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HIGHER ORDER THEORY FOR LONG-TERM BEHAVIOR OF EARTH AND LUNAR ORBITERS

Bernard Kaufman, et al

Naval Research Laboratory
Washington, D. C.

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BERNARD KAUFMAN
AND
ROBERT DASENBROCK

Operations Research Branch
Report 72-8
Mathematics and Information Sciences Division

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A semianalytical method capable of analyzing both lunar and earth orbiters is presented. Primary attention is focused on predicting the evolution of the orbit as affected by third-body perturbations and those of the rotating primary. The singly averaged (literal) equations of motion are expanded by machine to a high order in the parallax factor and the mean motion ratio. These equations are numerically integrated to yield the orbital evolution and stability for a wide range of initial conditions. In addition, a purely analytical method is introduced to yield the orbital lifetimes for a special class of orbits.

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HIGHER ORDER THEORY FOR LONG-TERM BEHAVIOR OF EARTH AND LUNAR ORBITERS

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Abstract: A semianalytical method capable of analyzing both lunar and earth orbiters is presented. Primary attention is focused on predicting the evolution of the orbit as affected by third-body perturbations and those of the rotating primary. The singly averaged (literal) equations of motion are expanded by machine to a high order in the parallax factor and the mean motion ratio. These equations are numerically integrated to yield the orbital evolution and stability for a wide range of initial conditions. In addition, a purely analytical method is introduced to yield the orbital lifetimes for a special class of orbits.

INTRODUCTION

The design of any mission to place a satellite in orbit about a body is a complex and time consuming investigation of the types of orbits that would meet stated scientific objectives. Orbiters in the earth-moon system are influenced by a very complex dynamic field. This system is a unique one because of the relative masses of the two bodies. A satellite in orbit about either of these bodies undergoes interacting perturbations that are difficult to model. If periapsis is low enough, then the asphericity of the central body and atmospheric drag become important; if apoapsis is high enough, perturbations caused by the third body will dominate. Coupling between the gravity field and the third-body perturbations can cause complicated behavior in certain characteristics of the orbit. Further, the effects of the sun as a perturbing force cannot be neglected.

Because of this complex dynamic field, the mission analysis phase of the study must, and will of necessity, include a time history of many orbits in order to gain the maximum scientific data from the final orbit chosen. Such a history should include not only lifetime predictions but also a reasonably accurate history of all of the orbital parameters for a variety of initial conditions. Because such a variety of conditions will be used, it also becomes essential that the model chosen to produce the time history be not only accurate but very fast. Any good n-body precision integration program of the Cowell or Encke type is capable of meeting the first of these criteria, but certainly not the second.

To speed up the computations involved, the standard approximation used has been to doubly average the disturbing function due to the presence of a third body. The disturbing function is first averaged over one orbit of the satellite, and then over one revolution of the central body about the disturbing body. This process eliminates all short and

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medium-period terms, leaving only the long-period perturbations for consideration. However, experience has shown that this model has limited use for the earth-moon system. Also, for a planet such as Mars where coupling between oblateness and third body can be strong, the doubly averaged system sometimes fails. A good discussion of this model can be found in Ref. 1.

The singly averaged equations of motion have been shown in Ref. 2 to be highly accurate for orbiters of Mars. Here only the short-period terms are averaged out and the equations of motion retain the medium- and long-period terms. Therefore the model is also valid for both near and far orbiters of the earth and moon. However, as shown in that reference, the expansion of the third-body disturbing function is essentially in terms of the parallax factor a/r'. For a 100,000-km-high earth orbiter, this parallax factor is about 0.25 for the moon as the disturbing body. The expansion in Ref. 2 was truncated to retain only terms of second order in the parallax factor, but for high earth or lunar orbiters this is not sufficient. The expansion must be to at least fourth order, and for high orbiters (100,000 km) should be carried even further.

When the third-body terms are averaged, the assumption is sometimes made that the disturbing body does not move significantly over one orbit of the satellite. However, for high orbits this assumption is clearly violated in the case of the earth-moon system. Thus, in carrying out the expansion, a time rate of change for all terms containing the third-body position must be included. This yields a further expansion of the disturbing function in terms of the mean motion ratio n'/n — the ratio of the mean motion of the disturbing body to that of the satellite.

To carry out the expansions in terms of the parallax factor and the mean motion ratio, and then to average the equations of motion over one orbit, requires an excessive amount of algebra for the higher order terms. It is almost impossible to proceed beyond fourth order in parallax using hand techniques. To aid in the algebraic computations, a general algebraic manipulation routine was developed which is now operational on a large-scale computer. This program was used to compute the averaged (literal) equations of motion to eighth order in the parallax factor and to second order in the mean motion ratio with the corresponding cross terms up to and including fifth order in parallax. The entire expansion required approximately 2 min of central processor time on a CDC 3800 computer. The output is FORTRAN compatible, card punched, and directly insertable into the variation-of-parameters program with no human interaction.

In addition to the third-body effects, the gravity harmonics of the rotating primary must be considered. For the moon, these equations may be averaged over the orbital period since the moon rotates slowly. However, for the more rapidly rotating earth, this analysis is invalid because the orbital mean motion may be nearly commensurate with the rotation of the primary. To avoid this problem, Gaussian quadratures are used, i.e., the equations of motion are numerically averaged from one-half orbit behind to one-half orbit ahead of the present position of the satellite. The Greenwich mean sidereal time enters explicitly in this analysis. These averaged rates are then used in the total variations of the elements. At present, a full 7X7 and 4X4 field is used for the earth and moon, respectively. When the tesseral harmonics are not required, only terms containing J_2, J_2^2, J_3, and J_4 are used, and the variational equations are calculated explicitly without using quadratures.

Finally, a model is also included for atmospheric drag and its effects on the semimajor axis and eccentricity. This model assumes a nonrotating atmosphere, no lift forces, and a drag force acting as a negative tangential component. The density is calculated only as a function of altitude, and the accelerations are averaged over one revolution of the satellite.

In addition to this semianalytic approach, a purely analytical model is also developed. This model concentrates on the long-period terms and applies to both lunar and high earth orbiters. To derive the long-period equations of motion, the medium-period terms must be removed. This can be done provided that the earth and moon move much faster on their respective paths than the line of apsides of the perturbed orbit under question. This condition is satisfied for all lunar orbiters for which the height does not exceed approximately four lunar radii. For the low lunar orbiters, the fastest periodic term in the equations of motion involves the earth angle, i.e., one revolution per month, whereas the nodal angle and perilune angle rotate with periods of not less than 6 months. Thus, these medium-period effects involving the earth angle can be removed by a von Zeipel transformation.

For earth orbiters, the analysis is more complicated. The line of apsides rotates with an angular velocity of about 8° per day for a low orbiter, whereas the sun and moon move along their respective paths with angular rates of 1° and 13° per day, respectively. Thus these medium-period terms cannot be removed for the low orbiters; however, the analysis is valid for the higher orbits and useful results can be obtained.

In the absence of oblateness, the solution to the equations of motion can be expressed in closed form using elliptic integrals. However, in certain special cases involving initially-near-circular orbits, the solution involves only the elementary functions. These special cases are extended to include oblateness by introducing a series expansion. The solutions are accurate and yield results applicable to initially-near-circular orbits. These solutions yield the long-period time history of eccentricity and pericenter position and give the orbital lifetime for unstable orbits. This procedure works well for high lunar orbits and can be extended to high earth orbiters.

MACHINE-AUTOMATED ALGEBRA

The use of machine-automated algebra in celestial mechanics has become increasingly popular in recent years. Due to the high probability of error which is introduced when hand methods are used, it was decided to develop the equations of motion to a high order by computer. To this end it was decided to construct an algebraic manipulation program compatible with the CDC 3800 computer presently in use at the Naval Research Laboratory. This program will manipulate the otherwise involved literal Poisson series occurring in classical perturbation theory. The computerized operations include the simplification, ordering, negation, addition, subtraction, multiplication, differentiation, and integration of the trigonometric series occurring in the theory. Other more specialized routines include a binomial and taylor series expansion. The program is written in FORTRAN and can be used on any machine possessing a FORTRAN compiler, with little or no modifications.

The equations of motion (to be described later) were developed entirely by computer. These literal equations were automatically card punched in FORTRAN-compatible form

and inserted directly into the variation of parameters program with no human interaction. Literally thousands of terms were involved in these expansions, with a savings in time of many months and possibly years. The method has the added advantage in that trivial algebraic and keypunching errors are eliminated as possible errors in the analysis. Remaining errors can be traced to those of concept and the programming of the literal expansions. A detailed description of the algebraic manipulation program may be found in Ref. 3.

VARIATION OF PARAMETERS

The equations for the variation of the keplerian elements are well known and are developed in any good textbook on celestial mechanics. Therefore the equations will be listed here, without derivation, in the two forms that were used in the computer program.

Lagrange's Planetary Equations

$$\left.\begin{aligned}
\frac{da}{dt} &= \frac{2}{na}\frac{\partial F}{\partial M} \\
\frac{de}{dt} &= \frac{(1-e^2)^{1/2}}{ena^2}\left((1-e^2)^{1/2}\frac{\partial F}{\partial M} - \frac{\partial F}{\partial \omega}\right) \\
\frac{d\Omega}{dt} &= \frac{1}{na^2(1-e^2)^{1/2}\sin i}\frac{\partial F}{\partial i} \\
\frac{di}{dt} &= -\frac{\csc i}{na^2(1-e^2)^{1/2}}\left(\frac{\partial F}{\partial \Omega} - \cos i\frac{\partial F}{\partial \omega}\right) \\
\frac{d\omega}{dt} &= \frac{(1-e^2)^{1/2}}{ena^2}\frac{\partial F}{\partial e} - \frac{\cos i}{na^2(1-e^2)^{1/2}\sin i}\frac{\partial F}{\partial i} \\
\frac{dM}{dt} &= n - \frac{2}{na}\frac{\partial F}{\partial a} - \frac{(1-e^2)}{nea^2}\frac{\partial F}{\partial e}.
\end{aligned}\right\} \tag{1}$$

Gauss' Form of the Equations

$$\left.\begin{aligned}
\frac{da}{dt} &= \frac{2}{n(1-e^2)^{1/2}}\left(R\,e\sin f + \frac{a(1-e^2)}{r}S\right) \\
\frac{de}{dt} &= \frac{(1-e^2)^{1/2}\sin f}{na}R + \frac{(1-e^2)^{1/2}}{ena^2}\left(\frac{a^2(1-e^2)-r^2}{r}\right)S \\
\frac{d\Omega}{dt} &= \frac{r\sin(\omega+f)}{na^2(1-e^2)^{1/2}\sin i}W \\
\frac{di}{dt} &= \frac{r\cos(\omega+f)}{na^2(1-e^2)^{1/2}}W
\end{aligned}\right\} \tag{2}$$

$$\left.\begin{aligned}
\frac{d\omega}{dt} &= -\frac{(1-e^2)^{1/2}\cos f}{ane}R + \frac{(1-e^2)^{1/2}\sin f}{ane}\left(1+\frac{r}{a(1-e^2)}\right)S \\
&\quad - \frac{r\sin(\omega+f)\cot i}{a^2 n(1-e^2)^{1/2}}W \\
\frac{dM}{dt} &= n + \left(\frac{(1-e^2)\cos f}{ane} - \frac{2r}{na^2}\right)R - \frac{(1-e^2)\sin f}{ane}\left(1+\frac{r}{a(1-e^2)}\right)S.
\end{aligned}\right\} \quad (2)$$

the disturbing force Δ is defined as

$$\Delta = R\mathbf{U}_r + S\mathbf{U}_\theta + W\mathbf{U}_A \quad (3)$$

and Δ is decomposed into components in the radial (R), transverse (S), and orbit plane normal (W) directions where

$$\mathbf{U}_A = \mathbf{U}_r \times \mathbf{U}_\theta .$$

THE THIRD-BODY DISTURBING FUNCTION

The acceleration experienced by a satellite under the influence of a point mass third body is

$$\ddot{\mathbf{r}}_I = \ddot{\mathbf{r}} - \ddot{\mathbf{r}}' = \nabla\left(\frac{\mu'}{|\mathbf{r}'-\mathbf{r}|}\right) \quad (4)$$

where μ' is the gravitational coefficient of the third body and $\ddot{\mathbf{r}}_I$ denotes the acceleration vector in inertial space. The position vectors to the third body and the satellite are $\mathbf{r}'$ and $\mathbf{r}$, respectively.

