) = |A|^{-1} \int_A h(x)^n \, dx \quad . \tag{31}$$

The probability $q(n)$ that a single defender fails to survive n independent and identically distributed shots is given by

$$\begin{aligned} q(n) &= 1 - p(n) \\ &= 1 - |A|^{-1} \int_A h(x)^n \, dx \\ &= |A|^{-1} \int_A [1 - h(x)^n] \, dx \quad , \end{aligned}$$

which is not equal, in general, to

$$1 - (1 - q(1))^n = 1 - (|A|^{-1} \int_A h(x) \, dx)^n \quad .$$

The expected attrition to m defenders whose distributions satisfy A4) is then given by

$$m \cdot q(n) = m|A|^{-1} \int_A (1 - h(x)^n) \, dx \equiv A(m, n) \; . \tag{32}$$

Let us again point out that A4) is satisfied and hence (32) is valid, without assuming that defenders' positions be independent of one another. For example, if a single defender chooses a uniformly distributed position and other defenders position themselves at the same distance from 0 in such a way as to make all angular distances between adjacent defenders equal, then A4) is satisfied. Or all defenders may be placed at a single point which is uniformly distributed on A, so a defender seeking to minimize his <u>expected attrition</u> gains nothing by dispersing his personnel, in a situation modeled by A1) - A4). The same would not be true, however, of a defender who tries to minimize the probability of annihilation of all his personnel (provided his forces are more than one).

2. Edge Effects

As will be discussed in considerable detail below, the difficulties in the failure of (29) arise in the fact that h is not constant over all of A : clearly $h(0) < 1$, while $h(x) = 1$ for x in the boundary $Fr(A)$ of A . The reason h is not constant is because shots are aimed so that no lethal area ever falls outside A , making it less likely that a shot whose center is uniformly distributed on S will kill a defender whose position is close to $Fr(A)$. This is an edge effect.

It will be proved in (38) that h is constant over most of A ; we will then consider the effect of neglecting the set where $h \neq h(0)$. Another alternative which makes h constant over all of A would be to make shot centers uniformly distributed on S_{1+r_2}, but if this is done then (28) is no longer true. The difficulties arise in "edge effects" near the boundary of A . One is tempted to conclude that edge effects can be neglected if r_2 is small; to a certain extent this is so, or that these edge effects are a result of the formulation of the model rather than the underlying physical process. We believe the second conclusion is unjustified. In physical models, from fluid dynamics to statistical mechanics, of finite systems it is precisely the boundary behaviors which must be considered most carefully and which make the theories valuable; we feel the same is true here.

3. Repositioning in the Same Area

We now proceed, as promised above, to a detailed analysis of the failure of (29) and one way of rectifying this difficulty (if one considers it to be a difficulty) by imposing an additional (but not very plausible) assumption.

Suppose we add to A1) - A4) the following hypothesis:

A5) Shots are made one at a time and between each pair of shots every defender repositions himself according to a uniform distribution on A , independent of the past positions of all defenders and of the entire process of incoming shots.

We begin by considering the survival probabilities for a single defender; suppose there are n shots that his positions (provided he lives long enough to occupy them) are $X_1, \ldots, X_n$ and that $(C_1, R_1), \ldots, (C_n, R_n)$ are the (center, radius) pairs of the incoming shots. Continue to assume g is of the form (25). Then for $k = 1, \ldots, n$ the probability of surviving the first k shots is

$$p(k) = P\left\{\bigcap_{j=1}^{k} \left\{X_j \notin S_{R_j}(C_j)\right\}\right\}$$

$$= \prod_{j=1}^{k} P\left\{X_j \notin S_{R_j}(C_j)\right\} = (p(1))^k \quad .$$

Hence under A5) and the assumptions of Proposition (27)

$$p(n) = \left(1 - \frac{E[\pi R^2]}{|A|}\right)^n , \qquad n = 1, 2, \ldots, \tag{33}$$

where R is a random variable with the same distribution as $R_1, \ldots, R_n$. From (33) it is evident that

$$q(n) = 1 - \left(1 - \frac{E[\pi R^2]}{|A|}\right)^n \tag{34}$$

$$= 1 - (1 - q(1))^n.$$

We emphasize that (34) is valid only under the restrictive and rather implausible hypothesis A5) and that A5) is meaningless in a situation where n shots are fired simultaneously and independently from different weapons.

For small values of r_2 and large values of n, one may make the approximation

$$p(n) \sim \exp\left(- nE[\pi R^2]/|A|\right) \tag{35}$$

although neither its desirability nor its necessity is clear, since, after all, exact expressions such as (33) are available. If (35) is employed, the expected attrition $A(m, n)$ to m defenders by n shots

is approximately

$$A(m,n) \sim m(1 - \exp(-nE[\pi R^2]/|A|)) \quad ;$$

the true value is

$$A(m, n) = m\left[1 - \left(1 - \frac{E[\pi R^2]}{|A|}\right)^n\right] . \tag{36}$$

It seems reasonable to define the quantity $nE[\pi R^2]$ as the (expected) "antipersonnel potential" of n shots. It is the word "potential" which is crucial: this is the expected lethal area of n shots with the same radius of lethality distribution whose areas of lethality do not intersect (which is the best situation the attacker can hope for).

Continuing to assume A1) - A5) and the hypothesis of (27), we see that the probability $p(n_1, \ldots, n_j)$ that a single defender survives n_1 independent shots from a type 1 weapon with expected area of lethality $E[\pi R_1^2]$, n_2 independent shots from a type 2 weapon that are independent of the shots from the type 1 weapon and have expected area of lethality $E[\pi R_2^2]$, and so on, is

$$p(n_1, \ldots, n_j) = \prod_{i=1}^{j} \left(1 - \frac{E[\pi R_i^2]}{|A|}\right)^{n_i} . \tag{37}$$

We may, provided $n_1, \ldots, n_j$ all be large, approximate the corresponding kill probability $q = 1 - p$ by

$$\begin{aligned} q(n_1, \ldots, n_j) &= 1 - p(n_1, \ldots, n_j) \\ &\sim 1 - \exp\left(-\frac{1}{|A|} \sum_{i=1}^{J} n_i E[\pi R_i^2]\right) . \end{aligned}$$

Again, it may be reasonable to interpret $\Sigma\, n_i E[\pi R_i^2]$ as antipersonnel potential. It should be noted, however, that no such scalar quantity appears in the exact equations (34) and (37), let alone the equations derived without the use of A5).

4. The Effect of Edge Effects

An alternative analysis of the failure of (29) is based on the following result. We assume until further notice that $r_2 < 1/2$.

(38) PROPOSITION. If $|x| < 1 - 2r_2$, then

$$h(x) = h(0)$$
$$= \frac{1}{(1 - r_2)^2}\left[(1 - 2r_2) + 2/3\,(r_2^2 + r_1 r_2 + r_1^2) - r_1(r_2 + r_1)\right] .$$

PROOF. For $|x| < 1 - 2r_2$ we have, from the computation of h given previously,

$$h(x) = \frac{1}{|S|}\Big[|\{u \in S : |x - u| > r_2\}|$$
$$+ \frac{1}{r_2 - r_1} \int_{\substack{r_1 \le |x-u| \le r_2 \\ u \in S}} (|x - u| - r_1)\, du\Big]$$
$$= \frac{1}{|S|}\Big[(|S| - \pi r_2^2)$$
$$+ \frac{1}{r_2 - r_1} \int_{\substack{r_1 \le |x-u| r_2 \\ |u| \le 1-r_2}} (|x - u| - r_1)\, du\Big] .$$

In the second term, make the change of variable $y = u - x$ (y is two-dimensional) to obtain

$$\frac{1}{|S|(r_2 - r_1)} \int_{\substack{r_1 \le |y| \le r_2 \\ |x+y| \le 1-r_2}} (|y| - r_1)dy .$$

For $|y| \le r_2$ and $|x| \le 1 - 2r_2$,

$$|x + y| \le |x| + |y| \le 1 - r_2 \quad ,$$

so the second restriction on the domain of integration is irrelevant. Hence for $|x| < 1 - 2r_2$,

$$h(x) = h(0) = \frac{1}{|S|}\Bigg[\pi(1 - r_2)^2 - \pi r_2^2 + \frac{1}{r_2 - r_1} \int_{r_1 \le |y| \le r_2} (|y| - r_1)\, dy\Bigg]$$

$$= \frac{1}{\pi(1 - r_2)^2}\left[\pi(1 - 2r_2) + \frac{1}{r_2 - r_1} \int_{r_1 \le |y| \le r_2} (|y| - r_1)\, dy\right] .$$