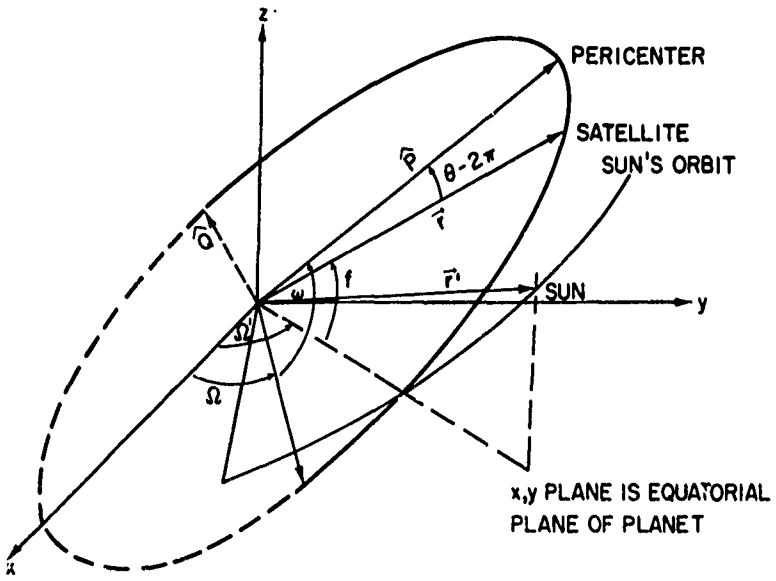

Fig. 1 — Planet-centered satellite geometry

Equation (4) may be expressed as

$$\ddot{\mathbf{r}} = -\mu' \nabla\left(\frac{1}{|\mathbf{r}' - \mathbf{r}|} - \frac{\mathbf{r}' \cdot \mathbf{r}}{r'^3}\right) \tag{5}$$

where

$$\ddot{\mathbf{r}}' = \frac{\mu'}{r'^3}\mathbf{r}' = \mu' \nabla\left(\frac{\mathbf{r}' \cdot \mathbf{r}}{r'^3}\right).$$

Equation (5) can be written in the form

$$\ddot{\mathbf{r}} = \nabla F(\mathbf{r}, \mathbf{r}') \tag{6}$$

where

$$F(\mathbf{r},\mathbf{r}') = \frac{\mu'}{r'}\left(\frac{1}{\left(1 - 2\frac{r}{r'}\cos S + \left(\frac{r}{r'}\right)^2\right)^{1/2}} - \left(\frac{r}{r'}\right)\cos S\right) \tag{7}$$

and

$$\cos S = \frac{\mathbf{r} \cdot \mathbf{r}'}{rr'}.$$

Now introduce the eccentric anomaly E directly into Eq. (7), i.e., let

$$\left(\frac{r}{r'}\right)\cos S = \delta A(\cos E - e) + \delta B(1 - e^2)^{1/2} \sin E$$

and

$$\left(\frac{r}{r'}\right)^2 = \delta^2(1 - e \cos E)^2,$$

where

$\delta = \frac{a}{r'}$ is the parallax factor,

$A = \hat{\mathbf{P}} \cdot \hat{\mathbf{r}}'$,

$B = \hat{\mathbf{Q}} \cdot \hat{\mathbf{r}}'$, where $\hat{\mathbf{P}}$ and $\hat{\mathbf{Q}}$ are as shown in Fig. 1, and

$\hat{\mathbf{r}}'$ is the unit vector to the third body.

The quantity $F(\mathbf{r}',\mathbf{r})$ was expanded directly to the eighth order in δ by using the algebraic manipulation program previously mentioned. Due to the direct nature of the expansion, the explicit expressions for the Legrendre polynomials were not required. The coefficients of each factor δ^n (where $2 \leqslant n \leqslant 8$) were automatically collected, and the disturbing function is then obtained in the form

$$F(\mathbf{r},\mathbf{r}') = \frac{\mu'}{r'}\sum_{n=2}^{8} \delta^n F_n(A,B,e,E) \tag{8}$$

where each $F_n(A,B,e,E)$ is of the form

$$F_n(A,B,e,E) = \sum_{m=0}^{n} C_{jk\ell m}\, A^j B^k e^\ell \cos mE + \sum_{m=1}^{n} S_{jk\ell m}\, A^j B^k e^\ell \sin mE \tag{9}$$

and both $C_{jk\ell m}$ and $S_{jk\ell m}$ are obtained as rational integer coefficients. The next step is to average the disturbing function over one orbital period, i.e.,

$$\overline{F}_n(A,B,e) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F_n\, d\ell = \frac{1}{2\pi}\int_{-\pi}^{\pi} F_n(1 - e\cos E)\, dE. \tag{10}$$

The equations of motion as determined by the Lagrangian planetary equations listed in Eq. (1) are

$$\dot{x}_i = f_i(x_i). \tag{11a}$$

Substituting $\overline{F}_n$ and its derivatives we then have the averaged equations of motion given by

$$\overline{\dot{x}}_i = f_i(A,B,\gamma,e) \tag{11b}$$

where

x_i is the Keplerian state vector,

$$\gamma = \begin{pmatrix}\gamma_1\\ \gamma_2\\ \gamma_3\end{pmatrix}\begin{cases}\gamma_1 = \hat{\mathsf{R}}\cdot\hat{\mathsf{r}}'\\ \gamma_2 = \mathsf{P}_1\cdot\hat{\mathsf{r}}'\\ \gamma_3 = \mathsf{Q}_1\cdot\hat{\mathsf{r}}'\end{cases}$$

with

$$\mathsf{P}_1 = \begin{pmatrix}-P(2)\\ P(1)\\ 0\end{pmatrix}$$

and

$$\mathsf{Q}_1 = \begin{pmatrix}-Q(2)\\ Q(1)\\ 0\end{pmatrix};\ \text{and}\ \hat{\mathsf{R}} = \hat{P}\times\hat{Q}.$$

In the first-order variation of parameters analysis it may be assumed that the orbital elements a, e, i, Ω, and ω are constant as the equations of motion are averaged. However, the third-body position cannot be held constant.

Therefore A, B, and γ are expanded about their nominal values, i.e.,

$$A = A_0 + \frac{\partial A}{\partial \theta'}\left(\frac{n'}{n}\right)(M - M_0) + \frac{1}{2}\frac{\partial^2 A}{\partial \theta'^2}\left(\frac{n'}{n}\right)^2 (M - M_0)^2 + \ldots \tag{12}$$

with similar expressions for B and γ. θ' is the third-body central angle and

$$\frac{\partial A}{\partial \theta'} = \frac{1}{n'} \hat{\mathbf{P}} \cdot \frac{d\mathbf{r}'}{dt}$$

where $d\mathbf{r}'/dt$ is given along with $\mathbf{r}'$ by an analytical ephemeris. The additional terms for Eq. (11a) representing the motion of the disturbing body then become

$$\delta \dot{x}_i = \frac{\partial f_i}{\partial A} \frac{\partial A}{\partial \theta'} \left(\frac{n'}{n}\right)(M - M_0) + \frac{\partial f_i}{\partial B} \frac{\partial B}{\partial \theta'} \left(\frac{n'}{n}\right)(M - M_0)$$

$$+ \frac{\partial f_i}{\partial \gamma} \frac{\partial \gamma}{\partial \theta'} \left(\frac{n'}{n}\right)(M - M_0) + \text{higher order terms.} \tag{13}$$

Averaging Eq. (13) as above and then adding Eq. (11b) yields the total averaged equations of motion.

Since the disturbing function is time dependent, the time rate of change of the semimajor axis has a nonzero average. Although no secular change results, small but not insignificant twice-monthly and twice-yearly fluctuations in the semimajor axis do appear. The final form of the equations of motion is obtained in the parallax and mean motion ratio as

$$\bar{\dot{x}}_i = \frac{\mu'}{r'} \left[\sum_{k=2}^{8} f_{k0} \left(\frac{a}{r'}\right)^k + \sum_{k=2}^{5} f_{k1} \left(\frac{a}{r'}\right)^k \left(\frac{n'}{n}\right) \right] \tag{14}$$

where the f_{k0} are the functions derived from the averaged disturbing function $\overline{F}_n$, and f_{k1} is the additional term due to the motion of the disturbing body. The derivatives of $\overline{F}_n$ necessary for f_{k0} are listed in Appendix A, and the derivatives required for f_{k1} are listed in Appendix B.

For a very high earth or lunar orbiter it is not enough to assume that the orbital elements a, e, i, Ω, and ω remain constant during the averaging process, as was done here. For these high orbits it is necessary to include the coupling between the short-period fluctuations in the elements with the short-periodic part of the disturbing function. This is especially important for high lunar orbiters. These additional expansions are being carried out but have not yet been implemented in the variation-of-parameters program.

GRAVITATIONAL FIELD ANALYSIS

For a satellite of the moon, the disturbing function could be averaged analytically; for most other planets such is not the case. Representing the gravitational field of the body by the standard expansion in spherical harmonics, we have

$$U = \frac{\mu}{r} + F = \frac{\mu}{r} \left[1 + \sum_{n=1}^{\infty} \sum_{m=0}^{n} \frac{R^n}{r^n} P_n^m(\sin \phi) \left(C_{nm} \cos m\lambda + S_{nm} \sin m\lambda\right) \right]. \tag{15}$$

Then in the Lagrangian equations we would normally substitute

$$\overline{F} = \frac{1}{\tau}\int_0^{\tau} F dt$$

for the disturbing function where, more explicitly,

$$\overline{F} = \frac{\mu(1-e^2)^{3/2}}{P^{n+1}} \frac{R^n}{2\pi} \int_0^{2\pi} (1 + e\cos f)^{n-1} \; P_n^m(\sin\phi)\, e^{im(\lambda-\theta)}\, \phi \, df \tag{16}$$

with $\theta = \dot{\theta} t$ ($\dot{\theta}$ is the rotation rate of the primary).

The integral in the above equation is evaluated with all quantities held constant, except f. For a satellite of the moon, the change in θ over one orbit is negligible to the first approximation, but for other planets, this change can be significant. It is for this reason that the disturbing accelerations caused by a nonspherical central body must be treated differently. Here the equations of motion are averaged numerically and these averaged equations are then integrated to obtain the variation in the orbital elements. This treatment was suggested by Uphoff in Ref. 4. He defines the averaged Lagrangian equations of motion in the following manner: Let $\dot{x}_i$ be the standard variational equation of any element. The averaged Lagrangian equation is then

$$\overline{\dot{x}}_i = \frac{1}{T}\int_0^{\tau} \dot{x}_i(\tau) dt = \frac{n}{2\pi}\int_0^{2\pi} \dot{x}_i(f) \frac{dt}{df} df. \tag{17}$$

But

$$\frac{df}{dt} = \frac{(\mu P)^{1/2}}{r^2};$$

therefore

$$\overline{\dot{x}}_i = \frac{nP^2}{2\pi\sqrt{\mu P}} \int_0^{2\pi} \frac{\dot{x}_i(f) df}{(1 + e\cos f)^2}. \tag{18}$$

Substituting the Gaussian form of the variational equations into Eq. (18) the following is obtained:

$$\left.\begin{aligned}
\overline{\dot{a}} &= \frac{a^2(1-e^2)}{\pi\sqrt{\mu a}} \int_{f_1}^{f_2} \frac{e\sin f\, R(f) + (1 + e\cos f) S(f)}{(1 + e\cos f)^2} df \\
\overline{\dot{e}} &= \frac{nP^2}{2\pi\mu} \int_{f_1}^{f_2} \left(\frac{\sin f}{(1 + e\cos f)^2} R(f) + \frac{2\cos f + e(1 + \cos^2 f)}{(1 + e\cos f)^3} S(f) \right) df \\
\overline{\dot{i}} &= \frac{nP^2}{2\pi\mu} \int_{f_1}^{f_2} \frac{\cos(\omega + f) W(f)}{(1 + e\cos f)^3} df
\end{aligned}\right\} \tag{19}$$

$$
\left.
\begin{aligned}
\bar{\dot{\omega}} &= \frac{nP^2}{2\pi\mu}\int_{f_1}^{f_2}\left(\frac{(2+e\cos f)\sin f}{e(1+e\cos f)^3}S(f) - \frac{\cos f}{e(1+e\cos f)^2}R(f)\right.\\
&\quad \left. - \frac{\sin(\omega+f)\cos i}{(1+e\cos f)^3\sin i}W(f)\right)df\\
\bar{\dot{\Omega}} &= \frac{nP^2}{2\pi\mu}\int_{f_1}^{f_2}\frac{\sin(\omega+f)}{(1+e\cos f)^3\sin i}W(f)d(f)\\
\bar{\dot{M}} &= n + \frac{nP^2(1-e^2)^{1/2}}{2\pi\mu}\int_{f_1}^{f_2}\left[\left(\frac{\cos f}{e} - \frac{2}{(1+e\cos f)}\right)\right.\\
&\quad \left. \times\frac{R(f)}{(1+e\cos f)^2} - \frac{(2+e\cos f)\sin f}{e(1+e\cos f)^3}S(f)\right]df.
\end{aligned}
\right\} \quad (19)
$$

The evaluation of the definite integrals in these equations is done using Gaussian quadratures with 24 points.