We may evaluate the integral changing to polar coordinates $y \to (r \cos \theta, r \sin \theta)$, where $r = |y|$ and $\theta = \tan^{-1}(y_2/y_1)$, with the result

$$\frac{1}{r_2 - r_1} \int_{r_1 \le |y| \le r_2} (|y| - r_1)\, dy$$

$$= \frac{1}{r_2 - r_1} \int_{r_1 \le r \le r_2} \int_0^{2\pi} (r - r_1)\, r\, dr\, d\theta$$

$$= \frac{2\pi}{r_2 - r_1} \int_{r_1 \le r \le r_2} (r^2 - r_1 r)\, dr$$

$$= \frac{2\pi}{r_2 - r_1}\left[1/3(r_2^3 - r_1^3) - \frac{r_1}{2}(r_2^2 - r_1^2)\right]$$

which completes the proof. ||

COROLLARY.

$$|\{x : h(x) \neq h(0)\}| = 4\pi(r_2 - r_2^2) \quad .$$

PROOF. This requires only noting that $h(x)$ is strictly greater than $h(0)$ if $|x| > 1 - 2r_2$ and a simple computation. $\square$

Since

$$(1 - q(1))^n = \left(\frac{1}{|A|} \int_A h(x)\, dx\right)^n$$

while

$$q(n) = 1 - \frac{1}{|A|} \int_A h(x)^n\, dx \quad ,$$

we see that the failure of $q(n)$ to equal $1 - (1 - q(1))^n$ under A1) - A4) and the hypotheses of (27) is due to the inequality

$$\frac{1}{|A|} \int_A h(x)^n\, dx \neq \left(\frac{1}{|A|} \int_A h(x)\, dx\right)^n \quad .$$

We know that if G is a probability measure and $f \equiv c$ is a constant function then

$$\int f^n\, dG = (\int f\, dG)^n = c^n$$

for all $n \geq 0$.

Since we showed in Proposition (38) that h is constant except over a subset of A of small Legesgue measure (provided r_2 is small) it seems plausible that $\left(\frac{1}{|A|} \int_A h\right)^n$ should serve as an approximation to $\frac{1}{|A|} \int_A h^n$. This, in fact, is so in two senses, which we note next.

(39) PROPOSITION. For any fixed n,

$$\lim_{r_2 \to 0} \left| \frac{1}{|A|} \int_A h(x)^n \, dx - \left(\frac{1}{|A|} \int_A h(x) \, dx \right)^n \right| = 0 \quad ,$$

and for any fixed $r_2 > 0$,

$$\lim_{n \to \infty} \left| \frac{1}{|A|} \int_A h(x)^n \, dx - \left(\frac{1}{|A|} \int_A h(x) \, dx \right)^n \right| = 0 \quad .$$

We omit the proof.

COROLLARY. For each fixed $n \geq 1$ and each $\delta \in [0, 1)$,

$$\frac{1}{|A|} \int_A h(x)^n \, dx = h(0)^n + o(r_2^\delta) \quad , \qquad r_2 \to 0$$

and

$$\left(\frac{1}{|A|} \int_A h(x) \, dx \right)^n = h(0)^n + o(r_2^\delta) \quad , \qquad r_2 \to 0 \quad .$$

The qualitative content of the corollary is that for fixed n and small r_2 the error in approximating either $q(n)$ or $1 - (1 - q(1))^n$ by $1 - h(0)^n$, and hence the error in the approximation

$$q(n) \sim 1 - (1 - q(1))^n \quad ,$$

is (roughly) proportional to r_2; estimates of the constant of proportionality may be made if required.

Let $f(x) = 1 - h(x)$, so that $f(x)$ is the probability that a defender at $x \in A$ is <u>killed</u> by a single shot whose center is uniformly distributed on S_{1-r_2} and is independent of the radius of lethality which is uniformly distributed on $[r_1, r_2]$. Then the approximation

$$q(n) \sim 1 - h(0)^n$$

may be written as

$$q(n) \sim 1 - (1 - f(0))^n \quad .$$

Since both sides here go to zero as $n \to \infty$, we may make the further approximation

$$q(n) \sim 1 - \exp(-nf(0))$$

which is valid for small r_2 <u>and</u> large n .

5. <u>Repositioning With Fixed Density</u>

We will next discuss a generalization of the model based on A1) - A4) which treats incoming shots as occurring one after another and requires that the defender reposition his forces after each shot in such a way as to maintain a fixed density of forces. Specifically, we will assume A1) - A4), that there are initially m defenders, and the following additional hypotheses:

A6) Let $\rho = m/|A|$. If after a given shot there are $k(1 \leq k \leq m)$ surviving defenders, each of those k defenders positions himself on the disk $S_{r(k)}$ with radius $r(k) = (k/\rho\pi)^{\frac{1}{2}} = (k/m)^{\frac{1}{2}}$ and center 0 , independent of the repositionings of the $(k - 1)$ other survivors, of the past positions of all defenders and of the entire process of incoming shots.

A7) Initial positions of the m defenders are independent of one another and of all incoming shots.

Hence the density of the repositioned defenders (over the smaller set $S_{r(k)}$ to which they have retreated) is $k/\pi r(k)^2 = \rho$ so the density of defenders (relative to the set over which they are uniformly distributed--not to A !) remains constant.

For simplicity we further assume that g is of the form (25), that all shots have lethality radius r almost surely, that the attacker is unaware of the pullback of defender personnel and

continues to fire shots whose centers are uniformly distributed on S_{1-r}, and that $r < 1/m$. Let T_n be the number of defenders remaining alive after the nth shot: The important result concerning the stochastic process $(T_n)_{n\geq 1}$ is the following.

(40) THEOREM. Under A1) - A4), A6), A7), and the additional assumptions stated in the preceding paragraphs, $\underset{\sim}{T} = (T_n)_{n\geq 1}$ is a Markov process with state space $E \doteq \{0, 1, \ldots, m\}$, initial distribution α given by

$$(41) \qquad \alpha(k) \equiv P\{T_1 = k\} = \binom{m}{k}(1 - r^2)^k (r^2)^{m-k} , \qquad k = 0, \ldots, m$$

and transition matrix P given by

$$(42a) \qquad P(0, j) = \delta(0, j) , \qquad j = 0, 1, \ldots, m$$

while for $1 \leq i < m$

$$(42b) \qquad P(i, j) = \begin{cases} \binom{i}{j} \dfrac{2}{(1 - r)^2} \displaystyle\int_0^{1-r} u(1 - f_i(u))^j \, f_i(u)^{1-j} \, du , & \text{if } j = 0, \ldots, i \\ 0 & \text{if } j = i + 1, \ldots, m , \end{cases}$$

where $S_r[(0, u)]$ is the disk with radius r and center $(0, u)$ and

$$f_i(u) = \frac{|S_{r(i)} \cap S_r[(0, u)]|}{\pi r(i)^2} ,$$

and finally

$$(42c) \qquad P(m, j) = \alpha(j) , \qquad j = 0, \ldots, m .$$

PROOF. As in the derivation of (31) one must guard against unwarranted independence assumptions. That (42a) holds is clear from physical interpretation.

If $X_1, \ldots, X_m$ are the positions of the defenders prior to the first shot and C_1 the position of the center of the first shot then

$$P\{T_1 = k\} = \binom{m}{k} P\{X_1 \notin S_r(C_1), \ldots, X_k \notin S_r(C_1),$$
$$X_{k+1} \in S_r(C_1), \ldots, X_m \in S_r(C_1)\}$$

(because defenders are indistinguishable)

$$= \binom{m}{k} E[P\{X_1 \notin S_r(C_1), \ldots, X_k \in S_r(C_1),$$
$$X_{k+1} \in S_r(C_1), \ldots, X_m \in S_r(C_1) | C_1\}]$$

$$= \binom{m}{k} \frac{1}{|S|} \int_{|S|} dy\, P\{X_1 \notin S_r(y)\}^k \, P\{X_{k+1} \in S_r(y)\}^{m-k}$$

(because $X_1, \ldots, X_m$ are independent of one another and of C_1)

$$= \binom{m}{k} \left(1 - \frac{\pi r^2}{\pi}\right)^k \left(\frac{\pi r^2}{\pi}\right)^{m-k}$$

proving (41) and (42c). A similarly pleasant simplification in the proof of (42b) is not possible because a shot uniformly distributed on S_{1-r} does not have a probability of killing a defender uniformly distributed on $S_{r(i)}$, which, given the center of the shot, is constant, as was true above, except for $i = m$.