The computation of the disturbing accelerations is accomplished using a technique devised by DeWitt in Ref. 5. This method was programmed to include any order of the zonals and tesserals desired. At present a full 7 X 7 and 4 X 4 field is used for the earth and moon, respectively. The accelerations are evaluated in cartesian coordinates and transformed to components in the radial (R), transverse (S), and orbit plane normal (W) directions by means of the rotation transformations

$$T_{R,S,W} = T_z(\omega + f)\, T_x(i)\, T_z(\Omega).$$

The transformed accelerations are then used directly in Eq. (19).

For central bodies other than the earth or moon, only terms containing J_2, J_2^2, J_3, and J_4 are used, and the variational equations are calculated explicitly without going through the above averaging and quadratures. The variational equations will simply be listed here. The equations in J_2 and J_2^2 are derived in detail in Ref. 6, with additional terms from Ref. 11, and those in J_3 and J_4 were taken from Ref. 7. This option is also used for high-speed analysis of earth and lunar orbiters where a full gravity field is not required.

Equations in J_2, J_2^2

$$
\left.
\begin{aligned}
\left(\overline{\frac{da}{dt}}\right)_{J_2} &\equiv 0\\
\left(\overline{\frac{de}{dt}}\right)_{J_2} &= -\frac{45nJ_2^2R_e^4}{32P^4}e(1-e^2)\left(\frac{14}{15} - \sin^2 i\right)\sin^2 i\,\sin 2\omega
\end{aligned}
\right\} \quad (20)
$$

$$
\left.
\begin{aligned}
\left(\overline{\frac{d\Omega}{dt}}\right)_{J_2} &= -\frac{3nJ_2R_e^2\cos i}{2P^2} - \frac{9nJ_2^2R_e^4\cos i}{4P^4}\left[\frac{3}{2} - \frac{5}{3}\sin^2 i\right. \\
&\quad + e^2\left(\frac{1}{6} + \frac{5}{24}\sin^2 i\right) + \frac{e^2\cos 2\omega}{4}\left(\frac{7}{3} - 5\sin^2 i\right) \\
&\quad \left. + (1-e^2)^{1/2}\left(1 - \frac{3}{2}\sin^2 i\right)\right] \\
\left(\overline{\frac{di}{dt}}\right)_{J_2} &= \frac{45}{64}\frac{J_2^2R_e^4}{P^4}ne^2\left(\frac{14}{15} - \sin^2 i\right)\sin 2i \sin 2\omega \\
\left(\overline{\frac{d\omega}{dt}}\right)_{J_2} &= \frac{3nJ_2R_e^2}{2P^2}\left(2 - \frac{5}{2}\sin^2 i\right) + \frac{3nJ_2^2R_e^4}{16P^4}\left[48 - 103\sin^2 i + \frac{215}{4}\sin^4 i\right. \\
&\quad + e^2\left(7 - \frac{9}{2}\sin^2 i - \frac{45}{8}\sin^4 i\right) - \cos 2\omega\left[\left(7 - \frac{15}{2}\sin^2 i\right)\sin^2 i\right. \\
&\quad \left. - e^2\left(7 - \frac{79}{2}\sin^2 i + \frac{135}{4}\sin^4 i\right)\right] \\
&\quad \left. + (1-e^2)^{1/2}(24 - 66\sin^2 i + 45\sin^4 i)\right]
\end{aligned}
\right\} \tag{20}
$$

where

R_e is the equatorial radius of the planet

$P = a(1 - e^2)$ is the semilatus rectum and the subscript J_2 means oblateness and J_2 terms only.

Equations in J_3

$$
\left.
\begin{aligned}
\left(\overline{\frac{dP}{dt}}\right)_{J_3} &= 2P\tan i\left(\frac{di}{dt}\right)_{J_3} \\
\left(\overline{\frac{de}{dt}}\right)_{J_3} &= -\frac{3}{8}\frac{nR_e^3J_3}{P^3}(1-e^2)(5\cos^2 i - 1)\cos\omega\sin i \\
\left(\overline{\frac{d\Omega}{dt}}\right)_{J_3} &= \frac{3nR_e^3J_3}{3P^3}e(15\cos^2 i - 11)\sin\omega\cot i \\
\left(\overline{\frac{di}{dt}}\right)_{J_3} &= \frac{3nR_e^3J_3}{8P^3}e(5\cos^2 i - 1)\cos\omega\cos i
\end{aligned}
\right\} \tag{21}
$$

$$\left(\overline{\frac{d\omega}{dt}}\right)_{J_3} = \frac{3nR_e^3J_3}{8P^3}\frac{(1+4e^2)}{e}(5\cos^2 i - 1)\sin\omega\sin i - \left(\overline{\frac{d\Omega}{dt}}\right)_{J_3}\cos i\Bigg\} \tag{21}$$

where

$$\left(\overline{\frac{da}{dt}}\right)_{J_3} = \frac{\left(\overline{\frac{dP}{dt}}\right)_{J_3} + 2ae\left(\overline{\frac{de}{dt}}\right)_{J_3}}{(1-e^2)} \tag{22}$$

Equations in J_4

$$\left.\begin{aligned}
\left(\overline{\frac{dP}{dt}}\right)_{J_4} &= 2P\tan i\left(\overline{\frac{di}{dt}}\right)_{J_4}\\
\left(\overline{\frac{de}{dt}}\right)_{J_4} &= -\frac{15nR_e^4J_4}{32P^4}e(1-e^2)(7\cos^2 i - 1)\sin 2\omega\sin^2 i\\
\left(\overline{\frac{d\Omega}{dt}}\right)_{J_4} &= \frac{15nR_e^4J_4}{32P^4}\cos i\,\{2(7\cos^2 i - 3)\\
&\quad + e^2[7\cos^2 i - 1 + 4\sin^2\omega(7\cos^2 i - 4)]\}\\
\left(\overline{\frac{di}{dt}}\right)_{J_4} &= \frac{15nR_e^4J_4}{64P^4}e^2(7\cos^2 i - 1)\sin 2\omega\sin 2i\\
\left(\overline{\frac{d\omega}{dt}}\right)_{J_4} &= -\frac{15nR_e^4J_4}{16P^4}\Bigg\{8 - 28\sin^2 i + 21\sin^4 i\\
&\quad - (7\cos^2 i - 1)\sin^2\omega\sin^2 i + e^2\Bigg[6 - 14\sin^2 i + \frac{63}{8}\sin^4 i\\
&\quad + \sin^2\omega\left(6 - 35\sin^2 i + \frac{63}{2}\sin^4 i\right)\Bigg]\Bigg\}
\end{aligned}\right\} \tag{23}$$

where a definition identical to Eq. (22) holds with the proper change in the subscripts.

DRAG

For the variations due to the presence of an atmosphere, a model has been assumed in which there are no lift forces present, the drag force acts as a negative tangential component, and the atmosphere is nonrotating. This results in variations only in the semimajor axis and the eccentricity. The variational equations are then averaged over a single orbit using Gaussian quadratures. The density is taken from several models and is calculated as a function of altitude. These models can be found in Ref. 8 and 9 for Mars and Venus, respectively. For the earth, no atmosphere was used in the present program, although it would be very easy to incorporate a model similar to the type used for Mars and Venus.

The detailed derivation may be found in Ref. 2 with only the results being listed here:

$$\bar{\dot{a}} = -\frac{C_D A a^2 (1 - e^2)^{3/2}}{2\pi\, m\mu} \int_{-\pi}^{\pi} \frac{\rho V^3}{(1 + e \cos f)^2}\, df \tag{24}$$

$$\bar{\dot{e}} = -\frac{C_D A (1 - e^2)^{3/2}}{2\pi\, m} \int_{-\pi}^{\pi} \frac{\rho V (e + \cos f)}{(1 + e \cos f)^2}\, df \tag{25}$$

where C_D is the aerodynamic drag coefficient, A is the cross-sectional area, and ρ is the density.

It is to be stressed that only a nonrotating atmosphere has been considered; otherwise di/dt and $d\Omega/dt$ would be nonzero. The complexities of assuming an exponential density profile have also been bypassed by averaging the effects over one revolution of the satellite.

SAMPLE CASES

The method described above has been programmed for a CDC 3800 computer in double-precision mode under the program name of POPLAR (Planetary Orbiter Prediction and Lifetime Analysis Routine). Figure 2 shows a comparison between POPLAR and an Encke n-body numerical integration program for a lunar orbiter. Most of the slight differences noticed here are due to the fact that mean elements were not used and the constants were different between the two programs. However, it is important to note that these differences do not appear to be growing with time. The numerical integration program took 4.6 min of 360/95 time, which translates to about 115 min of 3800 time. POPLAR took 3.65 min for the same case — a factor of over 30 to 1.

Figure 3 is a plot of eccentricity versus argument of pericenter for a lunar orbiter. Superimposed on this graph are contours of constant lifetime. The value of such a plot is that it allows initial conditions to be selected that will yield any given lifetime. For lunar orbiters, the inclination does not vary by more than a degree or two and, therefore, these curves may be used as a very good first approximation to a nominal orbit. The data shown in Fig. 3 is composed of thirteen different cases and took a total of about 15 min worth of 3800 CPU time.

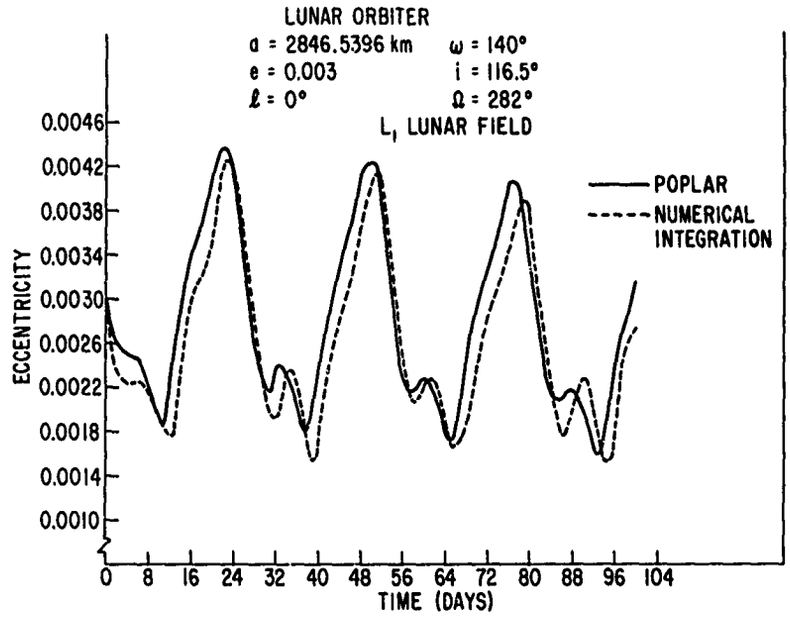

Fig. 2 — Comparison of POPLAR-generated and numerically integrated values of eccentricity vs time

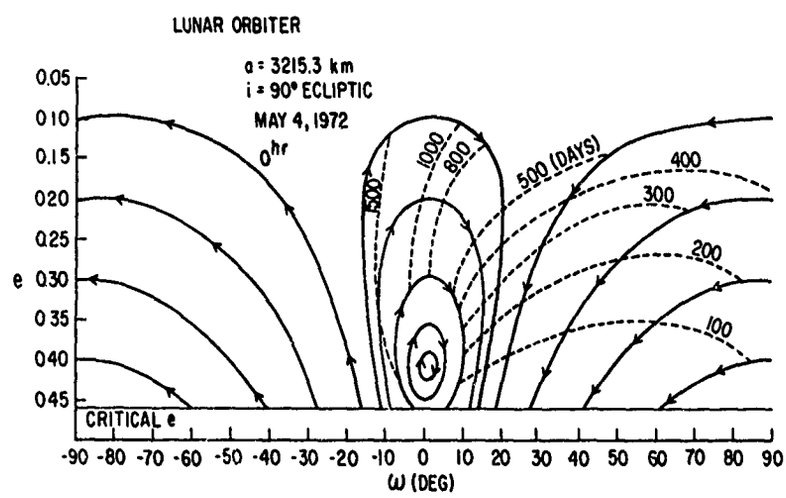

Fig. 3 — Eccentricity vs argument of perigee for a lunar orbiter

Figure 4 is a similar curve for an earth orbiter of 75° initial inclination to the ecliptic. For earth orbiters, the inclination does not remain as constant as for lunar orbiters. However, the change in inclination is only about 12°; therefore approximate initial conditions may still be obtained but must be finally checked by numerical integration. Total CPU time for Fig. 4 was about 25 min.

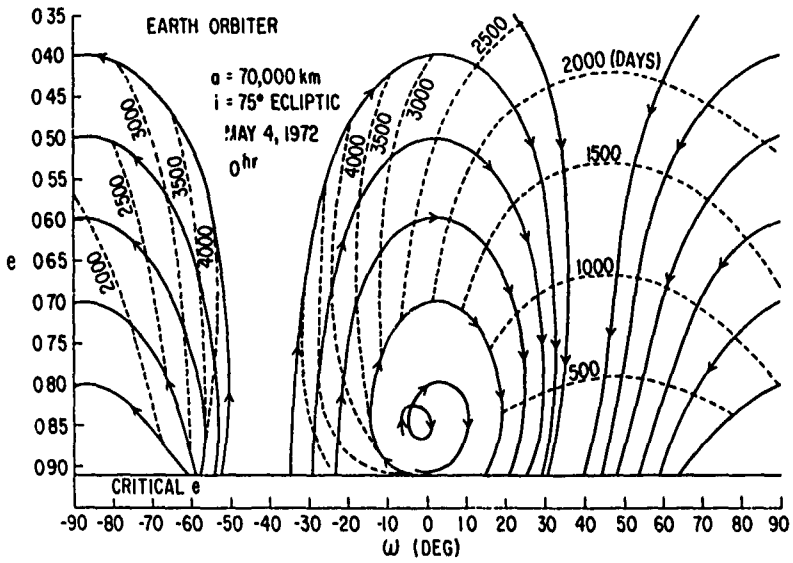

Fig. 4 — Eccentricity vs argument of perigee for an earth orbiter with i = 75°

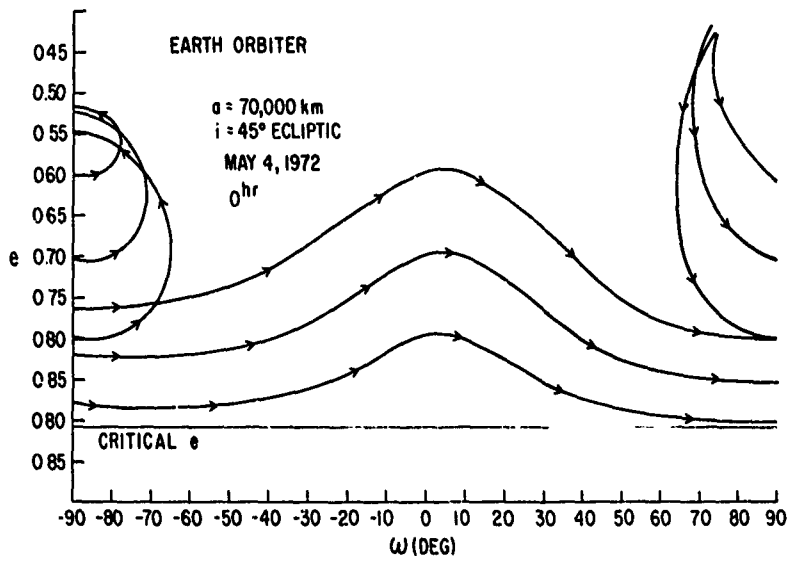

Fig. 5 — Eccentricity vs argument of perigee for an earth orbiter with i = 45°

Figure 5 is again for an earth orbiter inclined at 45° to the ecliptic and shows stable orbits as far as the lifetime is concerned. The intersections of the curves near $\omega = \pm 90°$ is indicative of the fact that inclination does not remain constant, varying as much as 15° for some of the curves in this figure. Again, however, first approximations to initial conditions may still be obtained. Approximately 20 min of CPU time was required for Fig. 5.

ANALYTICAL RESULTS

The results obtained thus far have been arrived at by numerically integrating the medium- and long-period equations of motion. For a lunar orbiter (and some earth orbiters) the medium-period terms may be averaged out, leaving the long-period equations of motion. For a low lunar orbiter these equations may be integrated analytically away from resonance by the method of successive approximations, as in Ref. 10. In the absence of oblateness the solution may be expressed in terms of elliptic integrals (Ref. 1). However, in the case of the initially-near-circular orbit, the solution to the latter problem may be expressed exactly in terms of the elementary functions. J_2 is considered by introducing a power series expansion in terms of the quantity $1-(1-e^2)^{1/2}$, which converges quickly for most values of e.

The long-period Hamiltonian for a moderately high lunar orbiter can be written as

$$F = \frac{\mu_M}{2L^2} + \frac{1}{4} J_2 \frac{\mu_M^4 R_M^2}{L^3 G^3}\left(1 - 3\frac{H^2}{G^2}\right) + \frac{n_e^2 a^2}{16}\left[\left(5 - 3\frac{G^2}{L^2}\right)\left(3\frac{H^2}{G^2} - 1\right)\right.$$

$$\left. + 15\left(1 - \frac{G^2}{L^2}\right)\left(1 - \frac{H^2}{G^2}\right)\cos 2g\right] \tag{26}$$

where L, G, H, and g are the usual Delaunay variables. For a Hamiltonian system

$$\dot{G} = \frac{\partial F}{\partial g}; \quad \dot{g} = -\frac{\partial F}{\partial G} \tag{27}$$

$$\dot{G} = -\frac{15}{8} n_e^2 a^2 \left(1 - \frac{G^2}{L^2}\right)\left(1 - \frac{H^2}{G^2}\right)\sin 2g. \tag{28}$$

A relationship between $\sin 2g$ and L, G, H must now be determined. This is accomplished by setting

$$F(L,G,H,g)|_{G=L} = F(L,G,H,g) \tag{29}$$

and solving for $\cos 2g$:

$$\cos 2g = \frac{1}{5}\frac{\left(1 - 5\frac{H^2}{G^2}\right)}{\left(1 - \frac{H^2}{G^2}\right)} + \frac{K_0}{1 - \frac{H^2}{G^2}}\left[\frac{1 - 3\frac{H^2}{L^2} - \frac{L^3}{G^3}\left(1 - 3\frac{H^2}{G^2}\right)}{\left(1 - \frac{G^2}{L^2}\right)}\right] \tag{30}$$

where

$$K_0 = \frac{4}{15}\left[\frac{\frac{\mu_M}{a} J_2 \left(\frac{R_M}{a}\right)^2}{n_e^2 a^2}\right].$$

K_0 is the ratio between the lunar oblateness effect and the terrestrial gravity effect.

Now let

$$f(G) = \frac{1}{G^3}\left(1 - 3\frac{H^2}{G^2}\right) \tag{31}$$

$$f(L) = \frac{1}{L^3}\left(1 - 3\frac{H^2}{L^2}\right) \tag{32}$$

and expand the expression for cos $2g$ up to the second power of $(1 - G/L) = X$:

$$\cos 2g = \frac{1}{5\left(1 - \frac{H^2}{G^2}\right)}\left[1 - 5\frac{H^2}{L^2}(1 + 2X + 3X^2 + \ldots)\right]$$

$$+ \frac{1}{2}\frac{K_0 L^3}{\left(1 - \frac{H^2}{G^2}\right)}(K_1 + K_2 X + K_3 X^2) \tag{33}$$

where

$$K_1 = \frac{15}{2}\frac{H^2}{L^2} - \frac{1}{2}$$

$$K_2 = \frac{105}{4}\frac{H^2}{L^2} - \frac{5}{4}$$

$$K_3 = \frac{525}{8}\frac{H^2}{L^2} - \frac{55}{24}$$

$$\cos 2g = \frac{A_0 + A_1 X + A_2 X^2}{\left(1 - \frac{H^2}{G^2}\right)} \tag{34}$$

and

$$A_0 = \frac{1}{5} - \frac{H^2}{L^2} + \frac{1}{2} K_0 L^3 K_1$$

$$A_1 = -2\frac{H^2}{L^2} + \frac{1}{2} K_0 K_2 L^3$$

$$A_2 = -3\frac{H^2}{L^2} + \frac{1}{2}K_0K_3L^3$$

$$\dot{G} = -\frac{15}{8}n_e^2a^2\left(1-\frac{G^2}{L^2}\right)(B_0 + B_1X + B_2X^2)^{1/2}. \tag{35}$$

Here

$$B_0 = 1 - \frac{H^2}{L^2} - A_0^2$$

$$B_1 = 2A_0A_1 - 4\frac{H^2}{L^2}\left(1-\frac{H^2}{L^2}\right)$$

$$B_2 = 4\frac{H^2}{L^4} - 6\frac{H^2}{L^2}\left(1-\frac{H^2}{L^2}\right) + 2A_0A_2 - A_1^2 .$$

In terms of X the equation for $\dot{G}$ becomes

$$\dot{X} = \frac{15}{16}\frac{n_e^2}{n^2}(C_0 + C_1X + C_2X^2)^{1/2} \tag{36}$$

where

$$C_0 = B_0$$

$$C_1 = B_1 - B_0$$

$$C_2 = \frac{B_0}{4} - B_1 + B_2.$$

Equation (36) may be integrated to obtain

$$\frac{1}{C_0^{1/2}}\log\left[\frac{(C_0 + C_1X + C_2X^2)^{1/2} + C_0^{1/2}}{X} + \frac{C_1}{2C_0}\right]\Bigg|_{x_0}^{x} = \frac{15}{16}\frac{n_e^2}{n^2}n(t-t_0). \tag{37}$$

Equation (37) gives a time history for the evolution of a lunar orbit on the boundary contour separating the librating and circulating orbits. The solution is valid for any value for the semimajor axis as long as the boundary situation occurs. The condition for the existence of an equilibrium point (and boundary line) is

$$0 < \left|\left(\frac{1}{5} - \frac{3}{2}K_0\right)\left(\frac{1-5\frac{H^2}{L^2}}{1-\frac{H^2}{L^2}}\right)\right| < 1. \tag{38}$$

These results can be summarized in Table 1. The lifetimes for six different lunar orbiters are computed using Eq. (37). As a check these same cases are computed using the program POPLAR which is described in a previous section. As can be seen from Table 1, the agreement is excellent considering the simplicity of the analysis.

Table 1
Comparison of Computed (Eq. 37) and POPLAR-Generated Lifetimes for Six Lunar Orbiters

A_0 (km)	e_0	i_0 (deg)	ω_0 (deg)	Ω_0 (deg)	Lifetime (yr)	
					Eq. (37)	POPLAR
5214	0.1	90	40	0	0.92	0.99
5214	0.1	75	40	0	0.98	1.05
5214	0.2	75	40	0	0.64	0.68
6952	0.1	90	40	0	0.65	0.70
6952	0.1	75	40	0	0.70	0.79
6952	0.2	75	40	0	0.48	0.55

SUMMARY

The inclusion of medium-period terms in the equations of motion, while not being new in the theory, has not until the present been available in a rapid program useful for the earth and moon because of the excessive algebra involved in carrying out the expansions in the parallax factor. The development of an algebraic manipulation routine has not only made this expansion possible, but has allowed the further inclusion of the motion of the disturbing body into the Lagrangian equations. Usually, the disturbing body has been held fixed during the averaging process. But for high orbiters this assumption is no longer valid. The algebraic program carries out, automatically, the expansion of the disturbing function, differentiation of the expansion, averaging of the equations of motion, and then punches out the resulting equations on computer cards in a FORTRAN-compatible mode. These cards are then inserted directly into the program.

Gravitational harmonics for the earth and moon are also included by averaging the variations in the elements by means of Gaussian quadratures over one orbit of the satellite. These averaged variations are then included in the total variation of the elements which are numerically integrated to yield a time history of the orbit.

Drag effects, while not used in the present examples for the earth, may be included in the perturbation model. At present, this is limited to a nonrotating atmosphere with no lift present and with the drag force acting as a negative tangential component. The atmospheric density ρ is calculated as a function of altitude.

The program has proven to be a very fast and accurate one. In its high-speed mode of calculating third-body perturbations and oblateness (J_2, J_3, J_4) only, it has reached speeds of greater than 500 to 1 over numerical integration. When the full gravitational

harmonics are included, speeds of 25 to 50 to 1 are still attainable. Such speeds and accuracy make the program extremely useful in parametric studies and in gaining insights into the behavior of planetary orbiters.

In the theoretical development, points of singularity exits at $i = 0°$ and $180°$, and $e = 0$. However, in the practical application of the theory, one may come arbitarily close to these values without encountering problems.

In special cases it has been shown that the long-period equations of motions may be solved analytically. The results yield lifetimes of unstable orbits accurate to about 10%.

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Appendix A

THE AVERAGED THIRD-BODY DISTURBING FUNCTION AND ITS DERIVATIVES

The derivatives listed below are those required for f_{k0} in Eq. (14).

$$\frac{\partial \overline{F}}{\partial \omega} = \frac{\partial \overline{F}}{\partial A}\frac{\partial A}{\partial \omega} + \frac{\partial \overline{F}}{\partial B}\frac{\partial B}{\partial \omega}$$

$$\frac{\partial \overline{F}}{\partial i} = \frac{\partial \overline{F}}{\partial A}\frac{\partial A}{\partial i} + \frac{\partial \overline{F}}{\partial B}\frac{\partial B}{\partial i}$$

$$\frac{\partial \overline{F}}{\partial \Omega} = \frac{\partial \overline{F}}{\partial A}\frac{\partial A}{\partial \Omega} + \frac{\partial \overline{F}}{\partial B}\frac{\partial B}{\partial \Omega}$$

$$\frac{\partial \overline{F}}{\partial a} = \frac{\text{order}}{a}\overline{F}$$

where

$$\frac{\partial A}{\partial \omega} = B$$

$$\frac{\partial B}{\partial \omega} = -A$$

$$\frac{\partial A}{\partial i} = \hat{\mathbf{R}} \cdot \hat{\mathbf{r}}' \sin \omega$$

$$\frac{\partial B}{\partial i} = \hat{\mathbf{R}} \cdot \hat{\mathbf{r}}' \cos \omega$$

and ...