Suppose $C_1, C_2, \ldots, C_{k+1}$ are the centers of the first k+1 shots and let $X_1, \ldots, X_{T_k}$ be the positions assumed by the T_k defenders surviving the first k shots. Since by A6) $X_1, \ldots, X_{T_k}$ are independent and identically uniformly distributed on $S_{r(T_k)}$, given T_k, and are independent of $C_1, \ldots, C_k$ and of all previous defender positions, it follows that $X_1, \ldots, X_{T_k}$ are conditionally independent of $T_1, \ldots, T_{k-1}$ given T_k, and hence that the Markov property holds:

$$P\{T_{k+1} = j | T_1, \ldots, T_k\} = P\{T_{k+1} = j | T_k\} \quad .$$

It remains to prove that

$$P\{T_{k+1} = j|T_k\} = P(T_k, j)$$

with P defined by (42); we may and do assume that $T_k \geq 1$. Noting that $P\{T_{k+1} = j|T_k\}$ is the conditional probability given T_k that exactly $T_k - j$ of the points $X_1, \ldots, X_{T_k}$ are within the disk $S_r(C_{k+1})$ and again using the indistinguishability of the defenders, we obtain

$$P\{T_{k+1} = j|T_k =$$

$$= \binom{T_k}{j} P\{X_1, \ldots, X_j \notin S_r(C_{k+1}), X_{j+1}, \ldots, X_{T_k} \in S_r(C_{k+1})|T_k\}$$

$$= \binom{T_k}{j} E[P\{X_1, \ldots, X_j \notin S_r(C_{k+1}), X_{j+1}, \ldots, X_{T_k} \in S_r(C_{k+1})|T_k, C_{k+1}\}|T_k]$$

$$= \binom{T_k}{j} \frac{1}{|S_{1-r}|} \int_{S_{1-r}} dy\, P\left\{X_1 \notin S_r(y)\right\}^j P\left\{X_1 \in S_r(y)\right\}^{T_k-j} .$$

By rotational symmetry of a uniform distribution on a disk, $P\{X_1 \in S_r(y)\}$ depends only on $|y|$ and since $|C_{k+1}|$ is distributed on $[0, 1 - r]$ with density $(1 - r)^{-2}2u$ the last expression becomes

$$\binom{T_k}{j} \frac{1}{(1 - r)^2} \int_0^{1-r} 2u du\, P\{X_1 \notin S_r[(0, u)]\}^j P\left\{X_1 \in S_r[0, u)]\right\}^{T_k-j} .$$

Since X_1 is uniformly distributed on $S_{r(i)}$, we have

$$P\{X_1 \in S_r[(0, u)]\} \equiv f_i(u) = \frac{|S_{r(i)} \cap S_r[(0, u)]|}{|S_{r(i)}|}$$

which completes the proof. ∎

Further remarks concerning the functions f_i are in Appendix A. The complexity of the expressions for the elements of P and the exact form of the f_i as given by (A.4) seem to preclude efficient computation of P on a computer.

Appendix B considers this process with an exogeneous time scale, rather than that determined by numbers of shots.

APPENDIX A

Let us consider in more detail the functions f_i defined in (40). To find the exact form of these functions one must compute $|S_{r(i)} \cap S_r[(0, u)]|$ for $u \in [0, 1 - r]$. It is evident that

$$|S_{r(i)} \cap S_r[(0, u)]| = \begin{cases} \pi r^2 , & \text{if } 0 \leq u \leq r(i) - r \\ 0 , & \text{if } u > r(i) + r , \end{cases}$$

so that only $u \in (r(i) - r, r(i) + r)$ remain to be considered. If one draws a picture with the upper half of the circle bounding $S_{r(i)}$, which has the equation

$$y = (r(i)^2 - x^2)^{1/2} , \tag{A.1}$$

and the circle which is the boundary of $S_r[(0, u)]$ and whose equation is

$$x^2 + (y - u)^2 = r^2 , \tag{A.2}$$

he sees that conventional methods of analytic geometry may be used to evaluate the area of $|S_{r(i)} \cap S_r[(0, u)]|$.

Solving (A.1) and (A.2) simultaneously shows that the two curves intersect at the points $(m_1(u), m_2(u))$, $(-m_1(u), m_2(u))$, where

$$\begin{aligned} m_1(u) &= \left[r(i)^2 - \left(\frac{r(i)^2 - r^2 + u^2}{2u} \right)^2 \right]^{1/2} \\ m_2(u) &= \frac{r(i)^2 - r^2 + u^2}{2u} . \end{aligned} \tag{A.3}$$

Let

$$\beta_1(x) = (r(i)^2 - x^2)^{\frac{1}{2}} \quad , \qquad - r(i) \leq x \leq r(i)$$

$$\beta_2(u, x) = u + (r^2 - x^2)^{\frac{1}{2}} \quad , \qquad - r \leq x \leq r$$

$$\beta_3(u, x) = u - (r^2 - x^2)^{\frac{1}{2}} \quad , \qquad - r \leq x \leq r \quad .$$

Note the implicit dependence of β_1 on i.