DISTURBING FUNCTION FOR ORDER 2

$\bar{F}$ =

```
 3*E**2*A**2
-3*E**2*B**2
 4

-3*E**2
 4

 3*A**2
 4

 3*B**2
 4

-1
 2
```

$\partial\bar{F}/\partial E$ FOR ORDER 2

```
 6*E**1*A**2

-3*E**1*B**2
 2

-3*E**1
 2
```

$\partial\bar{F}/\partial A$ FOR ORDER 2

```
 6*E**2*A**1

 3*A**1
 2
```

$\partial\bar{F}/\partial B$ FOR ORDER 2

```
-3*E**2*B**1
 2

 3*B**1
 2
```

DISTURBING FUNCTION FOR ORDER 3

```
F =    -25*E**3*A**3
         4

        75*E**3*A**1*B**2
        16

        45*E**3*A**1
        16

       -75*E**1*A**3
        16

       -75*E**1*A**1*B**2
        16

        15*E**1*A**1
         4
```

∂F̄/∂E FOR ORDER 3

```
       -75*E**2*A**3
         4

       225*E**2*A**1*B**2
        16

       135*E**2*A**1
        16

       -75*A**3
        16

       -75*A**1*B**2
        16

        15*A**1
         4
```

∂F̄/∂A FOR ORDER 3

```
       -75*E**3*A**2
         4

        75*E**3*B**2
        16

        45*E**3
        16
```

```
-225*E**1*A**2
  16

-75*E**1*B**2
  16

 15*E**1
  4
```

$\partial\overline{F}/\partial B$ FOR ORDER 3

```
 75*E**3*A**1*B**1
  8

-75*E**1*A**1*B**1
  8
```

DISTURBING FUNCTION FOR ORDER 4

```
F =    105*E**4*A**4
         8

      -315*E**4*A**2*B**2
        16

      -135*E**4*A**2
        16

       105*E**4*B**4
        64

        45*E**4*B**2
        32

        45*E**4
        64

       315*E**2*A**4
        16

       525*E**2*A**2*B**2
        32

      -615*E**2*A**2
        32

      -105*E**2*B**4
        32

        15*E**2*B**2
        32

        15*E**2
         8

       105*A**4
        64

       105*A**2*B**2
        32

       -15*A**2
         8

       105*B**4
        64

       -15*B**2
         8

        3
        8
```

$\partial\bar{F}/\partial E$ FOR ORDER 4

```
105*E**3*A**4
  2

-315*E**3*A**2*B**2
   4

-135*E**3*A**2
   4

 105*E**3*B**4
  16

  45*E**3*B**2
   8

  45*E**3
  16

 315*E**1*A**4
   8

 525*E**1*A**2*B**2
  16

-615*E**1*A**2
  16

-105*E**1*B**4
  16

  15*E**1*B**2
  16

  15*E**1
   4
```

$\partial\bar{F}/\partial A$ FOR ORDER 4

```
 105*E**4*A**3
   2

-315*E**4*A**1*B**2
   8

-135*E**4*A**1
   8

 315*E**2*A**3
   4

 525*E**2*A**1*B**2
  16
```

```
-615*E**2*A**1
  16

105*A**3
  16

105*A**1*B**2
  16

-15*A**1
   4
```

∂F̄/∂B FOR ORDER 4

```
-315*E**4*A**2*B**1
    8

105*E**4*B**3
  16

 45*E**4*B**1
  16

525*E**2*A**2*B**1
  16

-105*E**2*B**3
    8

 15*E**2*B**1
  16

105*A**2*B**1
  16

105*B**3
  16

-15*B**1
   4
```

DISTURBING FUNCTION FOR ORDER 5

```
F =  -441*E**5*A**5
       16

     2205*E**5*A**3*B**2
       32

      735*E**5*A**3
       32

    -2205*E**5*A**1*B**4
      128

     -735*E**5*A**1*B**2
       64

     -525*E**5*A**1
      128

    -2205*E**3*A**5
       32

    -2205*E**3*A**3*B**2
       64

     5145*E**3*A**3
       64

     2205*E**3*A**1*B**4
       64

     -735*E**3*A**1*B**2
       64

     -525*E**3*A**1
       32

    -2205*E**1*A**5
      128

    -2205*E**1*A**3*B**2
       64

      735*E**1*A**3
       32

    -2205*E**1*A**1*B**4
      128

      735*E**1*A**1*B**2
       32

     -105*E**1*A**1
       16
```

$\partial\bar{F}/\partial E$ FOR ORDER 5

```
-2205*E**4*A**5
   16

11025*E**4*A**3*B**2
   32

 3675*E**4*A**3
   32

-11025*E**4*A**1*B**4
   128

 -3675*E**4*A**1*B**2
    64

 -2625*E**4*A**1
   128

 -6615*E**2*A**5
    32

 -6615*E**2*A**3*B**2
    64

 15435*E**2*A**3
    64

  6615*E**2*A**1*B**4
    64

 -2205*E**2*A**1*B**2
    64

 -1575*E**2*A**1
    32

 -2205*A**5
   128

 -2205*A**3*B**2
    64

   735*A**3
    32

 -2205*A**1*B**4
   128

   735*A**1*B**2
    32

  -105*A**1
    16
```

$\partial\overline{F}/\partial A$ FOR ORDER 5

```
-2205*E**5*A**4
   16

 6615*E**5*A**2*B**2
   32

 2205*E**5*A**2
   32

-2205*E**5*B**4
  128

 -735*E**5*B**2
   64

 -525*E**5
  128

-11025*E**3*A**4
    32

 -6615*E**3*A**2*B**2
    64

 15435*E**3*A**2
    64

  2205*E**3*B**4
    64

  -735*E**3*B**2
    64

  -525*E**3
    32

-11025*E**1*A**4
   128

 -6615*E**1*A**2*B**2
    64

  2205*E**1*A**2
    32

 -2205*E**1*B**4
   128

  735*E**1*B**2
   32

 -105*E**1
   16
```

$\partial\bar{F}/\partial B$ FOR ORDER 5

```
 2205*E**5*A**3*B**1
   16

-2205*E**5*A**1*B**3
   32

 -735*E**5*A**1*B**1
   32

-2205*E**3*A**3*B**1
   32

 2205*E**3*A**1*B**3
   16

 -735*E**3*A**1*B**1
   32

-2205*E**1*A**3*B**1
   32

-2205*E**1*A**1*B**3
   32

  735*E**1*A**1*B**1
   16
```

DISTURBING FUNCTION FOR ORDER 6

```
F =     231*E**6*A**6
          4

       -3465*E**6*A**4*B**2
          14

        -945*E**6*A**4
          16

        3465*E**6*A**2*B**4
          32

         945*E**6*A**2*B**2
          16

         525*E**6*A**2
          32

       -1155*E**6*B**6
         256

        -945*E**6*B**4
         256

        -525*E**6*B**2
         256

        -175*E**6
         256

         3465*E**4*A**6
           16

        -4725*E**4*A**4
           16

       -51975*E**4*A**2*B**4
         256

        14175*E**4*A**2*B**2
          128

        23625*E**4*A**2
          256

         3465*E**4*B**6
          256

        -1575*E**4*B**2
          256

        -525*E**4
          128

         3465*E**2*A**6
           32
```

```
 51975*E**2*A**4*B**2
   256

-42525*E**2*A**4
   256

 10395*E**2*A**2*B**4
   128

-19845*E**2*A**2*B**2
   128

  4095*E**2*A**2
    64

 -3465*E**2*B**6
   256

  2835*E**2*B**4
   256

   315*E**2*B**2
    64

  -105*E**2
    32

  1155*A**6
   256

  3465*A**4*B**2
   256

  -945*A**4
   128

  3465*A**2*B**4
   256

  -945*A**2*B**2
    64

   105*A**2
    32

  1155*B**6
   256

  -945*B**4
   128

   105*B**2
    32

   -5
   16
```

$\partial\bar{F}/\partial E$ FOR ORDER 6

```
 693*E**5*A**6
   2

-10395*E**5*A**4*B**2
     8

-2835*E**5*A**4
     8

10395*E**5*A**2*B**4
   16

 2835*E**5*A**2*B**2
     8

 1575*E**5*A**2
   16

-3465*E**5*B**6
  128

-2835*E**5*B**4
  128

-1575*E**5*B**2
  128

 -525*E**5
  128

 3465*E**3*A**6
     4

-4725*E**3*A**4
     4

-51975*E**3*A**2*B**4
     64

 14175*E**3*A**2*B**2
     32

 23625*E**3*A**2
     64

  3465*E**3*B**6
     64

 -1575*E**3*B**2
     64

  -525*E**3
     32
```

```
  3465*E**1*A**6
    16

 51975*E**1*A**4*B**2
   128

-42525*E**1*A**4
   128

 10395*E**1*A**2*B**4
    64

-19845*E**1*A**2*B**2
    64

  4095*E**1*A**2
    32

 -3465*E**1*B**6
   128

  2835*E**1*B**4
   128

   315*E**1*B**2
    32

  -105*E**1
    16
```

$\partial\bar{F}/\partial A$ FOR ORDER 6

```
   693*E**6*A**5
     2

-3465*E**6*A**3*B**2
     4

 -945*E**6*A**3
     4

 3465*E**6*A**1*B**4
    16

  945*E**6*A**1*B**2
     8

  525*E**6*A**1
    16

10395*E**4*A**5
     8

-4725*E**4*A**3
     4
```

```
-51975*E**4*A**1*B**4
   128

 14175*E**4*A**1*B**2
    64

 23625*E**4*A**1
   128

 10395*E**2*A**5
    16

 51975*E**2*A**3*B**2
    64

-42525*E**2*A**3
    64

 10395*E**2*A**1*B**4
    64

-19845*E**2*A**1*B**2
    64

  4095*E**2*A**1
    32

  3465*A**5
   128

  3465*A**3*B**2
    64

  -945*A**3
    32

  3465*A**1*B**4
   128

  -945*A**1*B**2
    32

   105*A**1
    16
```

$\partial\bar{F}/\partial B$ FOR ORDER 6

```
-3465*E**6*A**4*B**1
    8

 3465*E**6*A**2*B**3
    8

  945*E**6*A**2*B**1
    8
```

```
-3465*E**6*B**5
  128

 -945*E**6*B**3
   64

 -525*E**6*B**1
  128

-51975*E**4*A**2*B**3
    64

 14175*E**4*A**2*B**1
    64

 10395*E**4*B**5
  128

 -1575*E**4*B**1
  128

 51975*E**2*A**4*B**1
  128

 10395*E**2*A**2*B**3
    32

-19845*E**2*A**2*B**1
    64

-10395*E**2*B**5
  128

  2835*E**2*B**3
    64

   315*E**2*B**1
    32

  3465*A**4*B**1
  128

  3465*A**2*B**3
    64

  -945*A**2*B**1
    32

  3465*B**5
  128

  -945*B**3
    32

   105*B**1
    16
```

DISTURBING FUNCTION FOR ORDER 7

```
F =  -3861*E**7*A**7
        32

      81081*E**7*A**5*B**2
        128

      18711*E**7*A**5
        128

     -135135*E**7*A**3*B**4
        256

     -31185*E**7*A**3*B**2
        128

     -14175*E**7*A**3
        256

     135135*E**7*A**1*B**6
       2048

      93555*E**7*A**1*B**4
       2048

      42525*E**7*A**1*B**2
       2048

      11025*E**7*A**1
       2048

     -81081*E**5*A**7
        128

      27027*E**5*A**5*B**2
         64

     126819*E**5*A**5
        128

     1756755*E**5*A**3*B**4
       2048

     -738045*E**5*A**3*B**2
       1024

     -855225*E**5*A**3
       2048

     -405405*E**5*A**1*B**6
       2048

      31185*E**5*A**1*B**4
       1024
```

```
 184275*E**5*A**1*B**2
   2048

  11025*E**5*A**1
    256

-135135*E**3*A**7
    256

-1756755*E**3*A**5*B**2
   2048

 1881495*E**3*A**5
   2048

-135135*E**3*A**3*B**4
   1024

 738045*E**3*A**3*B**2
   1024

 -57645*E**3*A**3
    128

 405405*E**3*A**1*B**6
   2048

-405405*E**3*A**1*B**4
   2048

  -2835*E**3*A**1*B**2
     64

   6615*E**3*A**1
    128

-135135*E**1*A**7
   2048

-405405*E**1*A**5*B**2
   2048

  31185*E**1*A**5
    256

-405405*E**1*A**3*B**4
   2048

  31185*E**1*A**3*B**2
    128

  -8505*E**1*A**3
    128

-135135*E**1*A**1*B**6
   2048
```

```
 31185*E**1*A**1*B**4
   256

-8505*E**1*A**1*B**2
   128

 315*E**1*A**1
  32
```

$\partial\bar{F}/\partial E$ FOR ORDER 7

```
-27027*E**6*A**7
    32

567567*E**6*A**5*B**2
   128

130977*E**6*A**5
   128

-945945*E**6*A**3*B**4
    256

-218295*E**6*A**3*B**2
    128

-99225*E**6*A**3
    256

945945*E**6*A**1*B**6
   2048

654885*E**6*A**1*B**4
   2048

297675*E**6*A**1*B**2
   2048

 77175*E**6*A**1
   2048

-405405*E**4*A**7
    128

135135*E**4*A**5*B**2
     64

634095*E**4*A**5
    128

8783775*E**4*A**3*B**4
   2048
```

```
-3690225*E**4*A**3*B**2
    1024

-4276125*E**4*A**3
    2048

-2027025*E**4*A**1*B**6
    2048

 155925*E**4*A**1*B**4
    1024

 921375*E**4*A**1*B**2
    2048

  55125*E**4*A**1
    256

 -405405*E**2*A**7
    256

-5270265*E**2*A**5*B**2
    2048

 5644485*E**2*A**5
    2048

 -405405*E**2*A**3*B**4
    1024

 2214135*E**2*A**3*B**2
    1024

 -172935*E**2*A**3
    128

 1216215*E**2*A**1*B**6
    2048

-1216215*E**2*A**1*B**4
    2048

  -8505*E**2*A**1*B**2
    64

  19845*E**2*A**1
    128

 -135135*A**7
    2048

 -405405*A**5*B**2
    2048
```

```
 31185*A**5
   256

-405405*A**3*B**4
  2048

 31185*A**3*B**2
   128

 -8505*A**3
   128

-135135*A**1*B**6
  2048

 31185*A**1*B**4
   256

 -8505*A**1*B**2
   128

  315*A**1
   32
```

$\partial\bar{F}/\partial A$ FOR ORDER 7

```
 -27027*E**7*A**6
     32

 405405*E**7*A**4*B**2
    128

  93555*E**7*A**4
    128

-405405*E**7*A**2*B**4
    256

 -93555*E**7*A**2*B**2
    128

 -42525*E**7*A**2
    256

 135135*E**7*B**6
   2048

  93555*E**7*B**4
   2048

  42525*E**7*B**2
   2048
```

```
   11025*E**7
    2048

 -567567*E**5*A**6
     128

  135135*E**5*A**4*B**2
      64

  634095*E**5*A**4
     128

 5270265*E**5*A**2*B**4
    2048

-2214135*E**5*A**2*B**2
    1024

-2565675*E**5*A**2
    2048

- 405405*E**5*B**6
    2048

   31185*E**5*B**4
    1024

  184275*E**5*B**2
    2048

   11025*E**5
     256

- 945945*E**3*A**6
     256

-8783775*E**3*A**4*B**2
    2048

 9407475*E**3*A**4
    2048

- 405405*E**3*A**2*B**4
    1024

 2214135*E**3*A**2*B**2
    1024

- 172935*E**3*A**2
     128

  405405*E**3*B**6
    2048
```

```
-405405*E**3*B**4
   2048

 -2835*E**3*B**2
    64

  6615*E**3
   128

-945945*E**1*A**6
   2048

-2027025*E**1*A**4*B**2
   2048

 155925*E**1*A**4
   256

-1216215*E**1*A**2*B**4
   2048

 93555*E**1*A**2*B**2
   128

-25515*E**1*A**2
   128

-135135*E**1*B**6
   2048

 31185*E**1*B**4
   256

 -8505*E**1*B**2
   128

  315*E**1
   32
```

$\partial\bar{F}/\partial B$ FOR ORDER 7

```
 81081*E**7*A**5*B**1
   64

-135135*E**7*A**3*B**3
   64

-31185*E**7*A**3*B**1
   64

405405*E**7*A**1*B**5
  1024

 93555*E**7*A**1*B**3
   512
```

```
 42525*E**7*A**1*B**1
  1024

 27027*E**5*A**5*B**1
    32

1756755*E**5*A**3*B**3
   512

-738045*E**5*A**3*B**1
   512

-1216215*E**5*A**1*B**5
  1024

 31185*E**5*A**1*B**3
   256

184275*E**5*A**1*B**1
  1024

-1756755*E**3*A**5*B**1
  1024

-135135*E**3*A**3*B**3
   256

 738045*E**3*A**3*B**1
   512

1216215*E**3*A**1*B**5
  1024

-405405*E**3*A**1*B**3
   512

 -2835*E**3*A**1*B**1
    32

-405405*E**1*A**5*B**1
  1024

-405405*E**1*A**3*B**3
   512

 31185*E**1*A**3*B**1
    64

-405405*E**1*A**1*B**5
  1024

 31185*E**1*A**1*B**3
    64

 -8505*E**1*A**1*B**1
    64
```

DISTURBING FUNCTION FOR ORDER 8

```
F =    32175*E**8*A**8
         128

     -225225*E**8*A**6*B**2
         128

      -45045*E**8*A**6
         128

      1126125*E**8*A**4*B**4
         512

       225225*E**8*A**4*B**2
         256

        86625*E**8*A**4
         512

     -1126125*E**8*A**2*B**6
        2048

     -675675*E**8*A**2*B**4
        2048

     -259875*E**8*A**2*B**2
        2048

      -55125*E**8*A**2
        2048

      225225*E**8*B**8
       16384

       45045*E**8*B**6
        4096

       51975*E**8*B**4
        8192

       11025*E**8*B**2
        4096

       11025*E**8
       16384

      225225*E**6*A**8
         128
```

```
- 675675*E**6*A**6*B**2
     256

-795795*E**6*A**6
     256

-5630625*E**6*A**4*B**4
    2048

 3828825*E**6*A**4*B**2
    1024

 3378375*E**6*A**4
    2048

 6531525*E**6*A**2*B**6
    4096

-2027025*E**6*A**2*B**4
    4096

-3066525*E**6*A**2*B**2
    4096

-1135575*E**6*A**2
    4096

- 225225*E**6*B**8
    4096

  -15015*E**6*B**6
    4096

   86625*E**6*B**4
    4096

   77175*E**6*B**2
    4096

    3675*E**6
     512

 1126125*E**4*A**8
     512

 5630625*E**4*A**6*B**2
    2048

-8783775*E**4*A**6
    2048

-8333325*E**4*A**4*B**4
    8192
```

```
-9594585*E**4*A**4*B**2
    4096

21008295*E**4*A**4
    8192

-6081075*E**4*A**2*B**6
    4096

 1936935*E**4*A**2*B**4
    1024

  426195*E**4*A**2*B**2
    4096

-240345*E**4*A**2
     512

  675675*E**4*B**8
    8192

-225225*E**4*B**6
    4096

-336105*E**4*B**4
    8192

    2205*E**4*B**2
     512

    6615*E**4
     512

 1126125*E**2*A**8
    2048

 6531525*E**2*A**6*B**2
    4096

-4639635*E**2*A**6
    4096

 6081075*E**2*A**4*B**4
    4096

-8963955*E**2*A**4*B**2
    4096

  377685*E**2*A**4
     512
```

```
 1576575*E**2*A**2*B**6
    4096

-4009005*E**2*A**2*B**4
    4096

   93555*E**2*A**2*B**2
     128

  -40005*E**2*A**2
     256

 -225225*E**2*B**8
    4096

  315315*E**2*B**6
    4096

   -3465*E**2*B**4
     512

   -5355*E**2*B**2
     256

     315*E**2
      64

  225225*A**8
   16384

  225225*A**6*B**2
    4096

  -15015*A**6
     512

  675675*A**4*B**4
    8192

  -45045*A**4*B**2
     512

   10395*A**4
     512

  225225*A**2*B**6
    4096

  -45045*A**2*B**4
     512

   10395*A**2*B**2
     256
```

```
  -315*A**2
    64

225225*B**8
 16384

-15015*B**6
   512

 10395*B**4
   512

  -315*B**2
    64

    35
   128
```

∂F̄/∂E FOR ORDER 8

```
  32175*E**7*A**8
     16

-225225*E**7*A**6*B**2
     16

 -45045*E**7*A**6
     16

1126125*E**7*A**4*B**4
     64

 225225*E**7*A**4*B**2
     32

  86625*E**7*A**4
     64

-1126125*E**7*A**2*B**6
     256

 -675675*E**7*A**2*B**4
     256

 -259875*E**7*A**2*B**2
     256

  -55125*E**7*A**2
     256
```

```
225225*E**7*B**8
  2048

 45045*E**7*B**6
   512

 51975*E**7*B**4
  1024

 11025*E**7*B**2
   512

 11025*E**7
  2048

675675*E**5*A**8
    64

-2027025*E**5*A**6*B**2
     128

-2387385*E**5*A**6
     128

-16891875*E**5*A**4*B**4
     1024

11486475*E**5*A**4*B**2
     512

10135125*E**5*A**4
     1024

19594575*E**5*A**2*B**6
     2048

-6081075*E**5*A**2*B**4
     2048

-9199575*E**5*A**2*B**2
     2048

-3406725*E**5*A**2
     2048

-675675*E**5*B**8
     2048

-45045*E**5*B**6
     2048
```

```
 259875*E**5*B**4
   2048

 231525*E**5*B**2
   2048

  11025*E**5
    256

1126125*E**3*A**8
    128

5630625*E**3*A**6*B**2
    512

-8783775*E**3*A**6
    512

-8333325*E**3*A**4*B**4
   2048

-9594585*E**3*A**4*B**2
   1024

21008295*E**3*A**4
   2048

-6081075*E**3*A**2*B**6
   1024

 1936935*E**3*A**2*B**4
    256

  426195*E**3*A**2*B**2
   1024

 -240345*E**3*A**2
    128

  675675*E**3*B**8
   2048

 -225225*E**3*B**6
   1024

 -336105*E**3*B**4
   2048

    2205*E**3*B**2
     128
```

```
   6615*E**3
    128

 1126125*E**1*A**8
   1024

 6531525*E**1*A**6*B**2
   2048

-4639635*E**1*A**6
   2048

 6081075*E**1*A**4*B**4
   2048

-8963955*E**1*A**4*B**2
   2048

  377685*E**1*A**4
    256

 1576575*E**1*A**2*B**6
   2048

-4009005*E**1*A**2*B**4
   2048

   93555*E**1*A**2*B**2
      64

  -40005*E**1*A**2
    128

 -225225*E**1*B**8
   2048

  315315*E**1*B**6
   2048

   -3465*E**1*B**4
    256

   -5355*E**1*B**2
    128

    315*E**1
     32
```

$\partial\bar{F}/\partial A$ FOR ORDER 8

```
  32175*E**8*A**7
     16

-675675*E**8*A**5*B**2
     64

-135135*E**8*A**5
     64

 1126125*E**8*A**3*B**4
    128

  225225*E**8*A**3*B**2
     64

   86625*E**8*A**3
    128

-1126125*E**8*A**1*B**6
    1024

- 675675*E**8*A**1*B**4
    1024

- 259875*E**8*A**1*B**2
    1024

  -55125*E**8*A**1
    1024

  225225*E**6*A**7
     16

-2027025*E**6*A**5*B**2
    128

-2387385*E**6*A**5
    128

-5630625*E**6*A**3*B**4
    512

 3828825*E**6*A**3*B**2
    256

 3378375*E**6*A**3
    512
```

```
 6531525*E**6*A**1*B**6
    2048

-2027025*E**6*A**1*B**4
    2048

-3066525*E**6*A**1*B**2
    2048

-1135575*E**6*A**1
    2048

 1126125*E**4*A**7
      64

16891875*E**4*A**5*B**2
    1024

-26351325*E**4*A**5
    1024

-8333325*E**4*A**3*B**4
    2048

-9594585*E**4*A**3*B**2
    1024

21008295*E**4*A**3
    2048

-6081075*E**4*A**1*B**6
    2048

 1936935*E**4*A**1*B**4
     512

  426195*E**4*A**1*B**2
    2048

-240345*E**4*A**1
     256

 1126125*E**2*A**7
     256

19594575*E**2*A**5*B**2
    2048

-13918905*E**2*A**5
    2048
```

```
 6081075*E**2*A**3*B**4
    1024

-8963955*E**2*A**3*B**2
    1024

  377685*E**2*A**3
     128

 1576575*E**2*A**1*B**6
    2048

-4009005*E**2*A**1*B**4
    2048

   93555*E**2*A**1*B**2
      64

  -40005*E**2*A**1
     128

  225225*A**7
    2048

  675675*A**5*B**2
    2048

  -45045*A**5
     256

  675675*A**3*B**4
    2048

  -45045*A**3*B**2
     128

   10395*A**3
     128

  225225*A**1*B**6
    2048

  -45045*A**1*B**4
     256

   10395*A**1*B**2
     128

    -315*A**1
      32
```