From (A.3) we see that $m_1(u)=r$ when $u=(r(i)^2-r^2)^{\frac{1}{2}}$ and that $m_1(u)<r$ for all other r in $[r(i)-r, r(i)+r]$.

Thus for $(r(i)^2 - r^2)^{\frac{1}{2}} \leq u \leq r(i) + r$,

$$|S_{r(i)} \cap S_r [(0, u)]| = \int_{-m_1(u)}^{m_1(u)} \beta_1(x)\, dx - \int_{-m_1(u)}^{m_1(u)} \beta_3(u, x)\, dx$$

while for $r(i) - r \leq u \leq (r(i)^2 - r^2)^{\frac{1}{2}}$

$$|S_{r(i)} \cap S_r [(0, u)]| = \int_{-m_1(u)}^{m_1(u)} \beta_2(u, x)\, dx - \int_{-m_1(u)}^{m_1(u)} \beta_1(x)\, dx \quad .$$

Using tables of integrals we obtain the following evaluation of the functions f_i .

(A.4)

$$f_i(u) = \frac{1}{\pi r(i)^2} \begin{cases} \pi r^2 & , \text{ if } 0 \le u \le r(i) - r \\ m_1(u)^2 + m_1(u)(r^2 - m_1(u)^2)^{\frac{1}{2}} & , \text{ if } r(i) - r \le u \le (r(i)^2 - r^2)^{\frac{1}{2}} \\ + r^2 \sin^{-1}\left(\frac{m_1(u)}{r}\right) & \\ - m_1(u)\,(r(i)^2 - m_1(u)^2)^{\frac{1}{2}} & \\ - r(i)^2 \sin^{-1}\left(\frac{m_1(u)}{r(i)}\right) & \\ m_1(u)\,(r(i)^2 - m_1(u)^2)^{\frac{1}{2}} & , \text{ if } (r(i)^2 - r^2)^{\frac{1}{2}} \le u \le r(i) + r \\ + r(i)^2 \sin^{-1}\left(\frac{m_1(u)}{r(i)}\right) & \\ - m_1(u)^2 + m_1(u)\,(r^2 - m_1(u)^2)^{\frac{1}{2}} & \\ + r^2 \sin^{-1}\left(\frac{m_1(u)}{r}\right) & \\ 0 & , \text{ if } r(i) + r \le u \le 1 - r \,. \end{cases}$$

APPENDIX B

The regrouping model suggested here is capable of handling random numbers of shots between repositionings of the defenders. We now give the theory and computations necessary to make this generalization.

Consider the Markov process $(T_n)_{n\geq 1}$ with transition matrix P.

Let U_n be the number of shots on day n; we assume

A9) Numbers of shots on various days are independent, identically distributed, bounded, positive, integer-valued random variables, independent of the positions and lethalities of all shots and of defender positions.

If we put

$$Y_n = U_1 + \dots + U_n \quad , \qquad n \geq 1 \quad .$$

Then $S_n = T_{Y_n}$ is the number of surviving defenders at the end of the nth day.

(B.1) PROPOSITION. Under the preceding assumptions, $(S_n)_{n\geq 1}$ is a Markov process with initial distribution μ given by

$$\mu(k) = P\{S_1 = k\}$$

(B.2) $$= \sum_{i=1}^{\ell_0} \varphi(i) \sum_{j=0}^{m} \alpha(j) P^{i-1}(j, k)$$

and transition matrix Q given by

(B.3) $$Q(i, j) = \sum_{m=1}^{\ell_0} \varphi(m) P^m(i, j) \quad .$$

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Here φ is the common distribution of $U_1, U_2, \ldots$ and ℓ_0 is the upper bound on the number of shots per day.

PROOF. According to the terminology of Feller (1966), (S_n) is subordinated to (T_n) by the distribution φ. The proof required here is a straightforward application the theory derived in Feller (1966). $\dashv$

Likely candidates for φ would be Poisson and geometric distributions.

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