$\partial\bar{F}/\partial B$ FOR ORDER 8

```
-225225*E**8*A**6*B**1
      64

 1126125*E**8*A**4*B**3
      128

  225225*E**8*A**4*B**1
      128

-3378375*E**8*A**2*B**5
     1024

- 675675*E**8*A**2*B**3
      512

- 259875*E**8*A**2*B**1
     1024

  225225*E**8*B**7
     2048

  135135*E**8*B**5
     2048

   51975*E**8*B**3
     2048

   11025*E**8*B**1
     2048

- 675675*E**6*A**6*B**1
      128

-5630625*E**6*A**4*B**3
      512

 3828825*E**6*A**4*B**1
      512

19594575*E**6*A**2*B**5
     2048

-2027025*E**6*A**2*B**3
     1024

-3066525*E**6*A**2*B**1
     2048
```

```
-225225*E**6*B**7
     512

 -45045*E**6*B**5
   2048

  86625*E**6*B**3
   1024

  77175*E**6*B**1
   2048

 5630625*E**4*A**6*B**1
   1024

-8333325*E**4*A**4*B**3
   2048

-9594585*E**4*A**4*B**1
   2048

-18243225*E**4*A**2*B**5
   2048

 1936935*E**4*A**2*B**3
     256

  426195*E**4*A**2*B**1
   2048

 675675*E**4*B**7
   1024

-675675*E**4*B**5
   2048

 -336105*E**4*B**3
   2048

    2205*E**4*B**1
     256

 6531525*E**2*A**6*B**1
   2048

 6081075*E**2*A**4*B**3
   1024

-8963955*E**2*A**4*B**1
   2048
```

```
 4729725*E**2*A**2*B**5
    2048

-4009005*E**2*A**2*B**3
    1024

   93555*E**2*A**2*B**1
      64

-225225*E**2*B**7
     512

 945945*E**2*B**5
   2048

  -3465*E**2*B**3
    128

  -5355*E**2*B**1
    128

 225225*A**6*B**1
   2048

 675675*A**4*B**3
   2048

 -45045*A**4*B**1
    256

 675675*A**2*B**5
   2048

 -45045*A**2*B**3
    128

  10395*A**2*B**1
    128

225225*B**7
  2048

-45045*B**5
    256

 10395*B**3
   128

  -315*B**1
    32
```

Appendix B

ADDITIONAL TERMS DUE TO THE MOTION OF THE THIRD BODY

The following are the derivatives required for Lagrange's planetary equations of motion listed in Eq. (1).

$$\overline{\frac{\partial F}{\partial \omega}} = (D1 * DA + D2 * DB) * CB$$

$$\overline{\frac{\partial F}{\partial M}} = (D3 * DA + D4 * DB) * CA$$

$$\overline{\frac{\partial F}{\partial e}} = [(D5 * DA + D7 * DB) * C + (D6 * DA + D8 * DB)/C] * CA$$

$$\overline{\frac{\partial F}{\partial i}} = [(D10 * DA + D11 * DB) \frac{\partial A}{\partial i}$$

$$+ D9 * S1 + (D11 * DA + D13 * DB) \frac{\partial B}{\partial i}$$

$$+ D12 * S2] * CB$$

$$\overline{\frac{\partial F}{\partial \Omega}} = [(D10 * DA + D11 * DB) \frac{\partial A}{\partial \Omega} + D9 * S3$$

$$+ (D11 * DA + D13 * DB) \frac{\partial B}{\partial \Omega} + D12 * S4] * CB$$

$$\overline{\frac{\partial F}{\partial a}} = \frac{\text{order}}{a} (D9 * DA + D12 * DB) * CB$$

where

$$C = (1 - e^2)^{1/2}$$

$$CA = \left(\frac{a'}{r'}\right)^3 \frac{n'}{n} \left(\frac{a}{r'}\right)^{(\text{order} - 2)} a^2 n'^2$$

$$CB = C * CA$$

$$S1 = \frac{1}{n'} \hat{R} \cdot \frac{d\hat{r}'}{dt} \sin \omega$$

$$S2 = \frac{1}{n'} \hat{\mathbf{R}} \cdot \frac{d\hat{r}'}{dt} \cos \omega$$

$$S3 = \frac{1}{n'} \begin{pmatrix} -P_y \\ P_x \\ 0 \end{pmatrix} \cdot \frac{d\hat{r}'}{dt}$$

$$S4 = \frac{1}{n'} \begin{pmatrix} -Q_y \\ Q_x \\ 0 \end{pmatrix} \cdot \frac{d\hat{r}'}{dt}$$

and

THE FOLLOWING IS FOR ORDER 2

```
D1 =    -9*E**2*A**1
         4

         8*E**1*A**1

         3*A**1
         2


D2 =     9*E**2*B**1
         4

        -8*E**1*B**1

        -3*B**1
         2


D3 =    -3*E**2*A**1

         6*E**1*A**1

         3*A**1
         2


D4 =     3*E**2*B**1
         2

        -3*B**1
         2
```

```
D5 =      3*E**1*B**1
          4

         -3*B**1
```

```
D6 =     -9*E**3*B**1
          8

          4*E**2*B**1

          3*E**1*B**1
          4
```

```
D7 =      3*E**1*A**1
          4

         -3*A**1
```

```
D8 =     -9*E**3*A**1
          8

          4*E**2*A**1

          3*E**1*A**1
          4
```

```
D9 =      9*E**2*B**1
           8

         -4*E**1*B**1

         -3*B**1
           4


D10 =     0


D11 =     9*E**2
           8

         -4*E**1

         -3
           4

D12 =     9*E**2*A**1
           8

         -4*E**1*A**1

         -3*A**1
           4


D13 =     0
```

THE FOLLOWING IS FOR ORDER 3

```
D1 =    45*E**3*A**2
         4

       -615*E**3*B**2
         64

        -21*E**3
         64

        -75*E**2*A**2
          2
         30*E**2*B**2

          3*E**2
          2

       -765*E**1*A**2
         64

        645*E**1*B**2
         64

          3*E**1
          8

        -15*A**2
          2

          3
          2


D2 =   -615*E**3*A**1*B**1
         32

         60*E**2*A**1*B**1

        645*E**1*A**1*B**1
         32
```

```
D3 =      45*E**3*A**2
           4

         -75*E**3*B**2
          16

         -21*E**3
          16

         -45*E**2*A**2
            2

           9*E**2
           2

        -135*E**1*A**2
          16

          75*E**1*B**2
          16

           3*E**1
           4

         -15*A**2
           2

           3
           2

D4 =     -75*E**3*A**1*B**1
           8

          75*E**1*A**1*B**1
           8


D5 =     -45*E**2*A**1*B**1
           4

          40*E**1*A**1*B**1

          15*A**1*B**1
           2
```

```
D6 =      15*E**4*A**1*B**1
           2

         -25*E**3*A**1*B**1

        -255*E**2*A**1*B**1
           32

          -5*E**1*A**1*B**1
```

```
D7 =     -45*E**2*A**2
            8

         135*E**2*B**2
           32

           9*E**2
           32

          20*E**1*A**2

         -10*E**1*B**2

          -2*E**1

          15*A**2
           4

          -3
           4
```

```
D8 =      15*E**4*A**2
           4

        -135*E**4*B**2
           64

         -21*E**4
           64
```

```
-25*E**3*A**2
  2

 5*E**3*B**2

 3*E**3
 2

-255*E**2*A**2
 64

 135*E**2*B**2
 64

  3*E**2
  8

 -5*E**1*A**2
  2

 -5*E**1*B**2

  3*E**1
  2
```

```
D9 =    -15*E**3*A**1*B**1
          2

         25*E**2*A**1*B**1

        255*E**1*A**1*B**1
         32

          5*A**1*B**1
```

```
D10 =   -15*E**3*B**1
          2

         25*E**2*B**1

        255*E**1*B**1
         32

          5*B**1
```

```
D11 =   -15*E**3*A**1
          2

         25*E**2*A**1

        255*E**1*A**1
         32

          5*A**1

D12     -15*E**3*A**2
          4

        135*E**3*B**2
         64

         21*E**3
         64

         25*E**2*A**2
          2

         -5*E**2*B**2

         -3*E**2
          2

        255*E**1*A**2
         64

       -135*E**1*B**2
         64

         -3*E**1
          8

          5*A**2
          2

          5*B**2

         -3
          2

D13 =   135*E**3*B**1
         32

       -10*E**2*B**1

      -135*E**1*B**1
         32

        10*B**1
```

THE FOLLOWING IS FOR ORDER 4

```
D1 =   -175*E**4*A**3
          4

       3255*E**4*A**1*B**2
         32

        135*E**4*A**1
         32

        140*E**3*A**3

       -294*E**3*A**1*B**2

        -18*E**3*A**1

        875*E**2*A**3
         16

      -4095*E**2*A**1*B**2
         32

       -165*E**2*A**1
         32

         84*E**1*A**3

        -42*E**1*A**1*B**2

        -30*E**1*A**1

        175*A**3
         16

       -105*A**1*B**2
         16

        -15*A**1
          4


D2 =   3255*E**4*A**2*B**1
         32

       -385*E**4*B**3
         16
```

```
-135*E**4*B**1
  32

-294*E**3*A**2*B**1

  56*E**3*B**3

  18*E**3*B**1

-4095*E**2*A**2*B**1
  32

 245*E**2*B**3
   8

 165*E**2*B**1
  32

 -42*E**1*A**2*B**1

 -56*E**1*B**3

  30*E**1*B**1

-105*A**2*B**1
  16

-105*B**3
  16

  15*B**1
   4
```

```
D3 =    -35*E**4*A**3

       315*E**4*A**1*B**2
         8

        75*E**4*A**1
         8

        70*E**3*A**3
```

```
-30*E**3*A**1

105*E**2*A**3
  4

-525*E**2*A**1*B**2
  16

-105*E**2*A**1
  16

 70*E**1*A**3

-30*E**1*A**1

175*A**3
 16

-105*A**1*B**2
  16

-15*A**1
  4

D4 =   315*E**4*A**2*B**1
         8

-105*E**4*B**3
  16

-45*E**4*B**1
  16

-525*E**2*A**2*B**1
  16

105*E**2*B**3
  8

-15*E**2*B**1
  16

-105*A**2*B**1
  16

-105*B**3
  16

15*B**1
  4
```

```
D5 =    315*E**3*A**2*B**1
          4

       -1085*E**3*B**3
          64

        -255*E**3*B**1
          64

        -525*E**2*A**2*B**1
          2

         119*E**2*B**3
          3

          41*E**2*B**1
          2

       -5355*E**1*A**2*B**1
          64

         525*E**1*B**3
          64

         135*E**1*B**1
          16

        -105*A**2*B**1
          2

         -35*B**3
          3

          25*B**1
          2

D6 =   -525*E**5*A**2*B**1
          16

         385*E**5*B**3
          64

         135*E**5*B**1
          64

         105*E**4*A**2*B**1

        -14*E**4*B**3

         -9*E**4*B**1
```

```
2625*E**3*A**2*B**1
  64

-245*E**3*B**3
  32

-165*E**3*B**1
  64

 63*E**2*A**2*B**1

 14*E**2*B**3

-15*E**2*B**1

525*E**1*A**2*B**1
 64


105*E**1*B**3
 64

-15*E**1*B**1
  8


D7 =    105*E**3*A**3
          4

       -3255*E**3*A**1*B**2
          64

        -255*E**3*A**1
          64

        -175*E**2*A**3
           2

         119*E**2*A**1*B**2

          41*E**2*A**1
           2

        -1785*E**1*A**3
          64

         1575*E**1*A**1*B**2
          64

          135*E**1*A**1
           16
```

```
-35*A**3
  2

-35*A**1*B**2

 25*A**1
  2
```

```
D8 =  -175*E**5*A**3
        16

     1155*E**5*A**1*B**2
        64

      135*E**5*A**1
       64

       35*E**4*A**3

      -42*E**4*A**1*B**2

       -9*E**4*A**1

      875*E**3*A**3
       64

     -735*E**3*A**1*B**2
       32

     -165*E**3*A**1
       64

       21*E**2*A**3

       42*E**2*A**1*B**2

      -15*E**2*A**1

      175*E**1*A**3
       64

      315*E**1*A**1*B**2
       64

      -15*E**1*A**1
        8
```

```
D9 =    525*E**4*A**2*B**1
         16

       -385*E**4*B**3
         64

       -135*E**4*B**1
         64

       -105*E**3*A**2*B**1

         14*E**3*B**3

          9*E**3*B**1

      -2625*E**2*A**2*B**1
         64

        245*E**2*B**3
         32

       165*E**2*B**1
        64

       -63*E**1*A**2*B**1

       -14*E**1*B**3

        15*E**1*B**1

      -525*A**2*B**1
        64

      -105*B**3
        64

        15*B**1
         8
```

```
D10 =   525*E**4*A**1*B**1
          8

       -210*E**3*A**1*B**1

      -2625*E**2*A**1*B**1
         32
```

```
-126*E**1*A**1*B**1

-525*A**1*B**1
  32
```

```
D11 =   525*E**4*A**2
         16

       -1155*E**4*B**2
          64

        -135*E**4
          64

        -105*E**3*A**2

          42*E**3*B**2

           9*E**3

       -2625*E**2*A**2
          64

        735*E**2*B**2
         32

        165*E**2
         64

        -63*E**1*A**2

        -42*E**1*B**2

         15*E**1

       -525*A**2
         64

       -315*B**2
         64

         15
          8
```

```
D12 =   175*E**4*A**3
         16

       -1155*E**4*A**1*B**2
          64

        -135*E**4*A**1
          64

         -35*E**3*A**3

          42*E**3*A**1*B**2

           9*E**3*A**1

        -875*E**2*A**3
          64

         735*E**2*A**1*B**2
          32

         165*E**2*A**1
          64

         -21*E**1*A**3


         -42*E**1*A**1*B**2

          15*E**1*A**1

        -175*A**3
          64

        -315*A**1*B**2
          64

          15*A**1
           8

D13 = -1155*E**4*A**1*B**1
         32

          84*E**3*A**1*B**1

         735*E**2*A**1*B**1
          16

         -84*E**1*A**1*B**1

        -315*A**1*B**1
          32
```

```
-126*E**1*A**1*B**1

-525*A**1*B**1
  32


D11 =   525*E**4*A**2
         16

       -1155*E**4*B**2
          64

        -135*E**4
          64

        -105*E**3*A**2

          42*E**3*B**2

           9*E**3

       -2625*E**2*A**2
          64

        735*E**2*B**2
         32

        165*E**2
         64

        -63*E**1*A**2

        -42*E**1*B**2

         15*E**1

       -525*A**2
         64

       -315*B**2
         64

         15
          8
```

THE FOLLOWING IS FOR ORDER 5

```
D1 =   4725*E**5*A**4
         32

     -82215*E**5*A**2*B**2
        128

      -3465*E**5*A**2
        128

      19005*E**5*B**4
        256

        175*E**5*B**2
          8

         95*E**5
        256

      -3675*E**4*A**4
          8

       1764*E**4*A**2*B**2

        441*E**4*A**2
          4

      -168*E**4*B**4

       -84*E**4*B**2

       -15*E**4
          8

     -23625*E**3*A**4
        128

     108675*E**3*A**2*B**2
        128

       3675*E**3*A**2
        128
```

```
-14175*E**3*B**4
   128

 -2625*E**3*B**2
   128

   -75*E**3
   128

 -2205*E**2*A**4
      4

   756*E**2*A**2*B**2

   567*E**2*A**2
      2

   168*E**2*B**4

  -196*E**2*B**2

   -25*E**2
      4

-34125*E**1*A**4
   256

 12915*E**1*A**2*B**2
    64

  8505*E**1*A**2
   128

  9345*E**1*B**4
   256

 -5985*E**1*B**2
   128

   -45*E**1
    32

  -315*A**4
     8

   105*A**2
     4

   -15
     8
```

```
D2 =   -27405*E**5*A**3*B**1
          64

        19005*E**5*A**1*B**3
          64

          175*E**5*A**1*B**1
            4

         1176*E**4*A**3*B**1

         -672*E**4*A**1*B**3

         -168*E**4*A**1*B**1

        36225*E**3*A**3*B**1
          64

       -14175*E**3*A**1*B**3
          32

        -2625*E**3*A**1*B**1
          64

          504*E**2*A**3*B**1

          672*E**2*A**1*B**3

         -392*E**2*A**1*B**1

         4305*E**1*A**3*B**1
           32

         9345*E**1*A**1*B**3
           64

        -5985*E**1*A**1*B**1
           64

D3 =     1575*E**5*A**4
           16

        -6615*E**5*A**2*B**2
           32

        -1365*E**5*A**2
           32

         2205*E**5*B**4
          128
```

```
 735*E**5*B**2
  64

 285*E**5
 128

-1575*E**4*A**4
    8

 525*E**4*A**2
   4

 -75*E**4
   8

-1575*E**3*A**4
   32

 6615*E**3*A**2*B**2
   64

 1365*E**3*A**2
   64

-2205*E**3*B**4
   64

 735*E**3*B**2
  64

 -75*E**3
  32

-1575*E**2*A**4
    4

 525*E**2*A**2
   2

 -75*E**2
   4

-14175*E**1*A**4
   128

 6615*E**1*A**2*B**2
   64

 1995*E**1*A**2
   32
```

```
 2205*E**1*B**4
  128

-735*E**1*B**2
  32

 -45*E**1
  16

-315*A**4
   8

 105*A**2
   4

 -15
   8
```

```
D4 =   -2205*E**5*A**3*B**1
          16

        2205*E**5*A**1*B**3
          32

         735*E**5*A**1*B**1
          32

        2205*E**3*A**3*B**1
          32

       -2205*E**3*A**1*B**3
          16

         735*E**3*A**1*B**1
          32

        2205*E**1*A**3*B**1
          32

        2205*E**1*A**1*B**3
          32

       -735*E**1*A**1*B**1
          16
```

```
D5 =      -1575*E**4*A**3*B**1
               4

           945*E**4*A**1*B**3
             4

           105*E**4*A**1*B**1
             2

          1260*E**3*A**3*B**1

          -546*E**3*A**1*B**3

          -238*E**3*A**1*B**1

          7875*E**2*A**3*B**1
            16

         -3045*E**2*A**1*B**3
            16

          -805*E**2*A**1*B**1
             8

           756*E**1*A**3*B**1

           210*E**1*A**1*B**3

          -322*E**1*A**1*B**1

          1575*A**3*B**1
            16

           945*A**1*B**3
            32

         -1365*A**1*B**1
            32


D6 =       945*E**6*A**3*B**1
             8

         -4095*E**6*A**1*B**3
            64

         -1155*E**6*A**1*B**1
            64
```

```
-735*E**5*A**3*B**1
   2

 147*E**5*A**1*B**3

 147*E**5*A**1*B**1
   2

-4725*E**4*A**3*B**1
   32

 5775*E**4*A**1*B**3
   64

 1225*E**4*A**1*B**1
   64

 -441*E**3*A**3*B**1

 -126*E**3*A**1*B**3

  189*E**3*A**1*B**1

-6825*E**2*A**3*B**1
   64

 -105*E**2*A**1*B**3
   4

 2835*E**2*A**1*B**1
   64

  -63*E**1*A**3*B**1
   2

  -21*E**1*A**1*B**3

   35*E**1*A**1*B**1
   2

D7 =  -1575*E**4*A**4
   16

 2835*E**4*A**2*B**2
   8

  105*E**4*A**2
   4

-2625*E**4*B**4
   64
```

```
-385*E**4*B**2
  32

 -65*E**4
  64

 315*E**3*A**4

-819*E**3*A**2*B**2

-119*E**3*A**2

  84*E**3*B**4

  35*E**3*B**2

   6*E**3

 7875*E**2*A**4
  64

-9135*E**2*A**2*B**2
  32

 -805*E**2*A**2
  16

 2625*E**2*B**4
  64

  35*E**2*B**2
   8

 105*E**2
  32

 189*E**1*A**4

 315*E**1*A**2*B**2

-161*E**1*A**2

 -84*E**1*B**4

  21*E**1*B**2

  10*E**1

 1575*A**4
  64
```

```
 2835*A**2*B**2
   64

-1365*A**2
   64

 -315*B**2
   64

 +  15
     8

D8 =    945*E**6*A**4
         32

-12285*E**6*A**2*B**2
   128

 -1155*E**6*A**2
   128

 2625*E**6*B**4
  256

  245*E**6*B**2
   64

   95*E**6
  256

 -735*E**5*A**4
    8

  441*E**5*A**2*B**2
    2

  147*E**5*A**2
    4

 -21*E**5*B**4

 -21*E**5*B**2
    2

 -15*E**5
    8

-4725*E**4*A**4
  128
```

```
17325*E**4*A**2*B**2
  128

 1225*E**4*A**2
  128

-2625*E**4*B**4
  128

 -175*E**4*B**2
  128

  -75*E**4
  128

 -441*E**3*A**4
    4

 -189*E**3*A**2*B**2

  189*E**3*A**2
    2

   42*E**3*B**4

   -7*E**3*B**2

  -25*E**3
    4

-6825*E**2*A**4
  256

 -315*E**2*A**2*B**2
    8

 2835*E**2*A**2
  128

 2625*E**2*B**4
  256

 -315*E**2*B**2
  128

  -45*E**2
   32

  -63*E**1*A**4
    8
```

```
      -63*E**1*A**2*B**2
        2

       35*E**1*A**2
        4

      -21*E**1*B**4

       35*E**1*B**2
        2

      -15*E**1
        8

D9 =  -945*E**5*A**3*B**1
        8

     4095*E**5*A**1*B**3
       64

     1155*E**5*A**1*B**1
       64

      735*E**4*A**3*B**1
        2

     -147*E**4*A**1*B**3

     -147*E**4*A**1*B**1
        2

     4725*E**3*A**3*B**1
       32

    -5775*E**3*A**1*B**3
       64

    -1225*E**3*A**1*B**1
       64

      441*E**2*A**3*B**1

      126*E**2*A**1*B**3

     -189*E**2*A**1*B**1

     6825*E**1*A**3*B**1
       64

      105*E**1*A**1*B**3
        4
```

```
-2835*E**1*A**1*B**1
   64

 63*A**3*B**1
  2

 21*A**1*B**3

-35*A**1*B**1
  2


D10 =  -2835*E**5*A**2*B**1
          8

 4095*E**5*B**3
   64

 1155*E**5*B**1
   64

 2205*E**4*A**2*B**1
    2

 -147*E**4*B**3

 -147*E**4*B**1
    2

14175*E**3*A**2*B**1
   32

-5775*E**3*B**3
   64

-1225*E**3*B**1
   64

 1323*E**2*A**2*B**1

 126*E**2*B**3

 -189*E**2*B**1

20475*E**1*A**2*B**1
   64

 105*E**1*B**3
   4
```

```
-2835*E**1*B**1
   64

 189*A**2*B**1
   2

 21*B**3

-35*B**1
   2


D11 =  -945*E**5*A**3
          8

      12285*E**5*A**1*B**2
          64

       1155*E**5*A**1
          64

        735*E**4*A**3
          2

       -441*E**4*A**1*B**2

       -147*E**4*A**1
          2

       4725*E**3*A**3
          32

      -17325*E**3*A**1*B**2
          64

      -1225*E**3*A**1
          64

        441*E**2*A**3

        378*E**2*A**1*B**2

       -189*E**2*A**1

       6825*E**1*A**3
          64

        315*E**1*A**1*B**2
          4
```

```
-2835*E**1*A**1
   64

 63*A**3
  2

 63*A**1*B**2

-35*A**1
  2


D12 =   -945*E**5*A**4
          32

       12285*E**5*A**2*B**2
         128

        1155*E**5*A**2
         128

       -2625*E**5*B**4
         256

        -245*E**5*B**2
          64

         -95*E**5
         256

         735*E**4*A**4
           8

        -441*E**4*A**2*B**2
           2

        -147*E**4*A**2
           4

          21*E**4*B**4

          21*E**4*B**2
           2

          15*E**4
           8

        4725*E**3*A**4
         128
```

```
-17325*E**3*A**2*B**2
  128

 -1225*E**3*A**2
  128

 2625*E**3*B**4
  128

  175*E**3*B**2
  128

   75*E**3
  128

  441*E**2*A**4
    4

  189*E**2*A**2*B**2

 -189*E**2*A**2
    2

  -42*E**2*B**4

    7*E**2*B**2

   25*E**2
    4

 6825*E**1*A**4
  256

  315*E**1*A**2*B**2
    8

-2835*E**1*A**2
  128

-2625*E**1*B**4
  256

  315*E**1*B**2
  128

   45*E**1
   32

   63*A**4
    8

   63*A**2*B**2
    2
```

```
          -35*A**2
            4

           21*B**4

          -35*B**2
            2

           15
            8

D13 =  12285*E**5*A**2*B**1
          64

        -2625*E**5*B**3
          64

         -245*E**5*B**1
          32

         -441*E**4*A**2*B**1

           84*E**4*B**3

           21*E**4*B**1

       -17325*E**3*A**2*B**1
          64

         2625*E**3*B**3
          32

          175*E**3*B**1
          64

          378*E**2*A**2*B**1

         -168*E**2*B**3

           14*E**2*B**1

          315*E**1*A**2*B**1
            4

        -2625*E**1*B**3
          64

          315*E**1*B**1
          64

           63*A**2*B**1

           84*B**3

          -35*B**1
```